Solution to an Open Question about Optimal Economic Growth Models

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Abstract. We prove that if the total factor productivity A of an aggregative economy is right at the barrier $\sigma + \lambda$, with σ being the growth rate of labor force and λ the real interest rate, then the unique policy to optimally control the economy is the same as the one for optimally controlling weak economies, where $A < \sigma + \lambda$. This result gives a complete answer for the interesting open question raised by Vu Thi Huong in her recent paper "*Optimal economic growth problems with high values of total factor productivity*" [Appl. Anal. 101, 1315–1329 (2022)].

Keywords: Economic growth model, Optimal control, Maximum principle, Bang–bang control, Singular control

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1 Introduction

The aim of this paper is to solve the open question raised in [12] about the optimal controls for an economy having the total factor productivity exactly equal to the sum of the growth rate of labor force and the real interest rate.

Let us briefly review some materials on optimal economic growth needed to understand the above question.

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In economics, one has to clarify relationships among capital, labor force, production technology, and national product of an economy. For this purpose, the models of economic growth due to Ramsey [24], Harrod [10], Domar [6], Solow [26], Swan [31], Romer [25], and Lucas [19] have been proposed. *Optimal economic growth problems* (see, e.g., [3, 18, 24]) aim at finding the saving curves that maximize certain targets of consumption satisfaction. For the origins and developments of optimal economic growth theories, we refer to the books [2] by Barro and Sala-i-Martin, [1] by Acemoglu, and the recent papers [27, 28, 29, 30] by Spear and Young.

The *total factor productivity* (TFP) of an aggregative economy is usually measured as the ratio of aggregate outputs, for example, the gross domestic product, to aggregate inputs. We will denote the TFP by A, the *growth rate of labor force* by σ, and the *real interest rate* by λ. The roles of these parameters in the related optimal control problem are explained in the next section. One has $A > 0$, $\sigma > 0$, and $\lambda \geq 0$.

Recently, Huong et al. [14] have obtained a new result on optimal economic growth problems, where the production function is an *AK function* (see, e.g., [2], [13, Section 4], and Section 2 below) and the utility function is a linear one. The authors have proved that if $A < \sigma + \lambda$, then the problem in question has a unique solution. This is obtained by using a solution existence theorem from [13] and a maximum principle for optimal control problems from [33]. The economic meaning of the result is as follows: *If the TFP is relatively small (the situation of a weak economy), then the expansion of the production facility does not lead to a higher total consumption satisfaction of the society.* If $A > \sigma + \lambda$ (the TFP is relatively large – the situation of a strong economy) then, as shown by Huong [12], *the optimal strategy depends not only on the data triplet* (A, σ, λ)*, but also on the length of the planning interval.* The optimal economic growth problem remains *unresolved* [12, Section 2] if $A = \sigma + \lambda$. In this case, one may think [12, p. 1319] that the problem can have some irregular optimal solutions (solutions with an infinite number of discontinuities on the fixed planning interval).

In the current paper, we will prove that if $A = \sigma + \lambda$, then the unique policy to optimally control the economy is the same as the one for optimally controlling weak economies. This result gives a complete answer to the open question raised in [12, Section 2]. Thus, in combination with the results of [12, 14], a comprehensive and explicit description of the unique solution of an arbitrary optimal economic growth problem where the production function is an AK function and the utility function is a linear one can be obtained. The proof of Theorem 4.2 below, which is our main result, relies not only on a solution existence theorem from [12, 13], and a maximum principle for optimal control problems from [33], but also on *some novel techniques*, including the ones in the proof of Proposition 4.2.

Concerning optimal economic growth models with nonlinear utility functions, note that a class of such problems has been studied recently by Huong et al. [15]. It is proved that the unique solution can be explicitly described via input parameters. Economic interpretations of the obtained results can be found in [15]. There is an open question (see [15, p. 594]) about the case where the TFP falls into a bounded open interval defined by the growth rate of labor force, the real interest rate, and the exponent of the utility function.

An optimal economic growth/consumption problem with a linear utility function and a per capita production function of the Cobb–Douglas type has been investigated by Miao and Vinter in their recent paper [20]. A detailed solution to the problem and an analysis of its structure, for arbitrary initial data, have been obtained by applying the maximum principle, the value function, the Hamilton-Jacobi-Bellman equation, and a nonstandard verification technique. The authors have observed that "The problem is of control theoretic interest, because the right side of the controlled differential equation and also the utility integrand are not uniformly Lipschitz continuous with respect to the state variable, owing to the presence of a fractional singularity."

As it has been noted, optimal economic growth models and AK/Cobb–Douglas functions are well-known in macroeconomic theory. However, as far as we know, there are no explicit solutions of optimal economic growth problems like (P) and (P1) below. Therefore, the results of [12, 14, 15, 20] and of this paper would be of interest to general audience, especially economists.

The organization of the present paper is as follows. In Sect. 2, we recall the relevant optimal economic growth models, the solution concept, and the economic meanings of the functions and parameters involved. The existing results and an open question are discussed in Sect. 3. A complete answer for the question is obtained in Sect. 4. The final section is devoted to some concluding remarks.

2 Optimal Economic Growth Models

For systematical expositions of optimal economic growth models, we refer to Takayama [32, Chapter 5], Pierre [22, Chapters 5, 7, 10, and 11], Chiang and Wainwright [4, Chapter 20], and Acemoglu [1, Chapters 7 and 8]. Herein, following [12, 13, 14, 15], we consider the problem of *optimal growth of an aggregative economy* as an optimal control problem:

$$
(\mathrm{P}) \quad \left\{
\begin{array}{lll}
\text{Maximize} & \displaystyle\int_{t_0}^{T} (1-s(t))^{\beta} k^{\alpha\beta}(t)\, e^{-\lambda t} dt & \\
\text{subject to} & \dot{k}(t) = A\, k^{\alpha}(t) s(t) - \sigma\, k(t)\,, & \text{a.e. } t \in [t_0, T] \\
 & k(t_0) = k_0\,, & \\
 & 0 \le s(t) \le 1\,, & \text{a.e. } t \in [t_0, T] \\
 & k(t) \ge 0\,, & \forall t \in [t_0, T]\,,
\end{array}
\right.
$$

with $\dot{k}(t) := dk/dt$, provided that the fixed parameters $T > t_0 \geq 0$, $k_0 > 0$, $\sigma > 0$, $\lambda \geq 0$, $A > 0$, and $\alpha, \beta \in (0, 1]$ are given.

The economic meanings of the functions and parameters involved in (P) were discussed in detail in [13, Subsection 2.1] and [14, Section 2]. Recall that a *solution* of (P) is a pair $(k(\cdot), s(\cdot))$ of an *absolutely continuous* real-valued function $k(\cdot)$ and a *Lebesgue measurable* function $s(\cdot)$ defined on $[t_0, T]$, which satisfies all the constraints and gives the objective function the maximum value. So, if $(k(\cdot), s(\cdot))$ is a solution of (P), then the Fréchet derivative $\dot{k}(t)$ of $k(\cdot)$ exists for almost every (a.e., for short) $t \in [t_0, T]$. For each t in the *planning interval* $[t_0, T]$, the values $k(t)$ and $s(t)$ respectively are the *capital-to-labor ratio* and the *propensity to save* at the time moment t. The parameter k_0 is the *initial capital-to-labor ratio*. The parameters σ and λ stand for the *growth rate of labor force* and the *real interest rate*, respectively. The parameters A, α, and β come from two typical functions defining the economy called the *per capita production function* $\phi(k) = Ak^{\alpha}$, $k \geq 0$ and the *utility function* $\omega(c) = c^{\beta}$, $c \geq 0$. The coefficient A expresses the *total factor productivity* (TFP). The exponent α (resp., $1 - \alpha$) refers to the *output elasticity of capital* (resp., the *output elasticity of labor*). Note that $\phi(k)$ is a *Cobb–Douglas function* when $\alpha \in (0, 1)$ and is an *AK function* when $\alpha = 1$ (see, e.g., [32] and [2]).

Observe that (P) is an optimal control problem with state constraints. It is of *Lagrange type* with functions $k(\cdot) : [t_0, T] \to \mathbb{R}$ and $s(\cdot) : [t_0, T] \to \mathbb{R}$ playing the roles of *state* and *control variables*, respectively.

If (k, s) is a solution of (P), then s is said to be an *optimal control* or an *optimal strategy.*

3 Existing Results and an Open Question

The existence of a solution to (P) has been established by Huong [13, Theorem 4.1] and [12, Theorem A.3]. Meanwhile, the problem of finding all the solutions of (P), i.e., getting a synthesis of solutions based on the data set $\{A, \alpha, \beta, \sigma, \lambda, t_0, T, k_0\}$, has been solved recently by Huong et al.[12, 14, 15] and very recently by Miao and Vinter in [20]. In the later paper, optimal strategies for the case where $\alpha \in (0, 1)$ and $\beta = 1$ (the production function is a Cobb-Douglas function and the utility function is a linear one) are described in a feedback form. For the case where $\alpha = \beta = 1$ (the production function is an AK function and the utility function is a linear one), such syntheses are given in [12, 14]. In this case, by [14, Lemma 4.1], the state constraint $k(t) \geq 0$ in (P) can be dropped because the conditions $k_0 > 0$, $s(t) \in [0, 1]$ for a.e. $t \in [t_0, T]$, and the differential equation in (P) imply that $k(t) > 0$ for all $t \in [t_0, T]$. So, for the case where $\alpha = \beta = 1$, one can rewrite (P) equivalently as the following optimal control problem with no state constraints:

$$
\text{(P1)} \quad \begin{cases} \text{Minimize} & \displaystyle\int_{t_0}^{T} (s(t)-1)\,k(t)\,e^{-\lambda t}dt \\ \text{subject to} & \dot{k}(t) = (A\,s(t)-\sigma)\,k(t)\,, \qquad \text{a.e. } t \in [t_0, T] \\ & k(t_0) = k_0\,, \\ & 0 \le s(t) \le 1\,, \qquad\qquad\qquad \text{a.e. } t \in [t_0, T]. \end{cases}
$$

We now recall the syntheses of solutions of (P1) from [14] and [12], which were obtained under the parametric condition that $A \neq \sigma + \lambda$.

Theorem 3.1. (See [14, Theorem 4.12].) *Suppose that $A < \sigma + \lambda$. Then,* (P1) *possesses a unique solution (k, s) given by*

$$
s(t) = 0, \quad \text{a.e. } t \in [t_0, T], \quad \text{and} \quad k(t) = k_0 e^{-\sigma(t-t_0)}, \quad \forall t \in [t_0, T]. \tag{3.1}
$$

Theorem 3.1 says that if the TFP is *smaller* than the sum of the growth rate of labor force and the real interest rate, then the unique optimal control is to keep the saving equal to 0 on the whole planning interval $[t_0, T]$.

Theorem 3.2. (See [12, Theorem 2.2].) *Suppose that $A > \sigma + \lambda$. Define*

$$
\rho = \frac{1}{\sigma+\lambda} \ln \frac{A}{A-(\sigma+\lambda)} \quad \text{and} \quad \bar{t} = T - \rho.
$$

Then, (P1) *has a unique solution (k, s), which can be explicitly described in the following way:*

(a) *If $T - t_0 \le \rho$ (i.e., $t_0 \ge \bar{t}$), then (k, s) is given by* (3.1)*;*

(b) *If $T - t_0 > \rho$ (i.e., $t_0 < \bar{t}$), then (k, s) is given by the formulas*

$$
s(t) = \begin{cases} 1, & \text{a.e.} \quad t \in [t_0, \bar{t}] \\ 0, & \text{a.e.} \quad t \in (\bar{t}, T] \end{cases} \quad \text{and} \quad k(t) = \begin{cases} k_0 e^{(A-\sigma)(t-t_0)}, & \forall t \in [t_0, \bar{t}] \\ k(\bar{t}) e^{-\sigma(t-\bar{t})}, & \forall t \in (\bar{t}, T]. \end{cases}
$$

Theorem 3.2 asserts that if the TFP is *higher* than the sum of the growth rate of labor force and the real interest rate, then the unique optimal control depends on the length $T - t_0$ of the planning interval. Namely, if $T - t_0 \le \rho$, then keeping the saving equal to 0 is the optimal control. In the opposite situation, where one has $T - t_0 > \rho$, the best strategy is to implement the maximum saving until the time instance $\bar{t} = T - \rho$, which belongs to (t_0, T), and switch the saving to 0 afterwards. So, *if the TFP is high enough but the planning period is rather short, then an expansion of the production facility does not lead to a higher* *total consumption satisfaction of the society.* Meanwhile, *if the TFP is high enough and the planning time is relatively long, then the highest total consumption satisfaction of the society is attained if and only if the largest expansion of the production facility is made until the time instance $\bar{t}$.*

Huong [12, Section 2] observes that (P1) has not yet been solved in the case where $A = \sigma + \lambda$. In other words, the following open question requires further investigation.

(**Q**) *What can we say about the optimal controls for an economy having the total factor productivity A exactly equal to the barrier $\sigma + \lambda$ formed by the growth rate of labor force σ and the real interest rate λ?*

Note that the barrier $\sigma+\lambda$ for the TFP appeared for the first time in [14, Theorem 4.12]. Following [14, Remark 4.14], we say that the economy under consideration is *weak* (resp., *strong*) if $A < \sigma + \lambda$ (resp., $A > \sigma + \lambda$). The results in Theorems 3.1 and 3.2 show that the behaviors of a weak economy and of a strong economy are *different*.

4 Solution to the Open Question

To resolve question (**Q**), we will rely on a Maximum Principle, which was a major tool in [14, 12] for solving (P1) when $A \neq \sigma + \lambda$. Among other things, we will see how the condition $A \neq \sigma + \lambda$ appears, and how we can deal with the situation when $A = \sigma + \lambda$ by proposing some interesting tricks. The Maximum Principle states necessary conditions for a pair of functions (k, s) to be a candidate for a solution of the optimal control problem (P1). Applied to (P1), the concepts of the Hamiltonian function and adjoint variable are as follows.

The *Hamiltonian function* $H : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [t_0, T] \to \mathbb{R}$ of (P1) is given by

$$H(k, s, \psi, t) = -\psi_0\,(s-1)\,k\,e^{-\lambda t} + \psi\,(A\,s - \sigma)\,k, \quad (k, s, \psi, t) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [t_0, T] \tag{4.2}$$

with ψ_0 being a real number. An *adjoint variable* of (P1) is an absolutely continuous function $\psi : [t_0, T] \to \mathbb{R}$ satisfying the differential equation

$$\dot{\psi}(t) = -\frac{\partial H}{\partial k}(k(t), s(t), \psi(t), t) = \psi_0\,(s(t)-1)\,e^{-\lambda t} - (A\,s(t) - \sigma)\,\psi(t) \tag{4.3}$$

for a.e. $t \in [t_0, T]$ and the *transversality condition*

$$\psi(T) = 0. \tag{4.4}$$

There exist excellent books where various forms of the Maximum Principle and their proofs can be found; see, for example, [23, Theorem 1], [11, Chapter 7], [33, Theorem 6.4.1], [21, Theorem 6.37], and [5, Theorem 22.2]. In the setting and notation of this paper, the Maximum Principle from [5, Theorem 22.2], which is the most suitable for our purpose, asserts the following.

Theorem 4.1 (Maximum Principle)**.** *If* (k, s)*, where* $k(\cdot) : [t_0, T] \to \mathbb{R}$ *is an absolutely continuous function and* $s(\cdot) : [t_0, T] \to \mathbb{R}$ *is a Lebesgue measurable function, is a solution of* (P1)*, then there exist a constant* $\psi_0 \geq 0$ *and an adjoint variable* $\psi(\cdot) : [t_0, T] \to \mathbb{R}$ *such that* $(\psi_0, \psi) \neq (0, 0)$ *and*

$$H(k(t), s(t), \psi(t), t) = \max_{0 \leq v \leq 1} H(k(t), v, \psi(t), t), \quad \text{a.e. } t \in [t_0, T]. \tag{4.5}$$

By (4.2) and the property that $k(t) > 0$ for all $t \in [t_0, T]$, condition (4.5) is equivalent to

$$(A\psi(t) - \psi_0 \, e^{-\lambda t})s(t) = \max_{0 \leq v \leq 1} (A\psi(t) - \psi_0 \, e^{-\lambda t}) \, v, \quad \text{a.e. } t \in [t_0, T].$$

Set $g(t) := A\psi(t) - \psi_0 \, e^{-\lambda t}$ for $t \in [t_0, T]$ and observe from the absolute continuity of the function ψ that $g : [t_0, T] \to \mathbb{R}$ is an absolutely continuous function. One sees that the optimal control satisfies the following condition for almost every $t \in [t_0, T]$:

$$s(t) = \begin{cases} 1, & \text{if } g(t) > 0 \\ 0, & \text{if } g(t) < 0 \\ \text{undetermined}, & \text{if } g(t) = 0. \end{cases} \tag{4.6}$$

One calls $g(\cdot)$ the *switching function*. Observe that the value $s(t)$ of the optimal control can be found by (4.6) for a.e. $t \in [t_0, T]$ if and only if the switching function vanishes just on a subset of zero Lebesgue measure of $[t_0, T]$.

Normality Problem (P1) and its solutions are said to be *abnormal* if $\psi_0 = 0$, as in that case the Maximum Principle is not sufficiently informative. Abnormal optimal control problems are not uncommon; see, for example, [16]. If $\psi_0 > 0$, then the problem and its solution is said to be *normal*. For more investigations on the normality of optimal control problems, the reader is referred to [7, 9, 8].

Problem (P1) is normal. This fact was observed in [14, Lemma 4.5] and can be justified in a simple way as follows. Suppose by contradiction that $\psi_0 = 0$. Then the initial value problem in (4.3) and (4.4) has the unique solution $\psi(t) = 0$ for all $t \in [t_0, T]$. Thus, one obtains $(\psi_0, \psi) = (0, 0)$, which is not allowed by the Maximum Principle, furnishing the required contradiction. By the normality, in the sequel we can admit that $\psi_0 = 1$.

Bang–bang and singular optimal controls If $g(t) \neq 0$ for a.e. $t \in [t', t'']$ with $t_0 \leq t' < t'' \leq T$, then the optimal control $s(\cdot)$ is called *bang–bang* in the interval $[t', t'']$. In this case, the optimal control might *switch* from $s(t) = 1$ to $s(t) = 0$, or vice versa, at a *switching time* $t_1 \in [t', t'']$. If $g(t) = 0$ for a.e. $t \in [t', t'']$ with $t_0 \leq t' < t'' \leq T$, then the optimal control $s(\cdot)$ is said to be *singular* in the interval $[t', t'']$.

Note that the optimal control might switch from a bang–bang arc to a singular arc, and vice versa.

Proposition 4.1. (See also [14, Lemma 4.10]) *If the optimal control $s(\cdot)$ is singular in some interval $[t', t''] \subseteq [t_0, T]$ with $t' < t''$, then $A = \sigma + \lambda$.*

Proof Suppose that the optimal control is singular in some interval $[t', t'']$ with $t_0 \leq t' < t'' \leq T$. Then one has $\psi(t) = e^{-\lambda t}/A$ for a.e. $t \in [t', t'']$. It follows that $\dot{\psi}(t) = -\lambda\, e^{-\lambda t}/A$ for a.e. $t \in [t', t'']$. So, the differential equation (4.3) implies that

$$-\lambda\, e^{-\lambda t}/A = (s(t) - 1)\, e^{-\lambda t} - (A\, s(t) - \sigma)\, e^{-\lambda\, t}/A$$

for a.e. $t \in [t', t'']$. Multiplying both sides of the last equality by $A\, e^{\lambda\, t}$ gives

$$-\lambda = A\,(s(t) - 1) - (A\, s(t) - \sigma)$$

for a.e. $t \in [t', t'']$. Clearly, cancellations and re-arranging yield $A = \sigma + \lambda$, as required. □

By Proposition 4.1, if $A \neq \sigma + \lambda$, then the optimal control cannot be singular in any interval $[t', t'']$ of $[t_0, T]$ with $t' < t''$. Using this key property and some technical lemmas, Huong et al. [14, Lemma 4.11] have shown that either the switching function has a fixed sign on the whole interval $[t_0, T]$, or it vanishes at just one point $\bar{t} \in [t_0, T]$. Then, using formula (4.6) to define values of the optimal control via the sign of the switching function, the authors of [14, 12] were able to solve (P1) under the condition $A \neq \sigma + \lambda$.

The proof of Proposition 4.1 tells us that the singularity of the optimal control may happen when $A = \sigma + \lambda$. If the optimal control $s(\cdot)$ is singular in some interval $[t', t'']$ of $[t_0, T]$ with $t' < t''$, then formula (4.6) does not yield any information about the optimal control on $[t', t'']$. To deal with this situation, one often uses a conventional approach (see, e.g., [22, Sections 7.3–7.5]) as follows: If $g(t) = 0$ for a.e. $t \in [t', t'']$, $t' < t''$, then one successively takes derivatives of the switching function $g(\cdot)$ until the control variable appears so that an expression for a candidate singular control is obtained. However, applying this approach to the switching function $g(t) = A\psi(t) - e^{-\lambda t}$, we have found that *no matter how many times the switching function is differentiated, the control variable does not appear!* This difficulty prompts us to conjecture that singularity does not occur when $A = \sigma + \lambda$. By employing the special structure of (P1), we will prove our conjecture stated as Proposition 4.2 below.

One should keep in mind, however, that for some problems, while successive differentiation of the switching function may never lead to the appearance of the control variable as a candidate for a singular optimal control, a singular optimal control can still exist; see for example [17, Section 3.2.2]. In the latter case, the singular optimal control, when exists, is said to be of infinite order.

Proposition 4.2. *If $A = \sigma + \lambda$, then the optimal control is a bang–bang control on $[t_0, T]$ (with no switching).*

Proof Let $\Sigma := \{t \in [t_0, T] \mid g(t) = 0\}$. Since the function $\psi(\cdot)$ is continuous, so is the function $g(t) = A\psi(t) - e^{-\lambda t}$. Hence, Σ is a compact set.

First, suppose that Σ is empty. Then one must have either $g(t) > 0$ for all $t \in [t_0, T]$, or $g(t) < 0$ for all $t \in [t_0, T]$. In the first situation, $s(t) = 1$ for a.e. $t \in [t_0, T]$. In the second situation, $s(t) = 0$ for a.e. $t \in [t_0, T]$. This means that the optimal control $s(\cdot)$ is a bang–bang control on $[t_0, T]$ (with no switching).

Next, suppose that Σ is nonempty. By the compactness of Σ, the number $\tau := \max\{t \mid t \in \Sigma\}$ is well defined. If $\tau = T$, then we have $A\psi(T) - e^{-\lambda T} = 0$. This is impossible, because $\psi(T) = 0$ by the transversality condition (4.4). Thus, $\tau < T$. By the definition of τ, we have $g(t) \neq 0$ for all $t \in (\tau, T]$. So, there are only two possibilities: (i) $g(t) > 0$ for all $t \in (\tau, T]$; (ii) $g(t) < 0$ for all $t \in (\tau, T]$.

If the situation (i) occurs, then $s(t) = 1$ for a.e. $t \in (\tau, T]$. So, (4.3) and (4.4) yield

$$\dot{\psi}(t) = -(A - \sigma)\,\psi(t) \quad \text{a.e. } t \in (\tau, T] \quad \text{and} \quad \psi(T) = 0,$$

which in turn results in $\psi(t) = 0$ for all $t \in [\tau, T]$. Therefore $g(t) = -e^{-\lambda t} < 0$ for all $t \in [\tau, T]$. But this contradicts our assumption.

If the situation (ii) occurs, then $s(t) = 0$ for a.e. $t \in (\tau, T]$. Hence, (4.3) and (4.4) imply that

$$\dot{\psi}(t) = -e^{-\lambda t} + \sigma\,\psi(t) \quad \text{a.e. } t \in (\tau, T] \quad \text{and} \quad \psi(T) = 0.$$

A direct computation using the condition $A = \sigma + \lambda$ shows that the unique solution of this Cauchy problem for a linear ordinary differential equation is given by

$$\psi(t) = -\frac{1}{A}\left[e^{\sigma t - (\sigma+\lambda)T} - e^{-\lambda t}\right], \qquad \forall t \in [\tau, T]. \tag{4.7}$$

Now, thanks to (4.7), we get

$$g(t) = A\psi(t) - e^{-\lambda t} = -\left[e^{\sigma t - (\sigma+\lambda)T} - e^{-\lambda t}\right] - e^{-\lambda t} = -e^{\sigma t - (\sigma+\lambda)T}$$

for all $t \in [\tau, T]$. In particular, $g(\tau) = -e^{\sigma\tau - (\sigma+\lambda)T} < 0$. As $\tau \in \Sigma$, we have arrived at a contradiction.

The proof is complete because the case $\Sigma \neq \emptyset$ has been excluded. □

We are now ready to give a complete answer for the open question **(Q)**.

Theorem 4.2. *If $A = \sigma + \lambda$, then* (P1) *possesses a unique solution* (k, s) *given by* (3.1).

Proof By the remark given at the beginning of Section 3, (P1) has a solution (k, s) with $k(\cdot) : [t_0, T] \to \mathbb{R}$ being an absolutely continuous function and $s(\cdot) : [t_0, T] \to \mathbb{R}$ being a measurable function. Since $A = \sigma + \lambda$, it follows from Proposition 4.2 that the optimal control is either the function $s(t) = 1$ for a.e. $t \in [t_0, T]$ or the function $s(t) = 0$ for a.e. $t \in [t_0, T]$. In the first situation, it is clear that the value of the objective function of (P1) is 0. Meanwhile, in the second situation, the value is $-\int_{t_0}^{T} k(t)\, e^{-\lambda t} dt$. Recalling that $k(t) > 0$ for all $t \in [t_0, T]$, we see that the latter is negative. So, the optimal control can only be $s(t) = 0$ for a.e. $t \in [t_0, T]$. Therefore, the constraints in Problem (P1) assert

$$\dot{k}(t) = -\sigma k(t) \ \text{ a.e. } t \in [t_0, T] \quad \text{and} \quad k(t_0) = k_0.$$

It follows that $k(t) = k_0 e^{-\sigma(t-t_0)}$ for all $t \in [t_0, T]$. We have thus proved that (P1) has a unique solution (k, s) given by (3.1). □

Giving Theorem 4.2 a similar interpretation to the one for Theorem 3.1, we can say that if the TFP is equal to the sum of the growth rate of labor force and the real interest rate, then keeping the saving equal to 0 is the optimal control. This means that, any expansion of the production facility does not lead to a higher total consumption satisfaction of the society. So, when A is right at the barrier $\sigma + \lambda$, the policy to control the economy in an optimal way is the same as that works for optimally controlling weak economies (see [14]).

5 Conclusions

Based on a solution existence theorem for optimal economic growth problems, a maximum principle for optimal control problems, and new arguments, we have solved the open question posed in [12, Section 2]. Combining our Theorem 4.2 with the preceding results yields a comprehensive description of the unique global solution of an arbitrary optimal economic growth problem where the utility function is linear and the production is described by an AK function.

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