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BOUNDEDNESS FOR MAXIMAL MULTILINEAR BOCHNER-RIESZ OPERATORS ON CERTAIN HARDY SPACES

ZHOU XIAOSHA

Abstract. In this paper, the boundedness for the maximal multilinear Bochner-Riesz operators on certain Hardy and Herz-Hardy spaces are obtained.

1. Introduction

Let m be a positive integer and A be a function on R^n. The multilinear Bochner-Riesz operators is defined by

$$B^A_{*,\delta}(f)(x) = \sup_{r>0} |B^A_{r,\delta}(f)(x)|,$$

where

$$B^A_{r,\delta}(f)(x) = \int_{R^n} \frac{R_{m+1}(A; x, y)}{|x-y|^m} B^\delta_r(x-y) f(y) dy,$$

$$R_{m+1}(A; x, y) = A(x) - \sum_{|\alpha|\le m} \frac{1}{\alpha!} D^\alpha A(y)(x-y)^\alpha,$$

and $(B^\delta_r(f))\hat{}(\xi) = (1 - r^2|\xi|^2)^\delta_+ \hat{f}(\xi)$. Set $B^\delta_r(f)(x) = f * B^\delta_r(x)$, where $B^\delta_r(x) = r^{-n}B^\delta(x/r)$ with $r > 0$ and $B^\delta(x)$ is the kernel (see [10]). We also define

$$B^\delta_*(f)(x) = \sup_{r>0} |B^\delta_r(f)(x)|,$$

which is the Bochner-Riesz operator (see [10]).

Note that when $m = 0$, $B^A_{*,\delta}$ is just the commutator of Bochner-Riesz operator (see [7]). It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [1-5]). In [8], Liu proved the boundedness of multilinear Littlewood-Paley operators on Hardy and Herz-Hardy spaces when $D^\alpha A$ are Lipschitz functions for $|\alpha| = m$. The main purpose of this paper is to establish the continuity of the multilinear Bochner-Riesz operator on certain Hardy and Herz-Hardy spaces when $D^\alpha A$ are BMO functions for $|\alpha| = m$.

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First, let us introduce some definitions (see [9-15]). Throughout this paper, $Q = Q(x_0, d)$ will denote the cube of R^n centered at x_0 with side-length d and sides parallel to the axes.

Definition 1.1. Let A be a function on R^n and m be a positive integer and $0 < p \leq 1$. A bounded measurable function a on R^n is said to be a $(p, D^m A)$ atom if
i) $\text{supp} a \subset Q = Q(x_0, d)$,
ii) $||a||_{L^\infty} \leq |Q|^{-1/p}$,
iii) $\int_{R^n} a(y)dy = \int_{R^n} a(y)D^\alpha A(y)dy = 0, |\alpha| = m$.

A temperate distribution f is said to belong to $H^p_{D^m A}(R^n)$, if, in the Schwartz distributional sense, it can be written as

$$f(x) = \sum_{j=0}^{\infty} \lambda_j a_j(x),$$

where the a_j are $(p, D^m A)$ atoms, $\lambda_j \in C$ and $\sum_{j=0}^{\infty} |\lambda_j|^p < \infty$. Moreover, $||f||_{H^p_{D^m A}} = \left(\sum_{j=0}^{\infty} |\lambda_j|^p\right)^{1/p}$.

Let $B_k = \{x \in R^n : |x| \leq 2^k\}, C_k = B_k \setminus B_{k-1}, k \in Z$. Denote $\chi_k = \chi_{C_k}$ for $k \in Z$ and $\chi_0 = \chi_{B_0}$, where χ_E is the characteristic function of the set E.

Definition 1.2. Let $0 < p$, $q < \infty$ and $\alpha \in R$.
(1) The homogeneous Herz space is defined by

$$\dot{K}^{\alpha,p}_q(R^n) = \{f \in L^q_{loc}(R^n \setminus \{0\}) : ||f||_{\dot{K}^{\alpha,p}_q} < \infty\},$$

where

$$||f||_{\dot{K}^{\alpha,p}_q} = \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} ||f\chi_k||^p_{L^q}\right]^{1/p}.$$

(2) The nonhomogeneous Herz space is defined by

$$K^{\alpha,p}_q(R^n) = \{f \in L^q_{loc}(R^n) : ||f||_{K^{\alpha,p}_q} < \infty\},$$

where

$$||f||_{K^{\alpha,p}_q} = \left[\sum_{k=0}^{\infty} 2^{k\alpha p} ||f\chi_k||^p_{L^q} + ||f\chi_0||^p_{L^q}\right]^{1/p}.$$

Definition 1.3. Let $\alpha \in R$, $0 < p < \infty$, $1 < q \leq \infty$, m be a positive integer and A be a function on R^n. A function $a(x)$ on R^n is called a central $(\alpha, q, D^m A)$-atom (or a central $(\alpha, q, D^m A)$-atom of restrict type), if
1) $\text{Supp} a \subset Q(0, d)$ for some $d > 0$ (or for some $d \geq 1$),
2) $||a||_{L^q} \leq |Q(0, d)|^{-\alpha/q}$,
3) $\int_{R^n} a(x)dx = \int_{R^n} a(x)D^\beta A(x)dx = 0$, $|\beta| = m$.

A temperate distribution f is said to belong to $H\dot{K}^{\alpha,p}_{q,D^m A}(R^n)$ (or $HK^{\alpha,p}_{q,D^m A}(R^n)$), if it can be written as $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$ (or $f = \sum_{j=0}^{\infty} \lambda_j a_j$) in the $S'(R^n)$ sense, where a_j is a central $(\alpha, q, D^m A)$-atom (or a central $(\alpha, q, D^m A)$-atom of restrict type) supported on $B(0,2^j)$ and $\sum_{j=-\infty}^{\infty}|\lambda_j|^p<\infty$ (or $\sum_{j=0}^{\infty}|\lambda_j|^p<\infty$), moreover, $||f||_{H\dot{K}^{\alpha,p}_{q,D^mA}}$ (or $||f||_{HK^{\alpha,p}_{q,D^mA}}$)$=\left(\sum_j|\lambda_j|^p\right)^{1/p}$.

2. Theorems and proofs

We begin with some preliminary lemmas.

Lemma 2.1 (see [3]). *Let A be a function on R^n and $D^\alpha A\in L^q(R^n)$ for $|\alpha|=m$ and some $q>n$. Then*

$$|R_m(A;x,y)|\le C|x-y|^m\sum_{|\alpha|=m}\left(\frac{1}{|\tilde{Q}(x,y)|}\int_{\tilde{Q}(x,y)}|D^\alpha A(z)|^q dz\right)^{1/q},$$

where $\tilde{Q}(x,y)$ is the cube centered at x and having side length $5\sqrt{n}|x-y|$.

Lemma 2.2. *Let $1<q<\infty$, $\delta>(n-1)/2$ and $D^\alpha A\in BMO(R^n)$ for $|\alpha|=m$. Then $B^A_{*,\delta}$ is bounded on $L^q(R^n)$, that is*

$$||B^A_{*,\delta}(f)||_{L^q}\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}||f||_{L^q}.$$

Proof. By the condition of B^δ_r (see [10]), we have

$$\begin{aligned}|B^\delta_r(x-y)|\le&Cr^{-n}(1+|x-y|/r)^{-(\delta+(n+1)/2)}\\=&C\left(\frac{r}{r+|x-y|}\right)^{\delta-(n-1)/2}\frac{1}{(r+|x-y|)^n}\\\le&C|x-y|^{-n},\end{aligned}$$

so that

$$B^A_{*,\delta}(f)(x)\le C\int_{R^n}\frac{|R_{m+1}(A;x,y)|}{|x-y|^{m+n}}|f(y)|dy,$$

thus, by [4, 5], we get

$$||B^A_{*,\delta}(f)||_{L^p}\le C||f||_{L^p}.$$

□

Theorem 2.3. *Let $\delta>(n-1)/2$, $1\ge p>2n/(2\delta+n+1)$ and $D^\beta A\in BMO(R^n)$ for $|\beta|=m$. Then $B^A_{*,\delta}$ is bounded from $H^p_{D^mA}(R^n)$ to $L^p(R^n)$.*

Proof. It suffices to show that there exists a constant $C>0$ such that for every (p,D^mA)-atom a,

$$||B^A_{*,\delta}(a)||_{L^p}\le C.$$

Let a be a (p,D^mA)-atom supported on a ball $Q=Q(x_0,d)$. We write

$$\begin{aligned}\int_{R^n}[B^A_{*,\delta}(a)(x)]^p dx=&\int_{|x-x_0|\le 2d}[B^A_{*,\delta}(a)(x)]^p dx+\int_{|x-x_0|>2d}[B^A_{*,\delta}(a)(x)]^p dx\\=&I+II.\end{aligned}$$

For I, taking $q>1$, by Hölder's inequality and the L^q-boundedness of $B^A_{*,\delta}$ (Lemma 2.2), we see that

$$I\le C||B^A_{*,\delta}(a)||^p_{L^q}\cdot|Q(x_0,2d)|^{1-p/q}\le C||a||^p_{L^q}|Q|^{1-p/q}\le C.$$

To obtain the estimate of II, we need to estimate $B^A_{*,\delta}(a)(x)$ for $x\in(2Q)^c$. Let $\tilde{Q}=5\sqrt{n}Q$ and $\tilde{A}(x)=A(x)-\sum\limits_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A)_{\tilde{Q}}\cdot x^\alpha$, where $(A)_Q$ denotes the mean values of A on Q. Then $R_{m+1}(A;x,y)=R_{m+1}(\tilde{A};x,y)$. We have, by the vanishing moment of a,

$$\begin{aligned}B^A_{r,\delta}(a)(x)\le&\int_Q\left|\frac{|B_{r,\delta}(x-y)|}{|x-y|^m}-\frac{|B_{r,\delta}(x-x_0)|}{|x-x_0|^m}\right||R_m(\tilde{A};x,y)||a(y)|dy\\&+\int_Q\frac{|B_{r,\delta}(x-x_0)|}{|x-x_0|^m}|R_m(\tilde{A};x,y)-R_m(\tilde{A};x,x_0)||a(y)|dy\\&+\left|\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_Q\frac{B_{r,\delta}(x-y)(x-y)^\alpha}{|x-y|^m}D^\alpha\tilde{A}(y)a(y)dy\right|\\=&II_1+II_2+II_3.\end{aligned}$$

Note that $|x-y|\sim|x-x_0|$ for $y\in Q$ and $x\in 2^{k+1}Q\setminus 2^kQ$. By Lemma 2.1 and the following inequality (see [15])

$$|b_{Q_1}-b_{Q_2}|\le C\log(|Q_2|/|Q_1|)||b||_{BMO}\text{ for }Q_1\subset Q_2,$$

we know that, for $y\in Q$ and $x\in 2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}$,

$$\begin{aligned}|R_m(\tilde{A};x,y)|\le&C|x-y|^m\sum_{|\alpha|=m}(||D^\alpha A||_{BMO}+|(D^\alpha A)_{\tilde{Q}(x,y)}-(D^\alpha A)_{\tilde{Q}}|)\\\le&Ck|x-y|^m\sum_{|\alpha|=m}||D^\alpha A||_{BMO}.\end{aligned}$$

By the formula (see [3]):

$$R_m(\tilde{A};x,y)-R_m(\tilde{A};x_0,y)=\sum_{|\beta|<m}\frac{1}{\beta!}R_{m-|\beta|}(D^\beta\tilde{A};x,x_0)(x-y)^\beta$$

and Lemma 2.1, we have

$$\begin{aligned}|R_m(\tilde{A};x,y)-R_m(\tilde{A};x_0,y)|\le&C\sum_{|\beta|<m}\sum_{|\alpha|=m}|x-x_0|^{m-|\beta|}|x-y|^{|\beta|}||D^\alpha A||_{BMO}\\\le&C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}|x-x_0||x-y|^{m-1}.\end{aligned}$$

We consider the following two cases:
Case 1: $0<r\le d$. Notice that [10]:

$$|B^\delta(x)|\le C(1+|x|)^{-(\delta+(n+1)/2)},$$

then

$$
\begin{aligned}
II_1 &\le Cr^{-n}|Q|^{-1/p}\int_Q \frac{|R_m(\tilde{A};x,y)|}{|x-x_0|^m}(1+|x-y|/r)^{-(\delta+(n+1)/2)}dy \\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}r^{-n}k|Q|^{-1/p}\int_Q |(1+|x-y|/r)^{-(\delta+(n+1)/2)}dy \\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}|Q|^{1-1/p}k|x-x_0|^{-(\delta+(n+1)/2)}r^{(\delta-(n-1)/2)}; \\
II_2 &\le Cr^{-n}\int_Q \frac{|R_m(\tilde{A};x,y)-R_m(\tilde{A};x_0,y)|}{|x_0-y|^m}(1+|x-y|/r)^{-(\delta+(n+1)/2)}|a(y)|dy \\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}r^{-n}\int_Q \frac{|x-x_0||a(y)|}{|x_0-y|}(1+|x-y|/r)^{-(\delta+(n+1)/2)}dy \\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\int_Q \frac{|x_0-y|}{|x-x_0|^{n+1}}|a(y)|dy \\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}|x-x_0|^{-n-1}|Q|^{1/n-1/p+1}.
\end{aligned}
$$

For II_3, we write

$$
\begin{aligned}
&\int_Q \frac{B_{r,\delta}(x-y)(x-y)^\alpha}{|x-y|^m}(D^\alpha A(y)-(D^\alpha A)_Q)a(y)dy \\
&= \int_Q \left[\frac{B_{r,\delta}(x-y)(x-y)^\alpha}{|x-y|^m}-\frac{B_{r,\delta}(x-x_0)(x-x_0)^\alpha}{|x-x_0|^m}\right] \\
&\qquad \times[D^\alpha A(y)-(D^\alpha A)_Q]a(y)dy,
\end{aligned}
$$

similar to the estimate of II_1, we obtain

$$
\begin{aligned}
II_3 \le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\Big(&|x-x_0|^{-(\delta+(n+1)/2)}|Q|^{(\delta-(n-1)/2)/n+1-/p} \\
&+|x-x_0|^{-n-1}|Q|^{1/n-1/p+1}\Big).
\end{aligned}
$$

Therefore, recall that $p>n/(n+1)$ and $\delta>n/p-(n+1)/2$,

$$
II \le \sum_{k=1}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}[B^A_{*,\delta}(a)(x)]^p dx
$$

$$\le C\sum_{k=1}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}(\sum_{|\alpha|=m}|D^\alpha A(x)-(D^\alpha A)_{2^{k+1}Q}|)^p$$
$$\times\Big[k^p|x-x_0|^{-p(\delta+(n+1)/2)}|Q|^{p(\delta-(n-1)/2)/n+p-1}$$
$$+|x-x_0|^{-p(n+1)}|Q|^{p(1+1/n-1/p)}\Big]\,dx$$
$$+C(\sum_{|\alpha|=m}||D^\alpha A||_{BMO})^p\sum_{k=1}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}(|x-x_0|^{-p(n+1)}|Q|^{p(1+1/n-1/p)}$$
$$+k^p|x-x_0|^{-p(\delta+(n+1)/2)}|Q|^{p(\delta-(n-1)/2)/n+p-1})dx$$
$$\le C\left(\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\right)^p\sum_{k=1}^{\infty}[k^p2^{k(n-p-pn)}+2^{k(n-p(\delta+(n+1)/2))}]$$
$$\le C\left(\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\right)^p.$$

Case 2: $r>d$. In this case, we choose δ_0 such that $(n-1)/2<\delta_0<\min(\delta,(n+1)/2)$, notice that (see [10])

$$|\nabla^l B^\delta(z)|\le C(1+|z|)^{-(\delta+(n+1)/2)}$$

for any $l=(l_1,\cdots,l_n)\in(N\cup\{0\})^n$, where $\nabla^l=(\partial/\partial x_1)^{l_1}\cdots(\partial/\partial x_n)^{l_n}$. Similar to the proof of Case 1, we obtain

$$II\le C\left(\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\right)^p\sum_{k=1}^{\infty}\sup_{r>0}(r/d)^{((n+1)/2-\delta_0)p}$$
$$\times\sum_{k=1}^{\infty}k^p[2^{-k((n+1)/2+\delta_0)p-n}+2^{k(n-p-pn)}]$$
$$\le C\left(\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\right)^p\sum_{k=1}^{\infty}k^p[2^{-k((n+1)/2+\delta_0)p-n}+2^{k(n-p-pn)}]$$
$$\le C\left(\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\right)^p,$$

which together with the estimate for I yields the desired result. □

Theorem 2.4. *Let $0<p<\infty$, $1<q<\infty$, $n(1-1/q)\le\alpha<n(1-1/q)+1$, $\delta>\alpha+n/q-(n+1)/2$ and $D^\beta A\in BMO(R^n)$ for $|\beta|=m$. Then $B^A_{*,\delta}$ is bounded from $H\dot{K}^{\alpha,p}_{q,D^mA}(R^n)$ to $\dot{K}^{\alpha,p}_q(R^n)$.*

Proof. Let $f \in H\dot{K}^{\alpha,p}_{q,D^m A}(R^n)$ and $f(x) = \sum_{j=-\infty}^{\infty} \lambda_j a_j(x)$ be the atomic decomposition for f as in Definition 1.3. We write

$$||B^A_{*,\delta}(f)||_{\dot{K}^{\alpha,p}_q} \le C \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k-3} |\lambda_j| ||B^A_{*,\delta}(a_j)\chi_k||_{L^q} \right)^p \right]^{1/p}$$

$$+ C \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=k-2}^{\infty} |\lambda_j| ||B^A_{*,\delta}(a_j)\chi_k||_{L^q} \right)^p \right]^{1/p} = I + II.$$

For II, by the boundedness of $B^A_{*,\delta}$ on $L^q(R^n)$(see Lemma 2.2), we have

$$\begin{aligned}
II = \ & C \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} (\sum_{j=k-2}^{\infty} |\lambda_j| ||B^{\vec{b}}_{\delta,*}(a_j)\chi_k||_{L^q})^p \right]^{1/p} \\
\le \ & C \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} (\sum_{j=k-2}^{\infty} |\lambda_j| ||a_j||_{L^q})^p \right]^{1/p} \\
\le \ & C \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} (\sum_{j=k-2}^{\infty} |\lambda_j| \cdot 2^{-j\alpha})^p \right]^{1/p} \\
\le \ & C \begin{cases} \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \sum_{j=k-2}^{\infty} |\lambda_j|^p 2^{-j\alpha p} \right]^{1/p}, & 0 < p \le 1 \\ \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} (\sum_{j=k-2}^{\infty} |\lambda_j|^p \cdot 2^{-j\alpha p/2}) \right. & \\ \left. \times (\sum_{j=k-2}^{\infty} 2^{-j\alpha p'/2})^{p/p'} \right]^{1/p}, & 1 < p < \infty \end{cases} \\
\le \ & C \begin{cases} \left[\sum_{j=-\infty}^{\infty} |\lambda_j|^p (\sum_{k=-\infty}^{j+2} 2^{(k-j)\alpha p}) \right]^{1/p}, & 0 < p \le 1 \\ \left[\sum_{j=-\infty}^{\infty} |\lambda_j|^p (\sum_{k=-\infty}^{j+2} 2^{(k-j)\alpha p/2}) \right]^{1/p}, & 1 < p < \infty \end{cases} \\
\le \ & C(\sum_{j=-\infty}^{\infty} |\lambda_j|^p)^{1/p} \le C||f||_{H\dot{K}^{\alpha,p}_{q,b_1}(R^n)}.
\end{aligned}$$

For I, similar to the proof of Theorem 2.3, we have, for $x \in C_k$, $j \le k-3$,

$$\begin{aligned}
B^A_{*,\delta}(a_j)(x) \le \ & C \left[|x|^{-n-m-1} |Q_j|^{1/n} + |x|^{-(m+\delta_0+(n+1)/2)} |Q_j|^{(\delta_0-(n-1)/2)/n} \right] \\
& \times \left(\int_{Q_j} |a_j(y)||R_m(\tilde{A}; x, y)| dy \right) \\
& + C \left[|x|^{-n-1} |Q_j|^{1/n} + |x|^{-(\delta_0+(n+1)/2)} |Q_j|^{(\delta_0-(n-1)/2)/n} \right] \\
& \times \sum_{|\alpha|=m} \int_{Q_j} |D^\alpha A(y) - (D^\alpha A)_{Q_j}||a_j(y)| dy,
\end{aligned}$$

thus

$$I \le C\Bigg[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \sum_{j=-\infty}^{k-3} |\lambda_j| \sum_{|\beta|=m} \left(\int_{B_k} |D^\beta A(x) - (D^\beta A)_{B_k}|^q dx\right)^{1/q}$$
$$\times \left(2^{-k(n+1)+j(1+n(1-1/q)-\alpha)} + 2^{-k(\delta_0+(n+1)/2)+j(\delta_0-(n-1)/2+n(1-1/q)-\alpha)}\right)^p\Bigg]^{1/p}$$
$$+ C\Bigg[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} 2^{kn/q} (\sum_{|\alpha|=m} ||D^\alpha A||_{BMO})^p$$
$$\times (\sum_{j=-\infty}^{k-3} |\lambda_j|(k-j)(2^{-k(n+1)+j(1+n(1-1/q)-\alpha)}$$
$$+ 2^{-k(\delta_0+(n+1)/2)+j(\delta_0-(n-1)/2+n(1-1/q)-\alpha)}))^p\Bigg]^{1/p}$$
$$= I_1 + I_2.$$

To estimate I_1 and I_2, we consider two cases.

Case 1: $0 < p \le 1$.

$$I_1 \le C\Bigg[(\sum_{|\beta|=m} ||D^\beta A||_{BMO})^p \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \sum_{j=-\infty}^{k-3} |\lambda_j|^p$$
$$\Big[2^{(-k(n+1)+j(1+n(1-\frac{1}{q})-\alpha))p}$$
$$+2^{(-k(\delta_0+(n+1)/2)+j(\delta_0-\frac{n-1}{2}+n(1-\frac{1}{q})-\alpha))p}\Big] 2^{\frac{knp}{q}}\Bigg]^{\frac{1}{p}}$$
$$= C \sum_{|\beta|=m} ||D^\beta A||_{BMO}$$
$$\times \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p \sum_{k=j+3}^{\infty} \left(2^{(j-k)(1+n(1-\frac{1}{q})-\alpha)p} + 2^{(j-k)(\delta_0+(n+1)-\frac{n}{q}-\alpha)p}\right)\right)^{\frac{1}{p}}$$
$$\le C \sum_{|\beta|=m} ||D^\beta A||_{BMO} \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p\right)^{\frac{1}{p}} \le C||f||_{H\dot{K}^{\alpha,p}_{q,D^m A}},$$

similarly,

$$I_2 \le C||f||_{H\dot{K}^{\alpha,p}_{q,D^m A}}.$$

Case 2: $p > 1$. By Hölder's inequality, we deduce that

$$I_1 \leq C \sum_{|\beta|=m} ||D^\beta A||_{BMO}$$

$$\sum_{j=-\infty}^{\infty} \left[\left(\sum_{j=-\infty}^{k-3} |\lambda_j|^p 2^{\frac{(j-k)p(1+n(1-\frac{1}{q})-\alpha)}{2}} \right)^{\frac{1}{p}} \left(\sum_{j=-\infty}^{k-3} 2^{\frac{(j-k)p'(1+n(1-\frac{1}{q})-\alpha)}{2}} \right)^{\frac{1}{p'}} \right.$$

$$\left. + \left(\sum_{j=-\infty}^{k-3} |\lambda_j|^p 2^{\frac{(j-k)p(\delta_0+\frac{(n+1)}{2}-\frac{n}{q}-\alpha)}{2}} \right)^{1/p} \left(\sum_{j=-\infty}^{k-3} 2^{\frac{(j-k)p'(\delta_0+\frac{(n+1)}{2}-\frac{n}{q}-\alpha)}{2}} \right)^{\frac{1}{p'}} \right]$$

$$\leq C \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p \right)^{1/p} \leq C||f||_{H\dot{K}^{\alpha,p}_{q,D^m A}},$$

similarly,

$$I_2 \leq C||f||_{H\dot{K}^{\alpha,p}_{q,D^m A}(R^n)}.$$

This finishes the proof of Theorem 2.4. □

Remark 2.5. Theorem 2.4 also holds for nonhomogeneous Herz-type spaces.

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Department of Mathematics,
Changsha University of Science and Technology,
Changsha 410077, P. R. of China.
E-mail address: zhouxiaosha57@126.com