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Subdifferentials of optimal value functions under metric qualification conditions

Vu Thi Huong · Duong Thi Viet An · Hong-Kun Xu

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Abstract In this paper, by revisiting intersection rules for normal cones, we give formulas for estimating or computing the Fréchet/Mordukhovich/Moreau–Rockafellar subdifferentials of optimal value functions of constrained parametric optimization problems under metric qualification conditions. The results are then applied to derive chain rules for composite functions in both convex and nonconvex situations. Illustrative examples and comparisons to existing results, including those of Mordukhovich et al. [Trans. Amer. Math. Soc. 348 (1996), 1235–1280; Math. Program. Ser. B 116 (2009), 369–396] and of An and Jourani [J. Optim. Theory Appl. 192 (2022), 82–96], are also addressed.

Keywords Optimal value functions · Metric qualification conditions · Normal cones · Intersection rules · Subdifferentials · Chain rules

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Dedicated to Professor Nguyen Dong Yen on the occasion of his 65th birthday.

Vu Thi Huong
School of Management, Hangzhou Dianzi University, Hangzhou, 310018, China; and Institute of Mathematics, Vietnam Academy of Science and Technology, Hanoi, 10072, Vietnam. E-mail: vthuong@hdu.edu.cn; vthuong@math.ac.vn

Duong Thi Viet An
School of Management, Hangzhou Dianzi University, Hangzhou, 310018, China; and Department of Mathematics and Informatics, Thai Nguyen University of Sciences, Thai Nguyen, 250000, Vietnam. E-mail: andtv@hdu.edu.cn; andtv@tnus.edu.vn

Hong-Kun Xu (Corresponding Author)
School of Science, Hangzhou Dianzi University, Hangzhou, 310018, China; College of Mathematics and Information Science, Henan Normal University, Xinxiang, 453007, China. E-mail: xuhk@hdu.edu.cn

1 Introduction

Stability and solution sensitivity of *parametric optimization problems* are among central topics in optimization and variational analysis. They allow us to capture behaviors of the *optimal value function* and of the *solution map* when parameters of the problem undergo perturbations. We refer the interested reader to the book by Bonnans and Shapiro [4] for a systematic view on these large topics.

Optimal value functions of parametric optimization problems are usually nonsmooth, even if the problem is given by smooth data. Thus, in order to obtain differential property of optimal value functions, one may need to evaluate *generalized derivatives*. As observed by Mordukhovich et al. [27], generalized derivatives can be sorted into two major types: directional derivatives/tangential approximations in *primal spaces* and subdifferentials/normal approximations in *dual spaces*. In some cases primal-space and dual-space constructions are equivalent. However, there are frameworks where the latter cannot be obtained via duality/polarity from any primal-space approximations. In this paper, we choose the dual-space approach to study generalized differential properties of optimal value functions. More precisely, in minimization problems with fully convex inputs, optimal value functions are convex; hence the *subdifferential in the sense of convex analysis* (the *Moreau–Rockafellar subdifferential or the Fenchel subdifferential*) will be considered. Otherwise, the *Fréchet subdifferential* (*the regular subdifferential*) and *the Mordukhovich subdifferential* (*the limiting subdifferential*) will be taken into account. It is worthy to emphasize that such a study on subdifferentials of optimal value functions has a wide range of applications. In variational analysis, basic calculus rules such as multiplier rules for constrained optimization problems, chain/sum/product and quotient rules for subdifferentials of functions, or calculus rules for coderivatives of compositions, sums and intersections of set-valued maps can be derived from results of optimal value functions (see, e.g., [16, 18, 26, 27, 28, 29]). In other fields, recent applications of optimal value functions are found in Moreau–type infimal convolution problems [17], optimal control problems of discrete/continuous/semilinear elliptic PDEs systems [36, 2, 32], consumption/production economics [12], and robust deep learning with nonsmooth activations [10, Section 5.4].

Qualification conditions are sufficient conditions for the validity of fundamental calculus rules in nonsmooth analysis. For example, in convex analysis, the well-known *Moreau–Rockafellar theorem* provides us subdifferential sum rules for two proper convex functions under the condition that one of these functions is continuous at a point belonging to the domain of the other. Meanwhile, *normal qualification conditions* stated in terms of normal cones or *metric qualification conditions* formulated by means of the distance function ensure intersection rules for normal cones of nonconvex subsets. Metric qualification conditions were first studied and developed in a series of papers by Ioffe, Penot, Jourani, and Thibault [14, 16, 18, 19] mainly to deal with subsets, functions, and set-valued maps between Banach spaces in the *approximate theory* pioneered by Ioffe in the late 1980s. Though there were certain observations therein for the *limiting theory* introduced earlier by Mordukhovich [22], it was in 2001 that Ngai and Théra [29] brought clearly the metric approach to study intersection rules for the Mordukhovich normal cone in Asplund spaces. Namely, by using the *fuzzy sum rule* for the Fréchet subdifferential, they showed that the metric qualification condition is generally weaker than the normal qualification condition used previously in [28, Corollary 4.5]. Then, they utilized this to establish chain rules for composite functions and necessary optimality conditions for non-Lipschitz constrained optimization problems.

Motivated by the above-mentioned work of Ngai and Théra, we devote this paper to examine the performance of metric qualification conditions in studying subdifferentials of optimal value functions of constrained parametric optimization problems. Upper estimates for the Fréchet/Mordukhovich subdifferentials and exact representations for the Moreau–Rockafellar subdifferential of the optimal value function are given in terms of subdifferentials of the objective function and coderivatives of the constraint map. We follow a conventional approach which transfers the constrained problem to an unconstrained one so that subdifferentials of the optimal value function can be estimated/computed via subdifferentials of the sum function formed by the objective and the constraint. Among other things, our novel contribution is decomposing the latter into corresponding subdifferentials of the objective function and coderivatives of the constraint map under a "good" qualification condition. The chosen qualification condition is (A_1) or (A_2) for problems without assuming convexity and is (A_1) or $(\hat{A}_1)$ for problems with convex inputs, and its variants (see Section 4 below). All of these conditions are described geometrically via distance functions to the epigraph of the objective function and to the graph of the constraint map. In the first situation, (A_1) and (A_2) are "good" in the sense that they are weaker than the existing one (A_0), which is a normal qualification condition in the well-recognized work of Mordukhovich et al. (see [27, Theorem 7 (i) and (ii)] and [28, Section 6.1]), under the framework of Asplund spaces. In the second situation, it turns out that conditions (A_1) and $(\hat{A}_1)$ can perform well without requiring the completeness of underlying spaces, the closedness of the constraint map, and the lower semicontinuity of the objective function, which were assumed in the very recent work by An and Jourani [1]. Full descriptions of results and other comparisons on subdifferentials of optimal value functions will be given in Section 4, followed by applications in deriving chain rules for composite functions in Section 5.

Technically, the results in Sections 4 and 5 mainly come from the ones in Section 3 where we revisit intersection rules for the Fréchet/Mordukhovich normal cones and the normal cone in the sense of convex analysis using the metric spirit in the approximate theory. Note that approximate constructions (*A-subdifferential*, *G-subdifferential*, *G-normal cone*)[1] are quite different from

[1] According to Ioffe [15, p. 534], the notations "A" and "G" stands respectively for "analytic" and "geometric".

those in the limiting theory when the underlying space is not *weakly compact generated*; see, Mordukhovich and Shao [28, Section 9]. Thus, the results in Section 3 for the Mordukhovich normal cone cannot be derived from existing ones in the approximate theory. This conclusion is valid also for the normal cone in the sense of convex analysis, because the latter will be treated as a consequence of our new study on the Fréchet normal cone, rather than of approximate calculus.

The remaining of the paper is as follows. In Section 2, concepts of normal cones to sets, subdifferentials of extended-real-valued functions, and coderivatives of set-valued maps are presented. In the next section, basic properties of the distance function are recalled and intersection rules for normal cones under metric qualification conditions are studied. Section 4 contains three subsections, corresponding to results for the Fréchet/Mordukhovich/Moreau–Rockafellar subdifferentials of optimal value functions. The paper ends with chain rules for composite functions in Section 5, as an application of the results in Section 4.

2 Preliminaries

Throughout the paper, the topological dual spaces of normed spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are denoted, respectively, by X^* and Y^*. For each $x^* \in X^*$ and $x \in X$, $\langle x^*, x\rangle$ stands for the value of the continuous linear functional x^* at x. We use $\mathbb{B}(x, r)$ and $\mathbb{B}_{X^*}$ to denote the open ball centered at x with radius $r > 0$ and the closed unit ball of X^*, respectively. The notation $x_k^* \to x^*$ means the norm convergence to x^* of the sequence $\{x_k^*\}_{k\in\mathbb{N}}$ with $\mathbb{N} := \{1, 2, ...\}$, while $x_k^* \xrightarrow{w^*} x^*$ indicates the convergence to x^* of $\{x_k^*\}_{k\in\mathbb{N}}$, in the weak* topology of X^*. For a set-valued map $F : X \rightrightarrows X^*$, the limiting construction

$$\operatorname*{Limsup}_{x\to\bar{x}} F(x) := \Big\{x^* \in X^* \,|\, \exists x_k \to \bar{x},\ \ x_k^* \xrightarrow{w^*} x^* \text{ with } x_k^* \in F(x_k), \forall k \in \mathbb{N}\Big\}$$

is known as the *Painlevé-Kuratowski outer/upper limit* of F as $x \to \bar{x}$ with respect to the norm topology of X and the weak* topology of X^*.

This section contains main materials on generalized differentiation widely used in what follows. Those are normal cones to sets, subdifferentials of extended-real-valued functions, and coderivatives of set-valued maps under settings adopted from the book by Penot [31]. Thus, *unless otherwise stated, all the considered spaces in this paper are assumed to be normed spaces.* For a systematical treatment on generalized differentiation in Banach spaces, the interested reader is referred to the book by Mordukhovich [23].

Let us start with the concepts of normal cones to sets.

Definition 2.1 (See [31, Definition 2.96]) Let $\Omega \subset X$ be a nonempty set and let $\bar{x} \in X$. The *Fréchet normal cone* or *the regular normal cone* to Ω at $\bar{x}$ is defined by

$$N_F(\bar{x};\Omega) := \Big\{x^* \in X^* \mid \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } \langle x^*, x-\bar{x}\rangle \le \varepsilon\|x-\bar{x}\|, \forall x \in \Omega \cap \mathbb{B}(\bar{x},\delta)\Big\}$$

if $\bar{x} \in \Omega$, and by $N_F(\bar{x};\Omega) := \emptyset$ if $\bar{x} \notin \Omega$.

When the set Ω is convex, one can show that $N_F(\bar{x};\Omega)$ coincides with the normal cone in the sense of convex analysis described below ([31, Exercise 6, p. 174]).

Definition 2.2 (See [31, Definition 2.95]) Let $\Omega \subset X$ be a nonempty convex set and let $\bar{x} \in X$. The *normal cone in the sense of convex analysis* to Ω at $\bar{x}$ is defined by

$$N(\bar{x};\Omega) := \{x^* \in X^* \mid \langle x^*, x-\bar{x}\rangle \le 0, \ \ \forall x \in \Omega\}$$

if $\bar{x} \in \Omega$, and by $N(\bar{x};\Omega) := \emptyset$ if $\bar{x} \notin \Omega$.

When X is an *Asplund space*, i.e., a Banach space that every continuous convex function defined on an open convex subset W of X is Fréchet differentiable on a dense G_δ subset D of W ([31, Definition 3.96]), the concept of the normal cone is given as follows.

Definition 2.3 (See [31, Definition 6.5]) Let X be an Asplund space, $\Omega \subset X$ a nonempty set, and $\bar{x} \in X$. The *Mordukhovich normal cone* or the *limiting normal cone* to Ω at $\bar{x}$ is given by

$$N_M(\bar{x};\Omega) := \operatorname*{Limsup}_{x \to \bar{x}} N_F(x;\Omega)$$

when $\bar{x} \in \Omega$, and by $N_M(\bar{x};\Omega) := \emptyset$ when $\bar{x} \notin \Omega$.

For any $\bar{x} \in \Omega$, by Definitions 2.1 and 2.3, it holds that $x^* \in N_M(\bar{x};\Omega)$ if and only if there exist $x_k \xrightarrow{\Omega} \bar{x}$ (i.e., $x_k \to \bar{x}$ with $x_k \in \Omega$ for all k), $x_k^* \xrightarrow{w^*} x^*$ such that $x_k^* \in N_F(x_k;\Omega)$, for all $k \in \mathbb{N}$. So one always has

$$N_F(\bar{x};\Omega) \subset N_M(\bar{x};\Omega), \tag{2.1}$$

and this inclusion may be strict. For example, take $X := \mathbb{R}^2$,

$$\Omega := \{(x_1,x_2) \in \mathbb{R}^2 \mid x_2 \ge -\,|x_1|\},$$

and $\bar{x} := (0,0)$. Then $N_F(\bar{x};\Omega) = \{(0,0)\}$ and

$$N_M(\bar{x};\Omega) = \{(v,v) \mid v \le 0\} \cup \{(v,-v) \mid v \ge 0\};$$

see, e.g., [23, pages 5 and 7]. Note also from this example that the Mordukhovich normal cone $N_M(\bar{x};\Omega)$ may be nonconvex. There is one situation that the above-mentioned inclusion becomes an equality and the Mordukhovich normal cone is convex, that is when the set Ω is convex. In this case, we have

$$N_M(\bar{x};\Omega) = N_F(\bar{x};\Omega) = N(\bar{x};\Omega);$$

see [31, Excercise 6, p.174 and Proposition 6.6].

Before presenting concepts of subdifferentials of an extended-real-valued function $f : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$, we recall that the *domain* and *epigraph* of f are given, respectively, by $\operatorname{dom} f := \{x \in X \mid f(x) \in \mathbb{R}\}$ and $\operatorname{epi} f := \{(x, \alpha) \in X \times \mathbb{R} \mid \alpha \geq f(x)\}$.

Definition 2.4 (See [31, Definition 4.1 and Proposition 4.18]) Let $f : X \to \overline{\mathbb{R}}$ and $\bar{x} \in X$ be given. Suppose that $f(\bar{x})$ is finite. The *Fréchet subdifferential* or *the regular subdifferential* $\partial_F f(\bar{x})$ and the *Fréchet singular subdifferential* or *the regular singular subdifferential* $\partial_F^\infty f(\bar{x})$ of f at $\bar{x}$ are the sets

$$\partial_F f(\bar{x}) := \Big\{ x^* \in X^* \mid \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } f(x) - f(\bar{x}) - \langle x^*, x - \bar{x}\rangle \geq -\varepsilon \|x - \bar{x}\|, \forall x \in \mathbb{B}(\bar{x}, \delta) \Big\},$$

$$\partial_F^\infty f(\bar{x}) := \{x^* \in X^* \mid (x^*, 0) \in N_F((\bar{x}, f(\bar{x})); \operatorname{epi} f)\}.$$

When $f(\bar{x}) \in \{-\infty, +\infty\}$, one sets $\partial_F f(\bar{x}) := \partial_F^\infty f(\bar{x}) := \emptyset$.

If f is Fréchet differentiable at $\bar{x}$, then $\partial_F f(\bar{x}) = \{f'(\bar{x})\}$. However, the situation that $\partial_F f(\bar{x})$ is singleton does not mean that f is Fréchet differentiable at $\bar{x}$ as shown in [31, Example p. 266]. Concerning the Fréchet singular subdifferential, it is worthy to notice that $\partial_F^\infty f(\bar{x})$ always contains 0 though $\partial_F f(\bar{x})$ can be empty, as the case when $f(x) := -|x|$ with $x \in \mathbb{R}$ and $\bar{x} := 0$. However, in some situations, both $\partial_F f(\bar{x})$ and $\partial_F^\infty f(\bar{x})$ are nontrivial. An illustrative example [31, Example p. 276] for this is when $X := \mathbb{R}$, $f(x) := x$ if $x < 0$ and $f(x) := \sqrt{x}$ if $x \geq 0$, and $\bar{x} := 0$. Here one has $\partial_F f(\bar{x}) = [1, +\infty)$ and $\partial_F^\infty f(\bar{x}) = [0, +\infty)$.

If f is convex, then the concept of the Fréchet subdifferential collapses to the subdifferential usually used in convex analysis that we are about to recall.

Definition 2.5 Let $f : X \to \overline{\mathbb{R}}$ be a convex function and $\bar{x} \in X$. Suppose that $f(\bar{x})$ is finite. The *Moreau–Rockafellar subdifferential* or the *Fenchel subdifferential* or the *subdifferential in the sense of convex analysis* $\partial_{MR} f(\bar{x})$ of f at $\bar{x}$ is given by

$$\partial_{MR} f(\bar{x}) := \{x^* \in X^* \mid \langle x^*, x - \bar{x}\rangle \leq f(x) - f(\bar{x}), \forall x \in X\},$$

while the *singular subdifferential* $\partial^\infty f(\bar{x})$ of f at $\bar{x}$ is defined by

$$\partial^\infty f(\bar{x}) := \{x^* \in X^* \mid (x^*, 0) \in N((\bar{x}, f(\bar{x})); \operatorname{epi} f)\}.$$

One puts $\partial_{MR} f(\bar{x}) := \partial^\infty f(\bar{x}) := \emptyset$ if $f(\bar{x}) \in \{-\infty, +\infty\}$.

Corresponding to the concept of Mordukhovich cone to sets, one has the following concepts of subdifferentials of functions defined on Asplund spaces.

Definition 2.6 (See [31, Definition 6.1]) Let $f : X \to \overline{\mathbb{R}}$ be a function defined on an Asplund space X and $\bar{x} \in X$. Suppose that $f(\bar{x})$ is finite. The set

$$\partial_M f(\bar{x}) := \operatorname*{Limsup}_{x \xrightarrow{f} \bar{x}} \partial_F f(x),$$

where $x \xrightarrow{f} \bar{x}$ means that $x \to \bar{x}$ and $f(x) \to f(\bar{x})$, is said to be the *Mordukhovich subdifferential* or the *limiting subdifferential* of f at $\bar{x}$. The set

$$\partial_M^\infty f(\bar{x}) := \{x^* \in X^* \mid (x^*, 0) \in N_M((\bar{x}, f(\bar{x})); \operatorname{epi} f)\}$$

is called the *Mordukhovich singular subdifferential* of f at $\bar{x}$. When $\bar{x} \in X$ is such that $f(\bar{x}) \in \{-\infty, +\infty\}$, one lets $\partial_M f(\bar{x}) := \partial_M^\infty f(\bar{x}) := \emptyset$.

From the definition of the Mordukhovich subdifferential, it follows that, for any $\bar{x} \in X$ with $f(\bar{x}) \in \mathbb{R}$, $x^* \in \partial_M f(\bar{x})$ if and only if there exist $x_k \xrightarrow{f} \bar{x}$ and $x_k^* \in \partial_F f(x_k)$ such that $x_k^* \xrightarrow{w^*} x^*$. Thus, one always has

$$\partial_F f(\bar{x}) \subset \partial_M f(\bar{x}). \tag{2.2}$$

The above inclusion is often strict and $\partial_M f(\bar{x})$ may be nonconvex (see [31, Example, p. 408]). Similarly, it follows from the definition of singular subdifferentials and the relation (2.1) that

$$\partial_F^\infty f(\bar{x}) \subset \partial_M^\infty f(\bar{x}). \tag{2.3}$$

Notice that this inclusion can be strict. For example, let $f : \mathbb{R} \to \mathbb{R}$ be given as [23, p. 95]

$$f(x) = \begin{cases} -\sqrt{x - \frac{1}{n}}, & \text{if } \ \frac{1}{n} \le x < \frac{1}{n} + \frac{1}{n^4},\ n \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

In this case, $\partial_F f(0) = \partial_M f(0) = \partial_F^\infty f(0) = \{0\}$, while $\partial_M^\infty f(0) = (-\infty, 0]$.

Recall that the function f is *Lipschitz continuous* around $\bar{x}$ (cf. [23, p. 19]) if there is a neighborhood U of $\bar{x}$ and a constant $\ell \ge 0$ such that

$$\|f(x) - f(u)\| \le \ell \|x - u\|,\ \forall x, u \in U.$$

Meanwhile, f is *calm* at $\bar{x}$ (cf. [34, p. 322]) if for some $\ell \ge 0$ and a neighborhood U of $\bar{x}$, one has $\|f(x) - f(\bar{x})\| \le \ell \|x - \bar{x}\|$ for all $x \in U$. The Mordukhovich singular subdifferential occurs to be useful for the study of non-Lipschitzian functions, since $\partial_M^\infty f(\bar{x}) = \{0\}$ if f is Lipschitz continuous around $\bar{x}$ (see [23, Corollary 1.81]). In addition, it can be utilized in establishing appropriate qualification conditions for subdifferential calculus rules involving non-Lipschitzian functions as in [23, Chapter 3]. While, the construction of the Fréchet singular subdifferential carries nontrivial information for functions that do not satisfy the calmness property (and hence the Lipschitz continuous property). Because when f satisfies the calmness condition at $\bar{x}$, the Fréchet singular subdifferential reduces to $\{0\}$ (see [20, Corollary 1.31.2]). Besides, the Fréchet normal cone to epigraph of a function f at $(\bar{x}, f(\bar{x}))$ is completely defined by the sets of Fréchet subdifferential $\partial_F f(\bar{x})$ and Fréchet singular subdifferential $\partial_F^\infty f(\bar{x})$ (see [20, Corollary 1.31.2] and [31, Propositions 4.18]).

If f is convex, the Fréchet subdifferential, the Mordukhovich subdifferential and the Moreau-Rockafellar subdifferential coincide (see [31, Propositions 4.9 and 6.17(b)]), i.e., $\partial_F f(\bar{x}) = \partial_M f(\bar{x}) = \partial_{MR} f(\bar{x})$.

Recall from [31, Subsections 3.2.1, 4.1.3, and 6.1.1] that concepts of normal cones to sets and subdifferentials of functions can be linked across. Namely, it holds for any $\bar{x} \in \Omega \subset X$ that

$$N_F(\bar{x}; \Omega) = \partial_F \delta_\Omega(\bar{x}) = \partial_F^\infty \delta_\Omega(\bar{x}),$$

$$N(\bar{x}; \Omega) = \partial_{MR} \delta_\Omega(\bar{x}) = \partial^\infty \delta_\Omega(\bar{x}),$$

and that

$$N_M(\bar{x}; \Omega) = \partial_M \delta_\Omega(\bar{x}) = \partial_M^\infty \delta_\Omega(\bar{x})$$

when X is an Asplund space. Here, $\delta_\Omega : X \to \mathbb{R} \cup \{+\infty\}$ stands for the *indicator function* associated with the set Ω, i.e., the function that takes the value 0 on Ω and the value $+\infty$ on $X \setminus \Omega$. Besides, for a function $f : X \to \overline{\mathbb{R}}$ and $\bar{x}$ with $f(\bar{x}) \in \mathbb{R}$, one has that

$$x^* \in \partial_F f(\bar{x}) \Leftrightarrow (x^*, -1) \in N_F((\bar{x}, f(\bar{x})); \operatorname{epi} f),$$

$$x^* \in \partial_{MR} f(\bar{x}) \Leftrightarrow (x^*, -1) \in N((\bar{x}, f(\bar{x})); \operatorname{epi} f), \tag{2.4}$$

and that ([31, Proposition 6.9])

$$x^* \in \partial_M f(\bar{x}) \Leftrightarrow (x^*, -1) \in N_M((\bar{x}, f(\bar{x})); \operatorname{epi} f) \tag{2.5}$$

when X is an Aslund space and f is a lower semicontinuous function.

Let $F : X \rightrightarrows Y$ be a set-valued map between normed spaces. The *domain* and *graph* of F are given, respectively, by $\operatorname{dom} F := \{x \in X \mid F(x) \neq \emptyset\}$ and $\operatorname{Gr} F := \{(x, y) \in X \times Y \mid x \in \operatorname{dom} F,\ y \in F(x)\}$. One says that the set-valued map F is *closed* (resp., *convex*) if $\operatorname{Gr} F$ is closed (resp., convex) in the product space $X \times Y$, which is endowed with the norm $\|(x, y)\| = \|x\| + \|y\|$ for any $(x, y) \in X \times Y$.

Definition 2.7 (See [31, Definitions 4.24 and 6.13]) Let $(\bar{x}, \bar{y}) \in \operatorname{Gr} F$.

(i) The *Fréchet coderivative* of F at $(\bar{x}, \bar{y})$ is the set-valued map $D_F^* F(\bar{x}, \bar{y}) : Y^* \rightrightarrows X^*$ with

$$D_F^* F(\bar{x}, \bar{y})(y^*) := \{x^* \in X^* \mid (x^*, -y^*) \in N_F((\bar{x}, \bar{y}); \operatorname{Gr} F)\}, \quad y^* \in Y^*.$$

(ii) When F is convex, the *coderivative* of F at $(\bar{x}, \bar{y})$ is the set-valued map $D^*F(\bar{x}, \bar{y}) : Y^* \rightrightarrows X^*$ with

$$D^*F(\bar{x}, \bar{y})(y^*) := \{x^* \in X^* \mid (x^*, -y^*) \in N((\bar{x}, \bar{y}); \operatorname{Gr} F)\}, \quad y^* \in Y^*.$$

(iii) When X and Y are Asplund spaces, the *Mordukhovich coderivative* or the *limiting coderivative* of F at $(\bar{x}, \bar{y})$ is the set-valued map $D^*_M F(\bar{x}, \bar{y}) : Y^* \rightrightarrows X^*$ with

$$D^*_M F(\bar{x}, \bar{y})(y^*) := \{x^* \in X^* \mid (x^*, -y^*) \in N_M((\bar{x}, \bar{y}); \operatorname{Gr} F)\}, \quad y^* \in Y^*.$$

If $(\bar{x}, \bar{y}) \notin \operatorname{Gr} F$ then we accept the convention that the sets $D^*_F F(\bar{x}, \bar{y})(y^*)$, $D^*F(\bar{x}, \bar{y})(y^*)$ and $D^*_M F(\bar{x}, \bar{y})(y^*)$ are empty for any $y^* \in Y^*$.

3 Metric qualification conditions and intersection rules for normal cones

Qualification conditions play vital roles in deriving intersection rules for normal cones, both in convex and nonconvex analysis. In this section, we will discuss qualification conditions which are formulated in terms of distance functions. Thus, let us first recall the concept of the distance function and relevant properties. Given a nonempty subset Ω of a normed space X, the *distance function* to Ω is defined by

$$d(x, \Omega) := \inf_{u \in \Omega} \|x - u\|, \quad x \in X.$$

Clearly, $d(\cdot, \Omega) : X \to \mathbb{R}$ is Lipschitz continuous with modulus one. Besides, it is a convex function if Ω is a convex set (see, e.g., [9, Section 2.4]). Moreover, at points inside the set Ω, subdifferentials of $d(\cdot, \Omega)$ can be represented via corresponding normal cones and vice versa, as depicted below.

Proposition 3.1 (See [31, Lemma 4.21]; see also [23, Corollary 1.96] for a version formulated for Banach spaces.) *For any $\bar{x} \in \Omega$, one has*

(i) $\partial_F d(\bar{x}, \Omega) = N_F(\bar{x}; \Omega) \cap \mathbb{B}_{X^*}$;
(ii) $N_F(\bar{x}; \Omega) = \bigcup_{\lambda \geq 0} \lambda \partial_F d(\bar{x}, \Omega)$.

When the set Ω is convex, relationships between the Moreau–Rockafellar subdifferential of the distance function and the normal cone are known, see, e.g., [8, Proposition 11], [13, Lemma 3] and [1, Proposition 1] for the case of X being a Banach space, [7, Theorem 1] and [4, Example 2.130] with Ω being closed in a normed/Banach space. However, by applying directly Proposition 3.1 to the convex set Ω and the convex function $d(\cdot, \Omega)$, we can obtain the next proposition under the normed space setting without the closedness of Ω.

Proposition 3.2 *Suppose that Ω is convex. Then we have, for any $\bar{x} \in \Omega$,*

(i) $\partial_{MR}d(\bar{x}, \Omega) = N(\bar{x}; \Omega) \cap \mathbb{B}_{X^*}$;
(ii) $N(\bar{x}; \Omega) = \bigcup_{\lambda \geq 0} \lambda \partial_{MR} d(\bar{x}, \Omega)$.

For the limiting case, only a similar version to the second representation in the above two propositions has been obtained.

Proposition 3.3 (See [29, Lemma 3.6], [23, Theorem 1.97], and [31, Proposition 6.8].) *Suppose that X is an Asplund space and Ω is closed. Then, one has*

$$N_M(\bar{x}; \Omega) = \bigcup_{\lambda > 0} \lambda \partial_M d(\bar{x}, \Omega)$$

for any $\bar{x} \in \Omega$.

We will need the next two propositions when deriving intersection rules for normal cones via metric qualification conditions.

Proposition 3.4 (See [31, Lemma 6.7]) *Suppose that X is an Asplund space, Ω is closed, and $\bar{x} \in \Omega$. Then $x^* \in \partial_M d(\bar{x}, \Omega)$ if and only if there exist sequences $x_n \to \bar{x}$, $x_n^* \xrightarrow{w^*} x^*$ such that $x_n \in \Omega$ and $x_n^* \in \partial_F d(x_n, \Omega)$ for all $n \in \mathbb{N}$.*

The following property has been observed in [29, Lemma 3.6] under the assumption that X is a Banach space and the set Ω is closed. However, we find that this result is actually true for any nonempty set in a normed space. For the reader's convenience, we include the proof with several detailed explanations added.

Proposition 3.5 *Let Ω be a nonempty subset of a normed space X and let $\bar{x} \in \Omega$. If $x^* \in N_F(\bar{x}; \Omega)$, then $x^* \in \lambda \partial_F d(\bar{x}, \Omega)$ for all $\lambda \geq \|x^*\| + 1$.*

Proof Take $1 > \varepsilon > 0$. Since $x^* \in N_F(\bar{x}; \Omega)$, by the definition of the Fréchet normal cone, there exists $\delta > 0$ satisfying $\langle x^*, x - \bar{x} \rangle \leq \varepsilon \|x - \bar{x}\|$ for all $x \in \Omega \cap \mathbb{B}(\bar{x}, \delta)$. Fix an $x \in \mathbb{B}(\bar{x}, \delta/2)$ (i.e., $\|x - \bar{x}\| < \delta/2$) and let $\{x_n\}$ be a sequence in Ω such that $\lim_{n \to \infty} \|x_n - x\| = d(x, \Omega)$. Since $\bar{x} \in \Omega$, we must have $d(x, \Omega) \leq \|x - \bar{x}\| < \delta/2$. Thus, with no loss of generality, we may assume that $\|x_n - x\| < \delta/2$ for all n. It then turns out that $\|x_n - \bar{x}\| < \delta$ (i.e., $x_n \in \mathbb{B}(\bar{x}, \delta)$) for all $n \in \mathbb{N}$. Therefore, we obtain

$$\begin{aligned} \langle x^*, x - \bar{x} \rangle &= \langle x^*, x - x_n \rangle + \langle x^*, x_n - \bar{x} \rangle \\ &\leq \|x^*\| \|x - x_n\| + \varepsilon \|x_n - \bar{x}\| \\ &\leq (\|x^*\| + 1\|)\|x_n - x\| + \varepsilon \|x - \bar{x}\| \end{aligned}$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ in the last relation, we obtain

$$\langle x^*, x - \bar{x} \rangle \leq \big(\|x^*\| + 1\|\big) d(x, \Omega) + \varepsilon \|x - \bar{x}\|,$$

which together with $\lambda \geq \|x^*\| + 1$ implies $\langle x^*/\lambda, x - \bar{x} \rangle \leq d(x, \Omega) + \varepsilon \|x - \bar{x}\|$. As this inequality holds for an arbitrary $x \in \mathbb{B}(\bar{x}, \delta/2)$, we get $x^* \in \lambda \partial_F d(\bar{x}, \Omega)$. This completes the proof. $\square$

The qualification conditions (Q_1) and (Q_2) below are key factors for the main results of this paper. They have appeared in the literature under several different names. For instance, Ngai and Théra called (Q_1) the *metric inequality* in [29], while Penot named it the *linear coherence condition* in [31, Theorems 4.75 and 6.41]). The condition (Q_2) was referred to as the *intersection formula* by Jourani in [18]. Due to their metric nature of distance functions, we group them as *metric qualification conditions*.

Definition 3.1 Let C and D be two nonempty sets in X and $\bar{x} \in C \cap D$. We say that

(i) the sets C and D satisfy the *metric qualification condition* (Q_1) at $\bar{x}$ if there exist two numbers $a > 0$ and $r > 0$ such that

$$d(x, C \cap D) \leq ad(x, C) + ad(x, D), \quad \forall x \in \mathbb{B}(\bar{x}, r); \qquad (\mathrm{Q}_1)$$

(ii) the sets C and D satisfy the *metric qualification condition* (Q_2) at $\bar{x}$ for $\partial \in \{\partial_F, \partial_{MR}, \partial_M\}$ if there exists a number $a > 0$ such that

$$\partial d(\bar{x}, C \cap D) \subset a\partial d(\bar{x}, C) + a\partial d(\bar{x}, D). \qquad (\mathrm{Q}_2)$$

Let us discover some relationships between (Q_1) and (Q_2). We first need the following lemma on a local behavior of the Fréchet subdifferential. In the general case, this property is called *homotone* (see [30, Definition 2.1] and [31, Proposition 4.11]).

Lemma 3.1 *Let $f, g : X \to \overline{\mathbb{R}}$ and $\bar{x} \in X$ be such that $f(\bar{x}) = g(\bar{x}) \in \mathbb{R}$. Then $\partial_F f(\bar{x}) \subset \partial_F g(\bar{x})$ if there exists some $\rho > 0$ satisfying*

$$f(x) \leq g(x), \quad \forall x \in \mathbb{B}(\bar{x}, \rho). \qquad (3.1)$$

Proof Let $x^* \in \partial_F f(\bar{x})$. Then, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$f(x) - f(\bar{x}) - \langle x^*, x - \bar{x} \rangle \geq -\varepsilon \|x - \bar{x}\|, \ \forall x \in \mathbb{B}(\bar{x}, \delta).$$

Choose $\eta := \min(\rho, \delta)$. Then, it holds for any $x \in \mathbb{B}(\bar{x}, \eta)$ that

$$f(x) \geq f(\bar{x}) + \langle x^*, x - \bar{x} \rangle - \varepsilon \|x - \bar{x}\|.$$

This together with (3.1) implies

$$g(x) \geq f(\bar{x}) + \langle x^*, x - \bar{x} \rangle - \varepsilon \|x - \bar{x}\| \quad \forall x \in \mathbb{B}(\bar{x}, \eta).$$

Now since $f(\bar{x}) = g(\bar{x})$, one gets

$$g(x) \geq g(\bar{x}) + \langle x^*, x - \bar{x} \rangle - \varepsilon \|x - \bar{x}\| \quad \forall x \in \mathbb{B}(\bar{x}, \eta).$$

This proves that $x^* \in \partial_F g(\bar{x})$. □

Proposition 3.6 *Let C and D be subsets of an Asplund X. Suppose that the set $C \cap D$ is closed. Then, the metric qualification condition (Q_1) implies the metric qualification condition (Q_2) for $\partial = \partial_M$.*

Proof Suppose that (Q_1) is fulfilled at $\bar{x} \in C \cap D$. Take $x^* \in \partial_M d(\bar{x}, C \cap D)$ arbitrarily. By Proposition 3.4, there exist $x_n \to \bar{x}$, $x_n^* \xrightarrow{w^*} x^*$ such that $x_n \in C \cap D$ and $x_n^* \in \partial_F d(x_n, C \cap D)$ for all $n \in \mathbb{N}$. Since $x_n \to \bar{x}$, one can assume that $x_n \in \mathbb{B}(\bar{x}, r/2)$ for every $n \in \mathbb{N}$, where $r > 0$ is the number in (Q_1). Fix $n \in \mathbb{N}$. For any $x \in \mathbb{B}(x_n, r/2)$, one has

$$\|x - \bar{x}\| = \|x - x_n + x_n - \bar{x}\| \leq \|x - x_n\| + \|x_n - \bar{x}\| < r,$$

which shows that $x \in \mathbb{B}(\bar{x}, r)$; and hence $\mathbb{B}(x_n, r/2) \subset \mathbb{B}(\bar{x}, r)$. Therefore, condition (Q_1) implies

$$d(x, C \cap D) \leq ad(x, C) + ad(x, D), \ \forall x \in \mathbb{B}\big(x_n, r/2\big).$$

Notice that $d(x_n, C \cap D) = ad(x_n, C) + ad(x_n, D) = 0$ as $x_n \in C \cap D$. Now, applying Lemma 3.1 with $f(\cdot) := d(\cdot, C \cap D)$ and $g(\cdot) := ad(\cdot, C) + ad(\cdot, D)$ at x_n, we get $\partial_F d(x_n, C \cap D) = \partial_F f(x_n) \subset \partial_F g(x_n)$. Thus, we have $x_n^* \in \partial_F g(x_n)$ for every $n \in \mathbb{N}$. This together with the facts $x_n \to \bar{x}$ and $x_n^* \xrightarrow{w^*} x^*$ imply $x^* \in \partial_M g(\bar{x})$. As $x^* \in \partial_M d(\bar{x}, C \cap D)$ is taken arbitrarily, we have derived that

$$\partial_M d(\bar{x}, C \cap D) \subset \partial_M g(\bar{x}). \tag{3.2}$$

Moreover, as $d(\cdot, C)$ and $d(\cdot, D)$ are Lipschitz continuous, by Proposition 6.17(d) and Theorem 6.22 in [31], we obtain

$$\partial_M g(\bar{x}) = \partial_M [ad(\cdot, C) + ad(\cdot, D)](\bar{x}) \subset a\partial_M d(\bar{x}, C) + a\partial_M d(\bar{x}, D). \tag{3.3}$$

Combining (3.2) and (3.3) yields $\partial_M d(\bar{x}, C \cap D) \subset a\partial_M d(\bar{x}, C) + a\partial_M d(\bar{x}, D)$, which means that (Q_2) is satisfied at $\bar{x}$ for $\partial = \partial_M$. $\square$

The above relationship between (Q_1) and (Q_2) also holds for the case of the Moreau–Rockafellar subdifferential.

Proposition 3.7 *Suppose that C and D are convex. Then, the metric qualification condition* (Q_1) *implies the metric qualification condition* (Q_2) *for* $\partial = \partial_{MR}$.

Proof Suppose that the condition (Q_1) is satisfied at $\bar{x} \in C \cap D$. Let us fix arbitrarily $x^* \in \partial_{MR} d(\bar{x}, C \cap D)$. By the definition of the Moreau–Rockafellar subdifferential, one has

$$\langle x^*, x - \bar{x} \rangle \leq d(x, C \cap D) - d(\bar{x}, C \cap D) = d(x, C \cap D), \quad \forall x \in X.$$

By using the condition (Q_1), the last relation implies

$$\langle x^*, x - \bar{x} \rangle \leq ad(x, C) + ad(x, D), \quad \forall x \in \mathbb{B}(\bar{x}, r),$$

or, equivalently, $0 \leq ad(x, C) + ad(x, D) + \langle -x^*, x - \bar{x} \rangle$, for all $x \in \mathbb{B}(\bar{x}, r)$. This shows that $\bar{x}$ is a local minimizer of the function mapping f defined on X by $f(x) := ad(x, C) + ad(x, D) + \langle -x^*, x - \bar{x} \rangle$. Since $ad(\cdot, C)$, $ad(\cdot, D)$ are convex (due to the convexity of C, D) and $\langle -x^*, \cdot - \bar{x} \rangle$ is an affine function, $f(\cdot)$ is convex. Thus, we can conclude that $\bar{x}$ is a global minimizer of $f(\cdot)$. So, the optimality characterization in [31, Proposition 3.21] yields $0 \in \partial_{MR} f(\bar{x})$. From this and applying the sum rule in [31, Theorem 3.39] for the convex functions $ad(\cdot, C)$, $ad(\cdot, D)$ and $\langle -x^*, \cdot - \bar{x}\rangle$, we get $0 \in a\partial_{MR} d(\bar{x}, C) + a\partial_{MR} d(\bar{x}, D) - x^*$. In other words, $x^* \in a\partial_{MR} d(\bar{x}, C) + a\partial_{MR} d(\bar{x}, D)$. From this it follows that

$$\partial_{MR} d(\bar{x}, C \cap D) \subset a\partial_{MR} d(\bar{x}, C) + a\partial_{MR} d(\bar{x}, D).$$

This turns out to be the condition (Q_2) at $\bar{x}$ for $\partial = \partial_{MR}$. □

The next example illustrates the relationship between (Q_1) and (Q_2) in Propositions 3.6 and 3.7. The setting of this example originates from [31, Exercise 7, p. 430] but with different purposes.

Example 3.1 Let $(W, \|\cdot\|_W)$ be a normed space and let $X := W \times \mathbb{R}$ be the product space of X and $\mathbb{R}$ equipped with the norm $\|x\|_X := \|w\|_W + |\alpha|$ for any $x := (w, \alpha) \in W \times \mathbb{R}$. Note that X is infinite-dimensional if and only if W is infinite-dimensional. Besides, $(X, \|\cdot\|_X)$ is a Banach space or an Asplund space if and only if so is $(W, \|\cdot\|_W)$. Consider two convex subsets of X, $C := \{0_W\} \times \mathbb{R}_-$ and $D := \{0_W\} \times \mathbb{R}_+$, where $\mathbb{R}_-$ (resp., $\mathbb{R}_+$) denotes the set of nonpositive (resp., nonnegative) real numbers. We will show that both condition (Q_1) and (Q_2) are satisfied at $\bar{x} := (0_W, 0_\mathbb{R}) \in C \cap D = \{(0_W, 0_\mathbb{R})\}$. Indeed, for each $x = (w, \alpha) \in X$, by direct computations, we have

$$d(x, C \cap D) = \|w\|_W + |\alpha|,$$

$$d(x, C) = \|w\|_W + \max\{0, \alpha\} \quad \text{and} \quad d(x, D) = \|w\|_W + \max\{0, -\alpha\}.$$

Thus, it holds that, for any $x = (w, \alpha) \in X$,

$$d(x, C \cap D) = \|w\|_W + |\alpha| \le 2\|w\|_W + |\alpha| = d(x, C) + d(x, D).$$

It follows that (Q_1) is fulfilled at $\bar{x}$ with $a = 1$ and an arbitrary $r > 0$. Besides, thanks to the convexity of C and D, [31, Proposition 4.36] and [35, Section 4.6], it is not hard to show that

$$\partial_M d(\bar{x}, C \cap D) = \partial_F d(\bar{x}, C \cap D) = \partial_{MR} d(\bar{x}, C \cap D) = \mathbb{B}_{W^*} \times [-1, 1],$$

$$\partial_M d(\bar{x}, C) = \partial_F d(\bar{x}, C) = \partial_{MR} d(\bar{x}, C) = \mathbb{B}_{W^*} \times [0, 1],$$

and

$$\partial_M d(\bar{x}, D) = \partial_F d(\bar{x}, D) = \partial_{MR} d(\bar{x}, D) = \mathbb{B}_{W^*} \times [-1, 0].$$

So, we have

$$\begin{aligned}\partial_{MR} d(\bar{x}, C \cap D) = \mathbb{B}_{W^*} \times [-1, 1] &\subset 2\mathbb{B}_{W^*} \times [-1, 1] \\ &= \partial_{MR} d(\bar{x}, C) + \partial_{MR} d(\bar{x}, D),\end{aligned}$$

which shows the validity of (Q_2) at $\bar{x}$ with $a = 1$ for $\partial = \partial_{MR}$. Consequently, (Q_2) is satisfied at $\bar{x}$ for both $\partial = \partial_M$ and for $\partial = \partial_F$ by virtue of the fact that the distance functions $d(\cdot, C), d(\cdot, D)$ and $d(\cdot, C \cap D)$ are convex.

Remark 3.1 Proposition 3.7 is not a special case of Proposition 3.6, although when C and D are convex, the Mordukhovich subdifferential and the Moreau–Rockafellar subdifferential of the distance functions $d(\cdot, C), d(\cdot, D), d(\cdot, C \cap D)$ coincide. Namely, in the convex case, X can be a normed space while $C \cap D$ is not necessarily closed.

Remark 3.2 The proof schemes of Propositions 3.6 and 3.7 are not applicable to the case of the Fréchet subdifferential. Thus, it is still mysterious to us whether or not the condition (Q_1) implies (Q_2) for $\partial = \partial_F$.

The following characterization of the metric qualification condition (Q_1) was given in the recent paper [1] by An and Jourani for the convex case. A sufficient condition for (Q_1), which is established in terms of *alliedness* for a general case without assuming the convexity, can be found in [31, Theorem 6.44]. At the end of this section, we will present another sufficient condition for (Q_1), for the purpose of connecting different qualification conditions ensuring intersection rules for normal cones.

Proposition 3.8 (See [1, Proposition 2].) *Suppose that X is a Banach space and the sets C, D are closed and convex. Then the metric qualification condition (Q_1) is satisfied at $\bar{x} \in C \cap D$ if and only if there exists $s > 0$ with*

$$\partial_{MR} d(x, C \cap D) \subset a\partial_{MR} d(x, C) + a\partial_{MR} d(x, D), \quad \forall x \in C \cap D \cap \mathbb{B}(\bar{x}, s).$$

We now present intersection rules for normal cones under the metric qualification conditions (Q_1) and (Q_2) in both convex and nonconvex cases.

Theorem 3.1 *Suppose that X is an Asplund space and the sets C and D are closed. If the condition (Q_2) is satisfied at $\bar{x} \in C \cap D$ for $\partial = \partial_M$, then*

$$N_M(\bar{x}; C \cap D) \subset N_M(\bar{x}; C) + N_M(\bar{x}; D). \tag{3.4}$$

Proof Suppose that the condition (Q_2) is fulfilled at $\bar{x} \in C \cap D$ for $\partial = \partial_M$. Let $x^* \in N_M(\bar{x}; C \cap D)$. On the one hand, by Proposition 3.3, we have

$$N_M(\bar{x}; C \cap D) = \bigcup_{\lambda > 0} \lambda \partial_M d(\bar{x}, C \cap D).$$

So, $x^* = \lambda u^*$ for some $\lambda > 0$ and $u^* \in \partial_M d(\bar{x}, C \cap D)$. On the other hand, by (Q_2), we get $u^* \in a\partial_M d(\bar{x}, C) + a\partial_M d(\bar{x}, D)$. Thus,

$$x^* = \lambda u^* \in \lambda a \partial_M d(\bar{x}, C) + \lambda a \partial_M d(\bar{x}, D).$$

This means that there exist $x_1^* \in \partial_M d(\bar{x}, C)$ and $x_2^* \in \partial_M d(\bar{x}, D)$ satisfying $x^* = \lambda a x_1^* + \lambda a x_2^*$. Note that as $\lambda a > 0$, $x_1^* \in \partial_M d(\bar{x}, C)$, and $x_2^* \in \partial_M d(\bar{x}, D)$, Proposition 3.3 implies that $\lambda a x_1^* \in N_M(\bar{x}; C)$ and $\lambda a x_2^* \in N_M(\bar{x}; D)$. Thus, the above representation of x^* yields $x^* \in N_M(\bar{x}; C) + N_M(\bar{x}; D)$, which verifies (3.4). □

The next theorem on the upper estimate (3.4) for the Mordukhovich normal cone $N_M(\bar{x}; C \cap D)$ under the condition (Q_1) is a straightforward consequence of Proposition 3.6 and Theorem 3.1. It appeared in [31, Theorem 6.41] and in [29, Theorem 3.8, (i) implies (iii)] by more direct proofs and by using the *fuzzy sum rule* for the Fréchet subdifferential in Asplund spaces. Let us notice that the assumption X being an Asplund space is indispensable, as shown by a counterexample in the latter paper.

Theorem 3.2 *Suppose that X is an Asplund space and the sets C and D are closed. If the condition (Q_1) is satisfied at $\bar{x} \in C \cap D$, then the intersection rule for the Mordukhovich normal cone (3.4) holds.*

For the case of the Fréchet normal cone, the condition (Q_2) serves as a characterization for the intersection rule.

Theorem 3.3 *If the condition (Q_2) is satisfied at $\bar{x} \in C \cap D$ for $\partial = \partial_F$, then*

$$N_F(\bar{x}; C \cap D) = N_F(\bar{x}; C) + N_F(\bar{x}; D). \tag{3.5}$$

Conversely, if the intersection rule for the Fréchet normal cone (3.5) is fulfilled at $\bar{x} \in C \cap D$, then we can find $a > 0$ such that (Q_2) is valid.

Proof First we note from the definition of the Fréchet normal cone that

$$N_F(\bar{x}; C) + N_F(\bar{x}; D) \subset N_F(\bar{x}; C \cap D) \tag{3.6}$$

always holds. Thus, to prove the first assertion, suppose that (Q_2) is satisfied at $\bar{x} \in C \cap D$ for $\partial = \partial_F$, we will show the validity of the reverse inclusion of (3.6). Take any $x^* \in N_F(\bar{x}; C \cap D)$. By Proposition 3.1(ii), we can find $\lambda \geq 0$ and $u^* \in \partial_F d(\bar{x}, C \cap D)$ such that $x^* = \lambda u^*$. As $u^* \in \partial_F d(\bar{x}, C \cap D)$, (Q_2) implies $u^* \in a\partial_F d(\bar{x}, C) + a\partial_F d(\bar{x}, D)$. This means that $u^* = au_1^* + au_2^*$ with $u_1^* \in \partial_F d(\bar{x}, C)$ and $u_2^* \in \partial_F d(\bar{x}, D)$. Now, we use the first assertion in Proposition 3.1 to get $u_1^* \in N_F(\bar{x}; C)$ and $u_2^* \in N_F(\bar{x}; D)$. Since $\lambda a \geq 0$ and since $N_F(\bar{x}; C)$, $N_F(\bar{x}; D)$ are cones, the last two inclusions imply that $\lambda a u_1^* \in N_F(\bar{x}; C)$ and $\lambda a u_2^* \in N_F(\bar{x}; D)$; and therefore

$$x^* = \lambda u^* = \lambda a u_1^* + \lambda a u_2^* \in N_F(\bar{x}; C) + N_F(\bar{x}; D).$$

To prove the converse assertion, suppose that (3.5) holds at $\bar{x} \in C \cap D$. Given $x^* \in \partial_F d(\bar{x}, C \cap D)$, Proposition 3.1(i) implies $x^* \in N_F(\bar{x}; C \cap D)$. Combining this with the inclusion (3.5) yields $x^* = x_1^* + x_2^*$ with $x_1^* \in N_F(\bar{x}; C)$ and $x_2^* \in N_F(\bar{x}; D)$. It follows from the last two inclusions and Proposition 3.5 that $x_1^* \in \lambda_1 \partial_F d(\bar{x}, C)$ and $x_2^* \in \lambda_2 \partial_F d(\bar{x}, D)$, where $\lambda_1 := \|x_1^*\| + 1$ and $\lambda_2 := \|x_2^*\| + 1$. Thus, by choosing $a := \max\{\lambda_1, \lambda_2\} > 0$, we get

$$x^* = x_1^* + x_2^* \in a\partial_F d(\bar{x}, C) + a\partial_F d(\bar{x}, D).$$

In other words, (Q_2) is satisfied at $\bar{x} \in C \cap D$ for $\partial = \partial_F$. The proof is complete. □

In the case where C and D are convex, we obtain a convex version of Theorem 3.3 as follows.

Theorem 3.4 *If the sets C and D are convex, then*

$$N(\bar{x}; C \cap D) = N(\bar{x}; C) + N(\bar{x}; D) \tag{3.7}$$

if and only if the condition (Q_2) *is satisfied at $\bar{x} \in C \cap D$ for $\partial = \partial_{MR}$.*

As a direct consequence of Proposition 3.7 and Theorem 3.4, the upcoming theorem ensures the validity of the intersection rule for the normal cone under the condition (Q_1).

Theorem 3.5 *If the sets C and D are convex and if the condition* (Q_1) *is valid at $\bar{x} \in C \cap D$, then the intersection rule* (3.7) *for the normal cone holds.*

We use again the setting in Example 3.1 to illustrate Theorems 3.1–3.5.

Example 3.2 Let X, C, D, and $\bar{x}$ be given as in Example 3.1 where we have shown the validity of conditions (Q_1) and (Q_2) for $\partial \in \{\partial_M, \partial_F, \partial_{MR}\}$ at the point $\bar{x} \in C \cap D$. Since $N(\bar{x}; C \cap D) = W^* \times \mathbb{R}$, $N(\bar{x}; C) = W^* \times \mathbb{R}_+$, and $N(\bar{x}; D) = W^* \times \mathbb{R}_-$, formula (3.7) holds. Because the sets C and D are convex, we also have (3.4) and (3.5). Thus, the conclusions of Theorems 3.1–3.5 are verified.

In order to present a connection to the existing theory on intersection rules to normal cones, let us recall from [31, Definition 6.30] that a subset Ω in X is said to be *sequentially normally compact* (SNC) at $\bar{x} \in \Omega$ if for any sequences $x_k \xrightarrow{\Omega} \bar{x}$ and $x_k^* \in N_F(x_k; \Omega)$ for all k, one has

$$\left[x_k^* \xrightarrow{w^*} 0\right] \Longrightarrow \left[||x_k^*|| \to 0\right] \ \text{ as } k \to \infty.$$

If Ω is *compactly epi-Lipschitzian* (CEL) in the sense of Borwein-Strójwas [6] at $\bar{x}$, then it is SNC at $\bar{x}$ (see [23, Theorem 1.26]). But the converse does not hold in general; see more discussions in [23, Remark 1.27]. If X is finite-dimensional, all subsets of X have the CEL property (in particular, SNC property) at all of their points (see [6, Proposition 2.4]. Several equivalences for the SNC property in normed spaces can be found in [31, Lemma 6.31] while a systematic treatment of the SNC property in Banach spaces or in Asplund spaces can be found in [23, Chapters 3 and 6]. In general, such a property of sets together with qualification conditions are the main ingredients of nonsmooth analysis in infinite-dimensional spaces.

Theorem 3.6 (See [28, Corollary 4.5] and [23, Corollary 3.5]) *Suppose that X is an Asplund space, the sets C, D are closed, and either C or D is SNC at $\bar{x} \in C \cap D$. If the following normal qualification condition*

$$\left[x_1^* \in N_M(\bar{x}; C), x_2^* \in N_M(\bar{x}; D),\ x_1^* + x_2^* = 0\right] \Longrightarrow x_1^* = x_2^* = 0 \tag{Q_0}$$

is satisfied, then the intersection rule (3.4) *for the Mordukhovich normal cone holds.*

Remark 3.3 It is worthy to note that the SNC assumption is vital for the fulfillment of the intersection rule (3.4), even for convex and norm–compact sets in infinite–dimensional spaces, as shown by an example given after [23, Corollary 3.5]. In other words, the normal qualification condition (Q_0) alone is generally not enough to ensure the intersection rule (3.4) for closed sets in infinite–dimensional Asplund spaces. Meanwhile, the metric qualification condition (Q_2) itself; in particular (Q_1), is sufficient for the validity of (3.4), thanks to Theorems 3.1 and 3.2.

Example 3.3 Let $W = \mathbb{R}^n$ and consider X, C, D, and $\bar{x}$ as given in Example 3.1. Then, $(X, \|\cdot\|_X)$ is a finite-dimensional Asplund space and both C and D are SNC at $\bar{x}$. Invoking the calculations in Example 3.2, we see that the normal qualification condition (Q_0) is invalid with $x_1^* = (0_{W^*}, 1)$ and $x_2^* = (0_{W^*}, -1)$. Thus, Theorem 3.6 is not applicable. Meanwhile, Theorems 3.1 and 3.2 apply, as shown in Example 3.2.

The following proposition is due to [29, Proposition 3.7]. It states that under the framework of Theorem 3.6, the metric qualification condition (Q_1) is weaker than the normal qualification condition (Q_0). A finite-dimensional version of this result can be found in [31, Proposition 6.46].

Proposition 3.9 (See [29, Proposition 3.7]) *Under the assumptions of Theorem 3.6, the condition* (Q_0) *implies the condition* (Q_1).

In connection with Proposition 3.9, a natural question arises: *Can one obtain the same conclusion for the case where X is a normed space, C and D are convex?* This question remains open. More precisely, we just know the following result from Henrion and Jourani [11], where X is Banach, C and D are closed, convex, and either C or D is CEL at $\bar{x}$. The closedness of the sets and the completeness of the space cannot be dropped due to using of the Ekeland's variational principle and Robinson-Ursescu theorem in the proof. The finite-dimensional version without the CEL property can be found in [1].

Proposition 3.10 (See [11, Lemma 4.2 and Theorem 4.3]) *Suppose that X is a Banach space, the sets C and D are closed, convex. Let one of the sets C or D be CEL at $\bar{x}$. Then the condition* (Q_0) *implies the condition* (Q_1).

From Theorem 3.5 and Proposition 3.10, one gets the following result.

Theorem 3.7 *Suppose all assumptions of Proposition 3.10 are satisfied. If the condition* (Q_0) *holds then the intersection rule* (3.7) *for the normal cone is valid.*

Remark 3.4 In the Banach space setting, Ioffe [14, Theorem 5.4] obtained the intersection rule for *G-normal cones* under the validity of the normal qualification condition (Q_0) using G-normal cones and either C or D being *epi-Lipschitz* at $\bar{x}$. It is worth pointing out that the G-normal cone coincides with the normal cone in the sense of convex analysis if the set in question is convex ([14, Proposition 3.1]). Meanwhile, the epi-Lipschitz property which was introduced by Rockafellar in [33] is stronger than CEL property (see [6, Proposition 2.4]). To sum up, the assumptions in [14, Theorem 5.4] are stronger than our assumptions in Theorem 3.7 for the convex setting.

We close this section by some examples showing that the condition (Q_1) is really weaker than (Q_0) in both convex and nonconvex cases. The first one with an infinite-dimensional setting was observed in [31, Exercise 7, p. 430].

Example 3.4 Let X, C, D, and $\bar{x}$ be given as in Example 3.1. Then the condition (Q_0) is invalid with $x_1^* = (0_{W^*}, 1)$ and $x_2^* = (0_{W^*}, -1)$ while the condition (Q_1) is valid as shown in Example 3.1.

Example 3.5 Let $X = \mathbb{R}^2$, $C := \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_2 \geq |x_1|\}$,

$$D := \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_1 = x_2,\ x_1 \geq 0,\ x_2 \geq 0\} \\ \cup \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_1 = -x_2,\ x_1 \leq 0, x_2 \geq 0\},$$

and $\bar{x} = (0, 0)$. On the one hand, by a direct computation, we have

$$N_M(\bar{x}; C) = \{u = (u_1, u_2) \in \mathbb{R}^2 \mid u_2 \leq -|u_1|\}.$$

On the other hand, from [24, Example 1.14], we get

$$N_M(\bar{x}; D) = \{u = (u_1, u_2) \in \mathbb{R}^2 \mid u_2 = |u_1|\} \\ \cup \{u = (u_1, u_2) \in \mathbb{R}^2 \mid u_2 \leq -|u_1|\}.$$

Let $x_1^* := (1, -1)$ and $x_2^* := (-1, 1)$. Then x_1^* and x_2^* are nonzero elements satisfying $x_1^* \in N_M(\bar{x}; C)$, $x_2^* \in N_M(\bar{x}; D)$, and $x_1^* + x_2^* = (0, 0)$. In the other words, (Q_0) is not satisfied. Meanwhile, as $C \cap D = D$, it is clear that (Q_1) holds for $a = 1$ and for every $r > 0$. Thus, it follows from Theorem 3.2 that the intersection rule (3.4) for the Mordukhovich normal cone is fulfilled. This can be easily checked by direct calculations.

4 Subdifferentials of optimal value functions

Throughout this section, let $F : X \rightrightarrows Y$ be a set-valued map between normed spaces X and Y, and let $\varphi : X \times Y \to \overline{\mathbb{R}}$ be an extended-real-valued function. Corresponding to each pair $\{F, \varphi\}$, consider the following *parametric optimization problem*

$$\min\{\varphi(x, y) \mid y \in F(x)\}, \quad x \in X. \tag{P}$$

The map F and the function φ are respectively called the *constraint map* and the *objective function* of the problem (P). The *optimal value function* (also called *marginal function* or *performance function*) $\mu : X \to \overline{\mathbb{R}}$ of the problem (P) is an extended-real-valued function defined by

$$\mu(x) := \inf\{\varphi(x, y) \mid y \in F(x)\}, \quad x \in X. \tag{4.1}$$

By the convention $\inf\emptyset = +\infty$, we set $\mu(x) = +\infty$ for any $x \notin \operatorname{dom} F$. The *solution map* $M : X \rightrightarrows Y$ of (P) is a set-valued map given by

$$M(x) := \{y \in F(x) \mid \mu(x) = \varphi(x, y)\}, \quad x \in \operatorname{dom} F. \tag{4.2}$$

In this section, we aim at estimating/computing the Fréchet subdifferential, the Mordukhovich subdifferential, and the Moreau–Rockafellar subdifferential of the optimal value function $\mu(\cdot)$ in (4.1) via corresponding coderivatives of the constraint map F and subdifferentials of the objective function φ. Note that $\mu(\cdot)$ is convex if F and φ are both convex (see [25, Theorem 2.129]).

If the constraint map $F : X \rightrightarrows Y$ is given by $F(x) = Y$ for all $x \in X$, then we have the following unconstrained version of the parametric optimization problem (P)

$$\min\{\varphi(x, y) \mid y \in Y\}, \quad x \in X. \tag{P$_1$}$$

As a parametric optimization problem with the parameter x appearing in the objective function, the optimal value function $\mu_1 : X \to \overline{\mathbb{R}}$ and the solution map $M_1 : X \rightrightarrows Y$ of (P$_1$) are given respectively by

$$\mu_1(x) := \inf\{\varphi(x, y) \mid y \in Y\}, \quad x \in X,$$

and

$$M_1(x) := \{y \in Y \mid \varphi(x, y) = \mu_1(x)\}, \quad x \in X.$$

Conversely, the constrained optimization problem (P) can be rewritten in the form (P$_1$) of the unconstrained one with $\varphi + \delta_{\operatorname{Gr} F} : X \times Y \to \overline{\mathbb{R}}$ being the objective function. Note that the epigraph of the latter can be represented as

$$\operatorname{epi}(\varphi + \delta_{\operatorname{Gr} F}) = \operatorname{epi}\varphi \cap (\operatorname{Gr} F \times \mathbb{R}). \tag{4.3}$$

4.1 Upper estimates for the Fréchet subdifferential

The following proposition is about upper estimates for the Fréchet subdifferential and Fréchet singular subdifferential of optimal value functions of unconstrained parametric optimization problems. The first estimate is presented in [31, Theorem 4.47], and the second one is new.

Proposition 4.1 *Let X and Y be normed spaces and consider the unconstrained problem* (P$_1$). *Suppose that the optimal value function $\mu_1(\cdot)$ is finite at $\bar{x} \in \operatorname{dom} M_1$ and $\bar{y} \in M_1(\bar{x})$. Then, one has the inclusions*

$$\partial_F \mu_1(\bar{x}) \subset \{x^* \in X^* \mid (x^*, 0) \in \partial_F \varphi(\bar{x}, \bar{y})\},$$

$$\partial_F^\infty \mu_1(\bar{x}) \subset \{x^* \in X^* \mid (x^*, 0) \in \partial_F^\infty \varphi(\bar{x}, \bar{y})\}. \tag{4.4}$$

Proof of (4.4) Take any $x^* \in \partial_F^\infty \mu_1(\bar{x})$, i.e., $(x^*, 0) \in N_F((\bar{x}, \mu_1(\bar{x}); \operatorname{epi} \mu_1)$ and fix an arbitrary $\varepsilon > 0$. Then, it follows from the definition of the Fréchet normal cone that there exists $\delta > 0$ satisfying

$$\langle x^*, x-\bar{x}\rangle \le \varepsilon\|x-\bar{x}\|+\varepsilon|\alpha-\mu_1(\bar{x})|, \ \forall (x,\alpha) \in \operatorname{epi}\mu_1 \cap \mathbb{B}((\bar{x}, \mu_1(\bar{x})), \delta). \quad (4.5)$$

Let $(x, y, \alpha) \in \operatorname{epi}\varphi \cap \mathbb{B}((\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y})), \delta)$. On the one hand, the inclusion $(x, y, \alpha) \in \operatorname{epi}\varphi$ yields $\alpha \ge \varphi(x, y) \ge \mu_1(x)$. So, we have $(x, \alpha) \in \operatorname{epi}\mu_1$. On the other hand,

$$\begin{aligned}\|x-\bar{x}\| + |\alpha - \mu_1(\bar{x})| &\le \|x-\bar{x}\| + \|y-\bar{y}\| + |\alpha - \mu_1(\bar{x})| \\ &= \|x-\bar{x}\| + \|y-\bar{y}\| + |\alpha - \varphi(\bar{x}, \bar{y})| \le \delta,\end{aligned}$$

as $(x, y, \alpha) \in \mathbb{B}((\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y})), \delta)$. Thus, we get $(x, \alpha) \in \mathbb{B}((\bar{x}, \mu_1(\bar{x})), \delta)$. Therefore, it follows from (4.5) that

$$\langle x^*, x-\bar{x}\rangle \le \varepsilon\|x-\bar{x}\| + \varepsilon|\alpha - \mu_1(\bar{x})| \le \varepsilon\|x-\bar{x}\| + \varepsilon\|y-\bar{y}\| + \varepsilon|\alpha - \varphi(\bar{x}, \bar{y})|.$$

Since the latter holds for any $(x, y, \alpha) \in \operatorname{epi}\varphi \cap \mathbb{B}((\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y})), \delta)$, we conclude that $(x^*, 0, 0) \in N_F((\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y})); \operatorname{epi}\varphi)$. In other words, $(x^*, 0) \in \partial_F^\infty \varphi(\bar{x}, \bar{y})$, which completes the proof of formula (4.4). □

Correspondingly to the above result for unconstrained parametric optimization problems, the next theorem gives us upper estimates for the Fréchet subdifferential/Fréchet singular subdifferential of optimal value functions of constrained parametric optimization problems. Upper estimates are established via the Mordukhovich coderivative of the map describing the constraint and the Mordukhovich subdifferential/Mordukhovich singular subdifferential of the objective function under the conditions (Q_1) or (Q_2) for $\partial = \partial_M$. The first estimate is different from the one in [27, Theorem 1], where the upper estimate for the Fréchet subdifferential of optimal value functions is constituted via the Fréchet coderivative of the constraint map and the *upper Fréchet subdifferential* of the objective function under an assumption on the nonemptiness of the upper Fréchet subdifferential of the objective function. Besides, this seems to be the first time, to the best of our knowledge, that upper estimates for the Fréchet singular subdifferential of optimal value functions are given.

Theorem 4.1 *Let X and Y be Asplund spaces and consider the constrained problem* (P). *Suppose that the constraint map F has closed graph, the objective function φ is lower semicontinuous on $X \times Y$, the optimal value function $\mu(\cdot)$ is finite at $\bar{x} \in \operatorname{dom} M$, and $\bar{y} \in M(\bar{x})$. In addition, assume that*

(A_2) *the sets* $\operatorname{epi}\varphi$ *and* $\operatorname{Gr} F \times \mathbb{R}$ *satisfy the condition* (Q_2) *for* $\partial = \partial_M$ *at* $(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y}))$*; in particular,*
(A_1) *the sets* $\operatorname{epi}\varphi$ *and* $\operatorname{Gr} F \times \mathbb{R}$ *satisfy the condition* (Q_1) *at* $(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y}))$.

Then, one has the inclusions

$$\partial_F \mu(\bar{x}) \subset \bigcup_{(x^*, y^*) \in \partial_M \varphi(\bar{x}, \bar{y})} \{x^* + D_M^* F(\bar{x}, \bar{y})(y^*)\}, \quad (4.6)$$

$$\partial_F^\infty \mu(\bar{x}) \subset \bigcup_{(x^*,y^*)\in\partial_M^\infty \varphi(\bar{x},\bar{y})} \{x^* + D_M^* F(\bar{x},\bar{y})(y^*)\}. \tag{4.7}$$

Proof First, by transforming the constrained problem (P) into the unconstrained one (P_1) with $\varphi + \delta_{\operatorname{Gr} F} : X \times Y \to \overline{\mathbb{R}}$ being the objective function and applying the results of Proposition 4.1, we get the following inclusions

$$\partial_F \mu(\bar{x}) \subset \{u^* \in X^* \mid (u^*, 0) \in \partial_F(\varphi + \delta_{\operatorname{Gr} F})(\bar{x},\bar{y})\},$$

$$\partial_F^\infty \mu(\bar{x}) \subset \{u^* \in X^* \mid (u^*, 0) \in \partial_F^\infty(\varphi + \delta_{\operatorname{Gr} F})(\bar{x},\bar{y})\}.$$

Combining these with the relationships between Fréchet subdifferentials and Mordukhovich subdifferentials in (2.2) and (2.3) yields

$$\partial_F \mu(\bar{x}) \subset \{u^* \in X^* \mid (u^*, 0) \in \partial_M(\varphi + \delta_{\operatorname{Gr} F})(\bar{x},\bar{y})\} \tag{4.8}$$

and

$$\partial_F^\infty \mu(\bar{x}) \subset \{u^* \in X^* \mid (u^*, 0) \in \partial_M^\infty(\varphi + \delta_{\operatorname{Gr} F})(\bar{x},\bar{y})\}. \tag{4.9}$$

[*Proof of* (4.6)] Take any $u^* \in \partial_F \mu(\bar{x})$. Then, it follows from (4.8) that $(u^*, 0) \in \partial_M(\varphi+\delta_{\operatorname{Gr} F})(\bar{x},\bar{y})$. On the one hand, since φ is lower semicontinuous on $X \times Y$ and F has closed graph, $\varphi+\delta_{\operatorname{Gr} F}$ is lower semicontinuous on $X \times Y$. So, by using the relation (2.5) between the Mordukhovich subdifferential and the Mordukhovich normal cone, we get $(u^*, 0) \in \partial_M(\varphi + \delta_{\operatorname{Gr} F})(\bar{x},\bar{y})$ if and only if

$$(u^*, 0, -1) \in N_M[(\bar{x},\bar{y},(\varphi + \delta_{\operatorname{Gr} F})(\bar{x},\bar{y})); \operatorname{epi}(\varphi + \delta_{\operatorname{Gr} F})]. \tag{4.10}$$

On the other hand, $(\varphi + \delta_{\operatorname{Gr} F})(\bar{x},\bar{y}) = \varphi(\bar{x},\bar{y}) \in \mathbb{R}$ because $\bar{y} \in M(\bar{x})$ and $\mu(\bar{x}) \in \mathbb{R}$. So, using formula (4.3), we have

$$\begin{aligned} &N_M[(\bar{x},\bar{y},(\varphi + \delta_{\operatorname{Gr} F})(\bar{x},\bar{y})); \operatorname{epi}(\varphi + \delta_{\operatorname{Gr} F})] \\ &= N_M[(\bar{x},\bar{y},\varphi(\bar{x},\bar{y})); \operatorname{epi}\varphi \cap (\operatorname{Gr} F \times \mathbb{R})]. \end{aligned} \tag{4.11}$$

Moreover, as $\operatorname{epi}\varphi$ and $\operatorname{Gr} F \times \mathbb{R}$ satisfy the condition (Q_2) for $\partial = \partial_M$ at $(\bar{x},\bar{y},\varphi(\bar{x},\bar{y}))$, Theorem 3.1 implies that

$$\begin{aligned} N_M[(\bar{x},\bar{y},\varphi(\bar{x},\bar{y})); \operatorname{epi}\varphi \cap (\operatorname{Gr} F \times \mathbb{R})] \subset & N_M[(\bar{x},\bar{y},\varphi(\bar{x},\bar{y})); \operatorname{epi}\varphi] \\ &+ N_M[(\bar{x},\bar{y},\varphi(\bar{x},\bar{y})); \operatorname{Gr} F \times \mathbb{R}]. \end{aligned} \tag{4.12}$$

Note that the above inclusion is also valid under the condition (Q_1), due to Proposition 3.6 and Theorem 3.2. Combining (4.10), (4.11) and (4.12) yields

$$(u^*, 0, -1) \in N_M[(\bar{x},\bar{y},\varphi(\bar{x},\bar{y})); \operatorname{epi}\varphi] + N_M[(\bar{x},\bar{y},\varphi(\bar{x},\bar{y})); \operatorname{Gr} F \times \mathbb{R}]. \tag{4.13}$$

Because of (4.13), we can find

$$(x^*, y^*, \alpha^*) \in N_M[(\bar{x},\bar{y},\varphi(\bar{x},\bar{y})); \operatorname{epi}\varphi] \tag{4.14}$$

and

$$(\tilde{x}^*, \tilde{y}^*, \tilde{\alpha}^*) \in N_M[(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y})); \operatorname{Gr} F \times \mathbb{R}] \tag{4.15}$$

such that $(u^*, 0, -1) = (x^*, y^*, \alpha^*) + (\tilde{x}^*, \tilde{y}^*, \tilde{\alpha}^*)$. The latter means that

$$\begin{cases} u^* = x^* + \tilde{x}^* \\ \tilde{y}^* = -y^* \\ -1 = \alpha^* + \tilde{\alpha}^*. \end{cases} \tag{4.16}$$

From (4.15), one has

$$(\tilde{x}^*, \tilde{y}^*) \in N_M((\bar{x}, \bar{y}); \operatorname{Gr} F) \tag{4.17}$$

and

$$\tilde{\alpha}^* \in N_M(\varphi(\bar{x}, \bar{y}); \mathbb{R}) = \{0\}. \tag{4.18}$$

Meanwhile, from the last identity of (4.16) and (4.18), one obtains $\alpha^* = -1$. Thus, it follows from (4.14) that $(x^*, y^*) \in \partial_M \varphi(\bar{x}, \bar{y})$. Besides, the second identity of (4.16) and (4.17) yield $(\tilde{x}^*, -y^*) \in N_M((\bar{x}, \bar{y}), \operatorname{Gr} F)$. In another word, we have $\tilde{x}^* \in D_M^* F(\bar{x}, \bar{y})(y^*)$. This inclusion and the first identity of (4.16) imply that $u^* \in x^* + D_M^* F(\bar{x}, \bar{y})(y^*)$. The latter means that (4.6) is valid.

[*Proof of* (4.7)] Let $u^* \in \partial_F^\infty \mu(\bar{x})$. Then, $(u^*, 0) \in \partial_M^\infty(\varphi + \delta_{\operatorname{Gr} F})(\bar{x}, \bar{y})$ because of (4.9). By the definition of the Mordukhovich singular subdifferential and by formula (4.3), the last inclusion is equivalent to

$$(u^*, 0, 0) \in N_M[(\bar{x}, \bar{y}, (\varphi + \delta_{\operatorname{Gr} F})(\bar{x}, \bar{y})); \operatorname{epi}(\varphi + \delta_{\operatorname{Gr} F})]. \tag{4.19}$$

Thus, it follows from (4.19), (4.11) and (4.12) that

$$(u^*, 0, 0) \in N_M[(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y})); \operatorname{epi} \varphi] + N_M[(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y})); \operatorname{Gr} F \times \mathbb{R}].$$

Consequently, by similar arguments as in the proof of (4.6), we can represent $u^* = x^* + \tilde{x}^*$ with $x^*, \tilde{x}^*$ in X^* and y^* in Y^* satisfying

$$(x^*, y^*, 0) \in N_M[(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y})); \operatorname{epi} \varphi] \quad \text{and} \quad (\tilde{x}^*, -y^*) \in N_M((\bar{x}, \bar{y}); \operatorname{Gr} F).$$

The last two inclusions mean that $(x^*, y^*) \in \partial_M^\infty \varphi(\bar{x}, \bar{y})$ and $\tilde{x}^* \in D_M^* F(\bar{x}, \bar{y})(y^*)$, respectively. Therefore, $u^* \in x^* + D_M^* F(\bar{x}, \bar{y})(y^*)$ and (4.7) is proved. The proof of the theorem is complete. □

The following result is another version of Theorem 4.1 by making use of the assumption (A_0) instead of (A_2) or (A_1). As we will see later in the next subsections, (A_0) and its variants are well-known in the theory of nonsmooth calculus. Note that when φ is Lipschitz continuous around $(\bar{x}, \bar{y})$, then $\operatorname{epi} \varphi$ is SNC at $(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y}))$ (see [23, p. 121]) and $\partial_M^\infty \varphi(\bar{x}, \bar{y}) = \{(0, 0)\}$ (see [23, Corollary 1.81]). Thus, (A_0) is automatically satisfied in this situation.

Theorem 4.2 *The conclusion of Theorem 4.1 still holds if the assumption* (A_2) *(in particular* (A_1)*) is replaced by*

(A_0) *either* $\operatorname{epi}\varphi$ *is SNC at* $(\bar{x},\bar{y},\varphi(\bar{x},\bar{y}))$ *or* $\operatorname{Gr}F$ *is SNC at* $(\bar{x},\bar{y})$*, and*

$$\partial_M^\infty\varphi(\bar{x},\bar{y})\cap(-N_M((\bar{x},\bar{y});\operatorname{Gr}F))=\{(0,0)\}. \tag{4.20}$$

Proof Clearly, (4.20) means that the sets $\operatorname{epi}\varphi$ and $\operatorname{Gr}F\times\mathbb{R}$ satisfy the normal qualification condition (Q_0) at $(\bar{x},\bar{y},\varphi(\bar{x},\bar{y}))$. Thus, the assumption (A_1) in Theorem 4.1 is guaranteed by the assumption (A_0) and Proposition 3.9. □

When the objective function of (P) does not depend on the parameter x, i.e., $\varphi(x,y):=g(y)$, for all $(x,y)\in X\times Y$, where $g:Y\to\overline{\mathbb{R}}$ is an extended-real-value function defined on Y, it is not hard to show that $\operatorname{epi}\varphi=X\times\operatorname{epi}g$, $\partial\varphi(\bar{x},\bar{y})=\{0\}\times\partial g(\bar{y})$, and $\partial^\infty\varphi(\bar{x},\bar{y})=\{0\}\times\partial^\infty g(\bar{y})$ with $\partial\in\{\partial_M,\partial_F,\partial_{MR}\}$. Thus, the next result is straightforward from Theorems 4.1 and 4.2.

Proposition 4.2 *Let* X *and* Y *be Asplund spaces and consider the constrained problem* (P)*. Suppose that the constraint map* F *has closed graph, the objective function* φ *is given by* $\varphi(x,y):=g(y)$ *for all* $(x,y)\in X\times Y$*, with* $g:Y\to\overline{\mathbb{R}}$ *being lower semicontinous on* Y*, the optimal value function* $\mu(\cdot)$ *is finite at* $\bar{x}\in\operatorname{dom}M$*, and* $\bar{y}\in M(\bar{x})$*. In addition, assume that*

(A_2') *the sets* $X\times\operatorname{epi}g$ *and* $\operatorname{Gr}F\times\mathbb{R}$ *satisfy the condition* (Q_2) *for* $\partial=\partial_M$ *at* $(\bar{x},\bar{y},g(\bar{y}))$*; in particular,*

(A_1') *the sets* $X\times\operatorname{epi}g$ *and* $\operatorname{Gr}F\times\mathbb{R}$ *satisfy the condition* (Q_1) *at* $(\bar{x},\bar{y},g(\bar{y}))$*; in particular,*

(A_0') *either* $\operatorname{epi}g$ *is SNC at* $(\bar{y},g(\bar{y}))$ *or* $\operatorname{Gr}F$ *is SNC at* $(\bar{x},\bar{y})$*, and*

$$(\{0\}\times\partial_M^\infty g(\bar{y}))\cap(-N_M((\bar{x},\bar{y});\operatorname{Gr}F))=\{(0,0)\}.$$

Then we have

$$\partial_F\mu(\bar{x})\subset\bigcup_{y^*\in\partial_M g(\bar{y})}D_M^*F(\bar{x},\bar{y})(y^*)\ \text{and}\ \partial_F^\infty\mu(\bar{x})\subset\bigcup_{y^*\in\partial_M^\infty g(\bar{y})}D_M^*F(\bar{x},\bar{y})(y^*).$$

4.2 Upper estimates for the Mordukhovich subdifferential

In order to give upper estimates for the Mordukhovich subdifferential and the Mordukhovich singular subdifferential of the optimal value function $\mu(\cdot)$ defined in (4.1), let us first recall relevant concepts and results from [23].

Definition 4.1 (See [23, Definition 1.63] and [27]) Let $M(\cdot):X\rightrightarrows Y$ be the solution map defined in (4.2) of the parametric optimization problem (P). One says that

(i) $M(\cdot)$ is *μ-inner semicontinuous* at $(\bar{x},\bar{y})\in\operatorname{Gr}M$ if for every sequence $x_k\xrightarrow{\mu}\bar{x}$ there exists a sequence $y_k\in M(x_k)$ converging to $\bar{y}$ as $k\to\infty$;

(ii) $M(\cdot)$ is *μ-inner semicompact* at $\bar{x}$ if for every sequence $x_k \xrightarrow{\mu} \bar{x}$ there is a sequence $y_k \in M(x_k)$ that contains a convergent subsequence as $k \to \infty$.

The properties of the solution map considered in the above definition extend the corresponding notions in [23, Definition 1.63] and adapt them to the solution map $M(\cdot)$ of (P). The only difference is that the condition $x_k \to \bar{x}$ in [23] is replaced by the weaker condition $x_k \xrightarrow{\mu} \bar{x}$. This causes no any effect to the conclusions of [23, Theorem 1.108], as observed in [27].

Proposition 4.3 (See [23, Theorem 1.108]) *Let X and Y be Banach spaces and consider the unconstrained problem* (P_1)*. Suppose that the optimal value function $\mu_1(\cdot)$ is finite at $\bar{x} \in \operatorname{dom} M_1$. Then, the following statements hold:*

(i) *If the solution map $M_1(\cdot)$ is μ_1-inner semicontinuous at $(\bar{x}, \bar{y}) \in \operatorname{Gr} M_1$, then one has the inclusions*

$$\partial_M \mu_1(\bar{x}) \subset \{x^* \in X^* \mid (x^*, 0) \in \partial_M \varphi(\bar{x}, \bar{y})\},$$

$$\partial_M^\infty \mu_1(\bar{x}) \subset \{x^* \in X^* \mid (x^*, 0) \in \partial_M^\infty \varphi(\bar{x}, \bar{y})\}.$$

(ii) *If the solution map $M_1(\cdot)$ is μ_1-inner semicompact at $\bar{x}$ and the objective function φ is lower semicontinuous on $X \times Y$, then one has the inclusions*

$$\partial_M \mu_1(\bar{x}) \subset \{x^* \in X^* \mid (x^*, 0) \in \bigcup_{\bar{y} \in M_1(\bar{x})} \partial_M \varphi(\bar{x}, \bar{y})\},$$

$$\partial_M^\infty \mu_1(\bar{x}) \subset \{x^* \in X^* \mid (x^*, 0) \in \bigcup_{\bar{y} \in M_1(\bar{x})} \partial_M^\infty \varphi(\bar{x}, \bar{y})\}.$$

Based on the above result for the unconstrained parametric optimization problem (P_1), we are able to formulate corresponding estimates for the Mordukhovich subdifferential and the Mordukhovich singular subdifferential of the optimal value function of the constrained one (P). In comparison with Theorem 4.1 (resp., Theorem 4.2, Proposition 4.2), in order to obtain upper estimates for Mordukhovich subdifferentials of $\mu(\cdot)$, Theorem 4.3 (resp., Theorem 4.4, Proposition 4.4) requires additional assumptions put on the solution map $M(\cdot)$.

Theorem 4.3 *Let X and Y be Asplund spaces and consider the constrained problem* (P)*. Suppose that the constraint map F has closed graph, the objective function φ is lower semicontinous on $X \times Y$, and the optimal value function $\mu(\cdot)$ is finite at $\bar{x} \in \operatorname{dom} M$. Then, the following assertions hold:*

(i) *If the solution map $M(\cdot)$ is μ-inner semicontinuous at $(\bar{x}, \bar{y}) \in \operatorname{Gr} M$ and if* (A_2) *(in particular,* (A_1)*) is satisfied, then*

$$\partial_M \mu(\bar{x}) \subset \bigcup_{(x^*, y^*) \in \partial_M \varphi(\bar{x}, \bar{y})} \{x^* + D_M^* F(\bar{x}, \bar{y})(y^*)\}, \tag{4.21}$$

$$\partial_M^\infty \mu(\bar{x}) \subset \bigcup_{(x^*, y^*) \in \partial_M^\infty \varphi(\bar{x}, \bar{y})} \{x^* + D_M^* F(\bar{x}, \bar{y})(y^*)\}. \tag{4.22}$$

(ii) *If the solution map* $M(\cdot)$ *is* μ*-inner semicompact at* $\bar{x}$ *and if* (A_2) *(in particular,* (A_1)*) is satisfied with any* $\bar{y} \in M(\bar{x})$*, then*

$$\partial_M \mu(\bar{x}) \subset \bigcup_{\bar{y}\in M(\bar{x})} \bigcup_{(x^*,y^*)\in\partial_M\varphi(\bar{x},\bar{y})} \{x^* + D^*_M F(\bar{x},\bar{y})(y^*)\}, \tag{4.23}$$

$$\partial^\infty_M \mu(\bar{x}) \subset \bigcup_{\bar{y}\in M(\bar{x})} \bigcup_{(x^*,y^*)\in\partial^\infty_M\varphi(\bar{x},\bar{y})} \{x^* + D^*_M F(\bar{x},\bar{y})(y^*)\}. \tag{4.24}$$

Proof The proof will be divided into two parts, corresponding to assertions (i) and (ii).

[*Proof of assertion* (i)] Since the solution map $M(\cdot)$ is μ-inner semicontinuous at $(\bar{x},\bar{y})$, applying Proposition 4.3(i) for the unconstrained problem (P_1) with $\varphi + \delta_{\mathrm{Gr}\,F}$ being the objective function yields the following inclusions

$$\partial_M \mu(\bar{x}) \subset \{u^* \in X^* \mid (u^*,0) \in \partial_M(\varphi + \delta_{\mathrm{Gr}\,F})(\bar{x},\bar{y})\},$$

$$\partial^\infty_M \mu(\bar{x}) \subset \{u^* \in X^* \mid (u^*,0) \in \partial^\infty_M(\varphi + \delta_{\mathrm{Gr}\,F})(\bar{x},\bar{y})\}.$$

Let $u^* \in \partial_M \mu(\bar{x})$ (resp., $u^* \in \partial^\infty_M \mu(\bar{x})$). Then, $(u^*,0) \in \partial_M(\varphi + \delta_{\mathrm{Gr}\,F})(\bar{x},\bar{y})$ (resp., $(u^*,0) \in \partial^\infty_M(\varphi + \delta_{\mathrm{Gr}\,F})(\bar{x},\bar{y})$). Therefore, by similar arguments as in the proof of Theorem 4.1, we can find $x^*, \tilde{x}^*$ in X^* and y^* in Y^* such that $(x^*,y^*) \in \partial_M\varphi(\bar{x},\bar{y})$ (resp., $(x^*,y^*) \in \partial^\infty_M\varphi(\bar{x},\bar{y})$), $\tilde{x}^* \in D^*_M F(\bar{x},\bar{y})(y^*)$, and $u^* = x^* + \tilde{x}^*$. This means that u^* belongs to the set on the right-hand side of (4.21) (resp., the set on the right-hand side of (4.22)).

[*Proof of assertion* (ii)] Applying Proposition 4.3(ii) to the unconstrained problem (P_1) with $\varphi+\delta_{\mathrm{Gr}\,F}$ being the objective function and arguing similarly as in the above proof, we get the desired inclusions (4.23) and (4.24).

The proof is completed. □

As shown in the proof of Theorem 4.2, the assumption (A_1) is guaranteed if (A_0) is satisfied. Thus, the next theorem follows directly from Theorem 4.3. This result was presented in [27, Theorem 7, (i) and (ii)] under certain assumptions on the closedness of the constraint map F and the lower semicontinuity of the objective function φ. Namely, these two properties are required *locally* around the point $(\bar{x},\bar{y})$ under consideration in the latter paper, while herein we assume that F has closed graph and φ is lower semicontinuous on $X \times Y$. The interested reader is referred to [28, Theorem 6.1] and [23, Theorem 3.38] for earlier versions of the result.

Theorem 4.4 (Cf. [27, Theorem 7 (i) and (ii)]) *The conclusions of Theorem 4.3 are still valid if the assumption* (A_2) *(in particular* (A_1)*) is replaced by* (A_0).

Applying Theorems 4.3 and 4.4 to the case where the objective function of (P) does not depend on the parameter x, we get the next result.

Proposition 4.4 *Let X and Y be Asplund spaces and consider the constrained problem* (P)*. Suppose that the constraint map F has closed graph, the objective function φ is given by $\varphi(x,y) := g(y)$ for all $(x,y) \in X \times Y$, with $g : Y \to \overline{\mathbb{R}}$ being lower semicontinuous on Y, the optimal value function $\mu(\cdot)$ is finite at $\bar{x} \in \operatorname{dom} M$. Then, following statements are valid:*

(i) *If the solution map $M(\cdot)$ is μ-inner semicontinuous at $(\bar{x},\bar{y}) \in \operatorname{Gr} M$ and if* (A'_2) *(in particular,* (A'_1) *or* (A'_0)*) is satisfied, then*

$$\partial_M \mu(\bar{x}) \subset \bigcup_{y^* \in \partial_M g(\bar{y})} D^*_M F(\bar{x},\bar{y})(y^*) \ \ \text{and} \ \ \partial^\infty_M \mu(\bar{x}) \subset \bigcup_{y^* \in \partial^\infty_M g(\bar{y})} D^*_M F(\bar{x},\bar{y})(y^*).$$

(ii) *If the solution map $M(\cdot)$ is μ-inner semicompact at $\bar{x}$ and if* (A'_2) *(in particular,* (A'_1) *or* (A'_0)*) is satisfied with any $\bar{y} \in M(\bar{x})$, then*

$$\partial_M \mu(\bar{x}) \subset \bigcup_{\bar{y} \in M(\bar{x})} \bigcup_{y^* \in \partial_M g(\bar{y})} D^*_M F(\bar{x},\bar{y})(y^*),$$

$$\partial^\infty_M \mu(\bar{x}) \subset \bigcup_{\bar{y} \in M(\bar{x})} \bigcup_{y^* \in \partial^\infty_M g(\bar{y})} D^*_M F(\bar{x},\bar{y})(y^*).$$

4.3 Representations for the Moreau–Rockafellar subdifferential

This subsection is devoted to studying the Moreau–Rockafellar subdifferentials of optimal value functions in the case that the constraint map and the objective function are both convex. Compared to results in the above subsections, herein we will work on normed spaces (or at most Banach spaces) rather than requiring them being Asplund. Besides, assumptions on the inner semicontinuity of the solution map will be dropped. Lastly but most importantly, we will obtain exact representations for the Moreau–Rockafellar subdifferentials of optimal value functions, instead of upper estimates. Keep in mind that such representations for the Fréchet subdifferential and the Mordukhovich subdifferential can be obtained but usually under extra assumptions on the differentiability of the objective function and on another special stability of the solution map (see, e.g, [27, Theorem 2 and Theorem 7 (iii)]).

Similarly as in previous subsections, we will start with a result for the unconstrained parametric optimization problem (P_1), which follows directly from [3, Theorem 4.2] because the regularity condition (a) therein is automatically satisfied.

Proposition 4.5 (Cf. [3, Theorem 4.2]) *Let X and Y be normed spaces and consider the unconstrained problem* (P_1)*. Suppose that the objective function φ is convex, the optimal value function $\mu_1(\cdot)$ is finite at $\bar{x} \in \operatorname{dom} M_1$, and $\bar{y} \in M_1(\bar{x})$. Then, one has the inclusions*

$$\partial_{MR} \mu_1(\bar{x}) = \{x^* \in X^* \mid (x^*,0) \in \partial_{MR} \varphi(\bar{x},\bar{y})\},$$

$$\partial^\infty \mu_1(\bar{x}) = \{x^* \in X^* \mid (x^*,0) \in \partial^\infty \varphi(\bar{x},\bar{y})\}.$$

Based on the above result, we are able to give exact formulas for computing the Moreau–Rockafellar subdifferential and the singular subdifferential of $\mu(\cdot)$ of the constrained parametric optimization problem (P). In the first two theorems, the condition (A_1) and the new condition $(\hat{\mathrm{A}}_1)$ both rely on the condition (Q_1) between sets. The difference is that the second one involves computations of sets $\operatorname{dom}\varphi$ and $\operatorname{Gr} F$ in $X \times Y$ which is somewhat easier than the first one conducting in $X \times Y \times \mathbb{R}$. Nevertheless, under (A_1), we can obtain representations for both the Moreau–Rockafellar subdifferential and the singular subdifferential while only the latter can be achieved if assuming $(\hat{\mathrm{A}}_1)$. In comparison with the result in [1, Theorem 1] where the formula (4.25) was obtained under the condition (A_1) by a different approach, herein we do not require that X, Y are Banach spaces, the constraint map has closed graph, and the objective function is lower semicontinuous on $X \times Y$.

Theorem 4.5 *Let X and Y be normed spaces and consider the constrained problem* (P) *with the constraint map F and the objective function φ being convex. Suppose that the optimal value function $\mu(\cdot)$ is finite at $\bar{x} \in \operatorname{dom} M$ and $\bar{y} \in M(\bar{x})$. If* (A_1) *holds, then*

$$\partial_{MR}\mu(\bar{x}) = \bigcup_{(x^*,y^*)\in\partial_{MR}\varphi(\bar{x},\bar{y})} \{x^* + D^*F(\bar{x},\bar{y})(y^*)\}, \tag{4.25}$$

$$\partial^{\infty}\mu(\bar{x}) = \bigcup_{(x^*,y^*)\in\partial^{\infty}\varphi(\bar{x},\bar{y})} \{x^* + D^*F(\bar{x},\bar{y})(y^*)\}. \tag{4.26}$$

Proof Since $\operatorname{Gr} F$ is convex and φ is convex, $\varphi + \delta_{\operatorname{Gr} F}$ is a convex function. Applying the results of Proposition 4.5 for the unconstrained problem (P_1) with $\varphi + \delta_{\operatorname{Gr} F}$ being the objective function, we get the following inclusions

$$\partial_{MR}\mu(\bar{x}) = \{u^* \in X^* \mid (u^*, 0) \in \partial_{MR}(\varphi + \delta_{\operatorname{Gr} F})(\bar{x}, \bar{y})\}, \tag{4.27}$$

$$\partial^{\infty}\mu(\bar{x}) = \{u^* \in X^* \mid (u^*, 0) \in \partial^{\infty}(\varphi + \delta_{\operatorname{Gr} F})(\bar{x}, \bar{y})\}. \tag{4.28}$$

Due to (4.27) (resp., (4.28)), it holds for any $u^* \in X^*$ that $u^* \in \partial_{MR}\mu(\bar{x})$ (resp., $u^* \in \partial^{\infty}\mu(\bar{x})$) if and only if $(u^*, 0) \in \partial_{MR}(\varphi + \delta_{\operatorname{Gr} F})(\bar{x}, \bar{y})$ (resp., $(u^*, 0) \in \partial^{\infty}(\varphi + \delta_{\operatorname{Gr} F})(\bar{x}, \bar{y})$). On the one hand, by the relation (2.4) between the Moreau–Rockafellar subdifferential and the normal cone, the last inclusion means

$$(u^*, 0, -1) \in N[(\bar{x}, \bar{y}, (\varphi + \delta_{\operatorname{Gr} F})(\bar{x}, \bar{y})); \operatorname{epi}(\varphi + \delta_{\operatorname{Gr} F})] \tag{4.29}$$

(resp.,

$$(u^*, 0, 0) \in N[(\bar{x}, \bar{y}, (\varphi + \delta_{\operatorname{Gr} F})(\bar{x}, \bar{y})); \operatorname{epi}(\varphi + \delta_{\operatorname{Gr} F})]). \tag{4.30}$$

On the other hand, we have $(\varphi + \delta_{\mathrm{Gr}\,F})(\bar{x}, \bar{y}) = \varphi(\bar{x}, \bar{y}) \in \mathbb{R}$ because of $\bar{y} \in M(\bar{x})$ and $\mu(\bar{x}) \in \mathbb{R}$. So, it follows from formula (4.3) that

$$\begin{aligned} N[(\bar{x}, \bar{y}, (\varphi+\delta_{\mathrm{Gr}\,F})(\bar{x}, \bar{y})); \mathrm{epi}\,(\varphi + \delta_{\mathrm{Gr}\,F})] \\ = N[(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y})); \mathrm{epi}\,(\varphi+\delta_{\mathrm{Gr}\,F})] \\ = N[(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y})); \mathrm{epi}\,\varphi \cap (\mathrm{Gr}\,F \times \mathbb{R})]. \end{aligned} \tag{4.31}$$

Moreover, thanks to the assumption (A_1) and Theorem 3.4, we get

$$\begin{aligned} N[(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y})); \mathrm{epi}\,\varphi \cap (\mathrm{Gr}\,F \times \mathbb{R})] \\ = N[(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y})); \mathrm{epi}\,\varphi] + N[(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y})); \mathrm{Gr}\,F \times \mathbb{R}]. \end{aligned} \tag{4.32}$$

Thus, it follows from(4.29) (resp., (4.30)), (4.31), and (4.32) that $u^* \in \partial_{MR}\mu(\bar{x})$ (resp., $u^* \in \partial^\infty \mu(\bar{x})$) if and only if

$$(u^*, 0, -1) \in N[(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y})); \mathrm{epi}\,\varphi] + N[(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y})); \mathrm{Gr}\,F \times \mathbb{R}]$$

(resp., $(u^*, 0, 0) \in N[(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y})); \mathrm{epi}\,\varphi] + N[(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y})); \mathrm{Gr}\,F \times \mathbb{R}]$). Therefore, by the same argument as in the proof of Theorem 4.1, the latter means that there exist $x^*, \tilde{x}^*$ in X^* and y^* in Y^* satisfying $(x^*, y^*) \in \partial_{MR}\varphi(\bar{x}, \bar{y})$ (resp., $(x^*, y^*) \in \partial^\infty \varphi(\bar{x}, \bar{y})$), $\tilde{x}^* \in D^*F(\bar{x}, \bar{y})(y^*)$, and $u^* = x^* + \tilde{x}^*$, proving (4.25) (resp., (4.26)). □

Theorem 4.6 *The representation* (4.26) *is also valid if the assumption* (A_1) *in Theorem 4.5 is replaced by*

$(\hat{\mathrm{A}}_1)$ *the sets* $\mathrm{dom}\,\varphi$ *and* $\mathrm{Gr}\,F$ *satisfy the metric qualification condition* (Q_1) *at* $(\bar{x}, \bar{y})$.

Proof By the convexity of the functions φ, $\varphi + \delta_{\mathrm{Gr}\,F}$ and [3, Proposition 4.1], we have the following equations

$$\partial^\infty \varphi(\bar{x}, \bar{y}) = N[(\bar{x}, \bar{y}); \mathrm{dom}\,\varphi], \tag{4.33}$$

$$\partial^\infty (\varphi + \delta_{\mathrm{Gr}\,F})(\bar{x}, \bar{y}) = N[(\bar{x}, \bar{y}); \mathrm{dom}\,(\varphi + \delta_{\mathrm{Gr}\,F})]. \tag{4.34}$$

Besides, from the convexity of the functions φ, $\delta_{\mathrm{Gr}\,F}$ and [37, Theorem 2.1.3(ii)], we get $\mathrm{dom}\,(\varphi + \delta_{\mathrm{Gr}\,F}) = \mathrm{dom}\,\varphi \cap \mathrm{dom}\,\delta_{\mathrm{Gr}\,F} = \mathrm{dom}\,\varphi \cap \mathrm{Gr}\,F$. Therefore, it follows from assumption $(\hat{\mathrm{A}}_1)$ and Theorem 3.4 that

$$\begin{aligned} N[(\bar{x}, \bar{y}); \mathrm{dom}\,(\varphi + \delta_{\mathrm{Gr}\,F})] = N[(\bar{x}, \bar{y}); \mathrm{dom}\,\varphi \cap \mathrm{Gr}\,F] \\ = N[(\bar{x}, \bar{y}); \mathrm{dom}\,\varphi] + N[(\bar{x}, \bar{y}); \mathrm{Gr}\,F]. \end{aligned} \tag{4.35}$$

Combining (4.33), (4.34), and (4.35) yields

$$\partial^\infty (\varphi + \delta_{\mathrm{Gr}\,F})(\bar{x}, \bar{y}) = \partial^\infty \varphi(\bar{x}, \bar{y}) + N[(\bar{x}, \bar{y}); \mathrm{Gr}\,F].$$

Consequently, formula (4.28) becomes

$$\partial^\infty \mu(\bar{x}) = \{u^* \in X^* \mid (u^*, 0) \in \partial^\infty \varphi(\bar{x}, \bar{y}) + N[(\bar{x}, \bar{y}); \mathrm{Gr}\,F]\}.$$

Thus, arguing similarly as in the proof of Theorem 4.1, we get (4.26). □

The next result, which was observed in [1, Corollary 1 and Theorem 2], is a consequence of Theorems 4.5 and 4.6.

Theorem 4.7 *Let X and Y be Banach spaces and consider the constrained problem* (P)*. Suppose that the constraint map F is convex and closed, the objective function φ is convex and lower semicontinous on $X\times Y$. In addition, suppose that the optimal value function $\mu(\cdot)$ is finite at $\bar{x} \in \operatorname{dom} M$ and $\bar{y} \in M(\bar{x})$. The following statements hold:*

(i) *If either $\operatorname{epi}\varphi$ is CEL at $(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y}))$ or $\operatorname{Gr} F$ is CEL at $(\bar{x}, \bar{y})$ and if*

$$\partial^\infty \varphi(\bar{x}, \bar{y}) \cap (-N((\bar{x}, \bar{y}); \operatorname{Gr} F)) = \{(0,0)\}, \tag{4.36}$$

then one has the representations (4.25) *and* (4.26)*.*

(ii) *If $\operatorname{dom}\varphi$ is CEL at $(\bar{x}, \bar{y})$ and if the condition* (4.36) *is satisfied, then one also has the representation* (4.26)*.*

Proof As demonstrated in the proof of Theorem 4.1, the condition (4.36) means that the sets $\operatorname{epi}\varphi$ and $\operatorname{Gr} F\times\mathbb{R}$ satisfy the condition (Q_0) at $(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y}))$. Thus, the assumptions of the assertion (i) and Proposition 3.10 imply that the assumption (A_1) is valid. Therefore, (4.25) and (4.26) follow from Theorem 4.5.

To prove (ii), note that the condition (4.36) can be rewritten as

$$N((\bar{x}, \bar{y}); \operatorname{dom}\varphi) \cap (-N((\bar{x}, \bar{y}); \operatorname{Gr} F)) = \{(0,0)\}$$

due to (4.33). This means that the sets $\operatorname{dom}\varphi$ and $\operatorname{Gr} F$ satisfy the condition (Q_0) at $(\bar{x}, \bar{y})$. The latter together with the assumption that $\operatorname{dom}\varphi$ is CEL at $(\bar{x}, \bar{y})$ and Proposition 3.10 yields $(\hat{\mathrm{A}}_1)$. Thus, one obtains (4.26) from Theorem 4.6. The proof is finished. □

Remark 4.1 Observe that $\operatorname{dom}\varphi$ is the image of the set $\operatorname{epi}\varphi$ under the projection $(x, y, \alpha) \mapsto (x, y)$. So, if $\operatorname{epi}\varphi$ is CEL at $(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y}))$, then $\operatorname{dom}\varphi$ is CEL at $(\bar{x}, \bar{y})$ by [5, Proposition 3.6]. Hence, the condition (ii) in Theorem 4.7 is weaker than the assumptions on the CEL property of $\operatorname{epi}\varphi$ and the validity of (4.36) in the condition (i).

Let us now formulate results corresponding to Theorems 4.5, 4.6, and 4.7 for the case where the objective function of (P) does not depend on the parameter x. The first proposition with a normed space setting is derived from Theorems 4.5 and 4.6. While, the second one, which requires the completeness of underlying spaces, the closedness of the constraint map, and the lower semicontinuity of the objective function, is obtained from Theorem 4.7.

Proposition 4.6 *Let X and Y be normed spaces and consider the constrained problem* (P)*. Suppose that the constraint map F is convex and the objective function φ is given by $\varphi(x, y) := g(y)$ for all $(x, y) \in X\times Y$, with $g: Y \to \overline{\mathbb{R}}$ being a convex function on Y. Moreover, assume that the optimal value function $\mu(\cdot)$ is finite at $\bar{x} \in \operatorname{dom} M$ and $\bar{y} \in M(\bar{x})$. The following assertions hold:*

(i) *If* (A_1') *is valid, then one has the following formulas*

$$\partial_{MR}\mu(\bar{x}) = \bigcup_{y^* \in \partial_{MR} g(\bar{y})} D^*F(\bar{x},\bar{y})(y^*), \tag{4.37}$$

$$\partial^{\infty}\mu(\bar{x}) = \bigcup_{y^* \in \partial^{\infty} g(\bar{y})} D^*F(\bar{x},\bar{y})(y^*). \tag{4.38}$$

(ii) *If* $X \times \operatorname{dom} g$ *and* $\operatorname{Gr} F$ *satisfy* (Q_1) *at* $(\bar{x},\bar{y})$*, then* (4.38) *holds.*

Proposition 4.7 *Let* X *and* Y *be Banach spaces and consider the constrained problem* (P)*. Suppose that the constraint map* F *is convex and closed, and the objective function* φ *is given by* $\varphi(x,y) := g(y)$ *for all* $(x,y) \in X \times Y$*, with* $g : Y \to \overline{\mathbb{R}}$ *being convex and lower semicontinuous on* Y*. In addition, assume that the optimal value function* $\mu(\cdot)$ *is finite at* $\bar{x} \in \operatorname{dom} M$ *and* $\bar{y} \in M(\bar{x})$*. The following statements hold:*

(i) *If either* $\operatorname{epi} g$ *is CEL at* $(\bar{y}, g(\bar{y}))$ *or* $\operatorname{Gr} F$ *is CEL at* $(\bar{x},\bar{y})$ *and if*

$$\{0\} \times \partial^{\infty} g(\bar{y}) \cap (-N((\bar{x},\bar{y}); \operatorname{Gr} F)) = \{(0,0)\}, \tag{4.39}$$

then one has (4.37) *and* (4.38)*.*

(ii) *If* $\operatorname{dom} g$ *is CEL at* $\bar{y}$ *and if* (4.39) *is fulfilled, then* (4.38) *is valid.*

5 Chain rules for subdifferentials

In this section, we will apply results for subdifferentials of optimal value functions obtained in Section 4 to derive chain rules for subdifferentials of composite functions.

Theorem 5.1 *Let* $f : X \to \mathbb{R}$ *be a lower semicontinous function on an Asplund space* X*, and* $g : \mathbb{R} \to \overline{\mathbb{R}}$ *be a lower semicontinuous, nondecreasing function. Suppose that* $g \circ f$ *is finite at* $\bar{x}$ *and* $\bar{y} := f(\bar{x})$*. Suppose further that*

(i) $X \times \operatorname{epi} g$ *and* $\operatorname{epi} f \times \mathbb{R}$ *satisfy the condition* (Q_2) *at* $(\bar{x}, \bar{y}, g(\bar{y}))$ *for* $\partial = \partial_M$*; in particular,*

(ii) $X \times \operatorname{epi} g$ *and* $\operatorname{epi} f \times \mathbb{R}$ *satisfy the condition* (Q_1) *at* $(\bar{x}, \bar{y}, g(\bar{y}))$*.*

Then one has the inclusions

$$\partial_F (g \circ f)(\bar{x}) \subset \bigcup_{\lambda \in \partial_M g(\bar{y}),\, \lambda > 0} \big[\lambda \partial_M f(\bar{x}) \bigcup \partial_M^{\infty} f(\bar{x})\big],$$

$$\partial_F^{\infty} (g \circ f)(\bar{x}) \subset \bigcup_{\lambda \in \partial_M^{\infty} g(\bar{y}),\, \lambda > 0} \big[\lambda \partial_M f(\bar{x}) \bigcup \partial_M^{\infty} f(\bar{x})\big].$$

In addition, if for every sequence $x_k \xrightarrow{g\circ f} \bar{x}$ *one has* $f(x_k) \to f(\bar{x})$, *then*

$$\partial_M(g\circ f)(\bar{x}) \subset \bigcup_{\lambda\in\partial_M g(\bar{y}),\, \lambda>0} \big[\lambda\partial_M f(\bar{x}) \bigcup \partial_M^\infty f(\bar{x})\big],$$

$$\partial_M^\infty(g\circ f)(\bar{x}) \subset \bigcup_{\lambda\in\partial_M^\infty g(\bar{y}),\, \lambda>0} \big[\lambda\partial_M f(\bar{x}) \bigcup \partial_M^\infty f(\bar{x})\big].$$

Proof Define $F : X \rightrightarrows \mathbb{R}$ by $F(x) := [f(x),\infty)$. Then, $\operatorname{Gr} F = \operatorname{epi} f$. Besides, since g is nondecreasing, one has $(g\circ f)(x) = \inf_{y\in F(x)} g(y)$ for any $x \in X$. This means that the composite $g\circ f$ is the optimal value function μ of the problem (P), where the objective function does not depend on the parameter x. Thus, to obtain the upper estimates for the Fréchet subdifferential and the Fréchet singular subdifferential of the composite $g\circ f$, we apply Proposition 4.2 to get the following inclusions

$$\partial_F(g\circ f)(\bar{x}) \subset \bigcup_{\lambda\in\partial_M g(\bar{y})} D_M^* F(\bar{x},\bar{y})(\lambda), \tag{5.1}$$

$$\partial_F^\infty(g\circ f)(\bar{x}) \subset \bigcup_{\lambda\in\partial_M^\infty g(\bar{y})} D_M^* F(\bar{x},\bar{y})(\lambda). \tag{5.2}$$

Meanwhile, under the additional condition $x_k \xrightarrow{g\circ f} \bar{x}$ implying $f(x_k) \to f(\bar{x})$, we see that the solution map is μ-inner semicontinuous. Thus, by applying Proposition 4.4, we obtain the inclusions

$$\partial_M(g\circ f)(\bar{x}) \subset \bigcup_{\lambda\in\partial_M g(\bar{y})} D_M^* F(\bar{x},\bar{y})(\lambda), \tag{5.3}$$

$$\partial_M^\infty(g\circ f)(\bar{x}) \subset \bigcup_{\lambda\in\partial_M^\infty g(\bar{y})} D_M^* F(\bar{x},\bar{y})(\lambda). \tag{5.4}$$

Let us compute $D_M^* F(\bar{x},\bar{y})(\lambda)$. By definition, $x^* \in D_M^* F(\bar{x},\bar{y})(\lambda)$ if and only if $(x^*,-\lambda) \in N_M((\bar{x},\bar{y}); \operatorname{Gr} F)$. Since $\operatorname{Gr} F = \operatorname{epi} f$ and $\bar{y} = f(\bar{x})$, the latter is equivalent to $(x^*,-\lambda) \in N_M((\bar{x}, f(\bar{x})); \operatorname{epi} f)$. If $\lambda = 0$, the latter means $x^* \in \partial_M^\infty f(\bar{x})$. If $\lambda \neq 0$, then it follows from [29, Proposition 2.1] that

$$(x^*,-\lambda) \in N_M((\bar{x}, f(\bar{x})); \operatorname{epi} f) \Longleftrightarrow \lambda > 0 \text{ and } x^* \in \partial_M(\lambda f)(\bar{x}).$$

Moreover, when $\lambda > 0$, [31, Proposition 6.17(d)] yields $\partial_M(\lambda f)(\bar{x}) = \lambda\partial_M f(\bar{x})$. In summary, we get

$$D_M^* F(\bar{x},\bar{y})(\lambda) = \lambda\partial_M f(\bar{x}) \bigcup \partial_M^\infty f(\bar{x}), \quad \lambda > 0. \tag{5.5}$$

Therefore, we receive upper estimates for the Fréchet subdifferential and the Fréchet singular subdifferential of $g\circ f$ from (5.1), (5.2), and (5.5), respectively.

Meanwhile, combining (5.3) with (5.5) (resp., (5.4) with (5.5)) gives the upper valuation for the Mordukhovich subdifferential (resp., the Mordukhovich singular subdifferential) of $g \circ f$.

The proof is complete. □

By Proposition 3.9, the next corollary follows directly from Theorem 5.1.

Corollary 5.1 *The conclusions of Theorem 5.1 still hold if the assumption of assertion* (i) *(in particular* (ii)*) is replaced by*

$$(\{0\} \times \partial_M^\infty g(\bar{y})) \cap (-N_M(\bar{x}, \bar{y}); \operatorname{epi} f)) = \{(0,0)\}. \tag{5.6}$$

Remark 5.1 In [29, Theorem 4.10], Ngai and Théra studied formulas for estimating the Mordukhovich subdifferential and the Mordukhovich singular subdifferential of the composite $g \circ f$, where $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, $f = (f_1, f_2, ..., f_n)$ with $f_i : X \to \mathbb{R} \cup \{+\infty\}$, $i = 1, 2, ..., n$. More precisely, besides necessary conditions, the authors employed a condition which implies (5.6) for the case $n = 1$. Although our results are established for a smaller class of problems, we require regularity assumptions which are weaker than those of [29, Theorem 4.10]. In addition, we derived upper estimates for the Fréchet subdifferential and the Fréchet singular subdifferential of the composite that have not been mentioned in the literature.

Let us recall some definitions from [21, Chapter 1]. Let X, Y be normed spaces and $K \subset Y$ be a closed convex cone. Define an order in Y with respect to the cone K by $a \leq_K b$ if and only if $b - a \in K$. A function $g : Y \to \overline{\mathbb{R}}$ is called K-*nondecreasing* if $y_1 \leq_K y_2$ implies $g(y_1) \leq g(y_2)$. For a given map $f : X \to Y$, *epigraph* of f is the set $\operatorname{epi} f := \{(x, y) \in X \times Y \mid f(x) \leq_K y\}$. The map f is said to be K-*convex* if and only if $f(\alpha x_1 + (1-\alpha)x_2) \leq_K \alpha f(x_1) + (1-\alpha) f(x_2)$ for every $x_1, x_2 \in X$ and $\alpha \in [0, 1]$. Corresponding to a fixed $y^* \in Y^*$, the *scalarization* of the map f is given by $(y^* \circ f)(x) = \langle y^*, f(x) \rangle$ for every $x \in X$.

The chain rules in ordered spaces in the upcoming theorem are derived from Proposition 4.6.

Theorem 5.2 *Let X, Y be normed spaces and $K \subset Y$ be a closed convex cone. Besides, let $f : X \to Y$ be a K-convex map and $g : Y \to \overline{\mathbb{R}}$ be a convex and K-nondecreasing function. Suppose that $g \circ f$ is finite at $\bar{x}$ and $\bar{y} := f(\bar{x})$.*

(i) *If $X \times \operatorname{epi} g$ and $\operatorname{epi} f \times \mathbb{R}$ satisfy the condition* (Q_1) *at $(\bar{x}, \bar{y}, g(\bar{y}))$, then*

$$\partial_{MR}(g \circ f)(\bar{x}) = \bigcup_{y^* \in \partial_{MR} g(\bar{y})} \partial_{MR}(y^* \circ f)(\bar{x}), \tag{5.7}$$

$$\partial^\infty(g \circ f)(\bar{x}) = \bigcup_{y^* \in \partial^\infty g(\bar{y})} \partial_{MR}(y^* \circ f)(\bar{x}). \tag{5.8}$$

(ii) *If $X \times \operatorname{dom} g$ and $\operatorname{epi} f$ satisfy the condition* (Q_1) *at $(\bar{x}, \bar{y})$, then* (5.8) *holds.*

Proof Note that the composition $g \circ f : X \to \mathbb{R}$ is a convex function because the map f is K-convex and the function g is convex and K-nondecreasing. Consider the set-valued map $F : X \rightrightarrows Y$ with $F(x) := \{y \in Y \mid f(x) \leq_K y\}$, $x \in X$. It is not hard to see that $\operatorname{Gr} F = \operatorname{epi} f$ and F is convex because f is K-convex. Besides, since g is K-nondecreasing, one has $(g \circ f)(x) = \inf_{y \in F(x)} g(y)$ for any $x \in X$. Thus, it follows from Proposition 4.6(i) that

$$\partial_{MR}(g \circ f)(\bar{x}) = \bigcup_{y^* \in \partial_{MR} g(\bar{y})} D^*F(\bar{x}, \bar{y})(y^*)$$

and, when the assumption (i) of this theorem is satisfied,

$$\partial^\infty(g \circ f)(\bar{x}) = \bigcup_{y^* \in \partial^\infty g(\bar{y})} D^*F(\bar{x}, \bar{y})(y^*). \tag{5.9}$$

Similarly, Proposition 4.6(ii) implies (5.9) under the assumption (ii) of this theorem. Therefore, to finish the proof, it remains to show that

$$D^*F(\bar{x}, \bar{y})(y^*) = \partial_{MR}(y^* \circ f)(\bar{x}), \quad \forall y^* \in \partial_{MR} g(\bar{y}) \cup \partial^\infty g(\bar{y}). \tag{5.10}$$

To prove (5.10), we fix $y^* \in \partial_{MR} g(\bar{y}) \cup \partial^\infty g(\bar{y})$. Let $x^* \in D^*F(\bar{x}, \bar{y})(y^*)$. Then, $(x^*, -y^*) \in N((\bar{x}, \bar{y}); \operatorname{Gr} F)$ by definition. As $\operatorname{Gr} F = \operatorname{epi} f$, the last inclusion means that $\langle x^*, x - \bar{x}\rangle - \langle y^*, y - \bar{y}\rangle \leq 0$ for every $(x, y) \in X \times Y$ with $f(x) \leq_K y$. Particularly, one has $\langle x^*, x - \bar{x}\rangle \leq \langle y^*, f(x) - \bar{y}\rangle \leq 0$ for all $x \in X$. Since $\bar{y} = f(\bar{x})$, the latter means that $\langle x^*, x - \bar{x}\rangle \leq \langle y^*, f(x) - f(\bar{x})\rangle \leq 0$ for any $x \in X$, which yields $x^* \in \partial_{MR}(g \circ f)(\bar{x})$. Conversely, let $x^* \in \partial_{MR}(g \circ f)(\bar{x})$, we need prove that $x^* \in D^*F(\bar{x}, \bar{y})(y^*)$, i.e.,

$$(x^*, -y^*) \in N((\bar{x}, \bar{y}); \operatorname{Gr} F) = N((\bar{x}, \bar{y}); \operatorname{epi} f).$$

Indeed, pick any $(x, y) \in X \times Y$ satisfying $f(x) \leq_K y$ and put $u := y - f(x)$. Since $f(\bar{x}) \leq_K f(\bar{x})$, one has $f(\bar{x}) - u \leq_K f(\bar{x})$. This implies that

$$g \circ (f(\bar{x}) - u) \leq (g \circ f)(\bar{x}), \tag{5.11}$$

because g is K-nondecreasing. To continue, we consider the following two cases:

Case 1: $y^* \in \partial_{MR}(g \circ f)(\bar{x})$. Then, one has $\langle y^*, v - f(\bar{x})\rangle \leq g(v) - g(f(\bar{x}))$ for any $v \in Y$. Therefore with $v := f(\bar{x}) - u$, one gets

$$\langle y^*, f(\bar{x}) - u - f(\bar{x})\rangle \leq g \circ (f(\bar{x}) - u) - (g \circ f)(\bar{x}),$$

which is equivalent to $\langle y^*, -u\rangle \leq g \circ (f(\bar{x}) - u) - (g \circ f)(\bar{x})$. From this and (5.11), one obtains $\langle y^*, -u\rangle \leq 0$.

Case 2: $y^* \in \partial^\infty(g \circ f)(\bar{x})$. Then, $y^* \in N(f(\bar{x}); \operatorname{dom} g)$ because g is a convex function. So, by definition, one has

$$\langle y^*, v - f(\bar{x}) \leq 0, \ \forall v \in Y, \ g(v) < +\infty. \tag{5.12}$$

Notice that $v := f(\bar{x}) - u \in \operatorname{dom} g$ due to the inequality (5.11) and the fact that $\bar{x} \in \operatorname{dom}(g \circ f)$. Thus, applying (5.12) for $v := f(\bar{x}) - u$ yields $\langle y^*, f(\bar{x}) - u - f(\bar{x})\rangle \le 0$; or equivalently, $\langle y^*, -u\rangle \le 0$.

We have just shown that $\langle y^*, -u\rangle \le 0$ for both cases $y^* \in \partial_{MR}(g\circ f)(\bar{x})$ and $y^* \in \partial^\infty(g \circ f)(\bar{x})$. Inserting $u = y - f(x)$ again into the last inequality yields $\langle y^*, f(x)\rangle \le \langle y^*, y\rangle$. By adding $\langle y^*, -f(\bar{x})\rangle$ to both sides of this inequality, we obtain

$$\langle y^*, f(x) - f(\bar{x})\rangle \le \langle y^*, y - f(\bar{x})\rangle. \tag{5.13}$$

Now using the fact that $x^* \in \partial_{MR}(y^* \circ f)(\bar{x})$, we obtain

$$\langle x^*, x - \bar{x}\rangle \le \langle y^*, f(x) - f(\bar{x})\rangle. \tag{5.14}$$

It follows from (5.13) and (5.14) that $\langle x^*, x - \bar{x}\rangle \le \langle y^*, y - f(\bar{x})\rangle$. Because this inequality holds for arbitrary $(x, y) \in X \times Y$ such that $f(x) \le_K y$, we conclude that $(x^*, -y^*) \in N((\bar{x}, \bar{y}); \operatorname{epi} f)$, where $\bar{y} = f(\bar{x})$.

The proof is complete. □

Thanks to Proposition 3.10, we get a consequence of Theorem 5.2 as follows.

Corollary 5.2 *In addition to the assumptions of Theorem 5.2, suppose that* X, Y *are Banach spaces and* $\operatorname{epi} f, \operatorname{epi} g$ *are closed sets.*

(i) *If either* $\operatorname{epi} f$ *is CEL at* $(\bar{x}, \bar{y})$ *or* $\operatorname{epi} g$ *is CEL at* $(\bar{y}, g(\bar{y}))$ *and*

$$(\{0\} \times \partial^\infty g(\bar{y})) \cap (-N(\bar{x}, \bar{y}); \operatorname{epi} f)) = \{(0, 0)\}, \tag{5.15}$$

then one has (5.7) *and* (5.8).

(ii) *If* $\operatorname{dom} g$ *is CEL at* $(\bar{y}, g(\bar{y}))$ *and* (5.15) *is valid, then one gets* (5.8).

Remark 5.2 Assuming that X, Y are locally convex topological vector spaces, Mordukhovich et al. [26, Theorem 7.6], [25, Theorem 4.60] obtained formula (5.7) under the supposition that there exists $x \in X$ with g being finite and continuous at some point $y \in Y$ satisfying $f(x) \le_K y$. Following the reasoning in [1, Section 6], we can see that if X and Y are finite-dimensional, then the latter assumption is stronger than the assumption of Corollary 5.2(i).

Remark 5.3 In the particular case when $Y = \mathbb{R}$ and $K := \mathbb{R}_+$, i.e., when $f : X \to \mathbb{R}$ is a convex function and $g : \mathbb{R} \to \overline{\mathbb{R}}$ is a nondecreasing convex function, formulas (5.7) and (5.8) respectively become

$$\partial_{MR}(g \circ f)(\bar{x}) = \bigcup_{\lambda \in \partial_{MR} g(\bar{y}),\ \lambda \ge 0} \lambda \partial_{MR} f(\bar{x}),$$

$$\partial^\infty(g \circ f)(\bar{x}) = \bigcup_{\lambda \in \partial^\infty g(\bar{y}),\ \lambda \ge 0} \lambda \partial_{MR} f(\bar{x}).$$

This follows from the fact that if $g : \mathbb{R} \to \overline{\mathbb{R}}$ is a nondecreasing convex function, then for a given number $\lambda \in \mathbb{R}$, $\lambda \in \partial_{MR} g(\bar{y})$ or $\lambda \in \partial^\infty g(\bar{y})$ only if $\lambda \geq 0$. Indeed, suppose first that $\lambda \in \partial_{MR} g(\bar{y})$. Then, $\lambda(y - \bar{y}) \leq g(y) - g(\bar{y})$ for any $y \in \mathbb{R}$. Thus, choosing $y := \bar{y} - 1$ and using the nondecreasing property of g yield $\lambda \geq 0$. Similarly, suppose now $\lambda \in \partial^\infty g(\bar{y})$. Then $\lambda \in \operatorname{dom} g(\bar{y})$ because of the convexity of g. Thus, $\lambda(y - \bar{y}) \leq 0$ for any $y \in \mathbb{R}$ with $g(y) < +\infty$. So, as before, choosing $y := \bar{y} - 1$ and using the nondecreasing property of g, we get $\lambda \geq 0$.

6 Statements and Declarations

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