Proportional Local Assignability of Dichotomy Spectrum of One-sided Continuous Time-Varying Linear Systems

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Abstract

We consider a local version of the assignment problem for the dichotomy spectrum of linear continuous time-varying systems defined on the half-line. Our aim is to show that uniform complete controllability is a sufficient condition to place the dichotomy spectrum of the closed-loop system in an arbitrary position within some Hausdorff neighborhood of the dichotomy spectrum of the free system using an appropriate time-varying linear feedback. Moreover, we assume that the norm of the matrix of the linear feedback should be bounded from above by the Hausdorff distance between these two spectra with some constant multiplier.

Keywords: `Linear time-varying control systems`, `Linear state feedback control`, `Assignment spectrum`, `Dichotomy spectrum`

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1. Introduction

The concept of the dichotomy of linear differential equations with variable coefficients has a long history, beginning with the work of O. Perron [21], then

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formalized, developed and summarized in [17], [18] and [8]. The effectiveness of the notion of exponential dichotomy and the corresponding spectrum both in the study of the asymptotics of solutions of nonlinear systems, the first approximation of which is exponentially dichotomous, and in its applications to dynamical systems analysis, has caused that it entered the theory of dynamical systems including control theory as a classical tool.

On the other hand, in control theory, one of the basic methods of designing controls for systems described by linear equations with constant coefficients is the pole placement method, also known as pole-shifting or spectrum assignment method [28]. This method selects the feedback so that the poles of the closed loop system have a predetermined position. The theoretical basis for this method is that the controllability of a linear time-invariant system is equivalent to the fact that for each set of complex numbers with cardinality equal to the dimension of the state vector and symmetric relative to the real axis, there is a stationary feedback such that the poles of the closed-loop system form this set [10].

There have long been attempts in the literature to generalize this methodology to systems with variable coefficients and it has not been completed yet (see [2], [14], [16], [19], [3] and [23]). Even the formulation of the problem for time-varying systems encountered many difficulties. Firstly, because for time-varying systems we have many non-equivalent concepts of controllability. Secondly, because we have no proper replacement for the concept of poles, but their role, to a certain extent, is played by some numerical characteristics as the Lyapunov and the Bohl exponents or the dichotomy spectrum. This work fits into this topic and examines the problem of the so-called local proportional assignability of the dichotomy spectrum.

The benefits of dichotomy spectrum placement come directly from the importance of the dichotomy spectrum in the qualitative theory of nonautonomous dynamical systems generated by time-varying differential equations. To mention only a few results of this theory, note firstly the linearized asymptotic stability theorem of nonlinear systems which holds if the dichotomy spectrum of the linearized equation is negative, see [4]. Secondly, the nonautonomous Hartman-Grobman theorem requires the fact that the spectrum of the linear part does not contain zero, see [20]. Finally, in [27] a version of nonautonomous normal form theory was established in which all non-resonant terms of the Taylor expansion of the vector field (defined in terms of the location of the dichotomy spectrum of the linear part) can be eliminated.

Here, we consider a local version of the dichotomy spectrum assignment problem for linear continuous time-varying systems, whereas in [3] a global version was investigated. Our aim is to obtain sufficient conditions to place the dichotomy spectrum of the closed-loop system in an arbitrary position within some neighborhood of the dichotomy spectrum of the free system, i.e. the free system, using some time-varying linear feedback. Moreover, we require that the norm of the feedback should be bounded from above by the Hausdorff distance between these two spectra, with some constant coefficient. We say that the dichotomy spectrum is proportionally locally assignable if all these requirements are satisfied. Our main result is to show that uniform complete controllability is a sufficient condition for proportional local assignability of the dichotomy spectrum.

The paper is organized as follows. In the rest of this section, we introduce the notation used in the work. In the next section we introduce the definitions of exponential dichotomy, dichotomy and repeated dichotomy spectrum. We also formulate and prove a reducibility theorem which is important for our further considerations. The third section contains a formal definition of the problem of proportional local assignability of the dichotomy spectrum and the formulation and a proof of the main result of this paper Theorem 11.

The following notations will be used throughout this paper: Let $\mathcal{K}$ denote the set of all compact subsets of $\mathbb{R}$. For $U, V \in \mathcal{K}$, the Hausdorff distance d_H is defined as

$$d_H(U,V) := \max\left\{ \max_{x\in U}\min_{y\in V}|x-y|, \max_{y\in V}\min_{x\in U}|x-y| \right\}.$$

For matrices $M_1 \in \mathbb{R}^{d_1\times d_1}, \ldots, M_k \in \mathbb{R}^{d_k\times d_k}$, let $\operatorname{diag}(M_1,\ldots,M_k)$ denote the square matrix of dimension $d_1 + \cdots + d_k$ of the form

$$\operatorname{diag}(M_1, \ldots, M_k) = \begin{pmatrix} M_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & M_k \end{pmatrix}.$$

Let $\mathbb{R}^d$ be endowed with the standard Euclidean norm. For $s, d \in \mathbb{N}$, let $\mathcal{KC}_{s,d}(\mathbb{R}^+)$ be the set of all bounded and piecewise continuous matrix-valued functions $M : \mathbb{R}^+ \to \mathbb{R}^{s\times d}$ such that

$$\|M\|_\infty := \sup_{t\in\mathbb{R}^+} \|M(t)\| < \infty$$

and $\mathcal{C}_{s,d}(\mathbb{R}^+)$ the set of all bounded continuous matrix-valued functions $M : \mathbb{R}^+ \to \mathbb{R}^{s\times d}$.

2. Repeated dichotomy spectra and reducibility for linear one-sided continuous time-varying systems

In this section, we consider a one-sided continuous time-varying linear system

$$\dot{x} = M(t)x \qquad \text{for } t \in \mathbb{R}^+, \tag{1}$$

where $M \in \mathcal{KC}_{d,d}(\mathbb{R}^+)$. Denote by $X_M(\cdot,\cdot) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^{d\times d}$ the *transition matrix* of (1), i.e. $X_M(\cdot, s)\xi$ solves (1) with the initial value condition $x(s) = \xi$. We now recall the notion of exponential dichotomy which is also known as uniform hyperbolicity for time-varying systems, see e.g. [7] and the notion of dichotomy spectrum, see e.g. [24].

Definition 1 (Exponential dichotomy and dichotomy spectrum)**.** *System* (1) *is said to admit an* exponential dichotomy *(ED) on $\mathbb{R}^+$ if there exist $K, \varepsilon > 0$ and an invariant family of projections $P : \mathbb{R}^+ \to \mathbb{R}^{d\times d}$, i.e. $P(t)X_M(t, s) = X_M(t, s)P(s)$ if $s, t \in \mathbb{R}^+$, satisfying the following inequalities*

$$\|X_M(t, s)P(s)\| \le Ke^{-\varepsilon(t-s)} \ \ \text{if } s \le t, s, t \in \mathbb{R}^+, \tag{2}$$

and

$$\|X_M(t,s)(I-P(s))\| \le Ke^{\varepsilon(t-s)} \quad \text{if } t \le s, s,t \in \mathbb{R}^+. \tag{3}$$

The dichotomy spectrum of (1) *is defined by*

$$\Sigma_{\mathrm{ED}}(M) := \left\{\gamma \in \mathbb{R} : \dot{x} = (M(t)-\gamma I)x \text{ has no ED on } \mathbb{R}^+\right\}.$$

It is known that $\Sigma_{\mathrm{ED}}(M)$ is the union of at most d disjoint compact intervals (called spectral intervals), see [15, Theorem 5.12]. We now state and prove a result on how to decouple, via Lyapunov transformations, the system (1) into a block diagonal system with blocks corresponding to these spectral intervals. This type of result was established in [26] for two-sided continuous time-varying systems. Before doing this, we recall the notions of Lyapunov transformations, asymptotic equivalence (also known in the literature as kinematic similarity, Lyapunov similarity or simply equivalence) and reducibility. We refer the readers to [1] and the references therein for details.

Definition 2 (Lyapunov transformations, asymptotic equivalence and reducibility)**.** Lyapunov transformations*: The linear transformation $y = T(t)x$, where $T : \mathbb{R}^+ \to \mathbb{R}^{d\times d}$, is called a Lyapunov transformation if T is piecewise continuously differentiable and T, T^{-1}, $\dot{T}$ are bounded.*
Asymptotic equivalence*: System* (1) *is said to be asymptotically equivalent to*

$$\dot{y} = N(t)y, \quad \text{where } N \in \mathcal{KC}_{d,d}(\mathbb{R}^+) \tag{4}$$

if there exists a Lyapunov transformation $y = T(t)x$, where $T : \mathbb{R}^+ \to \mathbb{R}^{d\times d}$, such that

$$\dot{T}(t) = N(t)T(t) - T(t)M(t) \qquad \text{for } t \in \mathbb{R}^+.$$

Reducibility*: System* (1) *is reducible if there exist $M_1 \in \mathcal{KC}_{d_1,d_1}(\mathbb{R}^+), M_2 \in \mathcal{KC}_{d_2,d_2}(\mathbb{R}^+)$ such that* (1) *is asymptotically equivalent to*

$$\dot{y} = \operatorname{diag}(M_1(t), M_2(t))y. \tag{5}$$

Remark 3. *Assume that* (1) *is asymptotically equivalent to* (4) *via the Lyapunov transformation $y = T(t)x$. Denote by X_M and X_N the transition matrices* *of* (1) *and* (4)*, respectively. Then, it is well known, for e.g. see [26, Lemma 2.1], that*

$$X_N(t,s)T(s) = T(t)X_M(t,s) \quad \text{for } t,s \in \mathbb{R}^+. \tag{6}$$

Theorem 4 (Spectral theory and reducibility)**.** *Suppose that $M \in \mathcal{KC}_{d,d}(\mathbb{R}^+)$. The dichotomy spectrum $\Sigma_{\mathrm{ED}}(M)$ of* (1) *is nonempty and consists of at most d disjoint closed intervals. Let*

$$\Sigma_{\mathrm{ED}}(M) = \bigcup_{i=1}^{k} [\alpha_i, \beta_i],$$

where $-\infty < \alpha_1 \le \beta_1 < \alpha_2 \le \beta_2 < \cdots < \alpha_k \le \beta_k < \infty$ and $k \le d$. Then, the following statements hold:

(i) Let $i \in \{0, 1, \ldots, k\}$ be arbitrary. Then, for any $\gamma \in (\beta_i, \alpha_{i+1})$, with the convention that $\beta_0 = -\infty, \alpha_{k+1} = \infty$, the subspace

$$\mathcal{W}_i^{\gamma}(s) := \left\{ \xi \in \mathbb{R}^d : \limsup_{t\to\infty} e^{-\gamma t} \|X_M(t,s)\xi\| < \infty \right\} \tag{7}$$

is independent of the choice of γ (then we can write $\mathcal{W}_i^{\gamma}(s)$ simply as $\mathcal{W}_i(s)$), invariant, i.e. $X_M(t,s)\mathcal{W}_i(s) = \mathcal{W}_i(t)$ and the dimension of $\mathcal{W}_i(t)$ is independent of $t \in \mathbb{R}^+$.

(ii) Let n_i be the dimension of the subspace $\mathcal{W}_i(t)$ for $t \in \mathbb{R}^+$. Define $d_i := n_i - n_{i-1}$ for $i = 1, \ldots, k$ (with the convention that $n_0 := 0$). Then, there exist $M_i \in \mathcal{KC}_{d_i,d_i}(\mathbb{R}^+)$ for $i = 1, \ldots, k$ such that system (1) *is asymptotically equivalent to*

$$\dot{y} = \mathrm{diag}(M_1(t), \ldots, M_k(t))y$$

and

$$\Sigma_{\mathrm{ED}}(M_i) = [\alpha_i, \beta_i], \qquad i = 1, ..., k.$$

Before going to the proof of the preceding theorem, we need a result on decoupling a linear system when this system has an invariant bounded family of projections. This result is stated in the Introduction of [6, Lemma 2] but without a proof. It is also proved in [26, Theorem 3.1] but in this paper the author considers wider classes of systems – systems with locally integrable coefficients and wider classes of Lyapunov transformations which are assumed there to be absolutely continuous. Moreover, in the last paper the dichotomy is considered on the whole line.

Proposition 5. *Suppose that $M \in \mathcal{KC}_{d,d}(\mathbb{R}^+)$ and there exists an invariant bounded family of projections $P(t)$, $t \in \mathbb{R}^+$ for* (1). *Then* (1) *is asymptotically equivalent to a system*

$$\dot{x} = \begin{pmatrix} M_1(t) & 0 \\ 0 & M_2(t) \end{pmatrix} x \qquad \text{for } t \in \mathbb{R}^+, \tag{8}$$

where $d_1 = \dim \operatorname{im} P(t)$ is independent of $t \in \mathbb{R}^+$, $M_1 \in \mathcal{KC}_{d_1,d_1}(\mathbb{R}^+)$ and $M_2 \in \mathcal{KC}_{d-d_1,d-d_1}(\mathbb{R}^+)$. Moreover, the Lyapunov transformation establishing the equivalence of (1) *and (8) may be chosen such that*

$$T(t)P(t)T^{-1}(t) = \begin{pmatrix} I_{d_1} & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{for } t \in \mathbb{R}^+. \tag{9}$$

The main ingredient of the proof of the proposition is from [6, Lemma 2]. In fact, this result has been shown for systems (1) with $M \in \mathcal{C}_{d,d}(\mathbb{R}^+)$ but the arguments may be repeated for $M \in \mathcal{KC}_{d,d}(\mathbb{R}^+)$.

Lemma 6. *Suppose that $M \in \mathcal{KC}_{d,d}(\mathbb{R}^+)$ and there exists a projector matrix $P \in \mathbb{R}^{d\times d}$ such that $X_M(t,0)PX_M(0,t)$ is bounded on $\mathbb{R}^+$. Then* (1) *is asymptotically equivalent to a system $\dot{x} = B(t)x$ for $t \in \mathbb{R}^+$, where $B \in \mathcal{KC}_{d,d}(\mathbb{R}^+)$ satisfies that $PB(t) = B(t)P$ for $t \in \mathbb{R}^+$.*

Proof of Proposition 5. Since $P(t)$, $t \in \mathbb{R}^+$ is an invariant family of projections for (1), it follows that $P = X_M(0,t)P(t)X_M(t,0)$ is a projector matrix that does not depend on $t \in \mathbb{R}^+$. Furthermore, by boundedness of $P(t)$ we have that $P(t) = X_M(t,0)PX_M(0,t)$ is bounded. Then, by Lemma 6 there exists a Lyapunov transformation $T : \mathbb{R}^+ \to \mathbb{R}^{d\times d}$ establishing the asymptotic equivalence of (1) and the system

$$\dot{x} = B(t)x \qquad \text{for } t \in \mathbb{R}^+, \tag{10}$$

where $B \in \mathcal{KC}_{d,d}(\mathbb{R}^+)$ satisfies that

$$PB(t) = B(t)P \qquad \text{for } t \in \mathbb{R}^+. \tag{11}$$

Let $S \in \mathbb{R}^{d\times d}$ be an invertible matrix such that

$$SPS^{-1} = \begin{pmatrix} I_{d_1} & 0 \\ 0 & 0 \end{pmatrix} =: P_0,$$

where $d_1 = \dim \operatorname{im} P$. Then $\overline{T} : \mathbb{R}^+ \to \mathbb{R}^{d\times d}$, $\overline{T}(t) = ST(t)$ for $t \in \mathbb{R}^+$, is a Lyapunov transformation that establishes the asymptotic equivalence of (1) and

$$\dot{x} = C(t)x \qquad \text{for } t \in \mathbb{R}^+,$$

where

$$C(t) = ST(t)M(t)T^{-1}(t)S^{-1} + S\dot{T}(t)T^{-1}(t)S^{-1} = SB(t)S^{-1}.$$

Moreover, (11) implies that $P_0C(t) = C(t)P_0$ for all $t \in \mathbb{R}^+$. This equality implies that

$$C(t) = \begin{pmatrix} C_1(t) & 0 \\ 0 & C_2(t) \end{pmatrix},$$

where $C_1 \in \mathcal{KC}_{d_1,d_1}(\mathbb{R}^+)$ and $C_2 \in \mathcal{KC}_{d-d_1,d-d_1}(\mathbb{R}^+)$. The proof is complete. $\square$

Proof of Theorem 4. (i) The fact that the dichotomy spectrum $\Sigma_{\mathrm{ED}}(M)$ is the union of at most d closed intervals can be seen in [15, Theorem 5.12]. For each $\gamma \in (\beta_i, \alpha_{i+1})$, the subspace $\mathcal{W}_i(s)$ coincides with the range of the projection $P_\gamma(s)$, where P_γ is an invariant family of projections corresponding to the ED of the shifted system

$$\dot{x} = (M(t) - \gamma I)x,$$

see [15, Proposition 5.5]. Hence, invariance of P_γ implies that $\mathcal{W}_i(s)$ is also invariant and hence the dimension of $\mathcal{W}_i(s)$ is independent of s. Finally, $\operatorname{rank} P_\gamma$ is independent on the choice of $\gamma \in (\beta_i, \alpha_{i+1})$, thus the dimension of $\mathcal{W}_i(s)$ is also independent on the choice of $\gamma \in (\beta_i, \alpha_{i+1})$ and thus by definition of $\mathcal{W}_i(s)$ we conclude that $\mathcal{W}_i(s)$ is independent of the choice of $\gamma \in (\beta_i, \alpha_{i+1})$.

(ii) Let $\gamma \in (\beta_{k-1}, \alpha_k)$. Then, the shifted system $\dot{x}(t) = (M(t) - \gamma I)x(t)$ has an ED, i.e. there exist $K, \varepsilon > 0$ and an invariant family of projections $P_\gamma : \mathbb{R}^+ \to \mathbb{R}^{d\times d}$ satisfying the following inequalities

$$\|X_{M-\gamma I}(t,s)P_\gamma(s)\| \le Ke^{-\varepsilon(t-s)} \text{ if } s \le t, s, t \in \mathbb{R}^+, \tag{12}$$

and

$$\|X_{M-\gamma I}(t,s)(I - P_\gamma(s))\| \le Ke^{\varepsilon(t-s)} \text{ if } t \le s, s, t \in \mathbb{R}^+. \tag{13}$$

In particular, $P_\gamma(s)$ is bounded with respect to $s \in \mathbb{R}^+$ and therefore by Proposition 5 there exists a Lyapunov transformation $T : \mathbb{R}^+ \to \mathbb{R}^{d\times d}$ such that

$$\dot{T}(t) = N(t)T(t) - T(t)\left(M(t) - \gamma I\right) \qquad \text{for } t \in \mathbb{R}^+, \tag{14}$$

where

$$N(t) = \begin{pmatrix} N_1(t) & 0 \\ 0 & N_2(t) \end{pmatrix} \text{ for } t \in \mathbb{R}^+,$$

where $N_1(t) \in \mathbb{R}^{n_{k-1}\times n_{k-1}}$ and

$$n_{k-1} := \dim \mathcal{W}_{k-1}(t) = \dim \operatorname{im} P_\gamma(t).$$

From (14) we have

$$\dot{T}(t) = \begin{pmatrix} N_1(t) + \gamma I & 0 \\ 0 & N_2(t) + \gamma I \end{pmatrix} T(t) - T(t)M(t) \qquad \text{for } t \in \mathbb{R}^+,$$

which shows that system (1) is asymptotically equivalent to system

$$\dot{x} = \begin{pmatrix} M_1(t) & 0 \\ 0 & M_2(t) \end{pmatrix} x \qquad \text{for } t \in \mathbb{R}^+,$$

where $M_1(t) = N_1(t) + \gamma I \in \mathbb{R}^{n_{k-1}\times n_{k-1}}$ and $M_2(t) = N_2(t) + \gamma I \in \mathbb{R}^{d_k \times d_k}$. Then, from Remark 3 we derive that

$$\begin{pmatrix} X_{M_1}(t,s) & 0 \\ 0 & X_{M_2}(t,s) \end{pmatrix} = T(t)X_M(t,s)T(s)^{-1}.$$

Note that by Proposition 5 the Lyapunov transformation T satisfies (9) with n_{k-1} instead of d_1 and $P_\gamma(t)$ instead of $P(t)$. Therefore, we have

$$\begin{pmatrix} X_{M_1}(t,s) & 0 \\ 0 & X_{M_2}(t,s) \end{pmatrix} \begin{pmatrix} I_{d_1} & 0 \\ 0 & 0 \end{pmatrix} = T(t)X_M(t,s)T(s)^{-1}T(s)P_\gamma(s)T^{-1}(s)$$

and hence

$$\begin{aligned} \begin{pmatrix} X_{M_1}(t,s) & 0 \\ 0 & 0 \end{pmatrix} &= T(t)X_M(t,s)P_\gamma(s)T^{-1}(s) \\ &= e^{\gamma(t-s)}T(t)X_{M-\gamma I}(t,s)P_\gamma(s)T^{-1}(s). \end{aligned}$$

Thus, the inequalities (12) and (13) imply

$$\|X_{M_1}(t,s)\| \le K\,\|T\|\,\left\|T^{-1}\right\|e^{(\gamma-\varepsilon)(t-s)} \qquad \text{if } s \le t, s, t \in \mathbb{R}^+,$$

and

$$\|X_{M_2}(t,s)\| \le K\,\|T\|\,\left\|T^{-1}\right\|e^{(\gamma+\varepsilon)(t-s)} \qquad \text{if } t \le s, s, t \in \mathbb{R}^+.$$

Thus

$$\Sigma_{\mathrm{ED}}(M_1) \subset (-\infty, \gamma) \text{ and } \Sigma_{\mathrm{ED}}(M_2) \subset (\gamma, \infty)\,. \tag{15}$$

On the other hand, it is known that

$$\Sigma_{\mathrm{ED}}(M) = \Sigma_{\mathrm{ED}}(M_2) \cup \Sigma_{\mathrm{ED}}(M_1) = \bigcup_{i=1}^{k}[\alpha_i, \beta_i],$$

see [5]. This together with (15) implies that

$$\Sigma_{\mathrm{ED}}(M_2) = [\alpha_k, \beta_k] \quad \text{ and } \quad \Sigma_{\mathrm{ED}}(M_1) = \bigcup_{i=1}^{k-1}[\alpha_i, \beta_i].$$

Using this procedure and reapplying it to subsystems, we complete the proof by induction. □

When we also want to emphasize the information of dimension of subspaces corresponding to the dichotomy spectral intervals, we arrive at the following definition of the repeated dichotomy spectrum. We refer the readers to [9, 12] for a similar definition of repeated Lyapunov spectrum with the same meaning.

Definition 7. *The* repeated dichotomy spectrum $\Sigma_{\mathrm{ED}}^r(M)$ *of* (1) *is defined by*

$$\Sigma_{\mathrm{ED}}^r(M) = \Big(\underbrace{[\alpha_1, \beta_1], \dots, [\alpha_1, \beta_1]}_{d_1 \, times}, \dots, \underbrace{[\alpha_k, \beta_k], \dots, [\alpha_k, \beta_k]}_{d_k \, times} \Big), \tag{16}$$

where $d_1, \dots, d_k$ are the dimensions of the subsystems corresponding to the spectral intervals $[\alpha_1, \beta_1], \dots, [\alpha_k, \beta_k]$, respectively.

Remark 8. *From Definition 7, two spectral intervals of a repeated dichotomy spectrum are either disjoint or the same. Then, a collection of d closed intervals $[\alpha_1, \beta_1], \dots, [\alpha_d, \beta_d]$ is said to be* admissible for repeated dichotomy spectrum of a linear continuous time-varying system on $\mathbb{R}^d$ *(for short admissible closed intervals) if for $i \neq j$*

$$[\alpha_i, \beta_i] = [\alpha_j, \beta_j] \quad or \quad [\alpha_i, \beta_i] \cap [\alpha_j, \beta_j] = \emptyset.$$

3. Proportional local assignability of repeated dichotomy spectrum

3.1. Time-varying control systems and the statement of the main result

Consider a linear time-varying control system described by the following equation

$$\dot{x} = A(t)x + B(t)u \qquad \text{for } t \in \mathbb{R}^+, \tag{17}$$

where $A \in \mathcal{KC}_{d,d}(\mathbb{R}^+)$, $B \in \mathcal{KC}_{d,m}(\mathbb{R}^+)$ and $u \in \mathcal{KC}_{m,1}(\mathbb{R}^+)$ is the control. For $(t_0, x_0) \in \mathbb{R}^+ \times \mathbb{R}^d$ the solution of system (17) satisfying $x(t_0) = x_0$, will be denoted by $x(\cdot, t_0, x_0, u)$. Now we will introduce the definition of uniform complete controllability, see e.g. [23] and the references therein.

Definition 9 (Uniform complete controllability)**.** *System (17) is called uniformly completely controllable on $\mathbb{R}^+$ if there exist $\alpha, K > 0$ such that for all $(t_0, \xi) \in \mathbb{R}^+ \times \mathbb{R}^d$ there exists a control $u \in \mathcal{KC}_{m,1}(\mathbb{R}^+)$ such that $x(t_0 + K, t_0, 0, u) = \xi$ and*

$$\|u(t)\| \leq \alpha \|\xi\| \qquad for \ t \in [t_0, t_0 + K].$$

If in system (17) we apply a control of the form

$$u(t) = F(t)x(t),$$

where the feedback $F \in \mathcal{KC}_{m,d}(\mathbb{R}^+)$, we obtain a so-called closed loop system

$$\dot{x} = (A(t) + B(t)F(t))x. \tag{18}$$

Our interest in this paper is to know the possibility of proportional local assigning of $\Sigma^r_{\mathrm{ED}}(A+BF)$. We have the following definition of proportional local assignability of dichotomy spectrum (cf. [23, Definition 16.2] for the definition of the proportional local assignability of an arbitrary Lyapunov invariant of linear time-varying control systems).

Definition 10. *Denote the repeated dichotomy spectrum of the free system*

$$\dot{x} = A(t)x \tag{19}$$

by $\Sigma^r_{\mathrm{ED}}(A) = \Big([a_1, b_1], \dots, [a_d, b_d]\Big)$, *where* $[a_1, b_1], \dots, [a_d, b_d]$ *are admissible closed intervals. The repeated dichotomy spectrum of* (18) *is called proportionally locally assignable if there exist* $\delta, \ell > 0$ *such that for arbitrary admissible closed intervals* $[\widehat{a}_1, \widehat{b}_1], \dots, [\widehat{a}_d, \widehat{b}_d]$ *with* $\max_{1\le i\le d} d_H([\widehat{a}_i, \widehat{b}_i], [a_i, b_i]) \le \delta$ *there exists* $F \in \mathcal{KC}_{m,d}(\mathbb{R}^+)$ *satisfying that* $\|F\|_\infty \le \ell \max_{1\le i\le d} d_H([\widehat{a}_i, \widehat{b}_i], [a_i, b_i]))$ *and*

$$\Sigma^r_{\mathrm{ED}}(A+BF) = \Big([\widehat{a}_1, \widehat{b}_1], \dots, [\widehat{a}_d, \widehat{b}_d]\Big).$$

We now state the main result of this paper about the fact that uniform complete controllability implies proportional local assignability of repeated dichotomy spectrum.

Theorem 11 (Proportional local assignability of repeated dichotomy spectrum)**.** *Suppose that system* (17) *is uniformly completely controllable. Then, the repeated dichotomy spectrum of* (18) *is proportionally locally assignable.*

3.2. Proportional local assignability of repeated dichotomy spectrum by additive perturbation

Together with system (19), we will consider the additively perturbed system

$$\dot{y} = (A(t) + Q(t))y \qquad \text{for } t \in \mathbb{R}^+. \tag{20}$$

The perturbation $Q \in \mathcal{KC}_{d,d}(\mathbb{R}^+)$ will be called an additive perturbation of the system (19). The following theorem from [22, Theorem 2] will play an important role in our further consideration.

Theorem 12. *If system* (17) *is uniformly completely controllable, then there exist $\beta > 0$ and $\ell_1 > 0$ such that for an arbitrary matrix $Q \in \mathcal{KC}_{d,d}(\mathbb{R})$, $\|Q\|_\infty \leq \beta$, there exists a control $F \in \mathcal{KC}_{m,d}(\mathbb{R}^+)$, $\|F\|_\infty \leq \ell_1\|Q\|_\infty$ providing the asymptotic equivalence of the system (20) and system* (18)*.*

Definition 13 (Proportional local assignability of spectrum by additive perturbation)**.** *The repeated dichotomy spectrum of (1) is called* proportionally locally assignable by additive perturbation *if there exist positive numbers $\delta, \ell > 0$ such that for arbitrary admissible closed intervals $[\widehat{\alpha}_1, \widehat{\beta}_1], \ldots, [\widehat{\alpha}_d, \widehat{\beta}_d]$ with $\max_{1\leq i\leq d} d_H([\widehat{\alpha}_i, \widehat{\beta}_i], [\alpha_i, \beta_i]) \leq \delta$ there exists a function $Q \in \mathcal{KC}_{d,d}(\mathbb{R}^+)$ such that*

$$\|Q\|_\infty \leq \ell \max_{1\leq i\leq d} d_H([\widehat{\alpha}_i, \widehat{\beta}_i], [\alpha_i, \beta_i]), \quad \Sigma^r_{\mathrm{ED}}(M + Q) = \Big([\widehat{\alpha}_1, \widehat{\beta}_1], \ldots, [\widehat{\alpha}_d, \widehat{\beta}_d]\Big). \tag{21}$$

In the following proposition, we show the persistence of proportional local assignability of repeated dichotomy spectrum by additive perturbation via asymptotic equivalence.

Proposition 14. *Proportional local assignment of repeated dichotomy spectrum by additive perturbation persists via asymptotic equivalence.*

Proof. Consider a system

$$\dot{y} = N(t)y \qquad \text{for } t \in \mathbb{R}^+$$

which is asymptotically equivalent to (1) via the Lyapunov transformation $T = (T(t))_{t\in\mathbb{R}^+}$, i.e.

$$\dot{T}(t) = N(t)T(t) - T(t)M(t) \qquad \text{for } t \in \mathbb{R}^+.$$

Suppose that the repeated dichotomy spectrum of (1) is proportionally locally assignable by additive perturbation with respect to δ, ℓ as in Definition 13. Let $[\widehat{\alpha}_1, \widehat{\beta}_1], \ldots, [\widehat{\alpha}_d, \widehat{\beta}_d]$ be arbitrary admissible closed intervals satisfying $\max_{1\le i\le d} d_H([\widehat{\alpha}_i, \widehat{\beta}_i], [\alpha_i, \beta_i]) \le \delta$. Then, by Definition 13 there exists a function $Q \in \mathcal{KC}_{d,d}(\mathbb{R}^+)$ satisfying $\|Q\|_\infty \le \ell \max_{1\le i\le d} d_H([\widehat{\alpha}_i, \widehat{\beta}_i], [\alpha_i, \beta_i])$ and

$$\Sigma^r_{\mathrm{ED}}(M + Q) = \Big([\widehat{\alpha}_1, \widehat{\beta}_1], \ldots, [\widehat{\alpha}_d, \widehat{\beta}_d]\Big). \tag{22}$$

Let

$$\widehat{Q}(t) = T(t)Q(t)T^{-1}(t) \qquad \text{for } t \in \mathbb{R}^+. \tag{23}$$

Then, we have the following claims

$$\|\widehat{Q}\|_\infty \le \ell \|T\|_\infty \|T^{-1}\|_\infty \max_{1\le i\le d} d_H([\widehat{\alpha}_i, \widehat{\beta}_i], [\alpha_i, \beta_i])$$

and

$$\Sigma^r_{\mathrm{ED}}(N + \widehat{Q}) = \Big([\widehat{\alpha}_1, \widehat{\beta}_1], \ldots, [\widehat{\alpha}_d, \widehat{\beta}_d]\Big).$$

The first claim follows from the inequality

$$\begin{aligned} \|\widehat{Q}\|_\infty &\le \|T\|_\infty \|Q\|_\infty \|T^{-1}\|_\infty \\ &\le \ell \|T\|_\infty \|T^{-1}\|_\infty \max_{1\le i\le d} d_H([\widehat{\alpha}_i, \widehat{\beta}_i], [\alpha_i, \beta_i]). \end{aligned}$$

The second one is deduced from (22) and the fact that $M(t) + Q(t)$ and $N(t) + \widehat{Q}(t)$ are asymptotically equivalent, since for $t \in \mathbb{R}^+$

$$\begin{aligned} & T^{-1}(t)\big(N(t) + \widehat{Q}(t)\big)T(t) - T^{-1}(t)\dot{T}(t) \\ = \;& T^{-1}(t)\big(N(t) + T(t)Q(t)T^{-1}(t)\big)T(t) - T^{-1}(t)\dot{T}(t) \\ = \;& T^{-1}(t)N(t)T(t) - T^{-1}(t)\dot{T}(t) + Q(t) \\ = \;& M(t) + Q(t). \end{aligned}$$

The proof is complete. $\square$

We now state and prove the main result of this subsection in which we describe a relation between proportional local assignability of the dichotomy spectrum of (20) by additive perturbation and proportional local assignability of (18).

Proposition 15. *Suppose that system (17) is uniformly completely controllable. If the repeated dichotomy spectrum of the associated free system (19) is proportionally locally assignable by additive perturbation, then the dichotomy spectrum of (18) is proportionally locally assignable.*

Proof. From the proportional local assignability of the dichotomy spectrum of (19) by additive perturbation, there exist $\delta_1, \ell_1 > 0$ such that for any admissible closed intervals $[\widehat{a}_1, \widehat{b}_1], \ldots, [\widehat{a}_d, \widehat{b}_d]$ with $\max_{1\leq i\leq d} d_H([\widehat{a}_i, \widehat{b}_i], [a_i, b_i]) \leq \delta_1$ there exists a function $Q \in \mathcal{KC}_{d,d}(\mathbb{R}^+)$ satisfying the estimate

$$\|Q\|_\infty \leq \ell_1 \max_{1\leq i\leq d} d_H([\widehat{a}_i, \widehat{b}_i], [a_i, b_i])$$

and providing the relation

$$\Sigma^r_{\mathrm{ED}}(A + Q) = \Big([\widehat{a}_1, \widehat{b}_1], \ldots, [\widehat{a}_d, \widehat{b}_d]\Big). \tag{24}$$

According to Theorem 12, there exist $\delta_2 > 0$ and $\ell_2 > 0$ such that for each system (20) with $Q \in \mathcal{KC}_{d,d}(\mathbb{R}^+), \|Q\|_\infty \leq \delta_2$ there exists a feedback control $F \in \mathcal{KC}_{m,d}(\mathbb{R}^+)$, such that $\|F\|_\infty \leq \ell_2\|Q\|_\infty$ and the corresponding closed-loop system (18) is asymptotically equivalent to system (20). Let

$$\delta := \min\left\{\frac{\delta_2}{\ell_1}, \delta_1\right\}, \qquad \ell := \ell_1\ell_2. \tag{25}$$

To conclude the proof, choose and fix arbitrary admissible closed intervals $[\widehat{a}_1, \widehat{b}_1], \ldots, [\widehat{a}_d, \widehat{b}_d]$ such that

$$\max_{1\leq i\leq d} d_H([\widehat{a}_i, \widehat{b}_i], [a_i, b_i]) \leq \delta.$$

By definition of δ and δ_1, there exists a function $Q \in \mathcal{KC}_{d,d}(\mathbb{R}^+)$ such that

$$\|Q\|_\infty \leq \ell_1 \max_{1\leq i\leq d} d_H([\widehat{a}_i, \widehat{b}_i], [a_i, b_i]) \leq \ell_1\delta \leq \delta_2$$

and (24) is satisfied. For this function Q and by definition of δ_2 there exists a feedback control $F \in \mathcal{KC}_{m,d}(\mathbb{R}^+)$ for system (18) such that

$$
\begin{aligned}
\|F\|_\infty \le \ell_2 \|Q\|_\infty &\le \ell_2 \ell_1 \max_{1\le i\le d} d_H([\widehat{a}_i, \widehat{b}_i], [a_i, b_i]) \\
&= \ell \max_{1\le i\le d} d_H([\widehat{a}_i, \widehat{b}_i], [a_i, b_i]),
\end{aligned}
$$

and such that systems (20) and (18) are asymptotically equivalent. Since equivalent systems have the same dichotomy spectrum the proof is completed. □

3.3. Proof of proportional local assignability by additive perturbation for systems with one dichotomy spectral interval

We now state and prove the main result of this subsection about proportional local assignability by additive perturbation for the system

$$
\dot{x} = A(t)x \tag{26}
$$

under the assumption that the dichotomy spectrum $\Sigma_{\mathrm{ED}}(A)$ consists of only one spectral interval.

Theorem 16. *Consider system* (26) *and suppose that its dichotomy spectrum consists of only one spectral interval. Then, the repeated dichotomy spectrum of* (26) *is proportionally locally assignable by additive perturbation.*

The main idea of the proof of the above theorem is to transform (26) into an upper triangular system and to use the following result on an explicit form of dichotomy spectrum of an upper-triangular system. A proof of (i) can be seen in [13] and a proof of (ii) can be seen in [5].

Proposition 17. *Consider an upper-triangular system*

$$
\dot{x} = U(t)x, \ \text{where } U(t) = \begin{pmatrix} u_{11}(t) & u_{12}(t) & \cdots & u_{1d}(t) \\ 0 & u_{22}(t) & \cdots & u_{2d}(t) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{dd}(t) \end{pmatrix} \in \mathcal{KC}_{d,d}(\mathbb{R}^+).
$$

Then, the following statements hold:

(i) The dichotomy spectrum $[\alpha_i, \beta_i] := \Sigma_{\mathrm{ED}}(u_{ii})$ *of the subsystem* $\dot{x}_i = u_{ii}(t)x_i$ *is given by*

$$\alpha_i = \liminf_{t-s\to\infty} \frac{1}{t-s}\int_s^t u_{ii}(\tau)d\tau \ \textit{and}\ \beta_i = \limsup_{t-s\to\infty} \frac{1}{t-s}\int_s^t u_{ii}(\tau)d\tau. \tag{27}$$

(ii) $\Sigma_{\mathrm{ED}}(U) = \bigcup_{i=1}^d \Sigma_{\mathrm{ED}}(u_{ii})$.

Proof of Theorem 16. It is known that there exists an upper triangular system

$$\dot{y} = U(t)y, \tag{28}$$

where $U \in \mathcal{KC}_{d,d}$, which is asymptotically equivalent to (26) (see e.g. [1, Theorem 3.3.1]). Since proportional local assignment of repeated dichotomy spectrum by additive perturbation persists via asymptotic equivalence (Proposition 14), it is sufficient to prove the proportional local assignment of repeated dichotomy spectrum for (28) under the assumption that $\Sigma_{\mathrm{ED}}(U) = [a, b]$, where $a \le b$. In what follows, we consider two separate cases:
Case 1: $a < b$. Let

$$\delta := \frac{b-a}{3+|a|+|b|} \qquad \text{and} \qquad \ell := \frac{\max(2, |a|+|b|)}{b-a}\big(1 + \max_{1\le i\le d}(\|u_{ii}\|_\infty)\big). \tag{29}$$

Now, let $[\widehat{\alpha}_1, \widehat{\beta}_1], \dots, [\widehat{\alpha}_d, \widehat{\beta}_d]$ be arbitrary admissible closed intervals satisfying that $\max_{1\le i\le d} d_H([\widehat{\alpha}_i, \widehat{\beta}_i], [a, b]) \le \delta$. By (29) and $d_H([\widehat{\alpha}_i, \widehat{\beta}_i], [a, b]) \le \delta$, we have $[\frac{2a+b}{3}, \frac{a+2b}{3}] \subset [\widehat{\alpha}_i, \widehat{\beta}_i]$ for all $i = 1, \dots, d$. Thus, by virtue of Remark 8 all intervals $[\widehat{\alpha}_1, \widehat{\beta}_1], \dots, [\widehat{\alpha}_d, \widehat{\beta}_d]$ coincide and let $\widehat{\alpha} := \widehat{\alpha}_i$ and $\widehat{\beta} := \widehat{\beta}_i$. Let

$$\eta := \frac{\widehat{\beta} - \widehat{\alpha}}{b-a}, \qquad \zeta := \frac{\widehat{\alpha} b - a\widehat{\beta}}{b-a}. \tag{30}$$

Define $Q \in \mathcal{KC}_{d,d}(\mathbb{R}^+)$ by

$$Q(t) := \mathrm{diag}((\eta-1)u_{11}(t) + \zeta, \dots, (\eta-1)u_{dd}(t) + \zeta) \qquad \text{for all } t \in \mathbb{R}^+.$$

By Definition 13, to complete the proof of the theorem in this case it is sufficient to show that

$$\|Q\|_\infty \le \ell \max\{|\widehat{\alpha} - a|, |\widehat{\beta} - b|\}, \qquad \Sigma_{\mathrm{ED}}(M+Q) = [\widehat{\alpha}, \widehat{\beta}]. \tag{31}$$

Concerning the estimate on $\|Q\|_\infty$, from the definition of Q we have

$$\|Q\|_\infty = \max_{1\le i\le d} |(\eta-1)\,\|u_{ii}\|_\infty + \zeta| \le \max_{1\le i\le d} (|\eta-1|\,\|u_{ii}\|_\infty + |\zeta|).$$

By (30), we have

$$|\eta-1| \le \frac{2}{b-a}\max\{|\widehat{\alpha}-a|, |\widehat{\beta}-b|\},\ |\zeta| \le \frac{|a|+|b|}{b-a}\max\{|\widehat{\alpha}-a|, |\widehat{\beta}-b|\}.$$

Thus,

$$\begin{aligned}
\|Q\|_\infty &\le \max(|\eta-1|, |\zeta|)(1+\max_{1\le i\le d}\|u_{ii}\|_\infty)\\
&\le \frac{\max(2,|a|+|b|)}{b-a}\left(1+\max_{1\le i\le d}\|u_{ii}\|_\infty\right) d_H([a,b],[\widehat{\alpha},\widehat{\beta}]),
\end{aligned}$$

which together with (29) proves the first part of (31). Concerning the remaining part of (31), by using Proposition 17 we obtain

$$\begin{aligned}
&\Sigma_{\mathrm{ED}}(U+Q)\\
=&\bigcup_{i=1}^d \Sigma_{\mathrm{ED}}(\eta u_{ii}+\zeta)\\
=&\bigcup_{i=1}^d \left[\liminf_{t-s\to\infty}\frac{1}{t-s}\int_s^t \eta u_{ii}(\tau)+\zeta\,d\tau, \limsup_{t-s\to\infty}\frac{1}{t-s}\int_s^t \eta u_{ii}(\tau)+\zeta\,d\tau\right]\\
=&\bigcup_{i=1}^d \left[\eta\liminf_{t-s\to\infty}\frac{1}{t-s}\int_s^t u_{ii}(\tau)\,d\tau+\zeta, \eta\limsup_{t-s\to\infty}\frac{1}{t-s}\int_s^t u_{ii}(\tau)\,d\tau+\zeta\right]\\
=&\bigcup_{i=1}^d \eta\Sigma_{\mathrm{ED}}(u_{ii})+\zeta = [\eta a+\zeta, \eta b+\zeta],
\end{aligned}$$

which together with the definition of η and ζ as in (30) shows that $\Sigma_{\mathrm{ED}}(U+Q) = [\widehat{\alpha},\widehat{\beta}]$. The proof of the theorem is complete in this case.

Case 2: $a=b$. By virtue of Proposition 28, we arrive at $\Sigma_{\mathrm{ED}}(u_{ii}) = \{a\}$ and

$$a = \lim_{t-s\to\infty}\frac{1}{t-s}\int_s^t u_{ii}(\tau)d\tau \quad \text{for } i=1,\dots,d. \tag{32}$$

Let $[\widehat{\alpha}_1,\widehat{\beta}_1],\dots,[\widehat{\alpha}_d,\widehat{\beta}_d]$ be arbitrary admissible closed intervals of the form

$$\Big(\underbrace{[a_1,b_1],\dots,[a_1,b_1]}_{d_1\text{times}},\dots,\underbrace{[a_k,b_k],\dots,[a_k,b_k]}_{d_k\text{times}}\Big)$$

satisfying that $\max_{1\le j\le k} d_H([a_j,b_j],\{a\}) \le 1$ and $b_1 < a_2, \ldots, b_{k-1} < a_k$. Let $j \in \{1,\ldots,k\}$ be arbitrary and define a function $p_j \in \mathcal{KC}_{1,1}(\mathbb{R}^+)$ by

$$p_j(t) := \begin{cases} b_j - a, & \text{if } t \in [(2m)^2, (2m+1)^2), \text{ where } m \in \mathbb{Z}_{\ge 0}; \\ a_j - a, & \text{if } t \in [(2m+1)^2, (2m+2)^2), \text{ where } m \in \mathbb{Z}_{\ge 0}. \end{cases} \tag{33}$$

By definition of p_j, we have

$$\limsup_{t-s\to\infty} \frac{1}{t-s}\int_s^t p_j(\tau)\, d\tau = b_j - a, \; \liminf_{t-s\to\infty} \frac{1}{t-s}\int_s^t p_j(\tau)\, d\tau = a_j - a. \tag{34}$$

We now define $Q(t) := \operatorname{diag}(q_1(t),\ldots,q_d(t))$ where

$$q_i(t) := p_j(t) \quad \text{ for } i \in \{d_1+\cdots+d_{j-1}+1, d_1+\cdots+d_j\}, j = 1,\ldots,k.^5$$

To conclude the proof, we will estimate $\|Q\|_\infty$ and compute $\Sigma^r_{\mathrm{ED}}(U+Q)$. Firstly, by definition of Q and (33) we have

$$\sup_{t\in\mathbb{R}^+} \|Q(t)\| \le \max_{1\le j\le k} d_H\left([a_j,b_j],\{a\}\right).$$

Finally, from (32) and (34) we derive that for $i \in \{d_1+\cdots+d_{j-1}+1, d_1+\cdots+d_j\}$, where $j \in \{1,\ldots,k\}$

$$\limsup_{t-s\to\infty} \frac{1}{t-s}\int_s^t u_{ii}(\tau)+q_i(\tau)\, d\tau = b_j, \liminf_{t-s\to\infty} \frac{1}{t-s}\int_s^t u_{ii}(\tau)+q_i(\tau)\, d\tau = a_j.$$

In view of Proposition 17(i), we have $\Sigma_{\mathrm{ED}}(u_{ii}+q_i) = [a_j,b_j]$ and thus

$$\Sigma^r_{\mathrm{ED}}(U+Q) = \Big(\underbrace{[a_1,b_1],\ldots,[a_1,b_1]}_{d_1\text{times}},\ldots,\underbrace{[a_k,b_k],\ldots,[a_k,b_k]}_{d_k\text{times}}\Big).$$

The proof is complete. ☐

3.4. Proof of the main results

Proof of Theorem 11. Thanks to Proposition 15, to show the proportional local assignability of the dichotomy spectrum of (18) it is sufficient to verify the

[5]Throughout the paper, we use the convention that $d_1+\cdots+d_{j-1} = 0$ when $j = 1$.

proportional local assignability of the dichotomy spectrum by additive perturbation of system (19). Let the repeated dichotomy spectrum $\Sigma^r_{\mathrm{ED}}(A)$ be of the following form

$$\begin{aligned}\Sigma^r_{\mathrm{ED}}(A) &= \Big([a_1,b_1],\dots,[a_d,b_d]\Big)\\ &= \Big(\underbrace{[a_1^*,b_1^*],\dots,[a_1^*,b_1^*]}_{d_1\text{times}},\dots,\underbrace{[a_k^*,b_k^*],\dots,[a_k^*,b_k^*]}_{d_k\text{times}}\Big),\end{aligned}$$

where

$$a_1 \le b_1 \le a_2 \le b_2 \le \cdots \le a_d \le b_d$$

and

$$a_1^* \le b_1^* < a_2^* \le b_2^* \cdots < a_k^* \le b_k^*.$$

Then, we have for all $i = 1,\dots,k$

$$[a_i^*,b_i^*] = [a_j,b_j] \qquad \text{for } d_1+\cdots+d_{i-1}+1 \le j \le d_1+\cdots+d_i. \tag{35}$$

In light of Theorem 4, system (19) is asymptotically equivalent to a block-diagonal system

$$\dot y = \operatorname{diag}(A_1(t),\dots,A_k(t))y \qquad \text{for } t\in\mathbb{R}^+, \tag{36}$$

where $A_i \in \mathcal{KC}_{d_i,d_i}(\mathbb{R}^+)$ for $i=1,\dots,k$ satisfies that

$$\Sigma_{\mathrm{ED}}(A_i) = [a_i^*,b_i^*] \qquad \text{for } i=1,\dots,k.$$

By Proposition 14, to conclude the proof we verify proportional local assignability of the dichotomy spectrum by additive perturbation of (36). Note that by virtue of Theorem 16, for $i=1,\dots,k$ the repeated dichotomy spectrum of each subsystem

$$\dot y_i = A_i(t)y_i, \tag{37}$$

is proportionally locally assignable by additive perturbation. This implies that for each $i=1,\dots,k$ there exist δ_i and ℓ_i such that for each admissible intervals $[a_1^i,b_1^i],\dots,[a_{d_i}^i,b_{d_i}^i]$ satisfying $\sup_{1\le j\le d_i} d_H([a_j^i,b_j^i],[a_i^*,b_i^*]) \le \delta_i$ there exists $Q_i\in\mathcal{KC}_{d_i,d_i}(\mathbb{R}^+)$ such that $\|Q_i\|_\infty \le \ell_i \max_{1\le j\le d_i} d_H([a_j^i,b_j^i],[a_i^*,b_i^*])$ and

$$\Sigma^r_{\mathrm{ED}}(A_i+Q_i) = \Big([a_1^i,b_1^i],\dots,[a_{d_i}^i,b_{d_i}^i]\Big).$$

Define

$$\delta := \min\left\{\min_{1\le i\le k}\delta_i, \min_{1\le i\le k-1}\frac{(a^*_{i+1}-b^*_i)}{3}\right\}, \quad \ell := \max_{1\le i\le k}\ell_i. \tag{38}$$

To complete the proof, let $[\widehat{a}_1,\widehat{b}_1],\dots,[\widehat{a}_d,\widehat{b}_d]$ be arbitrary admissible closed intervals satisfying that $\max_{1\le i\le d} d_H([\widehat{a}_i,\widehat{b}_i],[a_i,b_i]) \le \delta$. Using the fact that $\delta \le \min_{1\le i\le k-1}\frac{(a^*_{i+1}-b^*_i)}{3}$, for $i=1,\dots,k$, there exist exactly d_i intervals $[\widehat{a}_1,\widehat{b}_1],\dots,[\widehat{a}_d,\widehat{b}_d]$ whose Hausdorff distance to $[a^*_i,b^*_i]$ is smaller than δ. More precisely, for $i=1,\dots,k$ we have

$$\max_{d_1+\dots+d_{i-1}+1\le j\le d_1+\dots+d_i} d_H([\widehat{a}_j,\widehat{b}_j],[a^*_i,b^*_i]) \le \delta.$$

Since $\delta\le\delta_i$, it follows that there exists $Q_i(t)$ such that

$$\|Q_i\|_\infty \le \ell_i \max_{d_1+\dots+d_{i-1}+1\le j\le d_1+\dots+d_i} d_H([\widehat{a}_j,\widehat{b}_j],[a^*_i,b^*_i]) \tag{39}$$

and

$$\Sigma^r_{\mathrm{ED}}(A_i+Q_i) = \Big([\widehat{a}_{d_1+\dots+d_{i-1}+1},\widehat{b}_{d_1+\dots+d_{i-1}+1}],\dots,[\widehat{a}_{d_1+\dots+d_i},\widehat{b}_{d_1+\dots+d_i}]\Big). \tag{40}$$

Let $Q(t)=\mathrm{diag}(Q_1(t),\dots,Q_k(t))$. Then, by (39) and (35) we have

$$\begin{aligned}\|Q\|_\infty &\le \max_{1\le i\le k}\ell_i \max_{d_1+\dots+d_{i-1}+1\le j\le d_1+\dots+d_i} d_H([\widehat{a}_j,\widehat{b}_j],[a^*_i,b^*_i])\\ &\le \ell\max_{1\le j\le d} d_H([\widehat{a}_j,\widehat{b}_j],[a_j,b_j]).\end{aligned}$$

Furthermore, by (40) we have

$$\Sigma^r_{\mathrm{ED}}(A+Q) = \bigcup_{i=1}^k \Sigma^r_{\mathrm{ED}}(A_i+Q_i) = \Big([\widehat{a}_1,\widehat{b}_1],\dots,[\widehat{a}_d,\widehat{b}_d]\Big).$$

The proof is complete. □

4. Examples

In this section, we consider several time-varying linear planar control systems whose free systems have dichotomy spectrum consisting either of two spectral intervals (Example 18) or of one spectral interval (Example 19). When dealing these examples, we explain how to use the developed theoretical results in the previous section in constructing desired linear state feedbacks in the proportional local assigment of dichotomy spectrum problem.

Before going to these examples, we recall Kalman's characterization for uniform complete controllability, see e.g. [23], for linear time-varying control systems

$$\dot{x} = A(t)x + B(t)u \qquad \text{for } t \in \mathbb{R}^+. \tag{41}$$

The characterization is stated as that system (41) is uniformly completely controllable if and only if there exist positive constants ρ and ϑ such that the controllability matrix $W(t_0, t_0+\vartheta) := \int_{t_0}^{t_0+\vartheta} X_A(t_0,s)B(s)B(s)^{\mathrm{T}}X_A(t_0,s)^{\mathrm{T}}\, ds$ satisfies the inequality

$$\xi^{\mathrm{T}}W(t_0,t_0+\vartheta)\xi \geq \rho\|\xi\|^2 \quad \text{ for any } t_0 \in \mathbb{R}^+, \xi \in \mathbb{R}^d. \tag{42}$$

When we take $B(t) = I$ and the time-varying matrix $A(t)$ is bounded, then system (41) is uniformly complete controllable. To see this, let $m := \sup_{t\in\mathbb{R}^+}\|A(t)\|$. Then, we have $|X_A(t,s)| \leq e^{m|t-s|}$ for all $t, s \in \mathbb{R}^+$. Consequently, for a fixed $\vartheta > 0$ and for all $t_0 \in \mathbb{R}^+$ we have

$$\begin{aligned}
\xi^{\mathrm{T}}W(t_0,t_0+\vartheta)\xi &= \int_{t_0}^{t_0+\vartheta} \|X_A(t_0,s)\xi\|^2\, ds \\
&\geq \int_{t_0}^{t_0+\vartheta} e^{2m(t_0-s)}\, ds\|\xi\|^2 \\
&= \frac{1-e^{-2m\vartheta}}{2m}\|\xi\|^2,
\end{aligned}$$

which together with (42) shows the uniform complete controllability of (41) in this case.

Example 18. *Consider a linear time-varying control system of the following form*

$$\dot{x} = A(t)x + B(t)u \qquad \textit{for } t \in \mathbb{R}^+, \tag{43}$$

where

$$A(t) = \begin{pmatrix} \sin t & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \qquad B(t) = I.$$

By using the Proposition 17, the dichotomy spectrum of the free system can be computed explicitly as $\Sigma_{\mathrm{ED}}(A) = \{0, \frac{1}{2}\}$. We now apply the procedure in Case 2 in the proof of Theorem 16 to verify that the dichotomy spectrum of the free system is proportionally locally assignable with two constants $\delta = \frac{1}{6}$ and $\ell = 1$. Let $[a,b], [c,d]$ be arbitrary admissible closed intervals with $d_H([a,b], \{0\})$, $d_H([c,d], \{\frac{1}{2}\}) \leq \delta$. Then, since $\delta \leq \frac{1}{6}$, two intervals $[a,b]$ and $[c,d]$ are disjoint and $[a,b] \subseteq [-\delta, \delta]$ and $[c,d] \subseteq [\frac{1}{2} - \delta, \frac{1}{2} + \delta]$. We construct the linear state feedback $F \in \mathcal{KC}_{2,2}(\mathbb{R}^+)$ of the form $F(t) = diag(f_1(t), f_2(t))$, where

$$f_1(t) := \begin{cases} a, & \text{if } t \in [(2m)^2, (2m+1)^2), \text{ where } m \in \mathbb{Z}_{\geq 0}; \\ b, & \text{if } t \in [(2m+1)^2, (2m+2)^2), \text{ where } m \in \mathbb{Z}_{\geq 0}; \end{cases}$$

and

$$f_2(t) := \begin{cases} c - \frac{1}{2}, & \text{if } t \in [(2m)^2, (2m+1)^2), \text{ where } m \in \mathbb{Z}_{\geq 0}; \\ d - \frac{1}{2}, & \text{if } t \in [(2m+1)^2, (2m+2)^2), \text{ where } m \in \mathbb{Z}_{\geq 0}. \end{cases}$$

Then, $\|F\|_\infty \leq \max\{d_H([a,b], \{0\}), d_H([c,d], \{\frac{1}{2}\})\}$ and as it is shown in the proof of Theorem 16 the dichotomy spectrum $\Sigma_{\mathrm{ED}}(A + BF)$ of the closed loop system $\dot{x} = (A(t) + B(t)F(t))x$ is $[a,b] \cup [c,d]$.

Example 19. *Consider a linear time-varying control system of the following form*

$$\dot{x} = A(t)x + B(t)u \qquad \text{for } t \in \mathbb{R}^+, \tag{44}$$

where

$$A(t) = \begin{pmatrix} \sin(\log(1+t)) + \cos(\log(1+t)) & -1 \\ 0 & 0 \end{pmatrix}, \qquad B(t) = I.$$

The free system $\dot{x} = A(t)x$ is considered in [1, p. 95] and [11, Example 3.3] (after a shift of the time by 1). It is shown in these references that the Lyapunov exponents are unstable and the dichotomy spectrum of the free system is given by $\Sigma_{\mathrm{ED}}(A) = [-\sqrt{2}, \sqrt{2}]$. We now apply the construction in Case 1 in the proof of Theorem 16 to show that the dichotomy spectrum of the free system is *proportionally locally assignable with two positive constants (cf.* (29)*)*

$$\delta = \frac{2\sqrt{2}}{3+2\sqrt{2}} \qquad \text{and} \qquad \ell = 1+\sqrt{2}.$$

Let $[a,b],[c,d]$ *be arbitrary admissible closed intervals with*

$$d_H([a,b],[-\sqrt{2},\sqrt{2}]), \quad d_H([c,d],[-\sqrt{2},\sqrt{2}]) \le \delta.$$

Then, $[a,b] \equiv [c,d]$. *As was proved in Case 1 in the proof of Theorem 16, the linear state feedback* $F \in \mathcal{KC}_{2,2}(\mathbb{R}^+)$ *of the form* $F(t) = diag(f_1(t), f_2(t))$*, where*

$$\begin{aligned} f_1(t) &= \Big(\frac{b-a}{2\sqrt{2}} - 1\Big)\big(\sin(\log(1+t)) + \cos(\log(1+t))\big) + \frac{a+b}{2} \\ f_2(t) &= \frac{a+b}{2} \end{aligned}$$

satisfies that $\Sigma_{\mathrm{ED}}(A+BF) = [a,b]$ *and* $\|F\|_\infty \le \ell d_H([a,b],[-\sqrt{2},\sqrt{2}])$.

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[1] L. Ya. Adrianova, *Introduction to Linear Systems of Differential Equations*, Translated from the Russian by Peter Zhevandrov. Translations of Mathematical Monographs, **146**. American Mathematical Society, Providence, RI, 1995.

[2] B.D.O. Anderson, A. Ilchmann, F.R. Wirth, Stabilizability of linear time-varying systems, *Systems & Control Letters* **62** (2013), no. 9, 747-755.

[3] A. Babiarz, L.V. Cuong, A. Czornik, T. S. Doan, Necessary and sufficient conditions for assignability of dichotomy spectra of continuous time-varying linear systems, *Automatica* **125** (2021), 109466.

[4] L. Barreira and C. Vall, Stability of nonautonomous differential equations in Hilbert spaces, *Journal of Differential Equations* **217** (2005), no. 1, 204 248.

[5] F. Battelli, K. J. Palmer, Criteria for exponential dichotomy for triangular systems, *Journal of Mathematical Analysis and Applications* **428** (2015), no. 1, 525-543.

[6] W. A. Coppel, Dichotomies and reducibility, *Journal of Differential Equations* **3** (1967), no. 4, 500-521.

[7] W. A. Coppel, Dichotomies and reducibility (II), *Journal of Differential Equations* **4** (1968), no. 3, 386-398.

[8] W. A. Coppel, *Dichotomies in Stability Theory*, Lecture Notes in Mathematics **629**, Springer-Verlag Berlin Heidelberg 1978.

[9] H. Crauel, Lyapunov exponents of random dynamical systems on Grassmannians. Lyapunov exponents (Oberwolfach, 1990), 3850, *Lecture Notes in Math.*, **1486**, Springer, Berlin, 1991.

[10] B. Dickinson, On the fundamental theorem of linear state variable feedback, *IEEE Transactions on Automatic Control* **19** (1974), no. 5, 577–579.

[11] L. Dieci, E. S. Van Vleck, Lyapunov and Sacker–Sell spectral intervals, *Journal of Dynamics and Differential Equations* **19** (2007), no. 2, 265-293.

[12] T.S. Doan, On analyticity for Lyapunov exponents of generic bounded linear random dynamical systems, *Discrete Contin. Dyn. Syst. Ser. B* **22** (2017), no. 8, 31133126.

[13] T. S. Doan, K. J. Palmer, M. Rasmussen, The Bohl spectrum for linear nonautonomous differential equations, *J. Dynam. Differential Equations* **29** (2017), no. 4, 1459–1485.

[14] M. Ikeda, H. Maeda, S. Kodama, Stabilization of linear systems, *SIAM Journal on Control* **10** (1972), no. 4, 716-729.

[15] P.E. Kloeden, M. Rasmussen, *Nonautonomous Dynamical Systems*, Mathematical Surveys and Monographs, **176**, American Mathematical Society, Providence, RI, 2011.

[16] M. Kono, Eigenvalue assignment in linear periodic discrete-time systems, *International Journal of Control* **32** (1980), no. 1, 149-158.

[17] J. Massera, J. Schaeffer, Linear differential equations and functional analysis, *Annals of Mathematics* **67** (1958), 517–573.

[18] J. Massera, J. Schaeffer, Linear differential equations and functional analysis, iv *Mathematische Annalen* **139** (1960), 287–342.

[19] C.C. Nguyen, Arbitrary eigenvalue assignments for linear time-varying multivariable control systems, *International Journal of Control* **45** (1987), no. 3, 1051-1057.

[20] K.J. Palmer, A generalization of Hartman's linearization theorem, *J. Math. Anal. Appl.* **41** (1973), 753758.

[21] O. Perron, Die Stabilitätsfrage bei Differentialgleichungen, *Mathematische Zeitschrift* **32** (1930), 703–728.

[22] S. N. Popova, Global reducibility of linear control systems to systems of scalar type, *Differ. Equ.* **40** (2004), no. 1, 43–49.

[23] E. K. Makarov, S. N. Popova, *Controllability of Asymptotic Invariants of Time-dependent Linear Systems*, Belaruskaya Nauka, Minsk, 2012.

[24] R. J. Sacker, G. R. Sell, A spectral theory for linear differential systems, *Journal of Differential Equations* **27** (1978), no. 3, 320–358.

[25] S. Siegmund, Dichotomy spectrum for nonautonomous differential equations, *J. Dynam. Differential Equations* **14** (2002), no. 1, 243-258.

[26] S. Siegmund, Reducibility of nonautonomous linear differential equations, *J. London Math. Soc. (2)* **65** (2002), no. 2, 397-410.

[27] S. Siegmund, Normal forms for nonautonomous differential equations, *J. Differential Equations* **178** (2002), no. 2, 541573.

[28] E. Sontag, *Mathematical Control Theory: Deterministic Finite Dimensional Systems* Springer Science & Business Media 2013.