POLYLOGARITHM FUNCTIONS VIA NEVALLINNA THEORY

TA THI HOAI AN AND NGO QUOC HOAN

Abstract. In this page, we resee about some properties of the polylogarithm functions. To do those works, we base on the results in [5] and relate them to Nevallinna theory for the meromorphism functions. And then we present some other properties of the class of these functions.

Contents

1. Introduction

The polylogarithms are one of the interesting subjects of mathematics and physics. These functions were appeared within the functional expansions (which were common in physics as well as in engineering [21]) to represent the nonlinear dynamical systems in quantum electrodynanics and and have been developped by Tomonaga, Schwinger and Feynman [8]. They appeared then in the singular expansion of the solutions and their successive (ordinary or functional) derivations [12] of nonlinear differential equations with three singularities [1, 6, 17, 18] and then they also appeared in the asymptotic expansion of the Taylor coefficients (if it exists). The main challenge of these expansions lies in the divergences and leads to problems of *regularization* and *renormalization* which can be solved by combinatorial technics [3, 7, 6, 11, 12, 17, 18, 22]. For any $\mathrm{s} = (s_1, \ldots, s_r) \in \mathbb{C}^r$, $r \in \mathbb{N}_+$ and $z \in \mathbb{C}$ such that

$|z| < 1$, the *polylogarithm function* [1] at s $= (s_1, \ldots, s_r)$ is well-defined by

$$\mathrm{Li}_{s_1,\ldots,s_r}(z) := \sum_{n_1 > \ldots > n_r > 0} \frac{z^{n_1}}{n_1^{s_1} \ldots n_r^{s_r}}. \tag{1.1}$$

For example, if $r = 1$ then for each fixed complex s_1, the series in (1.1) defines an analytic function of z on the open disc $|z| < 1$. This series also converges on the disc $|z| < 1$, provided that $\Re(s_1) > 1$. In the special case, $z = 1$, we obtain the Riemann zeta value $\zeta(s_1) = \sum_{n=1}^{\infty} \frac{1}{n^{s_1}}$ at s_1. For other values of z, the value of polylogarithms is defined by analytic continuation.

In general case, in the same way, for any $r \in \mathbb{N}^*$, the polylogarithms are extended as the analytic functions on $\mathbb{C}$ by the analytic continuation. Then the Maclaurin's expansion of $\frac{\mathrm{Li}_{s_1,\ldots,s_r}(z)}{1-z}$ is given by

$$\frac{\mathrm{Li}_{s_1,\ldots,s_r}(z)}{1-z} = \sum_{N \geq 0} \mathrm{H}_{s_1,\ldots,s_r}(N)\, z^N, \tag{1.2}$$

where the coefficients $\mathrm{H}_{s_1,\ldots,s_r} : \mathbb{N} \longrightarrow \mathbb{Q}$ are the arithmetic functions which are called *the harmonic sums* (at s $= (s_1, \ldots, s_r)$). Moreover, the harmonic sum at s $= (s_1, \ldots, s_r)$ can be expressed as follows

$$\mathrm{H}_{s_1,\ldots,s_r}(N) := \sum_{N \geq n_1 > \ldots > n_r > 0} \frac{1}{n_1^{s_1} \ldots n_r^{s_r}}, \ N \in \mathbb{N}. \tag{1.3}$$

Setting now

$$\mathcal{H}_r = \{(s_1, \ldots, s_r) \in \mathbb{C}^r | \forall m = 1, \ldots, r, \Re(s_1) + \ldots + \Re(s_m) > m\}.$$

From the analytic continuation of polyzetas [14, 24][2], for any $(s_1, \ldots, s_r) \in \mathcal{H}_r$ after a theorem by Abel, one obtains the *polyzeta value* as follows

$$\lim_{z \to 1} \mathrm{Li}_{s_1,\ldots,s_r}(z) = \lim_{N \to \infty} \mathrm{H}_{s_1,\ldots,s_r}(N) = \zeta(s_1, \ldots, s_r).$$

On the other hand, being based on Picards and Borels theorems, in 1925, Nevanlinna [23] published his paper and evolved a theory entitled with his

[1]The notation $\mathrm{Li}_2(z)$ was introduced in Lewin (1981) for a function discussed in Euler (1768) and called the dilogarithm in Hill (1828). Other notations and names for $\mathrm{Li}_2(z)$ include $S_2(x)$ (Kölbig et al. (1970)), Spence function $Sp(z)$ (Hooft and Veltman (1979)), and $L_2(z)$ (Maximon (2003)). Moreover, for any $s \in \mathbb{N}^*$, the notation $\phi(z, s)$ was used for $\mathrm{Li}_s(z)$ in Truesdell (1945) for a series treated in Jonquire (1889), hence the alternative name Jonquires function.

[2]Here, the polyzeta value at $(s_1, \ldots, s_r) \in \mathbb{C}^r$ means the series

$$\zeta(s_1, \ldots, s_r) := \sum_{n_1 > \ldots > n_r > 0} \frac{1}{n_1^{s_1} \ldots n_r^{s_r}}. \tag{1.4}$$

name[3]. After, this theory has had many applications to the analyticity, growth, existence and unicity properties of meromorphic solutions to differential or functional equations.

In this page, basing on some properties of the meromorphic functions and the Nevallinna theory which was presented in [9], we would like to present another view on the polylogarithms to relook the properties of the polylogarithms which are understood as the meromorphism solutions of a non-linearly differential equation. Moreover, by this way, we present also some new proprieties about the polylogarithms.

2. Power series and Polylogarithms

In this page, we will denote by $\mathbb{K}$ an algebraically closed field of characteristic 0, complete with respect to an ultrametric absolute value $\| .\|$.

We will also denote by

$d(\alpha, R)$ the disk $\{x \in \mathbb{K} |\ |x-a| \leq R\}$,
$d(\alpha, R^-)$ the disk $\{x \in \mathbb{K} |\ |x-a| < R\}$,
$C(\alpha, R)$ the disk $\{x \in \mathbb{K} |\ |x-a| = R\}$,

for any $\alpha \in \mathbb{K},\ R \in \mathbb{R}_+^*$.

Moreover, we call the absolute value of $\mathbb{K}$ the set $\{|x|\ |x \in \mathbb{K}\}$, namely by $|\mathbb{K}|$.

Given a closed bounded subset D of $\mathbb{K}$. We denote by $\overline{D}$ the smallest closed disk containing D. Moreover, we showed that $\overline{D} \backslash D$ admits a partition of the form $\{d(a_i, r_i^-)\}_{i\in I}$ where the disk $d(a_i, r_i^-)$ is maximal (see in [9, 10]). The such disks $d(a_i, r_i^-), i \in I$ lying in the partition of $\overline{D} \setminus D$ are called the holes of D.

We denote now $R(D)$ by $\mathbb{K}-$algebra of rational functions without poles in closed bounded subset D of $\mathbb{K}$ provided with the norm of uniform convergence on D. Moreover, we calls $H(D)$ by the completion of $R(D)$ with respect to that norm[4].

Let's now take D to be a closed unbounded subset of $\mathbb{K}$. We call $R_b(D)$ the algebra of bounded rational functions having no pole in D, provide with the norm of uniform convergence on D. The completion of $R_b(D)$ is a $\mathbb{K}-$Banach algebra $H_b(D)$ again. The element of $H_b(D)$ is called bounded analytic elements in D. In particular, we denote by $H_0(D)$ the $\mathbb{K}-$Banach algebra of elements f such that $\lim\limits_{|x|\to+\infty,\ x\in D} f(x) = 0$.

Recall that an analytic element in $d(0, R^-)$ is a convergent power series in $d(0, R^-)$. Moreover, a convergent power series in $d(0, R^-)$ doesn't sure

[3] In fact, in 1925, Nevanlinna studied the value distribution of meromorphic functions on complex plan $\mathbb{C}$.

[4] In fact, we can be prove that $H(D)$ is a $\mathbb{K}-$ Banach algebra.

to be an analytic element in $d(0,R^-)$. However, an element in $d(0,R)$ is analytic if only if it is a convergent power series in $d(0,R)$.

Nextly, a power series $\sum\limits_{n=0}^{+\infty} a_n x^n$ is called entire function on $\mathbb{K}$ if the radius of convergence of $\sum\limits_{n=0}^{+\infty} a_n x^n$ is ∞. The set of entire function on $\mathbb{K}$ is denoted by $\mathcal{A}(\mathbb{K})$.

Given X to be a nonempty set and $\mathcal{F}$ is a set of subsets of X. The set $\mathcal{F}$ is a filter on X if only if $\mathcal{F}$ satisfies 3 following conditions:

(i) $\emptyset \notin \mathcal{F}$.
(ii) For any $n \in \mathbb{N}$, if $U_1, \ldots, U_n$ are the elements of $\mathcal{F}$ then $\bigcap\limits_{i\in I} U_i \in \mathcal{F}$.
(iii) If $U \in \mathcal{F}$ and $U \subset A \subset X$ then $A \in \mathcal{F}$.

Moreover, a filter $\mathcal{F}$ is called to be secant with a subset B of E if the family of set $\{H \cap B |\ H \in \mathcal{F}\}$ is a filter on B.

Let $a \in \mathbb{K}$ and let $r', r" \in \mathbb{R}$ be such that $0 < r' < r"$. We set

$$\begin{aligned} \Gamma(a, r', r") &= \{x \in \mathbb{K} | r' < |x-a| < r"\}, \\ \Delta(a, r', r") &= \{x \in \mathbb{K} | r' \leq |x-a| \leq r"\}. \end{aligned} \tag{2.1}$$

For any $a \in \mathbb{K}$ and $r \in \mathbb{R}_+$, we denote the circular filter of center a, of diameter r which admits for basis $\{\Gamma(b, r', r")|\ b \in d(a,r);\ r' < r < r"\}$. In special case, $\mathbb{K}$ isn't spherically complete, each decreasing family of disks $(D_n)_{n\in\mathbb{N}^*}$ such that $\bigcap\limits_{n\in\mathbb{N}^*} D_n = \emptyset$ also defines a filter which is called the circular filter of basis (D_n). Finally, for any $a \in \mathbb{K}$, the filter of neighborhoods of a is called circular filter of neighborhoods of a and such a cicular filter is said to be punctual.

Thanks for the works of B. Guennebaud and G. Garandel [15] which proved the important theorem as follows:

Theorem 1 ([15]). Each circular filter $\mathcal{F}$ on $\mathbb{K}$ defines a multiplicative semi-norm on $\mathbb{K}[x]$ which is a norm iff it is not punctual and the semi-norm is continuous with respect to the norm $\| .\|_D$ iff the filter is secant with D. Each circular filter $\mathcal{F}$ secant with D defines on $(R(D), \| \ .\ |_D)$ a continuous multiplicative semi-norm $\varphi_{\mathcal{F}}$ that has continuation to $H(D)$ and the mapping associating to each circular filter secant with D, its multiplicative semi-norm $\varphi_{\mathcal{F}}$ is a bijection from the set of circular filters secant with D onto the set of continuous multiplicative semi-norms on $H(D)$ and on $R(D)$.

Mult(H(D), $\| .\|_{\mathrm{D}}$) is compact with respect to the topology of pointwise convergence.

Let $R \in \mathbb{R}_+^*$ and let $f \in \mathcal{A}(d(a, R^-))$. Given $r \in]0, R[$, $f \in H(d(a, r))$, hence for any circular filter $\mathcal{F}$ secant with $d(a, r)$, $\varphi_{\mathcal{F}}(f)$ is defined. Moreover,

(i) If $s < R$ then we choice $b \in d(a, R)$.

(ii) If $\mathcal{F}$ is the circular filter of center b and diameter s, we put

$$\varphi_{b,s}(f) = \varphi_{\mathcal{C}}(f) = \| f \|_{d(a,r^-)}.$$

In special case, $a = 0$, we put

$$|f|(s) = \lim_{\mathcal{F}} |f(x)| = \lim_{|x| \to s, |x| \neq r} |f(x)|.$$

Noting that for any $r \in \mathbb{N}^*$ and $(s_1, \ldots, s_r) \in \mathbb{C}^r$, the polylogarithm at $(s_1, \ldots, s_r)$, namely by $\mathrm{Li}_{s_1,\ldots,s_r}(z)$, is defined by

$$\mathrm{Li}_{s_1,\ldots,s_r}(z) = \sum_{n_1 > \ldots > n_r > 0} \frac{z^{n_1}}{n_1^{s_1} \ldots n_r^{s_r}} \tag{2.2}$$

for any $|z| < 1$. Moreover, for any $|z| < 1$, the Maclaurin's expansion of $\dfrac{\mathrm{Li}_{s_1,\ldots,s_r}(z)}{1-z}$ is as follows:

$$\frac{\mathrm{Li}_{s_1,\ldots,s_r}(z)}{1-z} = \sum_{N \geq 0} \mathrm{H}_{s_1,\ldots,s_r}(N) z^N \tag{2.3}$$

where the coefficient $\mathrm{H}_{s_1,\ldots,s_r}(N)$ is called the harmonic sum at $(s_1, \ldots, s_r) \in \mathbb{C}^r$ and is well-defined by

$$\begin{aligned} \mathrm{H}_{s_1,\ldots,s_r}(0) &= 0, \\ \mathrm{H}_{s_1,\ldots,s_r}(N) &= \sum_{N \geq n_1 > \ldots > n_r > 0} \frac{1}{n_1^{s_1} \ldots n_r^{s_r}}, \quad \text{for any } N > 0. \end{aligned} \tag{2.4}$$

Then we get

$$|\mathrm{Li}_{s_1,\ldots,s_r}|(z) = \lim_{|x| \to z \neq 1} |\mathrm{Li}_{s_1,\ldots,s_r}(x)|. \tag{2.5}$$

This implies that, for any $q \in \mathbb{N}$ and $s_1, \ldots, s_r \in \mathbb{Z}$, then

$$\lim_{z \to +\infty} \frac{|\mathrm{Li}_{s_1,\ldots,s_r}|(z)}{z^q} = \lim_{|x| \to +\infty} \frac{|\mathrm{Li}_{s_1,\ldots,s_r}(x)|}{x^q} = \infty. \tag{2.6}$$

Recall that

Lemma 1 ([9, 10]). For any $f \in \mathcal{A}(\mathbb{K})$, the following statements are equivalent:

(i) $\displaystyle\lim_{r \to +\infty} \frac{|f|(r)}{r^q} = +\infty, \forall q \in \mathbb{N}$.

(ii) There doesn't exist $q \in \mathbb{N}$ such that $\displaystyle\lim_{r \to \infty} \frac{|f|(r)}{r^q} = 0$.

(iii) f is not a polynomial.

Then, as an immediate consequence of Lemma 1, we have

Proposition 1. For any $r \in \mathbb{N}^*$ and $s_1, \dots, s_r \in \mathbb{C}$, we have

(i) There doesn't exist $q \in \mathbb{N}$ such that $\lim_{r \to +\infty} \frac{|\operatorname{Li}_{s_1,\dots,s_r}|(r)}{r^q} = 0$.

(ii) $\operatorname{Li}_{s_1,\dots,s_r}(z)$ isn't a polynomial.

We can check some special cases for Proposition (1) as follows:

For any $s_1, \dots, s_r \in \mathbb{N}_+$, the polylogarithm at $(s_1, \dots, s_r)$ is not a polynomial. In fact, in this case, the polylogarithm can be presented like an algebraic combinatorial of $\{\log^n(z)\}_{n \in \mathbb{N}_+}$.
For any $s_1, \dots, s_r \in (\mathbb{Z} \backslash \mathbb{N}_+)$, the polylogarithm at $(s_1, \dots, s_r)$ is not also a polynomial with z but it is polynomial with rational coefficients on $\dfrac{1}{1-z}$.

On the other hand, each of polylogarithms is an analytic function on $d(0,1)$.

Proposition 2. For any $r \in \mathbb{R}$ such that $0 < r < 1$ and $s_1, \dots, s_k \in \mathbb{Z}$, we have

$$\lim_{N \to +\infty} |\mathrm{H}_{s_1,\dots s_k}(N) r^N| = 0.$$

Moreover, we also have

$$\| \operatorname{Li}_{s_1,\dots,s_r} \|_{d((0,r))} = \max_{n \in \mathbb{N}} |\frac{1}{n^{s_1}} \mathrm{H}_{s_2,\dots,s_r}(n)| r^n = \varphi_{\mathcal{F}}(\operatorname{Li}_{s_1,\dots,s_r}(z)) \tag{2.7}$$

where $\mathcal{F}$ is a circular filter secant with $d(0,r))$.

Proof. To prove this Proposition, we shall utilize the following results:

Lemma 2 ([9, 10]). Let $r \in \mathbb{R}_+^*$ and let $D = d(0,r)$. The set $H(D)$ is the set of power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ such that $\lim_{|n|_\infty \to \infty} |a_n| r^n = 0$ and

$$\|f\|_D = \max_{n \in \mathbb{N}} |a_n| r^n = \varphi_{\mathcal{F}}(f). \tag{2.8}$$

In particular, the norms $\| \,.\|_D$ and $\| \,.\|_{C(0,r)}$ are multiplicative and coincide on $H(C(0,r))$.

We now return to the proof of Proposition 2. Firtly, we will prove that

$$\lim_{N \to +\infty} |\mathrm{H}_{s_1,\dots s_k}(N) r^N| = 0$$

for any $r \in \mathbb{R}$ such that $0 < r < 1$ and $s_1, \dots, s_k \in \mathbb{N}$. From the definition of harmonic sums at mullti-indices, we need only consider the case $s_1, \dots, s_k \in$

$\mathbb{Z}_-$. Given $s_1, \ldots, s_k \in \mathbb{Z}_-$. Remarks that in this case, $\mathrm{H}_{s_1,\ldots,s_k}(N)$ is a polynomial of degree $m := -s_1 - \ldots - s_k + k$ on N. Suppose that

$$\mathrm{H}_{s_1,\ldots,s_k}(N) = \sum_{n=0}^{m} a_n N^n. \tag{2.9}$$

On the other hand, from $r \in (0,1)$, there is a constant $c > 0$ such that $r = \dfrac{1}{1+c}$. Then we get

$$r^N = \frac{1}{(1+c)^N} = \frac{1}{\sum\limits_{i=0}^{N} \binom{N}{i} c^i}. \tag{2.10}$$

Note that $\sum\limits_{i=0}^{N} \binom{N}{i} c^i$ is also a polynomial on N for any $i = 0, \ldots, N$. Then it is easily seen that

$$\lim_{N\to+\infty} |\mathrm{H}_{s_1,\ldots s_k}(N) r^N| = \lim_{N\to+\infty} \frac{\sum\limits_{t=0}^{m} a_t N^t}{\sum\limits_{i=0}^{N} \binom{N}{i} c^i} = 0. \tag{2.11}$$

On the other hand, for any $s_1, \ldots, s_r \in \mathbb{Z}$, we get

$$\mathrm{H}_{s_1,\ldots,s_r}(N) - \mathrm{H}_{s_1,\ldots,s_r}(N-1) = \frac{1}{N^{s_1}} \mathrm{H}_{s_2,\ldots,s_r}(N-1), \forall N \in \mathbb{N}^*.$$

Moreover, for any $z \in \mathbb{C}$ such that $|z| < 1$, we also have[5]

$$\begin{aligned}
\mathrm{Li}_{s_1,\ldots,s_r}(z) &= (1-z) \sum_{N\geq 0} \mathrm{H}_{s_1,\ldots,s_r}(N) z^N \\
&= \mathrm{H}_{s_1,\ldots,s_r}(0) + \sum_{N=1}^{\infty} \left[\mathrm{H}_{s_1,\ldots,s_r}(N) - \mathrm{H}_{s_1,\ldots,s_r}(N-1)\right] z^N
\end{aligned} \tag{2.12}$$

Hence, from the equation (2.11) and using Theorem 2, the other statements of Proposition 2 is proved. $\square$

A set $D \subset \mathbb{K}$ is said to be infraconnected if the closure of the set $\{|x-a| : x \in D\}$ is an interval for each $a \in \mathbb{K}$.

Let's D be an infraconnected subset of $\mathbb{K}$ and $f \in H(D)$. Let $\alpha \in D^\circ$ and let $r > 0$ such that $d(\alpha, r) \subset D$. Suppose that $f(x) = \sum\limits_{n=q}^{\infty} b_n (x-\alpha)^n$ whenever $x \in d(\alpha, r)$ with $b_q(\alpha) \neq 0$ and $q > 0$. Then α is called a zero of

[5] Remark that for any $s_1, \ldots, s_r \in \mathbb{Z}$ then $\mathrm{H}_{s_1,\ldots,s_r}(0) = 0$. However, $H_{1_{\mathbb{Z}}}(0) = 1$ where $1_{\mathbb{Z}}$ is called the empty index.

multiplicity order q or more simply, a zero of order q. In the same way, q is called the multiplicity order of α.

Corollary 1. The polylogarithms $\mathrm{Li}_{s_1,\ldots,s_k}(z)$ at $(s_1,\ldots,s_k)\in\mathbb{Z}_-^k$ is not invertible in $d(0,1^-)$.

Proof. Note that, for any $R\in\mathbb{R}_+$ and $f\in\mathcal{A}(d(a,R^-))$, the function f is invertible in $\mathcal{A}(d(a,R^-))$ if and only if f has no zero in $d(a,R^-)$ [9]. Moreover, since the definition of polylogarithms, $z=0$ is a solution of $\mathrm{Li}_{s_1,\ldots,s_k}(z)$. Thus, $\mathrm{Li}_{s_1,\ldots,s_k}(z)$ is not invertible in $\mathcal{A}(d(a,1^-))$. □

Recall that

Theorem 2. [9, 10] Let $R\in\mathbb{R}_+^*$. The $\mathbb{K}-$subalgebra $\mathcal{A}_b(d(0,R^-))$ of $\mathcal{A}(d(0,R^-))$ is a Banach $\mathbb{K}-$algebra with respect to the norm $\|\ .\|_{d(0,R^-)}$. Further, this norm is multiplicative and satisfies

$$\|f\|_{d(0,R^-)}=\lim_{r\to R}|f|(r)=\sup_{n\in\mathbb{N}}|a_n|R^n.$$

Let $f(x)=\sum\limits_{n=0}^{\infty}a_nx^n\in\mathcal{A}(d(0,R^-))$. Then f is bounded in $d(0,R^-)$ iff so is the sequence $(|a_n|R^n)_{n\in\mathbb{N}}$. Moreover, if f is bounded then $\|f\|_{d(0,R^-)}=\sup\limits_{n\in\mathbb{N}}|a_n|R^n$.

Using the statements as in the proof of Proposition 2 one again, for any $s_1,\ldots,s_r\in\mathbb{Z}$, we obtain that

$$\mathrm{Li}_{s_1,\ldots,s_r}(z)=\mathrm{H}_{s_1,\ldots,s_r}(0)+\sum_{N=1}^{\infty}\left[\mathrm{H}_{s_1,\ldots,s_r}(N)-\mathrm{H}_{s_1,\ldots,s_r}(N-1)\right]z^N$$

where, for any $R\in(0,1)$, we have

$$\lim_{N\to\infty}|\mathrm{H}_{s_1,\ldots,s_r}(N)-\mathrm{H}_{s_1,\ldots,s_r}(N-1)|\ R^N=0.$$

This means that the sequence $\left(|\mathrm{H}_{s_1,\ldots,s_r}(N)-\mathrm{H}_{s_1,\ldots,s_r}(N-1)|\ R^N\right)_{N\geq 1}$ is bounded. Then the polylogarithm function $\mathrm{Li}_{s_1,\ldots,s_r}(z)$ is bounded in $d(0,R^-)$ for any $R\in(0,1)$. Thus we get

$$\begin{aligned}\|\,\mathrm{Li}_{s_1,\ldots,s_r}\|_{d(0,R^-)} &= \lim_{r\to R}|\,\mathrm{Li}_{s_1,\ldots,s_r}|(r)\\ &= \sup_{n\in\mathbb{N}}(|a_n|\ R^n)\\ &= \sup_{n\in\mathbb{N}}\left(|\mathrm{H}_{s_1,\ldots,s_r}(N)-\mathrm{H}_{s_1,\ldots,s_r}(N-1)|\ R^N\right)=0.\end{aligned}$$

In fact, for $R = 1$, this is also valid if $\lim\limits_{N\to\infty} \mathrm{H}_{s_1,\ldots,s_r}(N) < \infty$. However, in the general case, this is false. For example, we have

$$\mathrm{H}_{-1}(N) = \sum_{n=1}^{N} n = \frac{N(N+1)}{2}$$

for any $N \in \mathbb{N}$. Hence

$$\lim_{N\to\infty} |\mathrm{H}_{-1}(N) - \mathrm{H}_{-1}(N-1)| = \lim_{N\to\infty} |\frac{N(N+1)}{2} - \frac{N(N-1)}{2}| = \lim_{N\to\infty} N = +\infty.$$

This means that we can not conclude the value of $\| \mathrm{Li}_{-1} \|_{d(0,1^-)}$ by using Theorem 2.

3. The Mittag-Leffler theorem and Polylogarithms

Let D to be a closed infraconnected subset of $\mathbb{K}$. We recall again now a following important result.

Theorem 3. [9] Let $f \in H(D)$. There is a unique sequence of holes $(T_n)_{n\in\mathbb{N}^*}$ of D and a unique sequence $(f_n)_{n\in\mathbb{N}}$ in $H(\mathbb{K}\backslash T_n)$ such that $f_0 \in H(\widetilde{D})$ for any $n > 0$, $\lim\limits_{n\to\infty} f_n = 0$ satisfying further

$$f = \sum_{n=0}^{\infty} f_n \text{ and } \|f\|_D = \sup_{n\in\mathbb{N}} \|f\|_D. \tag{3.1}$$

For every hole $T_n = d(a_n, r_n^-)$, we have

$$\|f_n\|_D = \|f_n\|_{\mathbb{K}\backslash T_n} = \varphi_{a_n,r_n}(f) \leq \|f\|_D. \tag{3.2}$$

If D is bounded and if $\widetilde{D} = d(a, r)$, we have

$$\|f_0\|_D = \|f_0\|_{\widetilde{D}} = \varphi_{a,r}(f_0) \leq \varphi_{a,r}(f) \leq \|f\|_D. \tag{3.3}$$

Let $D' = \widetilde{D}\backslash \left(\bigcup\limits_{n=1}^{\infty} T_n\right)$. Then f belongs to $H(D')$ and its decomposition in $H(D')$ is given again by (3.1) and f satisfies $\|f\|_{D'} = \|f\|_D$.

Using Theorem 3, for any $f \in H(D)$, we consider the series $\sum\limits_{n=1}^{\infty} f_n$ which is as in the equation (3.1). Each T_n is called a $f-$hole and

$f_n, n \in \mathbb{N}^*$ is called the Mittag - Leffler term of f associated to T_n.
f_0 is called the principal term of f.

For each $f-$hole T of D, the Mittag - Leffler term of f associated to T is denoted by $\overset{=}{f_T}$ whereas the principal term of f will be denoted by $\overset{=}{f_0}$.

The series $\sum\limits_{n=1}^{\infty} f_n$ is called the Mittag - Leffler term of f on the infraconnected set D.

Example 1. Given $f \in H(d(0,1^-))$ and denoted $(d(\alpha_m,1^-))_{m\in\mathbb{N}_+}$ to be the set of $f-$holes. Then we recall that f can be rewriten by

$$f = \sum_{n=0}^{+\infty} a_{n,0}x^n + \sum_{m,n\in\mathbb{N}_+} \frac{a_{n,m}}{(x-\alpha_m)^n} \tag{3.4}$$

where $\lim_{n\to\infty} a_{n,0} = 0$, $\lim_{n\to\infty} |a_{n,m}| = 0, \forall m \in \mathbb{N}_+$ and $\lim_{m\to\infty}(\sup_{n\in\mathbb{N}_+} |a_{n,m}|) = 0$. Moreover, we get

$$\|f\|_{d(0,1^-)} = \sup_{m\in\mathbb{N},n\in\mathbb{N}_+} |a_{n,m}|. \tag{3.5}$$

Conversely, every function f of the form (3.4), with α_m satisfying

$$|\alpha_m| = |\alpha_j - \alpha_m| = 1, \forall m \neq j,$$

belongs to $H(d(0,1^-))$. The norm $\| \,.\|_{d(0,1^-)}$ is multiplicative and equal to $\varphi^{0,1}$.

Taking $f(z) = \mathrm{Li}_P(z)$ for any $P \in (\mathbb{C}[x_0^*, x_1^*, (-x_0)^*], ш, 1_{X^*})$. Recall that for any $z \in \mathbb{C}$ such that $|z| < 1$, we have $\mathrm{Li}_{(-x_0)^*}(z) = z$, $\mathrm{Li}_{x_0^*}(z) = \dfrac{1}{z}$ and $\mathrm{Li}_{x_1^*}(z) = \dfrac{1}{1-z}$ [5]. Hence, for any such series P, we have

$$\mathrm{Li}_P(z) = \sum_{n\geq 0} a_{P,n}z^n + \sum_{n\in J\subset\mathbb{N}^*} \frac{a_{n,0}}{z^n} + \sum_{n\in I\subset\mathbb{N}^*} \frac{a_{n,1}}{(1-z)^n}$$

where I, J are the finite sets of indices. From $|1| = |0-1| = 1$, we obtain that

$$\| \mathrm{Li}_P \|_{d(0,1^-)} = \sup_{m\in\{0,1\},n\in\mathbb{N}_+} |a_{n,m}| = \max_{n\in\mathbb{N}_+}(\max(|a_{n,0}|, |a_{n,1}|)). \tag{3.6}$$

For example, taking $P = (-x_0)^* ш (4x_0)^* ш x_1^*$, then we have

$$\mathrm{Li}_P(z) \;=\; \mathrm{Li}_{(-x_0)^*}(z)\,\mathrm{Li}_{(4x_0)^*}(z)\,\mathrm{Li}_{x_1^*}(z) = z\frac{1}{z^4}\frac{1}{1-z} = \frac{1}{z^3(1-z)} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + \frac{1}{1-z}.$$

Hence $\| \mathrm{Li}_{(-x_0)^* ш (4x_0)^* ш x_1^*} \|_{d(0,1^-)} = 1$. Note that, from the definition of the shuffle product [5], in this case, we have $P = (3x_0 + x_1)^* = \sum_{n=0}^{\infty}(3x_0 + x_1)^n$.

In particular, for any $s_1, \ldots, s_r \in (\mathbb{Z}\backslash\mathbb{N}^*)$, the polylogarithm $\mathrm{Li}_{s_1,\ldots,s_r}(z)$ is a polynomial on $\dfrac{1}{1-z}$ of degrre $m = |s_1 + s_2 + \ldots + s_r| + r$ with coefficients in $\mathbb{Q}$ [5]. This means that, for any $z \in \mathbb{C}$ such that $|z| < 1$, we get

$$\mathrm{Li}_{s_1,\ldots,s_r}(z) = \sum_{k=0}^{m} \frac{a_{s_1,\ldots,s_r,k}}{(1-z)^k},$$

where $a_k \in \mathbb{Q}$ for any $k = 0, \ldots, m$. In fact, there was an algorithm to calculate the coefficients $a_{s_1,\ldots,s_r,k}, k \in \mathbb{N}$ [5]. And then we can give an useful algorithm to determine the value of norm $\| \mathrm{Li}_{s_1,\ldots,s_r} \|_{d(0,1^-)}$ for any $s_1, \ldots, s_r \in (\mathbb{Z}\backslash\mathbb{N}^*)$.

Definition 1. Let $f \in H_b(D)$ and let T be a hole of D. For any $a \in T$, let $f_T(z) = \sum_{n=0}^{\infty} \dfrac{b_n(a)}{(z-a)^n}$. We set $\mathrm{res}(f, T) = b_1(a)$ and call the residue of f on the hole T.

In fact, for any $f \in H_b(\mathbb{K}\backslash d(a, r^-)$ and for each $\alpha \in d(a, r^-)$, then we can rewriten $f(x)$ by $f(x) = \sum_{n=0}^{\infty} \dfrac{b_n(\alpha)}{(x-\alpha)^n}$. Moreover, the coefficient $b_1(\alpha)$ doesn't depend on α in $d(\alpha, r^-)$.

Example 2. Let s_1, s_2 be the non-positive integer. We showed that

$$\mathrm{Li}_{s_1,s_2}(z) = \sum_{k=0}^{|s_1+s_2|+2} \frac{a_k^{-s_1,-s_2}}{(1-z)^k} \tag{3.7}$$

where $a_k^{t_1,t_2}$ are the rational numbers which are calculated by the following algorithm:

$$\text{If } t_1 = 0 \text{ then } a_k^{t_1,t_2} = \begin{cases} a_{k-1}^{t_2} & \text{for any } k = t_1 + t_2 + 1 \\ a_{k-1}^{t_2} - a_k^{t_2} & \text{for every } 1 \le k \le t_1 + t_2 \\ -a_k^{t_2} & \text{for any } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$\text{If } t_1 > 0 \text{ then } a_k^{t_1,t_2} = \begin{cases} (k-1)a_{k-1}^{t_1-1,t_2} & \text{for any } k = t_1 + t_2 + r + 1 \\ (k-1)a_{k-1}^{t_1-1,t_2} - ka_k^{t_1-1,t_2} & \text{for any } 2 \le k \le t_1 + t_2 + r \\ -a_k^{t_1-1,t_2} & \text{for any } k = 1 \\ 0 & \text{otherwise.} \end{cases}$$

We denote () to be the empty index. Recalls that $a_0^{()} = 1$ and $a_k^{()} = 0, \forall k > 0$. Then we have

$$\mathrm{res}(\mathrm{Li}_{s_1,s_2}, T) = a_1^{|s_1|,|s_2|}$$

with $T = \{1\}$. Moreover, we get:

$$\| \mathrm{Li}_{s_1,s_2} \|_{d(0,1^-)} = \sup_{k \in (\mathbb{N} \cap [0, |s_1+s_2|+2])} |a_k^{|s_1|,|s_2|}| = \max_{k \in (\mathbb{N} \cap [0, |s_1+s_2|+2])} |a_k^{|s_1|,|s_2|}|.$$

In more general case, for any $s_1, \ldots, s_r \in (\mathbb{Z}\backslash\mathbb{N}^*)$, the polylogarithm $\mathrm{Li}_{s_1,\ldots,s_r}(z)$ is a polynomial of degree $|s_1 + \ldots + s_r| + r$ on $\dfrac{1}{1-z}$ with coefficients in $\mathbb{Q}$ [4, 5]. This means that

$$\mathrm{Li}_{s_1,\ldots,s_r}(z) = \sum_{k=0}^{|s_1+\ldots+s_r|+r} \frac{a_k^{|s_1|,\ldots,|s_r|}}{(1-z)^k} \tag{3.8}$$

where for any $t_1, \ldots, t_r \in \mathbb{N}$, the coefficients $a_k^{t_1,\ldots,t_r}$ are calculated by the following algorithm:

If $t_1 = 0$ then

$$a_k^{t_1,\ldots,t_r} := \begin{cases} a_{k-1}^{t_2,\ldots,t_r} & \text{for } k = t_1 + \ldots + t_r + r + 1, \\ a_{k-1}^{t_2,\ldots,t_r} - a_k^{t_2,\ldots,t_r} & \text{for } 1 \leq k \leq t_1 + \ldots + t_r + r, \\ -a_k^{t_2,\ldots,t_r} & \text{for } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

If $t_1 > 0$ then

$$a_k^{t_1,\ldots,t_r} := \begin{cases} (k-1)a_{k-1}^{t_1-1,t_2,\ldots,t_r} & \text{for } k = t_1 + \ldots + t_r + r + 1, \\ (k-1)a_{k-1}^{t_1-1,t_2,\ldots,t_r} - ka_k^{t_1-1,t_2,\ldots,t_r} & \text{for } 2 \leq k \leq t_1 + \ldots + t_r + r, \\ -a_k^{t_1-1,t_2,\ldots,t_r} & \text{for } k = 1, \\ 0 & \text{for otherwises.} \end{cases}$$

Proposition 3. Then we obtain that

$$\operatorname{res}(\mathrm{Li}_{s_1,\ldots,s_r}, T) = \max_{k \in (\mathbb{N} \cap [0, |s_1 + \ldots + s_r| + r])} |a_k^{|s_1|,\ldots,|s_r|}| \tag{3.9}$$

for any $s_1, \ldots, s_r \in (\mathbb{Z} \backslash \mathbb{N}^*)$ and $T = \{1\}$.

Note that, for any $n \in \mathbb{N}^*$ and $s_1; \ldots; s_r \in \mathbb{Z}$, we get

$$\mathrm{Li}_{s_1;s_2;\ldots;s_r}^{(n)}(z) = \frac{1}{z^n} \mathrm{Li}_{s_1-n;s_2;\ldots;s_r}(z). \tag{3.10}$$

By a result of the classical upper bound of $|f'|(r)$ in function of $|f|(r)$[9], we obtain that

Proposition 4. For any $n \in \mathbb{N}$; $t \in (0,1)$ and $s_1, \ldots; s_r \in \mathbb{Z}$, we have

$$|\mathrm{Li}_{s_1;\ldots;s_r}|(t) \geq \frac{t^n}{n!}|g|(t) \tag{3.11}$$

where $g(t) = \dfrac{1}{t^n} \mathrm{Li}_{s_1-n;\ldots;s_r}(t)$.

4. Polylogarithm as a meromorphism function

In this section, we will start from the lemma as follows:

Lemma 3. Let $E = \mathbb{K} \setminus d(a, r^-)$ with $a \in \mathbb{K}$ and $r > 0$. Let $f \in H(E)$ be invertible in $H(E)$. Then $f(x)$ is a Laurent series of the form $\sum_{n=-\infty}^{q} a_n(x-a)^n$ with $|a_q|r^q > |a_n|r^n$ for every $n < q$.

From Lemma 3, we obtain that

Theorem 4. Let $T = d(0, r^-)$ with $a \in \mathbb{K}$ and $r > 0$, let $E = \mathbb{K} \setminus T$ and take $b \in T$. For any $s_1, \ldots, s_r \in \mathbb{N}$, there exists $\lambda \in \mathbb{K}$, $q \in \mathbb{Z}$ and $h \in H(E)$ invertible in $H(E)$, satisfying $\|h - 1\|_E < 1$, $\lim_{|x| \to +\infty} h(x) = 1$ and $\mathrm{Li}_{-s_1,\ldots,-s_r}(x) = \lambda (x - b)^q h(x)$. Moreover, λ, q are respectively unique, satisfying those relations. Further, both λ, q do not depend on b in T.

Proof. For any $s_1, \ldots, s_r \in (\mathbb{Z} \backslash \mathbb{N}_+)$ and $z \in \mathbb{C} \cup d(0, 1^-)$, the polylogarithm functions $\mathrm{Li}_{s_1,\ldots,s_r}(z)$ was showed as a polylomial on $\dfrac{1}{z-1}$. Then we can extend these functions as a meromorphism functions on $E = \mathbb{C} \backslash d(0, 1)$.

In fact, for any $s_1, \ldots, s_r \in \mathbb{N}$, we get

$$\mathrm{Li}_{-s_1,\ldots,-s_r}(z) = \sum_{k=0}^{|s_1+\ldots+s_r|+r} \frac{a_k^{|s_1|,\ldots,|s_r|}}{(1-z)^k} \tag{4.1}$$

where the coefficients $a_k^{|s_1|,\ldots,|s_r|}$ are defined - well as in the equation (3.8). Setting now

$$k_0 = \min k | a_k^{|s_1|,\ldots,|s_r|} \neq 0.$$

Then we get $q = k_0$, $\lambda = a_{k_0}^{|s_1|,\ldots,|s_r|}$ and

$$h(z) = \sum_{k=0}^{|s_1+\ldots+s_r|+r} \frac{a_k^{|s_1|,\ldots,|s_r|}}{a_{k_0}^{|s_1|,\ldots,|s_r|}(1-z)^{k-k_0}}.$$

□

Example 3. For example, for any $|z| < 1$, we have

$$\mathrm{Li}_{-3}(z) = \frac{6}{(-1+z)^4} + \frac{12}{(-1+z)^3} + \frac{7}{(-1+z)^2} + \frac{1}{(-1+z)}. \tag{4.2}$$

Then we can extend this function to a function on $\mathbb{C}$ which has the same value with $\mathrm{Li}_{-3}(z)$ for any $|z| < 1$. Of course, this extension function is meromorphism and invertible on $E = \mathbb{C} \backslash d(0, 1)$. In fact, we have

$$Li_{-3}(z) = (z-1)\left[\frac{6}{(-1+z)^3} - \frac{12}{(-1+z)^2} + \frac{7}{(-1+z)} + 1\right] = (z-1)h(z)$$

where $h(z) = \dfrac{6}{(-1+z)^3} - \dfrac{12}{(-1+z)^2} + \dfrac{7}{(-1+z)} + 1$. It is easily seen that

$\lambda = q = 1$.

$\lim\limits_{|z| \to \infty} h(z) = \lim\limits_{|z| \to \infty} \left[\dfrac{6}{(-1+z)^3} - \dfrac{12}{(-1+z)^2} + \dfrac{7}{(-1+z)} + 1\right] = 1.$

$h(z) = \dfrac{6}{(-1+z)^3} - \dfrac{12}{(-1+z)^2} + \dfrac{7}{(-1+z)} + 1 = \dfrac{z^3 + 4z^2 + z}{(z-1)^3}$ is invertible in $H(E)$.

$\|h-1\|_E < 1$.

Hence, there exists $\lambda \in \mathbb{K}$, $q \in \mathbb{Z}$ and $h \in H(E)$ invertible in $H(E)$, satisfying $\|h-1\|_E < 1$, $\lim\limits_{|z|\to+\infty} h(z) = 1$ and

$$\mathrm{Li}_{-3}(z) = \lambda(z-1)^q h(z).$$

Definition 2. Let $\mathbb{K}\backslash d(a,r^-)$ with $a \in \mathbb{K}$ and $r > 0$. Let $f \in H(E)$ be invertible in $H(E)$ and let $\lambda(x-a)^q h(x)$ be the factorization given in Theorem 4. The integer q is called the index of f associated to $d(a,r^-)$, namely by $m(f, d(a,r^-))$.

If $\lambda = 1$ the element f is called a pure factor associated to $d(a,r^-)$. Let $\mathcal{G}^T$ be the group of invertible elements of $H(\mathbb{K}\backslash T)$.

It is easy seen that $\mathrm{Li}_{-3}(z)$ is a pure factor associated to $d(0,1^-)$. Moreover, for every $n \in \mathbb{N}$, the polylogarithm $\mathrm{Li}_{-n}(z)$ is a pure factor associated to $d(0,1^-)$.

Proposition 5. Let $T = d(a,r^-)$. The set of pure factors associated to T is a sub-multiplicative group of the group $\mathcal{G}^T$. Further, every element of $\mathcal{G}^T$ is of the form λh with h a pure factor associated to T and $\lambda \in \mathbb{K}^*$.

We denote by $\mathcal{M}(\mathbb{K})$ the field of fractions of $\mathcal{A}(\mathbb{K})$. The element of $\mathcal{M}(\mathbb{K})$ is called meromorphism function in $\mathbb{K}$.

Given $a \in \mathbb{K}$ and $r \in \mathbb{R}_+$. We denote by $\mathcal{M}(d(a,r^-))$ (resp. $\mathcal{M}_b(d(a,r^-))$ and $\mathcal{M}_u(d(a,r^-))$) field of fractions of $\mathcal{A}(d(a,r^-))$ (resp. $\mathcal{A}_b(d(a,r^-))$ and the set $\mathcal{M}(d(a,r^-))\backslash\mathcal{M}_b(d(a,r^-))$). The element of $\mathcal{M}(d(a,r^-))$ is called meromorphism function in $d(a,r^-)$.

In fact, each of polylogarithms

$$\mathrm{Li}_{s_1,\ldots,s_r}(z) = \sum_{n_1>n_2>\ldots>0} \frac{z^{n_1}}{n_1^{s_1}\ldots n_r^{s_r}}$$

is a meromorphism function in $d(0,1^-)$. Moreover, by the continuation extension, we can extend these functions as a meromorphism function in $\mathbb{C}$. For example, we have

$$\mathrm{Li}_{-2,-1}(z) := \sum_{n_2>n_1>0} \frac{z^{n_1}}{n_1^{s_1} n_2^{s_2}} = \frac{z^4+7z^3+4z^2}{(1-z)^5}, \forall |z| < 1. \tag{4.3}$$

Hence $\mathrm{Li}_{-2,-1}(z)$ is a meromorphism function in $d(0,1^-)$. Extending this function to $\mathbb{C}$ as a fraction function on $\mathbb{C}$, meaning

$$\mathrm{Li}_{-2,-1}(z) = \frac{z^4+7z^3+4z^2}{(1-z)^5}, \forall z \in \mathbb{C}.$$

Then $\mathrm{Li}_{-2,-1}(z)$ is a meromorphism function in $\mathbb{C}$.

Now, we define a divisor in $\mathbb{K}$ (resp. a divisor in a disk $d(a, R^-)$) to be a map $T : \mathbb{K} \to \mathbb{N}$ (resp. $T : d(a, R^-) \to \mathbb{N}$ whose support is countable and has a finite intersection with each disk $d(a, r), \forall r > 0$ (resp. $\forall\, r \in (0, R)$). This means that, for any $f \in \mathcal{M}(\mathbb{K})$ (resp. of $d(a, R^-)$), we can define the divisor $\mathcal{D}(f)$ of f as $\mathcal{D}(f)(\alpha) = 0$ whenever $f(\alpha) \neq 0$ and $\mathcal{D}(f)(\alpha) = s$ when f has a zero of order s at α.

Given $K \equiv \mathbb{R}$. For any $s_1, \ldots, s_r \in \mathbb{N}$ then $\mathrm{Li}_{-s_1,\ldots,-s_r}(z)$ is a polynomial of degree $s_1 + \ldots + s_r + r$ on $\dfrac{1}{1-z}$. Thus this function can be rewriten by

$$\mathrm{Li}_{-s_1,\ldots,-s_r}(z) = \frac{P(x)}{(1-z)^{s_1+\ldots+s_r+r}} \tag{4.4}$$

where $P(x) \in \mathbb{R}[x]$ such that $\deg(P(x)) = s_1 + \ldots + s_r + r - 1$. Since the number of roots of $P(x)$ is not exceeded $\deg(P(x))$, we have

$$\sharp\mathrm{supp}(\mathcal{D}(\mathrm{Li}_{-s_1,\ldots,-s_r})) \leq s_1 + \ldots + s_r + r - 1. \tag{4.5}$$

For example, we have

$$\mathrm{Li}_{-2,-3}(z) = \frac{z^6 + 29z^5 + 93z^4 + 53z^3 + 4z^2}{(1-z)^7}, \forall |z| < 1.$$

Setting by

$$P(z) = z^6 + 29z^5 + 93z^4 + 53z^3 + 4z^2 = z^2(z^4 + 29z^3 + 93z^2 + 53z + 4).$$

Then the set of real roots of $P(z)$ is $\{z_1, z_2, z_3, z_4, 0\}$ where $z_1 \in (-26; -25)$; $z_2 \in \left(-3; -\dfrac{5}{2}\right)$; $z_3 \in \left(-1; -\dfrac{1}{2}\right)$ and $z_4 \in \left(-\dfrac{1}{2}; 0\right)$. Hence, on the domain of $\mathrm{Li}_{-2,-3}(z)$, we get

$$\mathcal{D}(\mathrm{Li}_{-2,-3})(z) = \begin{cases} 2 & \text{if } z = 0, \\ 1 & \text{if } z = z_3 \text{ or } z = z_4, \\ 0 & \text{otherwise.} \end{cases} \tag{4.6}$$

On the other hand, we have

Lemma 4. Let $f \in \mathcal{M}(\mathbb{K})$. There exist $h \in \mathcal{A}(\mathbb{K})$ such that $\mathcal{D}(h) = \mathcal{D}(f)$ and then $l = \dfrac{h}{f} \in \mathcal{A}(\mathbb{K})$. Then $\mathcal{D}(\dfrac{1}{f}) = \mathcal{D}(l)$ and we can write f in the form $\dfrac{h}{l}$ with $h, l \in \mathcal{A}(\mathbb{K})$ having no common zero.

Since Lemma 4, it is easily seen that $P(1) \neq 0$ where the polynomial $P(x)$ is determinated as in (4.4).

Similarly, given an ideal I of $\mathcal{A}(\mathbb{K})$ (resp. of $\mathcal{A}(d(0, R^-))$), we denote by $\mathcal{D}(I)$ the lower bound of the $D(f)$ with $f \in I$ and $\mathcal{D}(I)$ is called the divisor of I.

To end this section, given $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}$ (resp. $f \in \mathcal{M}_u(d(0, R^-))$ which has a pole α of order q and let $f(x) = \sum\limits_{k=-q}^{-1} a_k(x-\alpha)^k + h(x)$ with $a_{-q} \neq 0$ and $h \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}_u(d(0, R^-))$ and h holomorphic at α. The coefficient a_{-1} is called residue of f at α and denoted by $\operatorname{res}(f, \alpha)$.

Theorem 5. Let $a \in \mathbb{K}$, $R \in \mathbb{R}_+^*$, $f \in \mathcal{M}(d(a, R^-))$ and $r \in (0, R)$. Let α_j, $j \leq j \leq q$ be the poles of f in $d(a, r)$, let $\rho \in (0, \min\limits_{i \neq j} |\alpha_i - \alpha_j|)$ and for each $j = 1, \ldots, q$, let $\rho_j \in (0, \rho)$ and setting $T_j = d(\alpha_j, \rho_j^-)$. Denote $D = d(\alpha, r) \setminus (\cup_{j=1}^q T_j)$. Then f belongs to $H(D)$ and $\operatorname{res}(f, \alpha_j) = \operatorname{res}(f, T_j)$ for any $j = 1, \ldots, q$.

As a direct consequence of Theorem 5, let $f \in H_b(D)$ be a meromorphism function in $T = d(b, r^-)$ and admits only one pole b inside T. Let q be the multiplicity order of b. Then the Mittag - Leffler term of f associated to T is of the form $\sum\limits_{j=1}^{q} \dfrac{a_j}{(x-b)^j}$ with $a_q \neq 0$ and also is of the form $\dfrac{P(x)}{(x-a_j)^q}$ where $P(x)$ is a polynomial of degree $s < q$. Moreover, it does not depend on r when r tends to 0.

5. Some other computations on Polylogarithms

Given a divisor $T = (a_n, q_n)_{n \in \mathbb{N}}$ with $0 < |a_n| \leq r$, $\forall n \in \mathbb{N}^*$, we denote by $\overline{T}$ the divisor $(a_n, 1)_{n \in \mathbb{N}}$. Let $f \in \mathcal{A}(d(a, R^-))$ and let $(a_n, q_n)_{n \in \mathbb{N}} = \mathcal{D}(f)$. Then $\omega_{a_n}(f) = q_n$ for every $n \in \mathbb{N}$ and $\omega_\alpha(f) = 0$ for every $\alpha \in d(a, R^-) \setminus \{a_n | n \in \mathbb{N}\}$.

For any $f = \dfrac{h}{l} \in \mathcal{M}(\mathbb{K})$ (resp. $f = \dfrac{h}{l} \in \mathcal{M}(d(a, R^-))$) and each $\alpha \in \mathbb{K}$ (resp. $f = \dfrac{h}{l} \in d(a, R^-)$), the number $\omega_\alpha(h) - \omega_\alpha(l)$ does not depend on the functions h, l choosed to make $f = \dfrac{h}{l}$. Thus we can generalize the notation by setting

$$\omega_\alpha(f) = \omega_\alpha(h) - \omega_\alpha(l).$$

Note that

If $\omega_\alpha(f)$ is an positive integer $q > 0$ then α is called a zero of f of order q.
If $\omega_\alpha(f)$ is an negative integer $q < 0$ then α is called a pole of f of order q.
If $\omega_\alpha(f) \geq 0$ then f is holomorphic at α.

Let $(a_n)_{1\leq n\leq \sigma(r)}$ be the finite sequence of zeros of f such that $0<|a_n|\leq r$, of respective order s_n. Setting now

$$Z(r,f)=\max(\omega_0(f),0)\log r+\sum_{n=1}^{\sigma(r)} s_n(\log r-\log|a_n|) \tag{5.1}$$

where we define that

$$\omega_0(f)=\begin{cases} q & \text{if } 0 \text{ is a multiple solution of order } q \text{ of } f \\ -q & \text{if } 0 \text{ is a tipping point of order } q \text{ of } f \end{cases}. \tag{5.2}$$

Moreover, $Z(r,f)$ is called the counting function of zeros of f in $d(0,r)$, counting multiplicity.

Example 4. We have that

$$\mathrm{Li}_{-1,-2,-2}(z)=\frac{z^7+34z^6+133z^5+100z^4+12z^3}{(1-z)^8}, \forall z\in\mathbb{C} \text{ such that } |z|<1.$$

Remark again that

$$z^7+34z^6+133z^5+100z^4+12z^3=z^3\left(z^4+34z^3+133z^2+100z+12\right).$$

This implies that

$$\omega_0(\mathrm{Li}_{-1,-2,-2}(z))=3.$$

By using the computer, it is easily seen to prove that the polynomial

$$z^4+34z^3+133z^2+100$$

has 2 real solutions $z\in\mathbb{R}$ such that $|z|<1$, namely by z_1,z_2 where $z_1\in\left(-1;-\frac{1}{2}\right)$ and $z_2\in\left(-\frac{1}{2};0\right)$. Thus, for any $r\in(0,1]$, we obtain that

$$Z(r,\mathrm{Li}_{-1,-2,-2}(z))=\begin{cases} 3\log r & \text{if} \quad r<|z_2| \\ 4\log r-\log(z_2) & \text{if} \quad |z_2|\leq r<|z_1|. \\ 5\log r-\log(z_1)-\log(z_2) & \text{if} \quad |z_1|\leq r<1 \end{cases}$$

Setting now

$$\overline{\omega_0}(f)=\begin{cases} 0 \text{ if } \omega_0(f)\leq 0 \\ 1 \text{ if } \omega_0(f)>0 \end{cases} \tag{5.3}$$

where f is a meromorphism function. In addition, we also denote by $\overline{Z}(r,f)$ the counting function of zeros of f without multiplicity:

$$\overline{Z}(r,f)=\overline{\omega_0}(f)\log r+\sum_{n=1}^{\sigma(r)}(\log r-\log|a_n|) \tag{5.4}$$

and this symbol is called the counting function of zeros of f in $d(0,r)$ ignoring multiplicity.

Example 5. By Example 4, for any $r \in (0,1]$, we obtain that

$$\overline{Z}\left(r,\ \mathrm{Li}_{-1,-2,-2}\right) = \begin{cases} \log r & \text{if} \quad r < |z_2| \\ 2\log r - \log(z_2) & \text{if} \quad |z_2| \le r < |z_1|. \\ 3\log r - \log(z_1) - \log(z_2) & \text{if} \quad |z_1| \le r \le 1 \end{cases}$$

Denote that the finite chain $\{b_n\}_{n=1}^{\tau(r)}$ of poles of function f such that $0 < |b_n| \le r$, with respective multiplicity order t_n. We call

$$N(r,f) = \max\left(-\omega_0(f), 0\right)\log r + \sum_{n=1}^{\tau(r)} t_n \left(\log r - \log|b_n|\right)$$

to be the counting function of the poles of f, counting multiplicity.

At the moment, one denotes that

$$\overline{\overline{\omega_0}}(f) = \begin{cases} 0 & \text{if } \omega_0(f) \ge 0 \\ 1 & \text{if } \omega_0(f) \le -1 \end{cases} \tag{5.5}$$

and the counting function of the poles of f, ignoring multiplicity is well-defined by

$$\overline{N}(r,f) = \overline{\overline{\omega_0}}(f)\ \log(r) + \sum_{n=1}^{\tau(r)} \left(\log(r) - \log(|\ b_n\ |)\right). \tag{5.6}$$

On the other hand, the Nevanlinna function[6] $T(r,f)$ in I or J is defined by

$$T(r,f) = \max\left(Z(r,f); N(r,f)\right). \tag{5.7}$$

Note that the functions Z, N, T aren't changed, up to an additive constant, if we change the origin.

Lemma 5 ([9]). Let $\widehat{\mathbb{K}}$ be a complete algebraically closed extension of $\mathbb{K}$ whose absolute value extends that of $\mathbb{K}$ and let $f \in \mathcal{M}(\mathbb{K})$ (resp. let $f \in \mathcal{M}(d(0,R^-))$). Let $\widehat{d}(0,R) = \left\{x \in \widehat{\mathbb{K}} \mid\ \mid x \mid < R\right\}$. The meromorphic function $\widehat{f}$ defined by f in $\widehat{d}(0,R)$ has the same Nevanlinna functions as f.

One notes also that

Lemma 6 ([9]). Let $\alpha_1, \dots, \alpha_n \in \mathbb{K}$ be pairwise distinct and $f \in \mathcal{M}\left(d(0,R^-)\right)$. Suppose that

$$P(u) = \prod_{i=1}^{n} \left(u - \alpha_i\right).$$

Then $Z\left(r, P(f)\right) = \sum\limits_{i=1}^{n} Z\left(r, f - \alpha_i\right)$ and $\overline{Z}\left(r, P(f)\right) = \sum\limits_{i=1}^{n} \overline{Z}\left(r, f - \alpha_i\right)$.

[6]It is called characteristic function of f.

Recall that, for any $n \in \mathbb{N}$,

$$\mathrm{Li}_{-n}(z) = \frac{z}{(1-z)^{n+1}} A_n(z). \tag{5.8}$$

Note that $A_{n+1}(z), \forall n \in \mathbb{N}$ is a polynomial of degree n. Moreover, this polynomial has only negative and simple real roots, a result due to Frobenius. By Lemma 6, for any $f \in \mathcal{M}\left(d(0, r^-)\right)$, $r \in \mathbb{R}^+$ and $n \in \mathbb{N}$,

$$Z\left(r, A_{n+1}(f)\right) = \sum_{i=1}^{n} Z\left(r, f - \alpha_i\right) \tag{5.9}$$

and

$$\overline{Z}\left(r, A_{n+1}(f)\right) = \sum_{i=1}^{n} \overline{Z}\left(r, f - \alpha_i\right) \tag{5.10}$$

where $\alpha_1, \ldots, \alpha_n$ are the real roots of polynomial $A_{n+1}(z)$.

In general case, for any $s \in \mathbb{N}^*$ and $n_1, \ldots, n_s \in \mathbb{N}$, we get

$$\mathrm{Li}_{-n_1,\ldots,-n_s}(z) = \frac{z^s}{(1-z)^{n_1+\ldots+n_s+s}} A_{n_1,\ldots,n_s}(z). \tag{5.11}$$

where $A_{n_1,\ldots,n_s}(z)$ is a polynomial of degree $n_1 + \ldots + n_s - 1$ on z with the rational coefficients. By the computers, we can see that these polynomials have also real roots. For example, the polynomial $A_{3,4}(z) = z^6 + 127z^5 + 1458z^4 + 3654z^3 + 2429z^2 + 387z + 8$, has 6 real roots $z_1 \in (-115; -114)$, $z_2 \in (-10; -9)$, $z_3 \in (-3; -2)$, $z_4 \in (-1; -\frac{1}{2})$, $z_5 \in (-\frac{1}{2}; -\frac{1}{10})$, $z_6 \in (-\frac{1}{10}; 0)$. The polynomial $A_{1,0,2}(z) = z^2 + 6z + 3$, has 2 real roots and the polynomial $A_{1,1,2}(z) = z^3 + 15z^2 + 26z + 6$ has 3 real roots. And then, using Lemma 6, we obtain that

Proposition 6. For any $n_1, \ldots, n_r \in \mathbb{N}$,

$$Z\left(r, A_{n_1,\ldots,n_s}(f)\right) = \sum_{i=1}^{n_1+\ldots+n_s-1} Z\left(r, f - \alpha_i\right) \tag{5.12}$$

and

$$\overline{Z}\left(r, A_{n_1,\ldots,n_s}(f)\right) = \sum_{i=1}^{n_1+\ldots+n_s-1} \overline{Z}\left(r, f - \alpha_i\right) \tag{5.13}$$

where $\alpha_1, \ldots, \alpha_{n_1+\ldots+n_s-1}$ are the real roots of polynomial $A_{n_1,\ldots,n_s}(z)$.

Suppose that $\mathcal{F}$ is a subset of meromorphic functions. Recall that a polynomial $P(z)$ is called a strong uniqueness polynomial for $\mathcal{F}$ if for any two non-constant meromorphic functions $f, g \in \mathcal{F}$, then

$$\left(P(f) = cP(g);\ c \neq 0\right) \Longrightarrow \left(f = g\right). \tag{5.14}$$

Theorem 6 ([16]). Let $P(z)$ be a polynomial satisfying $P(1)P'(1) \neq 0$. Then $P(z)$ is a strong uniqueness polynomial for $L-$functions.

Since the definition of the class of polynomials $A_{s_1,\ldots,s_r}(z)$, using Theorem 6, we obtain that

Corollary 2. For any $s_1, \ldots, s_r \in \mathbb{N}$, the polynomial $A_{s_1,\ldots,s_r}(z)$ is a strong uniqueness polynomial for $L-$functions.

Given f to be a meromorphic function in $\mathbb{C}$ and $a \in \mathbb{C} \cup \{\infty\}$. We denote by $E_f(a)$ the set of $a-$points of f counted with its multiplicities. Moreover, for any nonempty subset S of $\mathbb{C} \cup \{\infty\}$, set that

$$E_f(S) = \bigcup_{a \in S} E_f(a). \tag{5.15}$$

And then, the subset S is called a unique range set, counting multiplicities for $\mathcal{F}$ if for any $f, g \in \mathcal{F}$ then

$$(E_f(S) = E_g(S)) \Longrightarrow (f = g). \tag{5.16}$$

Thank so much the works of authors in [16] who sent to the important results about the set of roots of a strong uniqueness polynomial. As an immediate consequnce of those results, we obtain that

Theorem 7. For any $s_1, \ldots, s_r \in \mathbb{N}$, the set of roots of the polynomial $A_{s_1,\ldots,s_r}(z)$ is unique range set, counting multiplicities for $L-$functions.

Acknowledgements

The authors would like to thank anonymous colleagues for detailed reading of the manuscript and many valuable comments by which this paper could be improved.

References

[1] Bender C.M., Brody D. C., Meister B.K.– *Quantum field theory of partitions*, J. Math. Phys. **40**, pp. 3239–3245.

[2] Bui Van Chien, G.H.E. Duchamp, Ngo Quoc Hoan, V. Hoang Ngoc Minh, C. Tollu.– *(Pure) transcendence bases in $\phi-$deformed shuffle bialgebras*, Seminaire Lotharingien de Combinatoire, Article B74f(22 pages), 2018.

[3] A. Connes, D. Kreimer.– *Hopf algebras, renormalization and noncommutative geometry*, Comm. Math. Phys., 199, pp. 203242, 1998.

[4] G.H.E. Duchamp, V. Hoang Ngoc Minh, Ngo Quoc Hoan .–*Harmonic sums and polylogarithms at negative multi-indices* , J. Symbolic Computation, 83, pp. 166-186, 2017.

[5] G.H.E. Duchamp, V. Hoang Ngoc Minh, Ngo Quoc Hoan .–*Kleene stars of the plane, polylogarithms and symmetries*, Theoretical Computer Science, 800, pp. 52-72, 2019.

[6] G.H.E. Duchamp, Hoang Ngoc Minh, K.A. Penson, Ngo Quoc Hoan, P. Simonnet.– *Mathematical renormalization in quantum electrodynamics via noncommutative generating series*, Springer Proc. in Mathematics & Statistics, 198 , pp. 59-100, 2017.

[7] G.H.E. Duchamp, Hoang Ngoc Minh, A.I. Solomon, S. Goodenough.– *An interface between physics and number theory*, dans Journal of Physics, 284(1), pp. 012-023, 2011.

[8] F.J. Dyson.–*The radiation theories of Tomonaga, Schwinger and Feynman*, Physical Rev, vol 75, pp. 486-502, 1949.

[9] A. Escasut, Ta Thi Hoai An, *Classical p-adic Nevanlinna theory and Nevanlinna theory out a hole*, Contemporary Mathematics, 2010.

[10] A. Escasut, *Value distribution in $p-$adic Analysis*, WSCP Singapore, 2015.

[11] R.P. Feynman, A.R. Hibbs, *Quantum Mechanics and Path Integrals*, McGraw-Hill, New York, 1965.

[12] M. Fliess.–*Fonctionnelles causales non linéaires et indéterminées non commutatives,* Bull. SMF **109**, 3-40, 1981.

[13] M. Fliess.–*Réalisation locale des systèmes non linéaires, algèbres de Lie filtrées transitives et séries génératrices non commutatives.* Inven. Math. **71**(3), (1983), pp. 521-537.

[14] A. B. Goncharov.–*Multiple polylogarithms and mixed Tate motives.* ArXiv:math. AG/0103059 v4, pp 497516, 2001.

[15] B. Guennebaud, G. Garandel,

[16] Ha Huy Khoai, Vu Hoai An, *Determining an $L-$function in the extended Selberg class by its preimages of subsets*, The Ramanujan Journal, **58**, pp. 253-267; 2022.

[17] V. Hoang Ngoc Minh.– *Algebraic combinatoric aspects of asymptotic analysis of nonlinear dynamical system with singular inputs*, in Acta Academiae Aboensis, Ser. B **67**(2), 117-126, 2007.

[18] V. Hoang Ngoc Minh.– *On a conjecture by Pierre Cartier about a group of associators*, in Acta Mathematica Vietnamica, Vol. 3, 2013.

[19] V. Hoang Ngoc Minh.–*Structure of polyzetas and Lyndon words*, Vietnamese Math. J., 41, Issue 4, pp. 409-450, 2013.

[20] V. Hoang Ngoc Minh.–*Finite polyzêtas, Poly-Bernoulli numbers, identities of polyzêtas and noncommutative rational power series*, Proc. of 4th International Conference on Words, pages 232250, 2003.

[21] L. Lewin.–*Polylogarithms and associated functions*, North Holland, New York and Oxford, 1981.

[22] Le Tu Quoc Thang, J. Murakami.– *Kontsevich's integral for Kauffman polynomial*, Nagoya Math., 39-65 (1996)

[23] R. Nevanlinna, *Zur theorie der meromorphen funktionen*, Acta. Math. **46** , 1-99, 1925.

[24] J. Zhao.–*Analytic continuation of multiple zeta functions*, Proceedings of the American Mathematical Society 128 (5): 12751283.

Institute of Mathematics, Vietnam Academy of Science and Technology, 18 Hoang Quoc Viet Road, Cau Giay District, 10307 Hanoi, Vietnam
E-mail address: tthan@math.ac.vn

Hanoi University of Science and Technology, 1, Dai Co Viet, Hai Ba Trung, Ha Noi, Viet Nam
E-mail address: hoan.ngoquoc@hust.edu.vn