TANNAKIAN DUALITY AND GAUSS-MANIN CONNECTIONS FOR A FAMILY OF CURVES

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ABSTRACT. Let $X \longrightarrow \operatorname{Spec} R$ be a smooth family of smooth projective schemes parameterized by a smooth affine curve S over a field k of characteristic 0. For a flat connection on X/k we compare its (relative) de Rham cohomology equipped with the Gauss-Manin connection and the group cohomology determined in terms of Tannakian duality.

1. INTRODUCTION

Let X/k be a separated scheme, $x \in X \times_k \bar{k}$ be a $\bar{k}$-point of $X \times_k \bar{k}$, where $\bar{k}$ an algebraic closure of k. Grothendieck defines in [SGA1] the etale fundamental group $\pi^{\text{et}}(X, x)$ which is an algebraic analog of the topological fundamental group of a (arc-wise) connected topological space. This group is a profinite group which classifies Galois coverings of X (with a distinguished point above x). Then Grothendieck deduces the following exact sequences.

- The fundamental exact sequence which relates the etale fundamental groups of X, $\bar{X} := X \times_k \bar{k}$ and the Galois group $\operatorname{Gal}(\bar{k}/k)$:
$$1 \longrightarrow \pi^{\text{et}}(\bar{X}, x) \longrightarrow \pi^{\text{et}}(X, x) \longrightarrow \operatorname{Gal}(\bar{k}/k) \longrightarrow 1.$$
- The homotopy exact sequence which is associated to each proper map $f : X \longrightarrow S$ (of separated schemes over k) with connected fibers:
$$\pi^{\text{et}}(X_s, x) \longrightarrow \pi^{\text{et}}(X, x) \longrightarrow \pi^{\text{et}}(S, s) \longrightarrow 1,$$
where $s = f(x)$, $X_s = f^{-1}(s)$.

There is another natural algebraic replacement of the topological fundamental group motivated by the Riemann-Hilbert correspondence. Roughly speaking, on a complex manifold, there is a correspondence between systems of analytic linear differential equations, i.e. $\mathcal{O}_X$-coherent modules with flat connection, and local systems on X, i.e. finite dimensional complex linear representations of the topological fundamental group of X. Now for a smooth scheme X over a field k of characteristic 0, let x be a k-rational point of X. Consider the category of flat connections on X/k equipped with the fiber functor at x, i.e. to associate to each connection its fiber at x. Tannakian duality applied yields a pro-algebraic affine group scheme, called the *differential fundamental group of X at x*.

Let $f : X \longrightarrow \operatorname{Spec} R$ be a smooth, projective morphism of smooth schemes over k. Then there is an associated homotopy exact sequence of differential fundamental group schemes. The first aim of this work is to establish an analog of the fundamental exact sequence in this

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relative setting when S is the spectrum of a Dedekind ring (Theorem 3.2). Thus, let $\eta : S \longrightarrow X$ be a section to f, or, in other words, an R-point of X as an R-schemes (by means of f). Let s be a k-point of Spec$R$ and $x = \eta(s)$. Tannakian duality applied to the category of connections on X/k, R/k, and X/R equipped with natural fiber functors yields the fundamental groupoids $\Pi(X/k)$, $\Pi(R/k)$ and the relative fundamental group $\pi(X/R)$ (see Appendix A.2). These groupoids and group are put together in a sequence

$$\pi(X/R) \longrightarrow \Pi(X/k) \longrightarrow \Pi(S/k) \longrightarrow 1,$$

which is shown to be exact, see 2.1. We note that this result has appeard in the PhD thesis of Hugo Bay-Rousson [1], however some parts of the argument in the proof there is missing. We provide here a full account.

Given the exact sequences, our problem is to compare the de Rham cohomology and the group cohomology as well as expressed the Gauss-Manin connection in terms of group cohomology. Let L be the quotient group scheme of $\pi(X/R)$ such that the sequence below is exact:

$$1 \longrightarrow L \longrightarrow \Pi(X/k) \longrightarrow \Pi(S/k) \longrightarrow 1.$$

Then for a connection $\mathcal{V} \in \mathscr{C}(X/k)$ there is a natural map

$$H^n(L, V) \longrightarrow \mathrm{H}^n_{DR}(X/R, \inf(\mathcal{V}))$$

which is $\pi(R/k)$-equivariant (cf. Theorem 5.18), where $V = \eta^*(V)$ and $\inf(\mathcal{V})$ is the relative connection inflated from $\mathcal{V}$ (cf. 2.3.3). In general the above maps are far from an isomorphism. However for the case $f : X \longrightarrow$ Spec$R$ is a smooth family of curves of genus ≥ 1 we show they are isomorphisms in degre 0 and 1 (cf. Corollary 4.6).

2. Flat connections and the differential fundamental groupoid

Let k be a field of characteristic 0. Let R be a k-algebra, which is a Dedekind ring. Let $f : X \longrightarrow \mathrm{Spec}(R)$ be a smooth map with geometrically connected fibers.

2.1. **Connections.** Let $\Omega^1_{X/R}$ denote the sheaf of realtive Kähler differentials on X/R. By assumption on X, it is a locally free sheaf. A flat connection on a sheaf of $\mathcal{O}_X$-modules $\mathcal{V}$ on X is an $f^{-1}O_S$-linear map

$$\nabla : \mathcal{V} \longrightarrow \mathcal{V} \otimes \Omega^1_{X/S},$$

satisfying the Leibniz rule and is flat in the sense that the composed map $\nabla_1 \circ \nabla = 0$, where

$$\nabla_1 : \Omega^1_{X/S} \otimes \mathcal{V} \longrightarrow \Omega^2_{X/S} \otimes \mathcal{V}; \quad \omega \otimes e \longmapsto d\omega \otimes e - \omega \otimes \nabla(e).$$

(cf. [Ka70, (1.0)]).

IWhen no confusion may arise we shall address a sheaf with flat connection simply as a connection. The notation is $(\mathcal{V}, \nabla)$ and usually abbreviated it to $\mathcal{V}$.

We denote by MIC(X/k) the category of $\mathcal{O}_X$-quasi-coherent sheaves with k-linear flat connections. It is known that each $\mathcal{O}_X$-coherent sheaf with flat connection is locally free and the dual sheaf is equipped with a (dual) connection in a canonical way. Hence the full subcategory $\mathscr{C}(X/k)$ of coherent connections is a rigid tensor k-linear abelian category. Similarly we have the category $\mathscr{C}(S/k)$. We have the functor

$$f^*:\mathscr{C}(R/k)\longrightarrow\mathscr{C}(X/k).$$

We denote by MIC(X/R) the category of quasi-coherent $\mathscr{O}_X$-sheaves equipped with R-linear flat connections and by MIC$^\circ(X/R)$ the full subcategory of $\mathscr{O}_X$-locally free coherent sheaves. We let $\mathscr{C}(X/R)$ be the full subcategory of MIC(X/R) consisting of objects which can be presented as quotients of objects from MIC$^\circ(X/R)$. This is a Tannakian category over R in the sense of Saavedra (cf. [Sa72] or [DH18]).

There might be coherent sheaves with connection on X/R which do not belong to $\mathscr{C}(X/R)$ but we do not know any examples.

According to [DH18], an object of $\mathscr{C}(X/R)$ is locally free if (and only if) it is torsion free over R. We introduce the notion of a special subquotient of locally free connection as follows: Let $\mathcal{V}\in$ MIC$^\circ(X/R)$. Let $\alpha:\mathcal{V}'\to\mathcal{V}$ be a monomorphism in $\mathscr{C}(X/\mathrm{R})$. If Coker$(\alpha)$ is also locally free, we say that α is a *special monomorphism.* Call an object $\mathcal{V}''\in$ MIC$^\circ(X/R)$ a *special subquotient* of $\mathcal{V}$ if there exists a special monomorphism $\mathcal{V}'\to\mathcal{V}$ and an epimorphism $\mathcal{V}'\to\mathcal{V}''$. The category of all special sub-quotients of various $T^{a_1,b_1}(\mathcal{V})\oplus\cdots\oplus T^{a_m,b_m}(\mathcal{V})$ is denoted by $\langle\mathcal{V}\rangle^s_\otimes$.

2.2. **De Rham cohomology.** For a flat relative connection $(\mathcal{V},\nabla))$ on X/R, the sheaf of horizontal sections is define to be

$$\mathcal{V}^\nabla:=\mathrm{Ker}(\nabla:\mathcal{V}\to\Omega^1_{X/R}\otimes\mathcal{V}). \tag{1}$$

This is an R-linear sheaf. The 0-th de Rham cohomology of $\mathcal{V}$ is defined to to be

$$\mathrm{H}^0_{dR}(X/R,\mathcal{V}):=f_*\mathcal{V}^\nabla.$$

Moreover $\mathrm{H}^0_{DR}(X/R,\mathscr{M})$ can be identified with the hom-set of connections

$$\{\varphi:(\mathscr{O}_X,d)\longrightarrow(\mathcal{V},\nabla)\}.$$

Since MIC(X/R) is equivalence to categories of left modules on the sheaf of differential operators $\mathscr{D}_{X/R}$, it has enough injectives (cf. [Ka70]). Thus we can define the higher de Rham cohomologies to be the derived functors of the functor

$$\mathrm{H}^0_{DR}(X/R,-):\mathrm{MIC}(X/R)\longrightarrow\mathrm{Mod}_R.$$

This cohomologies can also be computed as the ext-groups of extensions in MIC(X/R). By definition, $\mathrm{Ext}^i_{\mathrm{MIC}(X/R)}(\mathscr{M},\mathscr{N})$ counts i-extensions in MIC(X/R), which are exact sequences

$$0\longrightarrow\mathscr{N}\longrightarrow\mathscr{N}_1\longrightarrow\ldots\longrightarrow\mathscr{N}_i\longrightarrow\mathscr{M}\longrightarrow 0,$$

upto an equivalence define as follows: two sequence are equivalent if there is a map of sequence between them induced by the identity maps on $\mathscr{M}$ and $\mathscr{N}$. Since MIC(X/R) has enough injective, $\mathrm{Ext}^i_{\mathrm{MIC}(X/R)}(\mathscr{M},-)$ are the right derived functors of the hom-functor

$$\mathrm{Hom}_{\mathrm{MIC}(X/R)}(\mathscr{M},-):\mathrm{MIC}(X/R)\longrightarrow\mathrm{Mod}_R.$$

Thus we have the following:

Lemma 2.1. *Let* $(\mathscr{M},\nabla)$ *be a flat connection in* $\mathrm{MIC}(X/R)$. *Then we have:*

$$\mathrm{Ext}^i_{\mathrm{MIC}(X/R)}(\mathscr{O}_X, d, \mathscr{M}) \cong \mathrm{H}^i_{DR}(X/R, \mathscr{M}).$$

2.3. **Fundamental groups and groupoids.** Let $\eta : \mathrm{Spec}(R) \longrightarrow X$ be an R-point of X (as an R-scheme). This section also yields a k-point x in the fiber X_s of X:

$$\begin{array}{ccc} \mathrm{Spec}(k) & \xrightarrow{s} & \mathrm{Spec}(R) \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle \eta} \\ X_s & \longrightarrow & X. \end{array}$$

These points yield various Tannakian groups and groupoids.

2.3.1. *The fundamental groups.* The fundamental group $\pi(X/k) = \pi(X/k, x)$ is the Tannakian dual of $\mathscr{C}(X/k)$ with respect to the fiber functor x^* (cf. Appendix A.6):

$$\mathrm{Rep}_k(\pi(X/k)) \cong \mathscr{C}(X/k).$$

The fundamental group $\pi(S/k) = \pi(S/k, s)$ is the Tannakian dual of $\mathscr{C}(X/k)$ with respect to the fiber functor s^*:

$$\mathrm{Rep}_k(\pi(S/k)) \cong \mathscr{C}(S/k).$$

The map $f : X \longrightarrow S$ induces a group homomorphism

$$f_* : \pi(X/k) \longrightarrow \pi(S/k).$$

This map is surjective as it admits a section induced from the section $\eta : S \longrightarrow X$.

2.3.2. *The fundamental groupoid.* Assume that $\mathrm{End}_{\mathscr{C}(X/k)}(R, d_{R/k}) = k$, then by [De90][Théorème 1.12] we have the absolute fundamental groupoid $\Pi(X/k) = \Pi(X/k, \eta)$ is the Tannakian dual of $\mathscr{C}(X/k)$ with respect to the fiber functor η^* (cf. Appendix A.8 for more details):

$$\mathrm{Rep}_f(R : \Pi(X/k)) \cong \mathscr{C}(X/k).$$

The absolute fundamental groupoid $\Pi(R/k)$ is the Tannakian dual of $\mathscr{C}(R/k)$ with respect to the forgetful functor $\mathrm{id} : (V, \nabla) \longmapsto V$:

$$\mathrm{Rep}_f(R : \Pi(R/k)) \cong \mathscr{C}(R/k).$$

The map $f : X \longrightarrow S$ induces a group homomorphism

$$f^* : \Pi(X/k) \longrightarrow \Pi(R/k).$$

This map is surjective as it admits a section induced from the section $\eta : S \longrightarrow X$.

We notice that $\pi(X/k)$ is the base change of $\Pi(X/k)$ with respect to the map $(s, s) : \mathrm{Spec}\, k \longrightarrow \mathrm{Spec}\ R \times \mathrm{Spec}\ R$.

2.3.3. *The relative fundamental group scheme.* The relative fundamental group scheme $\pi(X/R) = \pi(X/S,\eta)$ is the Tannakian dual of $\mathscr{C}(X/R)$ with respect to the fiber functor η^* (cf. Appendix A.10):

$$\mathrm{Rep}_{\mathrm{f}}(\pi(X/R)) \cong \mathscr{C}(X/R).$$

We have the *inflation functor*

$$\inf : \mathscr{C}(X/k) \longrightarrow \mathscr{C}(X/R)$$

which assigns to each connection $(\mathcal{V},\nabla)$ in $\mathscr{C}(X/k)$ the R-linear connection $(\mathcal{V},\nabla_{/R})$ in $\mathscr{C}(X/R)$:

$$\nabla_{/R} : \mathcal{V} \xrightarrow{\nabla} \Omega^1_{X/k} \otimes \mathcal{V} \longrightarrow \Omega^1_{X/R} \otimes \mathcal{V}.$$

This relative connection is called an *inflated* connection, and is denoted by $\inf(\mathcal{V})$ for short.

The functor inf induces a homomorphism

$$\pi(X/R) \longrightarrow \Pi(X/k),$$

which factors through the diagonal subgroup scheme $\Pi(X/k)^{\Delta}$ of $\Pi(X/k)$ (see Appendix A.2).

2.3.4. *The Ind-categories.* The Tannakian dualities mentioned above extend to the Ind-categories. Namely we have equivalences:

$$\begin{aligned} \mathrm{Rep}(\pi(X/k)) &\cong \text{Ind-}\mathscr{C}(X/k); \\ \mathrm{Rep}(\pi(X/R)) &\cong \text{Ind-}\mathscr{C}(X/R); \\ \mathrm{Rep}(R:\Pi(X/k)) &\cong \text{Ind-}\mathscr{C}(X/k), \end{aligned}$$

where the ind-categories on the right hand side are defined as the subcategories of connections which can be presented as the union of its coherent subconnections (or equivalently, as direct limits of coherent connections).

We notice that there are quasi-coherent connections which cannot be presented as union of coherent subconnections, for instance the sheaf of algebras of differential operators.

2.4. **The Gauss-Manin connection.** We briefly state the construction of Gauss-Manin connection which is based on [ABC20] and [Ka70]. If $\mathcal{V}$ is an inflated connection, $\mathcal{V} = \inf\mathcal{M}$, then the connection on $\mathcal{M}$ restricted to $\mathcal{M}^{\nabla_{/R}}$ yields a connection δ on $\mathrm{H}^0_{dR}(X/R, \inf\mathcal{M})$ over R/k, this is the *0-th Gauss-Manin connection*. The explicit construction is as follows.

The smoothness of f implies the following exact sequence of Kähler differentials:

$$0 \longrightarrow f^*\Omega^1_{R/k} \longrightarrow \Omega^1_{X/k} \longrightarrow \Omega^1_{X/R} \longrightarrow 0. \tag{2}$$

This filtration of $\Omega^1_{X/k}$ induces a filtration on $\Omega^2_{X/k}$ which is compatible with the connection and we obtain the following commutative diagram (cf. [Ka70, (3.2)]).

$$\begin{array}{ccccccc}
 & & & & & & \Omega^2_{R/k}\otimes\mathcal{M} \\
 & & & & & & \downarrow \\
 & & 0 & \longrightarrow & \Omega^1_{R/k}\otimes\mathcal{M} & \longrightarrow & \Omega^1_{R/k}\otimes\Omega^1_{X/k}\otimes\mathcal{M} \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{M} & \xrightarrow{\nabla} & \Omega^1_{X/k}\otimes\mathcal{M} & \xrightarrow{\nabla_1} & \Omega^2_{X/k}\otimes\mathcal{M} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{M} & \xrightarrow{\nabla_{/R}} & \Omega^1_{X/R}\otimes\mathcal{M} & \longrightarrow & \Omega^2_{X/R}\otimes\mathcal{M}.
\end{array}$$

Notice that $\nabla_1\circ\nabla=0$, hence diagram chasing yields a map

$$\delta:\mathcal{M}^{\nabla_{/R}}\longrightarrow\Omega^1_{R/k}\otimes\mathcal{M}^{\nabla_{/R}}.$$

Applying f_* we obtain a flat connection δ on $\mathrm{H}^0_{DR}(X/R,\mathrm{inf}\mathcal{M})$ over R/k. The resulting connection is denoted by $f_*(\mathcal{M},\nabla)$ or $f_*\mathcal{M}$ for short. Thus we have a left exact functor

$$f_*:\mathrm{MIC}(X/k)\longrightarrow\mathrm{MIC}(S/k).$$

The i-th derived functor of this functor is called the i-th Gauss-Manin connection of $(\mathcal{M},\nabla)$:

$$\mathrm{R}^i f_*:\mathrm{MIC}(X/k)\longrightarrow\mathrm{MIC}(X/k).$$

This functor can be computed by de Rham cohomology as follows. The sequence in (2) yields an exact sequences of complexes:

$$0\longrightarrow f^*\Omega^1_{R/k}\otimes(\Omega^{\bullet-1}_{X/R}\otimes\mathcal{M})\longrightarrow\Omega^\bullet_{X/k}\otimes\mathcal{M}\longrightarrow\Omega^\bullet_{X/R}\otimes\mathcal{M}\longrightarrow 0. \tag{3}$$

Applying the hyper-derived functor $\mathbf{R}^i f_*$ to the exact sequence above to get the long exact sequence:

$$0\longrightarrow\cdots\longrightarrow\mathbf{R}^i f_*(\Omega^\bullet_{X/R}\otimes\mathcal{M})\xrightarrow{d}\mathbf{R}^{i+1}f_*(f^*\Omega^1_{S/k}\otimes(\Omega^{\bullet-1}_{X/R}\otimes\mathcal{M}))\longrightarrow\cdots.$$

The connecting map

$$\delta_i:\mathbf{R}^i f_*(\Omega^\bullet_{X/R}\otimes\mathcal{M})\longrightarrow\mathbf{R}^{i+1}f_*(f^*\Omega^1_{R/k}\otimes(\Omega^{\bullet-1}_{X/R}\otimes\mathcal{M}))$$

can be written as:

$$\delta_i:\mathbf{R}^i f_*(\Omega^\bullet_{X/R}\otimes\mathcal{M})\longrightarrow\Omega^1_{R/k}\otimes\mathbf{R}^i f_*(\Omega^\bullet_{X/R}\otimes\mathcal{M})$$

by projection formula. As argued in [ABC20, 23.2.5], δ is the same as Gauss-Manin connection described above. Moreover, we get a long exact sequence:

$$\cdots\to\mathrm{H}^i_{DR}(X/k,(\mathcal{M},\nabla))\to\mathrm{H}^i_{DR}(X/R,\mathrm{inf}(\mathcal{M}))\xrightarrow{\delta_i}\Omega^1_{R/k}\otimes\mathrm{H}^i_{DR}(X/R,\mathrm{inf}(\mathcal{M}))\to\cdots \tag{4}$$

In conclusion we have the commutative diagram:

$$\begin{array}{ccc} \mathrm{MIC}(X/k) & \xrightarrow{\text{inf}} & \mathrm{MIC}(X/R) \\ {\scriptstyle \mathrm{R}^i f_*}\downarrow & & \downarrow{\scriptstyle \mathrm{H}^i_{DR}(X/k,-)} \\ \mathrm{MIC}(R/k) & \xrightarrow{\text{forget}} & \mathrm{Mod}(R), \end{array}$$

Lemma 2.2. *Let $f : X \to \operatorname{Spec} R$ be a proper smooth morphism of smooth k-schemes. Let $(\mathcal{V}, \nabla)$ be an object in $\mathscr{C}(X/k)$. Then $\mathrm{H}^i_{DR}(X/R, \inf(\mathcal{V}))$ is a finite projective R-module.*

Proof. We begin with the E_1-De Rham spectral sequence:

$$E_1^{ab} = R^a f_*(\Omega^b_{X/S} \otimes \mathrm{Inf}(\mathcal{V})) \Longrightarrow \mathbf{R}^{a+b} f_*(\Omega^\bullet_{X/S} \otimes \mathrm{Inf}(\mathcal{V})).$$

Combined with the coherent property is a strongly exact property and every term of

$$E_1^{ab} = R^a f_*(\Omega^b_{X/S} \otimes \mathrm{Inf}(\mathcal{V}))$$

is coherent, then this implies that $\mathbf{R}^{a+b} f_*(\Omega^\bullet_{X/S} \otimes \mathrm{Inf}(\mathcal{V}))$ coherent. As

$$(\mathbf{R}^{a+b} f_*(\Omega^\bullet_{X/S} \otimes \mathrm{Inf}(\mathcal{V})), GM) \in \mathrm{MIC}(S/k),$$

then we see that $\mathrm{H}^i_{DR}(X/S, \mathrm{Inf}(\mathcal{V}))$ is a locally free $\mathcal{O}_S$-module via [And01, Corollary 2.5.2]. □

Consequently, the functor $R^i f_*$ restrict to functors $\mathscr{C}(X/k) \longrightarrow \mathscr{C}(R/k)$ and we have the following commutative diagram:

$$\begin{array}{ccc} \mathscr{C}(X/k) & \xrightarrow{\text{inf}} & \mathscr{C}^\circ \\ {\scriptstyle \mathrm{R}^i f_*}\downarrow & & \downarrow{\scriptstyle \mathrm{H}^i_{DR}(X/k,-)} \\ \mathscr{C}(R/k) & \xrightarrow{\text{For}} & \mathrm{Mod}^\circ(R), \end{array}$$

where $\mathrm{Mod}^\circ R$ is the category of finite projective modules over R.

Lemma 2.3. *Let $(\mathcal{M}, \nabla)$ be an object of $\mathscr{C}(X/k)$. Then the connection $f^*(f_*\mathcal{M})$ is the maximal subobject of $(\mathcal{M}, \nabla)$ in $\mathscr{C}(X/k)$ with the property: its inflation to $\mathscr{C}(X/R)$ is a trivial connection.*

Proof. The connection on the pull-back

$$f^* f_* \mathcal{M} = f^* \mathrm{H}^0_{DR}(X/R, \inf \mathcal{M}) = \mathcal{O}_X \otimes \mathrm{H}^0_{DR}(X/R, \inf \mathcal{M})$$

is given by $\nabla(a \otimes e) = da \otimes e + a \otimes \delta e$. This implies that $(\mathcal{O}_X \otimes \mathrm{H}^0_{DR}(X/R, \inf \mathcal{M}), \nabla)$ is a subconnection of $\mathcal{M}$ with the property that its inflation is a trivial relative connection.

Actually, it has to be the maximal such, as any other $\mathcal{W} \subset \mathcal{M}$ would have the property that $(\nabla/R)|_{\mathcal{W}}$ is generated by horizontal sections. □

Remark 2.4. The Tannakian interpretation of Lemma 2.3 is as follows. Let $M = \eta^*(\mathscr{M})$ be the representation of $\Pi(X/k)$ that corresponds to M. Then $f_*\mathrm{inf}\mathscr{M}$ corresponds through η^* with a representation of $\Pi(R/k,\mathrm{id})$ and $f^*f_*\mathrm{inf}\mathscr{M}$ corresponds with a subreprsentation of M on which the action of $\Pi(X/k)$ factors through the action of $\Pi(R/k,\mathrm{id})$, or equivalently, the group L acts trivially, where L is the kernel of $f : \Pi(X/k) \longrightarrow \Pi(R/k,\mathrm{id})$, see Appendix A.2.4.

3. The fundamental exact sequence

3.1. **The homotopy exact sequence.** The following sequence is shown to be exact by L. Zhang [Zha14] in equal characteristics zero and J. P. dos Santos [dS15] in the general case and is called the homotopy exact sequence, it resembles the topological homotopy exact sequence. From now on, we assume more that $f : X \to S$ is the projective morphism and $\mathrm{End}_{\mathscr{C}(X/k)}(R, d_{R/k}) = k$.

Theorem 3.1. *The following sequence is exact*

$$\pi(X_s/k,x) \longrightarrow \pi(X/k,x) \longrightarrow \pi(S/k) \longrightarrow 1.$$

This exact sequence generalizes Grothendieck's homotopy exact sequence for etale fundamental groups [SGA1, Théorème IX.6.1].

3.2. **The fundamental exact sequence.** The functor f^* and the inflation functor induce the following sequence of homomorphisms of group and groupoid schemes:

$$\pi(X/R) \xrightarrow{\mathrm{inf}} \Pi(X/k) \xrightarrow{f} \Pi(S/k,\mathrm{id}) \longrightarrow 1.$$

We first notice that the composition $f \circ \mathrm{inf}$ corresponds to the functor taking pull-back of connections on S/k along f and then inflating to relative connections on X/R, hence sends any connection on S/k to a trivial relative connection on X/R. Therefore it is a trivial homomorphism. We aim to show that the above sequence is exact.

Theorem 3.2. *Let $f : X \longrightarrow \mathrm{Spec}R$ be a smooth projective map with geometrically connected fibers. Then the sequence*

$$\pi(X/R) \xrightarrow{\mathrm{inf}} \Pi(X/k) \xrightarrow{f} \Pi(S/k,\mathrm{id}) \longrightarrow 1$$

of flat affine group schemes and groupoids schemes over R is exact.

We notice that the exactness of this sequence amounts to the exactness of the following sequence of R-group schemes (see Appendix A.2.4):

$$\pi(X/R) \longrightarrow \Pi(X/k)^{\Delta} \longrightarrow \Pi(S/k,\mathrm{id})^{\Delta} \longrightarrow 1.$$

The name "fundamental exact sequence" is motivated by Grothendieck's fundamental exact sequences of the etale fundamental groups:

$$1 \longrightarrow \pi^{\mathrm{et}}(\overline{X},\bar{x}) \longrightarrow \pi^{\mathrm{et}}(X,\bar{x}) \longrightarrow \mathrm{Gal}(\bar{k}/k) \longrightarrow 1.$$

Indeed, the fundamental groupoid $\Pi(S/k,\mathscr{F})$ can be seen as a generalization of the Galois group $\mathrm{Gal}(\bar{k}/k)$, while the relative fundamental group plays the role of the geometric etale fundamental group. Notice that in our settings, the sequence is not left exact.

For the etale fundamental groups, Grothendieck used the fundamental exact sequence to deduce the homotopy exact sequence. For the differential fundamental groups, dos Santos provided direct proof using his criterion for the exact sequence of group schemes. In what follows we shall use 3.1 to deduce 3.2. The proof will be given in section 3.2.3.

3.2.1. *Some lemmas.* Combine Theorem 3.1 with the criterion for exact sequences of affine group schemes [EHS08, Theorem A.1] we deduce the following lemma (cf. [DHdS18, Theorem 9.1]).

Lemma 3.3. *Let $\mathscr{M}$ be an object of $\mathscr{C}(X/k)$.*

1. *The maximal trivial subobject of $\mathscr{M}|X_s$ is the restriction of a subobject $\mathscr{T}\longrightarrow\mathscr{M}$. Moreover, $\mathscr{T}$ is the pull-back to $\mathscr{C}(X/k)$ of an object of $\mathscr{C}(S/k)$;*
2. *If $\mathscr{N}$ belongs to $\langle\mathscr{M}|X_s\rangle_\otimes$, then there exists $\tilde{\mathscr{N}}$ in $\langle\mathscr{M}\rangle_\otimes$ and a monic $\mathscr{N}\longrightarrow\tilde{\mathscr{N}}|X_s$.*

Proof. According to Theorem 3.1, the following sequence

$$\pi(X_s/k)\longrightarrow\pi(X/k)\longrightarrow\pi(S/k)\longrightarrow 1.$$

is exact. Using the characterization of exactness presented in [EHS08, Theorem A.1], we immediately arrive at the desired conclusion. Moreover, the proof in loc. cit. shows that $\tilde{\mathscr{N}}$ can be chosen in $\langle\mathscr{M}\rangle_\otimes$. □

The following lemma is [DHdS18, Theorem 9.2], which generalizes a result of Deligne ([EH06, Theorem 5.10]).

Lemma 3.4. *Let $\mathscr{M}$ be an object of $\mathscr{C}(X/k)$ and let $\mathscr{V}\longrightarrow\inf(\mathscr{M})$ be a special subobject in $\mathscr{C}(X/R)$. Then there exists $\mathscr{N}\in\mathscr{C}(X/k)$ and an epimorphism* $\mathrm{Inf}(\mathscr{N})\longrightarrow\mathscr{V}$. *Moreover, $\mathscr{N}$ can be chosen in $\langle\mathscr{M}\rangle_\otimes$.*

Proof. In order to prove this lemma, we need the following claims.

Claim 1. Let $\mathscr{E}\to\inf(\mathscr{M})$ be a special monic in $\mathscr{C}(X/R)$. If

$$\mathrm{Hom}_{\mathscr{C}(X/R)}(\mathscr{E},\inf(\mathscr{M})/\mathscr{E})=0,$$

then the relative connection $\mathscr{E}$ extends to an absolute connection such that the arrow $\mathscr{E}\to\mathscr{M}$ is a morphism of $\mathscr{C}(X/k)$.

Verification. This is a special case of Theorem 9.4 in [DHdS18]. □

For a connection $\mathscr{V}$ which is is locally free of rank one as an $\mathscr{O}_X$-module we define the $\mathscr{V}$-socle series. The first $\mathscr{V}$-socle of $\inf(\mathscr{M})$,

$$\mathrm{Soc}_1(\mathscr{V})\subset\inf(\mathscr{M}),$$

is defined as the sum of all subobjects of $\inf(\mathscr{M})$ which are isomorphic to $\mathcal{V}$. For $i \geq 1$, we put

$$\mathrm{Soc}_{i+1}(\mathcal{V}) = \text{ inverse image in } \inf(\mathscr{M}) \text{ of } \mathrm{Soc}_1(\inf(\mathscr{M})/\mathrm{Soc}_i(\mathcal{V}),$$

that is

$$\mathrm{Soc}_{i+1}(\mathcal{V})/\mathrm{Soc}_i(\mathcal{V}) \cong \mathrm{Soc}_1(\inf(\mathscr{M})/\mathrm{Soc}_i(\mathcal{V})). \tag{5}$$

Claim 2. For $i \geq 1$, we have the following properties of the socle series.

(1) The subobject $\mathrm{Soc}_i(\mathcal{V})$ of $\inf(\mathscr{M})$ is special.
(2) In $\mathscr{C}(X/R)$ we have $\mathrm{Soc}_1(\mathcal{V}) \cong \mathcal{V}^{\oplus r}$ for some r.

Verification. We could assume that $\mathcal{V}$ is the trivial relative connection since the general case is treated by employing the relative connection $\mathcal{V}^\vee \otimes \inf(\mathscr{M})$. We see that the functor η^* induces an equivalence

$$\langle \inf(\mathscr{M})\rangle_\otimes^s \cong \mathrm{Rep}_R^\circ(G),$$

where G is an affine and flat group scheme over R (see Proposition B.8-(1) and Eq. (38)), and the result follows. □

We turn to prove the lemma keeping the assumption that $\mathcal{V}$ is locally free of rank one as an $\mathscr{O}_X$-module. Since $\mathrm{Soc}_i(\mathcal{V}) \subset \inf(\mathscr{M})$ is special (Claim 2-(1)), there exists $r \in \mathbb{N}$ such that

$$\mathrm{Soc}_r(\mathcal{V}) = \mathrm{Soc}_{r+1}(\mathcal{V}) = \dots$$

Due to the definition of the socle, we conclude that there are no submodules of

$$\inf(\mathscr{M})/\mathrm{Soc}_r(\mathcal{V})$$

isomorphic to $\mathcal{V}$. This implies that all arrows

$$\mathcal{V} \to \inf(\mathscr{M})/\mathrm{Soc}_r(\mathcal{V})$$

in $\mathscr{C}(X/R)$ to be null since the assumption on rank of $\mathcal{V}$ and by Prop. 5.1.1 in [DH18]. Thus, we see that any arrow

$$\mathrm{Soc}_r(\mathcal{V}) \longrightarrow \inf(\mathscr{M})/\mathrm{Soc}_r(\mathcal{V})$$

is null because the following sheaves

$$\mathrm{Soc}_1(\mathcal{V}),\ \frac{\mathrm{Soc}_2(\mathcal{V})}{\mathrm{Soc}_1(\mathcal{V})}, \dots$$

are all isomorphic to direct sums of $\mathcal{V}$ (Claim 2-(2) and Eq. (5)). By *Claim 1*, the relative connection $\mathrm{Soc}_r(\mathcal{V})$ is an inflation, i.e. there exists an absolute connection $\mathscr{N}$ together with a monomorphism $\mathscr{N} \to \mathscr{M}$ in $\mathscr{C}(X/k)$ such that

$$\inf(\mathscr{N}) \longrightarrow \inf(\mathscr{M})$$

is our special monic $\mathrm{Soc}_r(\mathcal{V}) \to \inf(\mathscr{M})$. Since $\mathcal{V}$ is a quotient of $\mathrm{Soc}_r(\mathcal{V}) = \inf(\mathscr{N})$ and since $\mathscr{N}$ is a subobject of $\mathscr{M}$, the result is as follows.

The general case follows from the fact that if $m = \mathrm{rank}(\mathcal{V})$, then

$$\bigwedge^{m-1} \mathcal{V}^\vee \otimes \det(\mathcal{V}) \cong \mathcal{V},$$

which show that $\mathcal{V}$ is a quotient of $\inf(\mathscr{M})^\vee \otimes \inf(\mathscr{N})$. □

3.2.2. *The group schemes H.* We first provide a Tannakian description of the group scheme H which is the image of the map

$$\mathrm{inf}: \pi(X/R) \longrightarrow \Pi(X/k)^{\Delta}.$$

Definition 3.5 (The category $\mathscr{C}$)**.** Let $\mathscr{C}$ be the full subcategory of $\mathscr{C}(X/R)$ of objects which can be presented as subquotient of objects of the form $\mathrm{Inf}(\mathscr{M})$, $\mathscr{M} \in \mathscr{C}(X/k)$.

Combining Lemmas 3.4 and 3.3 we deduce the following property of the category $\mathscr{C}$.

Lemma 3.6 (cf. [DHdS18, Theorem 8.2])**.** *Let $\mathscr{M}$ be an absolute connection on X/k. Then each locally free relative connection in $\langle \mathrm{inf}(\mathscr{M})\rangle_{\otimes}$ is indeed a* special *subobject of a tensor generated object from* $\mathrm{inf}(\mathscr{M})$.

The proof of this lemma requires techniques of flat affine group scheme and will be given in the Appendix, see Theorem B.9.

Lemma 3.7. *The category $\mathscr{C}$ defined above together with the fiber functor η^* is a Tannakian category and its Tannakian group H is the image of the homomorphism $\pi(X/R) \longrightarrow \Pi(X/k)^{\Delta}$. More precisely, the canonical map $\pi(X/R) \longrightarrow H$ is surjective (faithfully flat) and the map $H \longrightarrow \Pi(X/k)^{\Delta}$ is a closed immersion.*

Proof. We begin by proving that $\mathscr{C}$ is the Tannakian category in the sense of Definition A.9. The category $\mathscr{C}$ is trivially closed under taking tensor product. Moreover, $\mathscr{C}$ is R-linear tensor subcategory of $\mathscr{C}(X/R)$. It remains to show that $\mathscr{C}$ is abelian and each object of $\mathscr{C}$ is dominated by its subcategory of definition $\mathscr{C}^{\circ}$. Let $f: \mathscr{N} \to \mathscr{P}$ be a morphism in $\mathscr{C}$. The kernel and cokernel exist in the category $\mathscr{C}(X/R)$, we can check that these objects have the form of subquotient of $\mathrm{inf}(\mathscr{M})$. The natural map $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is still an isomorphism in $\mathscr{C}$ implies that $\mathscr{C}$ is abelian. Moreover, every object of $\mathscr{C}$ can be presented as the quotient of $\mathscr{C}^{\circ}$ by its construction. We conclude that $\mathscr{C}$ is the Tannakian category and we denote its Tannakian group as H.

We now turn to prove that H is the image of the homomorphism $\pi(X/R) \longrightarrow \Pi(X/k)^{\Delta}$. We use Theorem A.1. By definition of $\mathscr{C}$ we have immediately that H is a quotient of $\pi(X/R)$. On the other hand, every object of $\mathscr{C}^{\circ}$ is isomorphic to a special subquotient of an object of the form $\mathrm{inf}(\mathscr{M})$ via Lemma 3.6, this again, my means of Theorem A.1, implies that $H \longrightarrow \Pi(X/k)^{\Delta}$ is a closed immersion. ☐

3.2.3. *Proof of 3.2.* Our strategy is to show that the group H defined in the previous subsection is equal to the kernel L of the map $f: \Pi(X/k) \longrightarrow \Pi(R/k, \mathrm{id})$. Notice that L is equalt to the kernel of $\Pi(X/k)^{\Delta} \longrightarrow \Pi(S/k, \mathscr{F})^{\Delta}$ (cf. Appendix A.2.4).

By construction, we have the closed immersion $H \subset L$, which induces a functor $\mathscr{F}: \mathrm{Rep}_R(L) \longrightarrow \mathrm{Rep}_R(H)$. This functor is faithful by construction. We will show it is full and essentially surjective.

Step 1. We show that $\mathscr{F}^{\circ}: \mathrm{Rep}^{\circ}_R(L) \longrightarrow \mathscr{C}^{\circ}$ is full.

Let U_0, U_1 be objects in $\mathrm{Rep}^{\circ}_R(L)$ and $\phi: \mathscr{F}^{\circ}(U_0) \to \mathscr{F}^{\circ}(U_1)$ a $\mathscr{C}$-morphism, that is, ϕ is a R-linear map $U_0 \to U_1$, which is H-linear. We show that ϕ is in fact L-linear. Indeed, by Corollary

5.6 there are L-linear morphism $\pi : V_0 \twoheadrightarrow U_0$ and $i : U_1 \hookrightarrow V_1$, where V_0 and V_1 are object in $\mathrm{Rep}(R : \Pi(X/k))$. We set $\psi = i\phi\pi$ and then we have have the following diagram

$$\begin{array}{ccc} U_0 & \xrightarrow{\ \phi\ } & U_1 \\ {\scriptstyle\pi}\uparrow & & \downarrow{\scriptstyle i} \\ V_0 & \xrightarrow[\ \psi\ ]{} & V_1. \end{array}$$

Thus ϕ is L-linear if and only ψ is. Thus one is led to show that

$$\mathrm{Hom}_L(V_0, V_1) = \mathrm{Hom}_H(V_0, V_1),$$

for any $V_0, V_1 \in \mathrm{Rep}(R : \Pi(X/k))$, which amounts to showing $V^L = V^H$ for $V = V_0^\vee \otimes V_1$, where V^L is the submodule of V of elements stable under the action of L.

Now, through Tannakian duality Lemma 2.3 tells us that (see Remark 2.4), for $V \in \mathrm{Rep}(R : \Pi(X/k))$ that correspond to the connection $\mathcal{V} \in \mathscr{C}(X/k)$,

$$V^H = V^{\pi(X/R)} = \mathrm{H}^0_{DR}(X/R, \inf(V)) \subset V^L.$$

On the other hand we obviously have $V^H = V^L$. Thus we have an equality.

Step 2. We show that $\mathscr{F}$ is essentially surjective. Let $\mathcal{V}$ be an object in $\mathscr{C}^\circ$. Then, according to Lemma 3.6, there exists an object $\mathscr{M}$ in $\mathscr{C}(X/k)$ such that $\mathcal{V}$ is a special subobject of $\inf(\mathscr{M})$. According to Lemma 3.4, $\mathcal{V}$ can be realized as the image of a morphism φ between inflated objects. By the above discussion, φ is also in the image of $\mathscr{F}$, hence so is $\mathrm{Im}(\varphi)$. It means that $\mathcal{V}$ corresponds to a representation of L and we finish the proof. □

4. A Comparison Theorem

We continue to assume that $f : X \longrightarrow R$ is a smooth proper morphism of smooth k-schemes of geometrically connected fibers. We shall assume moreover that the fibers are of dimension 1 and the genus is at least 1.

We aim to compare, through Tannakian duality, the de Rham cohomology of connections on X/R and the group cohomology of $\pi(X/R)$. In general they are not equal due to the fact that representations of the fundamental group constitute only a part of connections (cf. subsection 2.3.4). But in the case that all fibers are equidimensional of dimension 1, they may be equal. This will be done in Theorem 4.3.

4.1. **The Poincaré duality for relative de Rham cohomology.** The connection imposes a strong condition on the underlying sheaf. For instance, on a smooth scheme over a *field*, a *coherent* connection is automatically locally free.

By the assumption that $f_*\mathscr{O}_X = R$, there exists an R-module isomorphism

$$t : \mathrm{H}^1(X, \Omega^1_{X/R}) \longrightarrow R$$

(cf. [Stack, Tag 0G8I]) with the following property: for all $p, q \in \{0,1\}$, the pairing

$$\mathrm{H}^q(X, \Omega^p_{X/R}) \times \mathrm{H}^{1-q}(X, \Omega^{1-p}_{X/R}) \longrightarrow R$$

given by the relative cup product composed with t is a perfect pairing. We define the genus of X/R to be the rank of the projective module

$$\mathrm{H}^0(X, \Omega^1_{X/R}) \cong \mathrm{H}^1(X, \mathscr{O}_X).$$

Now given $(\mathscr{V}, \nabla) \in \mathscr{C}(X/R)$ with $\mathscr{V}$ a locally free sheaf and $\mathrm{H}^i_{DR}(X/R, (\mathscr{V}, \nabla))$ is R-flat for $i = 0,1,2$. By adapting the proof of [Stack, Tag 0FVY], the pairing

$$\mathrm{H}^i(X, \Omega^1_{X/R} \otimes_{\mathscr{O}_X} \mathscr{V}^\vee) \times \mathrm{H}^{1-i}(X, \mathscr{V}) \longrightarrow R, \tag{6}$$

are perfect for $i = 0,1$.

Proposition 4.1. *Let $f : X \longrightarrow \mathrm{Spec} R$ be a smooth proper morphism of smooth k-schemes of geometrically connected fibers of dimension 1. Let $(\mathscr{V}, \nabla) \in \mathscr{C}(X/R)$ such that $\mathscr{V}$ is locally free sheaf of finite rank and $\mathrm{H}^i_{DR}(X/R, (\mathscr{V}, \nabla))$ is R-flat for $i = 0,1,2$. Then*

$$\begin{aligned} \mathrm{H}^2_{dR}(X/R, (\mathscr{V}, \nabla)) &\cong \mathrm{H}^0_{dR}(X/R, (\mathscr{V}, \nabla)^\vee)^\vee \\ \mathrm{H}^1_{dR}(X/R, (\mathscr{V}, \nabla)) &\cong \mathrm{H}^1_{dR}(X/R, (\mathscr{V}, \nabla)^\vee)^\vee. \end{aligned} \tag{7}$$

Proof. We consider the first page of Hodge to de Rham spectral sequence for $\mathrm{H}^*_{dR}(X, \Omega^1_{X/R} \otimes \mathscr{V})$:

$$\mathrm{H}^1(X, \mathscr{V}) \xrightarrow{\mathrm{H}^1(\nabla)} \mathrm{H}^1(X, \Omega^1_{X/R} \otimes \mathscr{V})$$

$$\mathrm{H}^0(X, \mathscr{V}) \xrightarrow{\mathrm{H}^0(\nabla)} \mathrm{H}^0(X, \Omega^1_{X/R} \otimes \mathscr{V}),$$

so the second page of this spectral sequence as follows

$$\begin{matrix} \ker \mathrm{H}^1(\nabla) & \mathrm{coker} \mathrm{H}^1(\nabla) \\ \ker \mathrm{H}^0(\nabla) & \mathrm{coker} \mathrm{H}^0(\nabla), \end{matrix}$$

this implies that the spectral sequence degenerates at E^2. Analogously, we also have the second page of Hodge to de Rham spectral sequence for $\mathrm{H}^*_{dR}(X, \Omega^1_{X/R} \otimes \mathscr{V}^\vee)$:

$$\begin{matrix} \ker \mathrm{H}^1(\nabla^\vee) & \mathrm{coker} \mathrm{H}^1(\nabla^\vee) \\ \ker \mathrm{H}^0(\nabla^\vee) & \mathrm{coker} \mathrm{H}^0(\nabla^\vee). \end{matrix}$$

Notice that by assumption, every term of the above spectral sequences is projective with finite rank. Thus, we prove (7) by showing that

$$\begin{aligned} \operatorname{coker} \mathrm{H}^1(\nabla) &\cong \ker \mathrm{H}^0(\nabla^\vee)^\vee \\ \ker \mathrm{H}^1(\nabla) \oplus (\operatorname{coker} \mathrm{H}^0(\nabla)) &\cong (\operatorname{coker} \mathrm{H}^0(\nabla^\vee))^\vee \oplus \ker \mathrm{H}^1(\nabla^\vee)^\vee. \end{aligned} \tag{8}$$

We first prove $\operatorname{coker} \mathrm{H}^1(\nabla) \cong \ker \mathrm{H}^0(\nabla^\vee)^\vee$. Using (6), we have

$$\begin{array}{ccc} \mathrm{H}^1(X, \Omega^1_{X/R} \otimes \mathcal{V}) & & \mathrm{H}^1(X, \mathcal{V}) \\ \big\downarrow \cong & & \big\downarrow \cong \\ \mathrm{H}^0(X, \mathcal{V}^\vee)^\vee & & \mathrm{H}^0(X, \Omega^1_{X/R} \otimes \mathcal{V}^\vee)^\vee \end{array} .$$

If we can prove that $\mathrm{H}^1(\nabla)^\vee = \mathrm{H}^0(\nabla^\vee)$, then the following diagrams

$$\begin{array}{ccc} \mathrm{H}^1(X, \Omega^1_{X/R} \otimes \mathcal{V}) & \xleftarrow[\mathrm{H}^1(\nabla)]{} & \mathrm{H}^1(X, \mathcal{V}) \\ \big\downarrow \cong & & \big\downarrow \cong \\ \mathrm{H}^0(X, \mathcal{V}^\vee)^\vee & \xleftarrow[H^0(\nabla^\vee)^\vee]{} & \mathrm{H}^0(X, \Omega^1_{X/R} \otimes \mathcal{V}^\vee)^\vee, \end{array}$$

$$\mathrm{H}^0(X, \mathcal{V}^\vee) \xrightarrow{\mathrm{H}^0(\nabla^\vee)} \mathrm{H}^0(X, \Omega^1_{X/R} \otimes \mathcal{V}^\vee),$$

implies that $\operatorname{coker} \mathrm{H}^1(\nabla) \cong \ker \mathrm{H}^0(\nabla^\vee)^\vee$. To show that $\mathrm{H}^1(\nabla)$ is dual to $\mathrm{H}^0(\nabla^\vee)$, we show that the following diagram

$$\begin{array}{ccc} \mathrm{H}^1(X, \mathcal{V}) \times \mathrm{H}^0(X, \mathcal{V}^\vee) & \xrightarrow{\mathrm{H}^1(\nabla) \times id} & \mathrm{H}^1(X, \mathcal{V} \otimes \Omega^1_{X/R}) \times \mathrm{H}^0(X, \mathcal{V}^\vee) \\ \big\downarrow id \times \mathrm{H}^0(\nabla^\vee) & & \big\downarrow \cup \\ \mathrm{H}^1(X, V) \times \mathrm{H}^0(X, \Omega^1_{X/R} \otimes \mathcal{V}^\vee) & \xrightarrow{\cup} & \mathrm{H}^1(X, \Omega^1_{X/R} \otimes \mathcal{V}^\vee \otimes \mathcal{V}) \\ & & \big\downarrow ev \\ & & \mathrm{H}^1(X, \Omega^1_{X/R}) \cong R \end{array} \tag{9}$$

is commutative, that is, $ev \circ \cup \circ \mathrm{id} \times \mathrm{H}^0(\nabla^\vee) = ev \circ \cup \circ \mathrm{H}^1(\nabla) \times \mathrm{id}$. Since the differential map $d_{X/R}$ restricted to $\mathrm{H}^0(X, \mathcal{O}_X)$ is zero, so

$$\nabla^\vee(\ell) = d_{X/R} \circ \ell - (1 \otimes \ell) \circ \nabla = -(1 \otimes \ell) \circ \nabla,$$

when restricted to $\mathrm{H}^0(X, \mathcal{O}_X)$. This implies that the Diagram (9) is commutative since this diagram describes the dual of free R-module of finite rank.

Using the same argument, we get the second equation of (8). □

Propositions 2.2 and 7 imply:

Corollary 4.2. *Let X be a smooth projective curve over principal ideal domain R such that* $\mathrm{H}^0(X,\mathcal{O}_X) = R$. *Let* $(\mathcal{V},\nabla) \in \mathcal{C}(X/k)$. *Then*

$$\mathrm{H}^2_{dR}(X/R,\inf(\mathcal{V})) \cong \mathrm{H}^0_{dR}(X/R,\inf(\mathcal{V})^\vee)^\vee$$
$$\mathrm{H}^1_{dR}(X/R,(\inf(\mathcal{V})) \cong \mathrm{H}^1_{dR}(X/R,\inf(\mathcal{V})^\vee)^\vee.$$

4.2. **A comparison theorem.** Let k be a field of characteristic 0. Let R be a k-algebra, which is a Dedekind ring. Let $f : X \longrightarrow S = \mathrm{Spec}(R)$ be a smooth map with fiber geometrically connected. We recall from Subsection **??** that the fibre functor η^* induces

$$\mathrm{Rep}_{\mathrm{f}}(\pi(X/R)) \cong \mathcal{C}(X/R).$$

By taking the ind-category of these categories, we have

$$\mathrm{Rep}(\pi(X/R)) \cong \text{Ind-}\mathcal{C}(X/R).$$

It follows from the Tannakian duality mentioned above and the description of de Rham cohomology as ext-group (cf. subsection 2.2) that for any $\mathcal{V} \in \text{Ind-}\mathcal{C}(X/R)$ and $V = \eta^*(\mathcal{V})$ we have isomorphisms

$$\mathrm{H}^0_{DR}(X/R,\mathcal{V}) \cong \mathrm{H}^0(\pi(X/R),V) \tag{10}$$
$$\mathrm{H}^1_{DR}(X/R,\mathcal{V}) \cong \mathrm{H}^1(\pi(X/R),V). \tag{11}$$

The point is that, for any exact sequence

$$0 \longrightarrow \mathcal{V} \longrightarrow \mathcal{V}' \longrightarrow \mathcal{O}_X \longrightarrow 0,$$

in $\mathrm{MIC}(X/R)$, if $\mathcal{V}$ is in $\text{Ind-}\mathcal{C}(X/R)$, then so is $\mathcal{V}'$.

We are now ready to prove:

Theorem 4.3. *Let $\mathcal{V}$ be an object in $\mathcal{C}(X/R)$ which is locally free. Denote $V := \eta^*(\mathcal{V})$. Then the map*

$$\delta^2 : \mathrm{H}^2(\pi(X/R),V)) \cong \mathrm{H}^2_{dR}(X/R,\inf(\mathcal{V}))$$

is is injective. If X has genus $g \geq 1$ and $\mathcal{V}$ is an inflated connection then δ^2 is an isomorphism.

Proof. Let J be the injective envelope of V in $\mathrm{Rep}(\pi(X/R))$ and let $(\mathcal{J},\nabla_{\mathcal{J}})$ be the corresponding connection. Since J is injective in $\mathrm{Rep}(\pi(X/R))$, one has

$$\mathrm{H}^1_{dR}(X/R,(\mathcal{J},\nabla_{\mathcal{J}})) \cong \mathrm{H}^1(\pi(X/R),J) = 0.$$

Hence, the long exact sequences associated with the exact sequences $0 \to V \to J \to J/V \to 0$ and $0 \to (\mathcal{V},\nabla_{\mathcal{V}}) \to (\mathcal{J},\nabla) \to (\mathcal{J}/\mathcal{V},\nabla) \to 0$ yield

$$H^2(\pi(X/R),V) \cong H^1(\pi(X/R),J/V) \cong \mathrm{H}^1_{dR}(X/R,\mathcal{J}/\mathcal{V}) \hookrightarrow \mathrm{H}^2_{dR}(X/R,\mathcal{V}). \tag{12}$$

Let $(\mathcal{V},\nabla) \in \mathcal{C}(X/k)$. Corollary 4.2 gives us that $\mathrm{H}^2_{\mathrm{DR}}(X/R,\inf(\mathcal{V}))$ is a Poincaré dual to $\mathrm{H}^0_{\mathrm{DR}}(X/R,\inf(\mathcal{V})^\vee)$. Recall that $f^*f_*(\inf(\mathcal{V})^\vee)$ is the maximal trivial subobject of $\inf(\mathcal{V})^\vee$ as relative connection, and the inclusion induces an isomorphism on $\mathrm{H}^0_{\mathrm{DR}}(X/R,-)$. Consequently the dual map, which is surjective, induces an isomorphism on $\mathrm{H}^2_{\mathrm{DR}}(X/R,-)$. Thus, it suffices to show the surjectivity for the trivial connection $(\mathcal{O}_X, d)$.

As for the last claim, if $g \geq 1$ then the module $\mathrm{H}^0(X, \Omega^1_X)$ non-zero. Consequently, the first de Rham cohomology $\mathrm{H}^1_{DR}(X/R, \mathscr{O}_X)$ is non-zero. The Poincaré duality for de Rham cohomology tells us that the paring

$$\mathrm{H}^1_{DR}(X/R, \mathscr{O}_X) \times \mathrm{H}^1_{DR}(X/R, \mathscr{O}_X) \longrightarrow \mathrm{H}^2_{DR}(X/R, \mathscr{O}_X)$$

is perfect. Since de Rham cohomology commute with base change, we conclude that the above map is surjective (cf. [Stack, Tag 0FM0]). Hence there are two classes $\alpha, \beta \in \mathrm{H}^1_{DR}(X/R, (\mathscr{O}_X, d))$, so that $0 \neq \alpha \cup \beta$ is a generator of the module $\mathrm{H}^2_{DR}(X/R, (\mathscr{O}_X, d)) \cong R$. Thus there is a diagram of extensions in $\mathscr{C}(X/R)$:

$$\alpha: \quad 0 \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{E} \longrightarrow \mathscr{O} \longrightarrow 0$$
$$\beta: \quad 0 \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{F} \longrightarrow \mathscr{O} \longrightarrow 0.$$

The cup product $\alpha \cup \beta$ is the sequence obtained by compose the above sequences:

$$\alpha \cup \beta: \quad 0 \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{E} \longrightarrow \mathscr{F} \longrightarrow \mathscr{O} \longrightarrow 0 \tag{13}$$

The corresponding sequence $\mathrm{Rep}_{\mathrm{f}}(\pi(X/R))$ shows that δ^2 is surjective. □

4.3. **The universal extension theorem.** Our next aim is to compare the cohomology of H and of $\pi(X/R)$. By the nature of H, we will need the following result, which is an adaption of [EH06, Thm. 4.2]. The purpose of this subsection is to prove the following theorem:

Proposition 4.4 (*Universal extension*)**.** *Let $\mathscr{V}$ be an object in $\mathscr{C}(X/k)$, then there exists an extension in $\mathscr{C}(X/k)$:*

$$0 \longrightarrow \mathscr{V} \longrightarrow \mathscr{W} \longrightarrow f^*(\mathrm{R}^1 f_*(\mathscr{V}, \nabla)) \longrightarrow 0$$

with the property that the connecting morphism in the long exact sequence of de Rham cohomology on X/R:

$$\mathrm{H}^0_{DR}(X/R, \inf(f^*(\mathrm{R}^1 f_*(\mathscr{V}, \nabla))) = \mathrm{H}^1_{DR}(X, \inf(\mathscr{V})) \overset{\text{connecting}}{\longrightarrow} \mathrm{H}^1_{DR}(X/R, \inf(\mathscr{V}))$$

is the identity map.

Proof. Let $\mathscr{Z} = f^*(\mathrm{R}^1 f_*(\mathscr{V}, \nabla))$ (see subsection 2.4). Then

$$\inf(Z) = \mathscr{O}_X \otimes \mathrm{H}^1_{DR}(X/R, \inf(\mathscr{V})).$$

Let

$$\mathscr{W} = \mathscr{V} \otimes \mathscr{Z}^\vee$$

as objects in $\mathscr{C}(X/k)$. Then we have isomorphisms in $\mathscr{C}(X/k)$ of cohomologies Gauss-Manin connections:

$$\begin{aligned} \mathrm{H}^1_{DR}(X/R, \inf(\mathscr{W})) &\cong \mathrm{Ext}^1_{\mathrm{MIC}(X/R)}(\mathscr{O}_X, \inf(\mathscr{V}) \otimes \inf \mathscr{Z}^\vee) \\ &\cong \mathrm{Ext}^1_{\mathrm{MIC}(X/R)}(\inf(\mathscr{Z}), \inf(\mathscr{V})) \\ &\cong \mathrm{H}^1_{DR}(X/R, \inf(\mathscr{V}))^\vee \otimes \mathrm{Ext}^1_{\mathrm{MIC}(X/R)}(\mathscr{O}_X, \inf(\mathscr{V})) \\ &\cong \mathrm{H}^1_{DR}(X/R, \inf(\mathscr{V}))^\vee \otimes \mathrm{H}^1_{DR}(X/R, \inf(\mathscr{V})) \\ &\cong \mathrm{End}_R(\mathrm{H}^1_{DR}(X/R, \inf(\mathscr{V}))). \end{aligned}$$

Let ε be the element in $\mathrm{Ext}^1_{\mathrm{MIC}(X/R)}(\inf(\mathscr{Z}),\inf(\mathscr{V}))$, the image of which the identity map in $\mathrm{End}_R(\mathrm{H}^1_{DR}(X/R,\inf(\mathscr{V})))$. As the identity map is killed by the Gauss-Manin connections, so is ε.

Now consider the exact sequence of complexes:

$$0 \longrightarrow f^*\Omega^1_{R/k}\otimes_R(\Omega^{\bullet-1}_{X/R}\otimes\mathscr{W}) \longrightarrow \Omega^\bullet_{X/k}\otimes\mathscr{W} \longrightarrow \Omega^\bullet_{X/R}\otimes\mathscr{W} \longrightarrow 0.$$

In our situation, our base scheme is an affine scheme, so we get a the long exact sequence (4):

$$\cdots \longrightarrow \mathrm{H}^1_{DR}(X/k,\mathscr{W}) \longrightarrow \mathrm{H}^1_{DR}(X/R,\inf(\mathscr{W})) \longrightarrow \Omega^1_{R/k}\otimes_R \mathrm{H}^1_{DR}(X/R,\inf(\mathscr{W})) \longrightarrow \cdots$$

Now the homomorphism:

$$\delta_1 : \mathrm{H}^1_{DR}(X/R,\inf(\mathscr{W})) \longrightarrow \Omega^1_{R/k}\otimes_R \mathrm{H}^1_{DR}(X/R,\inf(\mathscr{W}))$$

is the Gauss-Manin connection. Hence, $\varepsilon\in\mathrm{Ker}\delta_1$ and thus it is lifted to $\tilde{\varepsilon}\in\mathrm{H}^1_{DR}(X/k,(\mathscr{W},\nabla_{\mathscr{W}}))$ by the long exact sequence. Consequently there exists an extension of connections in $\mathscr{C}(X/k)$:

$$\tilde{\varepsilon}:\quad 0\longrightarrow(\mathscr{W},\nabla_{\mathscr{W}})\longrightarrow(\mathscr{T}',\nabla_{\mathscr{T}'})\longrightarrow(O_X,d)\longrightarrow 0.$$

Notice that the inflation of this sequence to $\mathscr{C}(X/R)$ is ε. Hence the induced connecting map is the identity map by construction of ε. □

With theorem 4.4, we have a statement for cohomology comparison.

Corollary 4.5. *Let $(\mathscr{V},\nabla_{\mathscr{V}})\in\mathscr{C}(X/k)$, $V=\eta^*(\mathscr{V})$ the corresponding representation of $\pi(X/R)$. Let H be the group scheme in subsection 3.2.2. Then we have:*

$$\mathrm{H}^1(H,V)\cong\mathrm{H}^1(\pi(X/R),V)\overset{\text{Theorem 4.3}}{\cong}\mathrm{H}^1_{DR}(X/R,\inf(\mathscr{V})).$$

Proof. The category $\mathscr{C}\cong\mathrm{Rep}_f(H)$ is a full subcategory of $\mathscr{C}(X/R)$, so the restriction homomorphism

$$\mathrm{H}^1(H,V)\longrightarrow\mathrm{H}^1(\pi(X/R),V)$$

is injective.

We prove the inverse, which, through Tannakian duality, amounts to saying that each exension

$$e:\quad 0\longrightarrow\inf(\mathscr{V})\longrightarrow(\mathscr{V}',\nabla_{\mathscr{V}'}))\longrightarrow O_X\longrightarrow 0$$

is in $\mathscr{C}$, that is $\mathscr{V}'\in\mathscr{C}$.

Let $\varepsilon\in\mathrm{H}^1(\pi(X/R),V)\cong\mathrm{H}^1_{DR}(X,\inf(\mathscr{V}))$ be the universal extension of Proposition 4.4. Then $e\in\mathrm{H}^1_{DR}(X/R,\inf(\mathscr{V}))$, seen as a map in $\mathscr{C}(X/R)$

$$e:\mathscr{O}_X\longrightarrow\mathscr{O}_X\otimes\mathrm{H}^1_{DR}(X/R,\mathrm{Inf}(\mathscr{V})),$$

fits in the commuative diagram

$$\begin{array}{ccccccccc} e: & 0 & \longrightarrow & \inf(\mathcal{V}) & \longrightarrow & (\mathcal{V}', \nabla_{\mathcal{V}'}) & \longrightarrow & O_X & \longrightarrow & 0 \\ & & & \downarrow{=} & & \downarrow & & \downarrow{e} & & \\ \tilde{\varepsilon}: & 0 & \longrightarrow & \inf(\mathcal{V}) & \longrightarrow & \inf(\mathcal{W}) & \longrightarrow & \inf(\mathcal{Z}) & \longrightarrow & 0. \end{array}$$

Applying long exact sequence of $\mathrm{H}^0_{DR}(X/R, -)$ we get:

$$\begin{array}{ccccc} \longrightarrow & \mathrm{H}^1_{DR}(X/R, O_X) & \xrightarrow{d} & \mathrm{H}^1_{DR}(X/R, \inf(\mathcal{V})) \\ & \downarrow{\epsilon} & & \downarrow{Id} \\ \longrightarrow & \mathrm{H}^1_{DR}(X/R, \inf(\mathcal{V})) & \xrightarrow{Id} & \mathrm{H}^1_{DR}(X/R, \inf(\mathcal{V})), \end{array}$$

Thus $(\mathcal{V}', \nabla_{\mathcal{V}'})$ is the subconnection of $\inf(\mathcal{W})$ hence is in $\mathscr{C}$. This completes the proof. □

Corollary 4.6. *Let L be the kernel of $\Pi(X/k)^{\Delta} \longrightarrow \Pi(R/k)^{\Delta}$. Let V be a finite representation of $\Pi(X/k)$. Then we have isomorphism*

$$\mathrm{H}^i(L, V) \cong \mathrm{H}^i(\pi(X/R), V) \text{ for } i = 0, 1.$$

Proof. This is because $L = H$, cf. Theorem 3.2. □

5. The Gauss-Manin connection from the Tannakian viewpoint

This section aims to prove the Theorem 5.18. To do that we extend the theory of the cohomology of group scheme [Ja87, Chapter 4] to the cohomology of groupoid scheme.

5.1. **Cohomology of groupoid schemes.**

Let $S = \mathrm{Spec}(R)$ be an k-affine scheme and G be an affine k-groupoid scheme acting transitively on S.

5.1.1. *The injective object in the category* $\mathrm{Rep}(S : G)$. Although we know that $\mathrm{Rep}(S : G)$ is the ind-category of the category $\mathrm{Rep}_f(S : G)$ (see A.2.2), we also recast that $\mathrm{Rep}(S : G)$ have enough injectives. The advantages of this work make us give details of injective objectives in this category which need for the aim of this section.

Lemma 5.1. *The following statements are true:*

(1) *The category $\mathrm{Rep}(S : G)$ have enough injectives.*
(2) *A G-module V is injective if and only if there is an injective R-module I such that V is isomorphic to a direct summand of $I \otimes_t \mathcal{O}(G)$ with I regards as a trivial G-module.*
(3) *If V and M are G-module with V is projective R-module and M is injective G-module, then $V \otimes M$ is injective G-module.*

Proof. Before giving the proof of (1) we need the following claim.

Claim. Let U, V be a G-module. We have the following functorial isomorphism

$$\mathrm{Hom}_G(U, V \otimes_t \mathcal{O}(G) \cong \mathrm{Hom}_R(U, V).$$

Verification. We have the following map

$$\begin{aligned}\mathrm{Hom}_G(U, V \otimes_t \mathcal{O}(G) &\to \mathrm{Hom}_R(U, V)\\ f &\mapsto (\mathrm{id} \otimes \varepsilon) f.\end{aligned}$$

On the other hand, we have the inverse map as follows

$$\begin{aligned}\mathrm{Hom}_R(U, V) &\to \mathrm{Hom}_G(U, V \otimes_t \mathcal{O}(G)\\ g &\mapsto \rho_V \circ g,\end{aligned}$$

where ρ_V is the coaction of $\mathcal{O}(G)$ on V. □

Let V be a G-module. Since V is quasi-coherent as R-module, then we can embed R-module V in some injective module I. We can consider I as G-module with trivial action. Then we have injective G-map as follows:

$$V \xrightarrow{\rho_V} V \otimes_t \mathcal{O}(G) \hookrightarrow I \otimes_t \mathcal{O}(G).$$

We shows that $I \otimes_t \mathcal{O}(G)$ is injective G-module. Indeed, we consider the following G-module:

$$\begin{array}{ccccc} 0 & \longrightarrow & M & \xrightarrow{\ i\ } & M' \\ & & \big\downarrow f & \swarrow g & \\ & & I \otimes_t \mathcal{O}(G) & & \end{array} .$$

We will show that there exists map g. By the above claim, we have

$$\begin{array}{ccc} \mathrm{Hom}_G(M', I \otimes_t \mathcal{O}(G)) & \xrightarrow{\ i^*\ } & \mathrm{Hom}_G(M, I \otimes_t \mathcal{O}(G) \\ \big\downarrow \cong & & \big\downarrow \cong \\ \mathrm{Hom}_R(M', I) & \xrightarrow{\ i^*\ } & \mathrm{Hom}_R(M, I). \end{array} \tag{14}$$

Since R-module I is injective then the map i^* at the bottom is surjective. Hence, take $f \in \mathrm{Hom}_G(N, I \otimes_t \mathcal{O}(G))$ we can find g satisfies Diagram (14). Therefore, $I \otimes_t \mathcal{O}(G)$ is injective G-module and the result is as follows.

Proof of (2). We have proved that V can embed into some $I \otimes_t \mathcal{O}(G)$. Since G-module V is injective then V is a direct summand of $I \otimes_t \mathcal{O}(G)$. The reverse direction is true because every direct summand of injective G-module is injective G-module.

Proof of (3). If M is a direct summand of $I \otimes_t \mathcal{O}(G)$ as in (2), then $V \otimes M$ is a direct summand of $V \otimes I \otimes_t \mathcal{O}(G)$. Since V is R-flat module, then $V \otimes I \otimes_t \mathcal{O}(G)$ is isomorphic to $V_{\mathrm{tr}} \otimes I \otimes_t \mathcal{O}(G)$ (M_{tr} is G-module with trivial action). We see that $V \otimes M$ is injective G-module by (2). □

Definition 5.2. Let G be an affine k-groupoid schemes acting on S. We call H is *subgroupoid scheme* if H is closed subscheme of G such that the morphism $(m|_H, \varepsilon|_H, \iota|_H)$ makes H becomes k-groupoid scheme acting on S. We call H is *discrete subgroupoid scheme* if H is closed subcheme of the diagonal group scheme G^Δ.

An affine k-groupoid scheme G acting on S is called *flat* if the morphisms $s, t : G \longrightarrow S$ are flat.

Remark 5.3. Lemma 5.1 is also true for G is flat affine k-groupoid schemes acting on S.

5.1.2. *Fixed point functor.* Let G be a k-groupoid scheme acting on S and V be a G-module. We define the set of fixed points by

$$V^G = \{v \in (V) | \rho_V(v) = v \otimes 1\}.$$

If $\phi : V \to V'$ is a homomorphism of G-modules, then $\phi(V^G) \subset V'^G$. Thus, we have the fixed point functor

$$\mathrm{Rep}(S:G) \longrightarrow \mathrm{Vec}_k$$
$$V \mapsto V^G.$$

We see that if G is flat, then the fixed point functor is left exact. The category $\mathrm{Rep}(S:G)$ has enough injectives, so have the derived functor

$$V \mapsto \mathrm{H}^n(G, V),$$

and call $\mathrm{H}^n(G, V)$ the *n-cohomology group* of V.

5.1.3. *The induction of groupoid schemes.* We extend the definition of induction functor in A.2.5 as follows: Let G be an affine k-groupoid scheme acting transitively on $S = \mathrm{Spec}(R)$ and H be its subgroupoid scheme. For any representation $V \in \mathrm{Rep}_R(H)$, set

$$\mathrm{Ind}_H^G(V) := (V \otimes_t \mathcal{O}(G))^H,$$

where H acts on V as usual and on $\mathcal{O}(G)$ through the right regular action of G on $\mathcal{O}(G)$ (i.e. $\mathcal{O}(G)$ is a right G-module). On this invariant space, G acts through the left regular action on $\mathcal{O}(G)$. Thus, Ind_H^G is a functor $\mathrm{Rep}_R(H) \to \mathrm{Rep}(R:G)$.

Lemma 5.4 (Frobenius Reciprocity)**.** *Let G be a flat affine k-groupoid scheme acting on $S =$ $\mathrm{Spec}(R)$ and H be its flat subgroupoid scheme. There exists a functorial isomorphism*

$$\mathrm{Hom}_G(V, \mathrm{Ind}_H^G(W)) \cong \mathrm{Hom}_H(V, W), \tag{15}$$

$V \in \mathrm{Rep}(R:G), W \in \mathrm{Rep}_R(H)$, i.e., Ind_H^G is the right adjoint to the functor restricting G-representations to H.

Proof. The map is given by composing with the canonical projection

$$\mathrm{Ind}_H^G(W) \longrightarrow W$$
$$v \otimes h \mapsto v\epsilon(h).$$

The converse map is given by $f \mapsto (f \otimes id)\rho_W$. □

Proposition 5.5. *Let L be a kernel of $f : \Pi(X/k)^\Delta \longrightarrow \Pi(S/k, \mathscr{F})^\Delta$. Then the induction functor $\mathrm{Ind}_L^{\Pi(X/k)^\Delta}$ is faithfully exact.*

Proof. We have the following map

$$\Pi(X/k)^\Delta \twoheadrightarrow \Pi(X/k)^\Delta/L \hookrightarrow \Pi(S/k,\mathscr{F})^\Delta.$$

Since f is surjective, the above map implies that $\Pi(X/k)^\Delta/L \cong \Pi(S/k,\mathscr{F})^\Delta$. We now can apply Proposition 2.5 in [DHH17] for

$$\Pi(X/k)^\Delta \twoheadrightarrow \Pi(X/k)^\Delta/L$$

to see that $\mathrm{Ind}_L^{\Pi(X/k)^\Delta}$ is faithfully exact. □

Corollary 5.6. *Any L-representation is a quotient of a $\Pi(X/k)$-representation. Consequently, any R-projective representation of L is also a special subobject of a $\Pi(X/k)$-representation.*

Proof. Using Proposition 5.5, Theorem A.3 combined with the following equality

$$\mathrm{Ind}_{\Pi(X/k)^\Delta}^{\Pi(X/k)} \circ \mathrm{Ind}_L^{\Pi(X/k)^\Delta} \cong \mathrm{Ind}_L^{\Pi(X/k)},$$

we see that $\mathrm{Ind}_L^{\Pi(X/k)}$ is faithfully exact. Hence, using the same argument in Corollary A.4 the result is as follows. □

5.1.4. *Shapiro's lemma.* Grothendieck's spectral sequence is standard in Homological algebra, we recall it here for the reader's sake. One can find proof in the book of Weilbel [We94].

Grothendieck's spectral sequence. Let $\mathscr{C},\mathscr{C}',\mathscr{C}''$ be abelian categories with $\mathscr{C},\mathscr{C}'$ having enough injectives. Suppose now that $\mathscr{F}:\mathscr{C}\to\mathscr{C}'$ and $\mathscr{F}':\mathscr{C}'\to\mathscr{C}''$ are additive (covariant) functors. If $\mathscr{F}'$ is left exact and if $\mathscr{F}$ maps injectite objects in $\mathscr{C}$ to objects acyclic for $\mathscr{F}'$, then there is a spectral sequence for each object M in $\mathscr{C}$ with differentials d_r of bidegree $(r,1-r)$, and

$$E_2^{p,q} = (R^p\mathscr{F}')(R^q\mathscr{F})M \Rightarrow R^{p+q}(\mathscr{F}'\circ\mathscr{F})M. \tag{16}$$

Remark 5.7. We remark on two facts.

(1) If $\mathscr{F}'$ is exact, then $\mathscr{F}'\circ R^q\mathscr{F} \simeq R^q\left(\mathscr{F}'\circ\mathscr{F}\right)$ for all $n\in\mathbb{N}$.
(2) If $\mathscr{F}$ is exact and maps injective objects to objects acyclic for $\mathscr{F}'$, then

$$(R^n\mathscr{F}')\circ\mathscr{F} \simeq R^n(\mathscr{F}'\circ\mathscr{F})$$

for all $n\in\mathbb{N}$.

Lemma 5.8. *Let G be a flat affine k-groupoid scheme acting on $S=\mathrm{Spec}(R)$. Let M,N,V be G-modules. If V is finitely generated and projective as an R-module, then we have for all $n\in\mathbb{N}$ a canonical isomorphism*

$$\mathrm{Ext}_G^n(M,V\otimes N) \simeq \mathrm{Ext}_G^n\left(M\otimes V^*,N\right).$$

Proof. We have a canonical isomorphism

$$\mathrm{Hom}(M,V\otimes N) \simeq \mathrm{Hom}\left(M\otimes V^*,N\right) \tag{17}$$

sending any φ to the map $m\otimes\alpha \mapsto (\alpha\otimes id_N)(\varphi(m))$. Indeed, we have the inverse map

$$\mathrm{Hom}(M,V\otimes N) \to \mathrm{Hom}(M\otimes V^*,N)$$

sending any ψ to the map $m \mapsto \alpha \otimes \psi(m \otimes \alpha^*)$ where $\alpha \in V$ such that $\alpha^*(\alpha) = 1$. We see that (17) is functorial in N and can be interpreted as an isomorphism of functors

$$\mathrm{Hom}_G(M, -) \circ (V \otimes -) \cong \mathrm{Hom}_G(M \otimes V^*, -).$$

We see that the functor $(V \otimes -)$ is exact and maps injective G-module to injective G-module (see Lemma 5.1-(3)). Therefore, we can apply Remark 5.7-(2). ☐

Remark 5.9. Let G be a flat affine k-groupoid scheme acting on $S = \mathrm{Spec}(R)$ and H be its flat subgroupoid scheme. Then $\mathrm{Ind}_H^G(-)$ is left exact functor. Thus, we can take the right derived functor $R^n\mathrm{Ind}_H^G(-)$.

Lemma 5.10. *Let G be a flat affine k-groupoid scheme acting on $S = \mathrm{Spec}(R)$ and H be its flat subgroupoid scheme. Let W be an H-module. There is a spectral sequence with*

$$E_2^{p,q} = \mathrm{H}^p\left(G, R^q\mathrm{Ind}_H^G W\right) \Rightarrow \mathrm{H}^{p+q}(H, W).$$

Proof. (1). Lemma 5.4 can be interpreted as an isomorphism of functors (choose $V = R$)

$$\mathrm{Hom}_G(R, -) \circ \mathrm{Ind}_H^G \simeq \mathrm{Hom}_H(R, -).$$

Since Ind_H^G is right adjoint to the exact functor restricting G-module to H, the functor Ind_H^G maps injective H-modules to injective G-modules. Hence, we can apply Grothendieck's spectral sequence. ☐

Definition 5.11. Let G be an affine k-groupoid scheme acting on $S = \mathrm{Spec}(R)$ and H be its flat subgroupoid scheme. We call H *exact* in G if Ind_H^G is an exact functor.

Using Lemma 5.10, we have the following result.

Proposition 5.12 (Shapiro's lemma)**.** *Let G be a flat affine k-groupoid scheme acting on $S =$ $\mathrm{Spec}(R)$ and H be its flat subgroupoid scheme. Suppose that H is exact in G. Let W be an H-module. For each $n \in \mathbb{N}$, there is an isomorphism*

$$\mathrm{H}^n\left(G, \mathrm{Ind}_H^G W\right) \simeq \mathrm{H}^n(H, W).$$

Since $\mathrm{Ind}_1^G = - \otimes_t \mathcal{O}(G)$ is an exact functor, we have the following result.

Corollary 5.13. *Let G be a flat affine k-groupoid scheme acting on $S = \mathrm{Spec}(R)$ and H be its flat subgroupoid scheme. Let $n \in \mathbb{N}$. We have for each G-module V:*

$$\mathrm{H}^n(G, V \otimes_t \mathcal{O}(G)) \simeq \begin{cases} V & \text{if} \quad n = 0, \\ 0 & \text{if} \quad n > 0. \end{cases}$$

Lemma 5.14. *Let G be a flat affine k-groupoid scheme acting on $S = \mathrm{Spec}(R)$ and H be its flat subgroupoid scheme. We have for each H-module V and each $n \in \mathbb{N}$ an isomorphism of k-vector spaces*

$$\mathrm{H}^n(H, V \otimes_t \mathcal{O}(G)) \simeq \left(R^n\mathrm{Ind}_H^G\right) V.$$

Proof. The proof is based on the way we define the induction functor. Indeed, the definition of Ind_H^G yields an isomorphism of functors

$$\mathrm{For} \circ \mathrm{Ind}_H^G \simeq (-)^H \circ (- \otimes_t \mathcal{O}(G)),$$

where For is the forgetful functor from $\mathrm{Rep}(R:G)$ to R-modules. Since the functor $(-\otimes_t \mathcal{O}(G))$ is exact and maps injective H-modules to modules acyclic for the fixed points functor (see Corollary 5.13-(1)), we can apply Remark 5.7-(1),(2) and the result is as follows. □

5.1.5. *Induction functors and injective objects.*

Proposition 5.15. *Let G be a flat affine k-groupoid scheme acting on $S = \mathrm{Spec}(R)$ and H be its flat subgroupoid scheme. If $\mathcal{O}(G)$ is an injective H-module, then H is exact in G.*

Proof. Since $\mathcal{O}(G)$ is an injective H-module, there exist H-module V_1 such that $\mathcal{O}(G)$ is a direct summand of $V_1 \otimes_t \mathcal{O}(G)$ (see Lemma 5.1-(2)). Hence, $V \otimes_t \mathcal{O}(G)$ is direct summand of $V_2 \otimes_t \mathcal{O}(G)$ for V, V_2 are H-module. By Lemma 5.14 and Corollary 5.13, we have

$$\mathrm{H}^n(H, V \otimes_t \mathcal{O}(G)) \simeq \left(R^n \mathrm{Ind}_H^G\right) V \simeq \begin{cases} V & \text{if} \quad n = 0, \\ 0 & \text{if} \quad n > 0. \end{cases}$$

This implies that H is exact in G. □

Remark 5.16. In the argument of the proof of Corollary 5.6, we see that $\mathrm{Ind}_L^{\Pi(X/k)}$ is exact.

The other direction in Corollary 5.15 is not true in general. However, we have the following result.

Lemma 5.17. *Let L be a kernel of $f : \Pi(X/k) \longrightarrow \Pi(S/k, \mathscr{F})$. Then $\mathrm{Res}(\mathcal{O}(\Pi(X/k)))$ is an injective L-module.*

Proof. We assume that V is a finite L-module. By Lemma 5.8, Lemma 5.14, Remark 5.16

$$\mathrm{Ext}_L^n(V, \mathcal{O}(\Pi(X/k))) \simeq \mathrm{Ext}_L^n\left(R, V^\vee \otimes_t \mathcal{O}(\Pi(X/k))\right) \simeq H^n\left(L, V^\vee \otimes_t \mathcal{O}(\Pi(X/k))\right) = 0$$

for all $n > 0$. Therefore, the functor $\mathrm{Hom}_L(-, \mathcal{O}(\Pi(X/k)))$ is exact when restricted to finite L-modules. Since L is Tannaka duality over dedekind ring (Theorem 3.2), each L-module is the direct limit of finite L-modules. This implies the exactness on all L-modules, that is, $\mathrm{Res}(\mathcal{O}(\Pi(X/k))))$ is an injective L-module. □

5.2. **The Gauss-Manin connection from the Tannaka viewpoint.** Let L be a kernel of $f : \Pi(X/k) \longrightarrow \Pi(S/k, \mathscr{F})$. We consider an absolute connection $(\mathcal{V}, \nabla) \in \mathrm{Ob}(\mathscr{C}(X/k))$ together with its fiber functor $V = \eta^*(\mathcal{V}) \in \mathrm{Ob}(\mathrm{Mod}_R)$. We claim that the finite R-module $\mathrm{H}^0(L, V)$ is a $\Pi(R/k)$ representation in a natural way. Indeed, for

$$(a, b) : T \to \mathrm{Spec}(R) \times_k \mathrm{Spec}(R)$$

and $g_{ab} \in \Pi(R/k)(T)$, consider $\tilde{g}_{ab} \in \Pi(X/k)(T)$ a preimage. Then (see Appendix A.2.2)

$$\tilde{g}_{ab}^{-1} \circ \tilde{g}_{ab} : a^* V \to b^* V \to a^* V$$

is the identity on $a^*\left(V^L\right)$ as $\tilde{g}_{ab}^{-1} \circ \tilde{g}_{ab} \in L(T)_{aa}$. Thus the lifting $\tilde{g}_{ab}$ yields a well-defined action of $\Pi(R/k)$ on $\mathrm{H}^0(L, V)$.

One considers the following diagram of functors:

$$\begin{array}{ccc} \mathrm{Rep}(R:\Pi(X/k)) & \xrightarrow{\mathrm{H}^0(L,V)} & \mathrm{Rep}(R:\Pi(R/k)) \\ \downarrow & & \downarrow \\ \mathrm{MIC}(X/k) & \xrightarrow{\mathrm{H}^0_{dR}(X,\mathrm{Inf}(V))} & \mathrm{MIC}(R/k). \end{array}$$

According to Lemma 2.3, the canonical morphism

$$\mathrm{H}^0(L,V) \longrightarrow \mathrm{H}^0_{DR}(X/R,\mathrm{Inf}(\mathcal{V}))$$

is an isomorphism. Thus the above diagram is commutative. As a consequence, we obtain, canonical morphisms

$$\mathrm{H}^n(\Pi(X/k),\mathrm{Res}_L^{\Pi(X/k)}(V)) = \mathrm{R}^n_{\Pi(X/k)}\mathrm{H}^0(L,V) \longrightarrow \mathrm{R}^n_{DR}\mathrm{H}^0_{DR}(X/R,(\mathrm{Inf}(\mathcal{V}))$$

where, on the left-hand side, the derived functor is taken in $\mathrm{Rep}(R:\Pi(X/k))$ and on the right-hand side, the derived functor is taken in $\mathrm{MIC}(X/k)$. We know that the right-hand side is the nth relative de Rham cohomology,

$$\mathrm{H}^{\mathrm{n}}_{\mathrm{DR}}(X/R,(\mathrm{Inf}(\mathcal{V})) = \mathrm{R}^{\mathrm{n}}_{\mathrm{MIC}}\mathrm{H}^{\mathrm{o}}_{\mathrm{DR}}(\mathrm{X},(\mathrm{Inf}(\mathcal{V})).$$

Theorem 5.18. *Let $W \in \mathrm{Rep}_f(R:G)$ and V be the restriction of W to $\mathrm{Rep}^\circ_R(L)$. For $n \geq 0$. The canonical homomorphism*

$$\mathrm{H}^n(\Pi(X/k),\mathrm{Res}_L^{\Pi(X/k)}(V)) \longrightarrow \mathrm{H}^n(L,V)$$

is an isomorphism. Consequently, it induces a representation of $\Pi(R/k)$ on $\mathrm{H}^n(L,V)$ which has the property that the canonical homomorphism

$$\mathrm{H}^i(L,V) \longrightarrow \mathrm{H}^i_{DR}(X,\mathrm{Inf}(\mathcal{V})) \quad i = 0,1,2;$$

is $\Pi(R/k)$-equivariant.

Proof. We first construct the morphism $\mathrm{H}^n(\Pi(X/k),\mathrm{Res}_L^{\Pi(X/k)}(V)) \longrightarrow \mathrm{H}^n(L,V)$. We can take the injective resolution for G-module W as follows:

$$0 \longrightarrow W \longrightarrow F_1 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow \cdots \tag{18}$$

Apply the restriction functor to the above resolution to get

$$0 \longrightarrow V \longrightarrow \mathrm{Res}(F_1) \longrightarrow \mathrm{Res}(F_2) \longrightarrow \mathrm{Res}(F_3) \longrightarrow \cdots \tag{19}$$

We will show that $\mathrm{Res}(F_i)$ is injective L-module for $i > 0$. This implies that restriction functor from Resolution (18) to Resolution (19) give us the canonical map from $\mathrm{H}^n(\Pi(X/k),\mathrm{Res}_L^{\Pi(X/k)}(V))$ to $\mathrm{H}^n(L,V)$. Moreover, this canonical map is isomorphism via the Remark 5.7-(2). Before proving $\mathrm{Res}(F_i)$ is injective L-module for $i > 0$, we need the following claim.

Claim. Let R be a dedekind domain. Let M and N be injective R-modules. Then $M \otimes N$ is injective R-module.

Verification. We prove this claim by Baer's criterion, i.e., we show that

$$\mathrm{Ext}^1_R(R/I, M \otimes N) = 0$$

for all ideal I of R. Since R/I is finite generated R-module, the extension functor Ext^1 commute with localization at every prime ideal $\mathfrak{P}$ of R, that is,

$$\mathrm{Ext}^i_R(R/I, M\otimes N)_{\mathfrak{P}} \cong \mathrm{Ext}^1_{R_{\mathfrak{P}}}((R/I)_{\mathfrak{P}}, M_{\mathfrak{P}}\otimes_R N_{\mathfrak{P}})$$

for all $\mathfrak{P}\in \mathrm{Spec}(R)$. Using Baer's criterion again we see that $M\otimes N$ is injective R-module if and only if $(M\otimes N)_{\mathfrak{P}}$ is injective $R_{\mathfrak{P}}$-module for all $\mathfrak{P}\in R$. We have two cases to consider:

(1) In case of $\mathfrak{P}=0$. We have $(M\otimes N)_{\mathfrak{P}}$ is injective $R_{\mathfrak{P}}$-module since $R_{\mathfrak{P}}$ is field.
(2) In case of $\mathfrak{P}\neq 0$. The localization ring $R_{\mathfrak{P}}$ is DVR so the module $M_{\mathfrak{P}}\otimes N_{\mathfrak{P}}$ is injective as $R_{\mathfrak{P}}$-module (It is well-known that tensor of injective modules over PID is injective module).

We conclude that $M\otimes N$ is injective R-module. □

We turn to prove that restriction of injective G-module to L-module is injective. Let F be injective G-module. By Lemma 5.1-(2), F is a direct summand of some $I\otimes_t \mathcal{O}(G)$ where I is G-module with trivial action and injective as R-module. Since restriction functor commute with direct sum and tensor product, we have $\mathrm{Res}(F)$ is direct summand of $I\otimes \mathrm{Res}(\mathcal{O}(G))$. Using Lemma 5.17 and Lemma 5.1-(2), we see that $\mathrm{Res}(F)$ is direct summand of $I\otimes J\otimes \mathcal{O}(L)$ where J is G-module with trivial action and injective as R-module. The above claim show that $I\otimes J$ is R-injective module, so Lemma 5.1-(2) implies that $\mathrm{Res}(F)$ is injective L-module.

The rest of the theorem true via Corollary 4.6. □

Remark 5.19. Theorem 5.18 show that the cohomology of groupoid scheme with coefficient restrict to suitable group scheme equal to cohomology of group scheme. This is analogue to the case de Rham cohomology of inflation connection.

APPENDIX A. AFFINE GROUP SCHEMES AND GROUPOID SCHEMES

A.1. **Affine group schemes and representations over a Dedekind domain.** Our reference is [DH18] and [De90], see also [Ja87, Sa72].

Let k be a field of characteristic 0. Let R be a k-algebra, which is a Dedekind ring.

A.1.1. *Representations.* Let G be a flat affine group scheme over R. We denote by $\mathrm{Rep}_R(G)$ the category of finite G-representations in R-modules, that is, finite R-modules equipped with a (rational) action of G. As G is flat, this is an abelian category, further, it is a tensor category. The full subcategory consisting of R-projective representations will be denoted by $\mathrm{Rep}^\circ_R(G)$. This is an R-linear, additive, rigid tensor category and is a subcategory of definition, i.e. each object of $\mathrm{Rep}_R(G)$ is a quotient of an object in $\mathrm{Rep}^\circ_R(G)$.

As R is a Dedekind ring, torsion free, flat and projective finite R-modules are the same. We say that $M\subset N$ is a special subobject in $\mathrm{Rep}_R(G)$ if the quotient N/M is R-flat. A special subquotient is *a special sub of a quotient* or, equivalently, *a quotient of a special sub.* This can be seen from the following diagram, where the left square is a push-out or, equivalently, a pull-back:

$$\begin{array}{ccccc} Q & \hookrightarrow & P & \longrightarrow & P/Q \\ \downarrow & \square & \downarrow & & \downarrow \simeq \\ M & \hookrightarrow & N & \longrightarrow & M/N. \end{array}$$

Thus, if $N/M \cong P/Q$ is R-flat then M is a special subquotient of P: it is a quotient of a special sub Q and a special sub of N.

A.1.2. *Morphisms of flat affine group schemes.* We study in this subsection flat affine group schemes and morphisms between them. Let $f: G \to G'$ be a homomorphism of flat affine group schemes over R. We say that f is surjective or a *quotient map* if it is faithfully flat.

Theorem A.1 (Theorem 4.1.2 [DH18]). *Let $f: G \longrightarrow G'$ be a homomorphism of affine flat groups over R, and ω_f° be the corresponding functor* $\mathrm{Rep}_R^\circ(G') \longrightarrow \mathrm{Rep}_R^\circ(G)$.

(1) *f is faithfully flat if and only if $\omega_f{}^\circ: \mathrm{Rep}_R^\circ(G') \longrightarrow \mathrm{Rep}_R^\circ(G)$ is fully faithful and its image is closed under taking subobjects.*
(2) *f is a closed immersion if and only if every object of $\mathrm{Rep}_R^\circ(G)$ is isomorphic to a special subquotient of an object of the form $\omega_f(X')$, $X' \in \mathrm{Rep}_R^\circ(G')$.*

A.1.3. *Exact sequence of flat affine group schemes.* Let $G \longrightarrow A$ be a homomorphism of affine group schemes over R. The kernel of this map is defined to be

$$L := G \times_A \mathrm{Spec}(R).$$

This is a closed subgroup of G. Let I_A be the kernel of counit $\epsilon: \mathcal{O}(A) \longrightarrow R$, i.e. the augmentation ideal of $\mathcal{O}(A)$, and let $I_A\mathcal{O}(G)$ be the ideal generated by the image of I_A in $\mathcal{O}(G)$. Then coordinate ring of L is isomorphic to $\mathcal{O}(G)/I_A\mathcal{O}(G)$. The sequence

$$1 \longrightarrow L \xrightarrow{q} G \xrightarrow{p} A \longrightarrow 1$$

is said to be exact if p is a quotient map with kernel L. We will provide a criterion for the exactness in terms of the functors

$$\mathrm{Rep}_R^\circ(A) \xrightarrow{p^*} \mathrm{Rep}_R^\circ(G) \xrightarrow{q^*} \mathrm{Rep}_R^\circ(L). \tag{20}$$

Theorem A.2 (Theorem 4.2.2 [DH18]). *Let us be given a sequence of homomorphisms*

$$L \xrightarrow{q} G \xrightarrow{p} A$$

with q a closed immersion and p faithfully flat. Then this sequence is exact if and only if the following conditions are fulfilled:

(a) *For an object $V \in \mathrm{Rep}_R^\circ(G)$, $q^*(V)$ in $\mathrm{Rep}_R^\circ(L)$ is trivial if and only if $V \cong p^*U$ for some $U \in \mathrm{Rep}_R^\circ(A)$.*
(b) *Let W_0 be the maximal trivial subobject of $q^*(V)$ in $\mathrm{Rep}_R^\circ(L)$. Then there exists $V_0 \subset V \in \mathrm{Rep}_R^\circ(G)$, such that $q^*(V_0) \cong W_0$.*
(c) *Any $W \in \mathrm{Rep}_R^\circ(L)$ is a quotient in (hence, by taking duals, a subobject of) $q^*(V)$ for some $V \in \mathrm{Rep}_R^\circ(G)$.*

A.2. **Groupoid schemes.** Our reference [De90], Section 3.

Let S be a k-scheme. An affine k-groupoid scheme acting on S is an $S\times_k S$-affine scheme G (with the structure maps being $s,t: G\to S$, which are called the source and the target maps), together with the following data:

(i) a map $m: G\,{}_s\!\times_t G \to G$, called the product of G, satisfying the following associativity property:
$$m(m\,{}_s\!\times_t \mathrm{id}_G) = m(\mathrm{id}_G\,{}_s\!\times_t m)$$
(ii) a map $\varepsilon: S\to G$, called the unit element map, satisfying:
$$m(\varepsilon\,{}_s\!\times_t \mathrm{id}_G) = m(\mathrm{id}_G\,{}_s\!\times_t m) = \mathrm{id}_G$$
(iii) a map $\iota: G\to G$, called the inverse map, satisfying:
$$\iota\circ s = t;\quad \iota\circ t = s$$
$$m(\iota\,{}_s\!\times_t \mathrm{id}_G) = \varepsilon\circ s,\quad m(\mathrm{id}_G\,{}_s\!\times_t \iota) = \varepsilon\circ t,$$

where ${}_s\!\times_t$ denotes the fiber product over S with respect to the maps s and t.

The groupoid scheme G said to be acting transitively on S if for any pair of morphism $(a,b): T\times U\to S$, if there is a faithfully flat quasi-compact map $\phi: W\to T\times U$ such that the set
$$\mathrm{Mor}_{S\times S}(W,G)\neq\emptyset.$$
This implies that $(s,t): G\to S\times S$ is a faithfully flat map.

A.2.1. *The diagonal group scheme.* Define the diagonal group scheme G^Δ of G as the pull-back of G along the diagonal map $\Delta: S\longrightarrow S\times S$.

$$\begin{array}{ccc} G^\Delta & \longrightarrow & G \\ \downarrow & & \downarrow \\ S & \longrightarrow & S\times_k S. \end{array}$$

A.2.2. *Representations.* Let V be a quasi-coherent sheaf on S. A representation of G in V is an operation ρ, that assigns to each k-schema T and each morphism $\phi: T\to G$ a T-isomorphism

$$\rho(\phi): a^*V\to b^*V \tag{21}$$

where $(a,b)=(s,t)\phi$, the source and the target of ϕ, and a^* (resp. b^*) denotes the pull-back of V along a (resp. b). One requires that this operation be compatible with the composition law of the groupoid $(S(T),G(T))$ and with the base change. The latter means: for any morphism $r: T'\to T$

$$\rho(r^*\phi) = r^*\rho(\phi). \tag{22}$$

A representation is called finite if the underlying sheaf is coherent. We denote this category by $\mathrm{Rep}(S:G)$ and denote the full subcategory of finite representations by $\mathrm{Rep}_f(S:G)$.

Assume that G acts transitively on S, then finite representations of G are locally free as sheaves on S. Moreover, they form a k-linear rigid tensor abelian category. Being equipped with a fiber functor – the forgetful functor to quasi-coherent sheaves on S, $\mathrm{Rep}(S:G)$ is a (non-neutral) Tannakian category. Each object of $\mathrm{Rep}(S:G)$ is a filtered union of its finite rank sub-representations.

A.2.3. *The coordinate ring.* We let S be affine, $S = \mathrm{Spec}(R)$ and let G be affine, its coordinate ring is denoted $\mathcal{O}(G)$. The groupoid structure on G induces the structures of a Hopf algebroid on $\mathcal{O}(G)$. The source and the target map for G induce algebra maps $s, t : R \to \mathcal{O}(G)$. The transitivity of G on S can be rephrased by saying that $\mathcal{O}(G)$ is faithfully flat over $R \otimes_k R$ with respect to the base map $t \otimes_k s : R \otimes_k R \to \mathcal{O}(G)$.

The composition law for G induces an $R \otimes_k R$-algebra map

$$\Delta : \mathcal{O}(G) \longrightarrow \mathcal{O}(G) \,{}_s\!\otimes_t \mathcal{O}(G). \tag{23}$$

satisfying $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$. The unit element of G induces a $R \otimes_k R$-algebra map

$$\varepsilon : \mathcal{O}(G) \longrightarrow R \tag{24}$$

where $R \otimes_k R$ acts on R diagonally (i.e., $(\lambda \otimes_k \mu)\nu = \lambda\mu\nu$). One has

$$m(\varepsilon \otimes \mathrm{id})\Delta = m(\mathrm{id} \otimes \varepsilon)\Delta = \mathrm{id} \tag{25}$$

Finally, the operation which consists of taking the inverse in G induces an automorphism ι of $\mathcal{O}(G)$ which interchanges the actions t and s:

$$\iota(t(\lambda)s(\mu)h) = s(\lambda)t(\mu)\iota(h), \tag{26}$$

and satisfies the following equations:

$$m(\iota \otimes \mathrm{id})\Delta = s \circ \varepsilon \quad m(\mathrm{id} \otimes \iota)\Delta = t \circ \varepsilon. \tag{27}$$

Since $S = \mathrm{Spec}(R)$, quasi-coherent sheaves on S are R-modules and coherent sheaves are finite R-module. The category of representations of G in S is also denoted by $\mathrm{Rep}(R : G)$. A representation ρ of G in V induces a map $\rho : V \to V \otimes_t \mathcal{O}(G)$, called coaction of $\mathcal{O}(G)$ on V, such that

$$(\mathrm{id}_V \otimes \Delta)\rho = (\rho \otimes \mathrm{id}_V), \quad (\mathrm{id}_V \otimes \varepsilon)\rho = \mathrm{id}_V. \tag{28}$$

An R-module equipped with such an action is called $\mathcal{O}(G)$-comodule. Conversely, any coaction of $\mathcal{O}(G)$ on an R-module V defines a representation of G in V. In fact, we have an equivalence between the category of G-representations and the category of $\mathcal{O}(G)$-comodules. The discussion in the previous subsection shows that V is projective over R.

In particular, the coproduct on $\mathcal{O}(G)$ can be considered as a coaction of $\mathcal{O}(G)$ on itself and hence defines a representation of G in H, called the right regular representation.

A.2.4. *Morphisms.* A morphims of k-groupoid schemes acting on a k-scheme S is a morphism of the underlying k-schemes which is compatible with all structure maps. We define the kernel of a homomorphism $f : G_1 \to G$ as the fiber product $\ker f := S \times_G G_1$:

$$\begin{array}{ccc} \ker f & \xrightarrow{f} & G \\ \downarrow & & \downarrow \\ S & \longrightarrow & S \times S. \end{array}$$

Thus $\ker f$ is a group scheme over S. Taking the diagonal group schemes, we see that $\ker f$ is isomorphic to the kernel of the homomorphism $G_1^\Delta \to G^\Delta$ of group schemes:

$$\begin{array}{ccccc} \ker f & \longrightarrow & G_1 & \xrightarrow{f} & G \\ \| & & \uparrow & & \uparrow \\ \ker f^\Delta & \longrightarrow & G_1^\Delta & \xrightarrow{f^\Delta} & G^\Delta. \end{array}$$

Indeed, the canonical map $\ker f^\Delta \longrightarrow \ker f$ comes from the definition of $\ker f$ and the (outer) commutative diagram

$$\begin{array}{ccccc} \ker f^\Delta & \longrightarrow & G_1^\Delta & \longrightarrow & G_1 \\ \downarrow & & \downarrow & & \downarrow f \\ S & \longrightarrow & G^\Delta & \longrightarrow & G. \end{array}$$

And the canonical map $\ker f \longrightarrow \ker f^\Delta$ comes from the map $\ker f \longrightarrow G_1^\Delta$ which satisfies the commutative diagram

$$\begin{array}{ccc} \ker f & \longrightarrow & G_1^\Delta \\ \downarrow & & \downarrow \\ S & \longrightarrow & G^\Delta. \end{array}$$

A.2.5. *The functor* $\mathrm{Ind}_{G^\Delta}^G$. In this subsection, we extend the result of [EH06].

We continue to assume S and G are affine, $S = \mathrm{Spec}(R)$. For any representation $W \in \mathrm{Rep}_R(G^\Delta)$, set

$$\mathrm{Ind}_{G^\Delta}^G(W) := (W \otimes_t \mathscr{O}(G))^{G^\Delta} \tag{29}$$

where G^Δ acts on W as usual and on $\mathscr{O}(G)$ through the right regular action of G on $\mathscr{O}(G)$ (i.e $\mathscr{O}(G)$ is a right G-module). On this invariant space, G acts through the left regular action on $\mathscr{O}(G)$. Thus $\mathrm{Ind}_{G^\Delta}^G$ is a functor $\mathrm{Rep}_R(G^\Delta) \to \mathrm{Rep}(S : G)$.

The space $\mathrm{Ind}_{G^\Delta}^G(W)$ can also be given as the equalizer of the maps

$$\begin{aligned} p &: W \otimes_t \mathscr{O}(G) \xrightarrow{\rho_W \otimes \mathrm{id}} W \otimes \mathscr{O}(G^\Delta) \otimes_t \mathscr{O}(G) \\ q &: W \otimes_t \mathscr{O}(G) \xrightarrow{\mathrm{id} \otimes \Delta} W \otimes_t \mathscr{O}(G)\, {}_s\!\otimes_t \mathscr{O}(G) \xrightarrow{\pi} W \otimes \mathscr{O}(G^\Delta) \otimes_t \mathscr{O}(G) \end{aligned} \tag{30}$$

where $\rho_W : W \to W \otimes \mathscr{O}(G^\Delta)$ is the coaction of $\mathscr{O}(G^\Delta)$ on W, Δ is the coproduct on $\mathscr{O}(G)$.

There exists a functorial isomorphism

$$\operatorname{Hom}_G(V, \operatorname{Ind}_{G^\Delta}^G(W)) \cong \operatorname{Hom}_{G^\Delta}(V, W), \tag{31}$$

$V \in \operatorname{Rep}(S:G)$, $W \in \operatorname{Rep}_R(G^\Delta)$, i.e., $\operatorname{Ind}_{G^\Delta}^G$ is the right adjoint to the functor restricting G-representations to G^Δ.

Theorem A.3 (Cf. [EH06, Remark 6.7])**.** *The functor* $\operatorname{Ind}_{G^\Delta}^G$ *is faithfully exact. Hence the canonical map*

$$\operatorname{Ind}_{G^\Delta}^G(W) \longrightarrow W$$

is surjective for any G-representation W.

Proof. Before giving the proof of this theorem, we discuss the algebra $\mathcal{O}(G^\Delta)$. By definition of G^Δ, we have

$$\mathcal{O}\left(G^\Delta\right) \cong \mathcal{O}(G) \otimes_{R\otimes_k R} R$$

where $R \otimes_k R \to R$ is the product map. Then $J := \operatorname{Ker}(R \otimes_k R \to R)$ is generated by elements of the form $\lambda \otimes 1 - 1 \otimes \lambda, \lambda \in R$. Since $\mathcal{O}(G)$ is faithful over $R \otimes_k R$, tensoring the exact sequence $0 \to J \to R \otimes_k R \to R \to 0$ with $\mathcal{O}(G)$ over $R \otimes_k R$, one obtains an exact sequence

$$0 \longrightarrow J\mathcal{O}(G) \longrightarrow \mathcal{O}(G) \xrightarrow{\pi} \mathcal{O}\left(G^\Delta\right) \longrightarrow 0. \tag{32}$$

That is, we can identify $J \otimes_{R\otimes_k R} \mathcal{O}(G)$ with its image $J\mathcal{O}(G)$ in $\mathcal{O}(G)$. In order to prove the faithfully exactness of $\operatorname{Ind}_{G^\Delta}^G$, we need the following claim.

Claim. Let us use the following notation of Sweedler for the coproduct on $\mathcal{O}(G)$:

$$\Delta(g) = \sum_{(g)} g_{(1)} \otimes g_{(2)}.$$

The following map:

$$\begin{aligned} \varphi : \mathcal{O}(G) \otimes_{R\otimes_k R} \mathcal{O}(G) &\longrightarrow \mathcal{O}\left(G^\Delta\right) \otimes_t \mathcal{O}(G), \\ g \otimes h &\longmapsto \sum_{(g)} \pi\left(g_{(1)}\right) \otimes g_{(2)} h, \end{aligned} \tag{33}$$

is an isomorphism, where π is defined in the formula 32.

Verification. We define the inverse map to this map. Let

$$\bar{\psi} : \mathcal{O}(G)_s \otimes_t \mathcal{O}(G) \longrightarrow \mathcal{O}(G) \otimes_{R\otimes_k R} \mathcal{O}(G)$$

be the map that maps $g \otimes h \mapsto \sum_{(g)} g_{(1)} \otimes \iota\left(g_{(2)}\right) h$. We have for $\lambda \in R$ and for $t, s : R \to \mathcal{O}(G)$

$$\begin{aligned} \bar{\psi}\left(t(\lambda) g \,{}_s\!\otimes_t h\right) &= \sum_{(g)} g_{(1)} \otimes \iota\left(t(\lambda) g_{(2)}\right) h \\ &= \sum_{g} g_{(1)} \otimes_{R\otimes_k R} s(\lambda) \iota\left(g_{(2)}\right) h \quad \text{by (26)} \\ &= s(\lambda) \sum_{g} g_{(1)} \otimes_{R\otimes_k R} \iota\left(g_{(2)}\right) h \\ &= \bar{\psi}\left(s(\lambda) g \,{}_s\!\otimes_t h\right). \end{aligned}$$

Thus $\bar{\psi}$ maps $J\mathcal{O}(G)\ {}_s\otimes_t\mathcal{O}(G)$ to 0, hence factors through a map

$$\psi:\mathcal{O}\left(G^{\Delta}\right)\otimes_t\mathcal{O}(G)\to\mathcal{O}(G)\otimes_{R\otimes_k R}\mathcal{O}(G).$$

Checking $\varphi\psi=\mathrm{id},\psi\varphi=\mathrm{id}$ can be easily done using the property (27) of ι.

We now prove that the functor $\mathrm{Ind}_{G^{\Delta}}^{G}$ is faithfully exact. Let $W\in\mathrm{Rep}_R(G^{\Delta})$. Tensoring the isomorphism in (33) with W and applying the functor $(-)^{G^{\Delta}}$, we obtain the following map

$$\begin{aligned}\Phi:\mathrm{Ind}_{G^{\Delta}}^{G}(W)\otimes_{R\otimes_k R}\mathcal{O}(G)&\xrightarrow{\cong}W\otimes_t\mathcal{O}(G),\\ w\otimes g\otimes h&\longmapsto w\otimes gh.\end{aligned}\tag{34}$$

The above map is isomorphism since we have its inverse as follows:

$$\begin{aligned}&W\otimes_t\mathcal{O}(G)\longrightarrow W\otimes\mathcal{O}\left(G^{\Delta}\right)\otimes\mathcal{O}(G)\xrightarrow{\Psi}W\otimes_t\mathcal{O}(G)\otimes_{R\otimes_k R}\mathcal{O}(G),\\ &\Psi=\left(\mathrm{id}_W\otimes\psi\right)\left(\rho_W\otimes\mathrm{id}\right).\end{aligned}$$

According to (34), the functor

$$\mathrm{Ind}_{G^{\Delta}}^{G}(-)\otimes_{R\otimes_k R}\mathcal{O}(G)\cong(-)\otimes_t\mathcal{O}(G)$$

hence is exact. Since $\mathcal{O}(G)$ is faithfully flat over $R\otimes_k R$, the functor $\mathrm{Ind}_{G^{\Delta}}^{G}$ is faithfully exact.

We now prove the last part of the theorem. Setting $V=\mathrm{Ind}_{G^{\Delta}}^{G}(W)$ in (31), we define the canonical $u_W:\mathrm{Ind}_{G^{\Delta}}^{G}(W)\to W$ as follows:

$$\begin{aligned}\mathrm{Hom}_G(\mathrm{Ind}_{G^{\Delta}}^{G}(W),\mathrm{Ind}_{G^{\Delta}}^{G}(W))&\longrightarrow\mathrm{Hom}_{G^{\Delta}}(\mathrm{Ind}_{G^{\Delta}}^{G}(W),W)\\ \mathrm{id}&\longmapsto u_W.\end{aligned}$$

The map u_W is nonzero whenever W is nonzero. Indeed, since $\mathrm{Ind}_{G^{\Delta}}^{G}$ is faithfully exact, $\mathrm{Ind}_{G^{\Delta}}^{G}(W)$ is nonzero whenever W is nonzero. Thus, if $u_W=0$, then both sides of (31) are zero for any V. On the other hand, the right-hand side contains the identity map. This shows that u_W cannot vanish.

We now turn to show that u_W is surjective. Let $\mathrm{U}=\mathrm{Im}(u_W)$ and let $\mathrm{T}=W/\mathrm{U}$. We have the following diagram:

$$\begin{array}{ccccccccc}0&\longrightarrow&\mathrm{Ind}_{G^{\Delta}}^{G}(U)&\longrightarrow&\mathrm{Ind}_{G^{\Delta}}^{G}(W)&\longrightarrow&\mathrm{Ind}_{G^{\Delta}}^{G}(T)&\longrightarrow&0\\ &&\downarrow&&\downarrow&&\downarrow&&\\ 0&\longrightarrow&U&\longrightarrow&W&\longrightarrow&T&\longrightarrow&0.\end{array}$$

The composition $\mathrm{Ind}_{G^{\Delta}}^{G}(W)\to\mathrm{Ind}_{G^{\Delta}}^{G}(T)\to T$ is 0, therefore $\mathrm{Ind}_{G^{\Delta}}^{G}(T)\to T$ is a zero map, implying $T=0$. □

Corollary A.4. *Any G^{Δ}-representation is a quotient of a G-representation. Consequently, any R-projective representation of G^{Δ} also a special subobject of a G-representation.*

Proof. Let W be a representation of G^{Δ} and $u_W:\mathrm{Ind}_{G^{\Delta}}^{G}(W)\longrightarrow W$ be the canonical map in Theorem A.3. This theorem implies that W is a quotient of $\mathrm{Ind}_{G^{\Delta}}^{G}(W)$.

We now prove the rest of the corollary. Since $\mathrm{Ind}_{G^\Delta}^G(W)$ is a union of its finite subrepresentations, we can therefore find a finite G-subrepresentation $W_0(W)$ of $\mathrm{Ind}_{G^\Delta}^G(W)$, which still maps surjectively on W. In order to obtain the statement on the embedding of R-projective representation W, one repeat the above argument for $(W^\vee)$ to get the surjective map $W_0(W^\vee) \twoheadrightarrow W^\vee$, and dualizes to $W \hookrightarrow (W_0(W^\vee))^\vee$. □

A.3. **Tannakian duality over a field.** Reference for this subsection is [DM82] and [De90].

Definition A.5. A rigid abelian tensor category $\mathscr{C}$ equipped with an exact faithful k-linear tensor functor $\omega : \mathscr{C} \to \mathrm{Vec}_k$ is called a *neutral Tannakian category* over k if. The functor ω is called a *fibre functor* for $\mathscr{C}$.

Theorem A.6. *Let $(\mathscr{C}, \omega)$ be a neutral Tannakian category. Then, there exists a k-group scheme G, such that ω induces an equivalence between $\mathscr{C}$ and* $\mathrm{Rep}_k(G)$.

The group scheme G above is called the Tannakian group of the category $(\mathscr{C}, \omega)$.

An example of a Tannakian category is the category of finite dimensional representations of an affine group scheme G over k, equipped with the forgetful functor of k-vector spaces. The resulting Tannakian group is isomorphic to G.

Another example, which is an object of this work, is the category of connections defined subsequently.

Definition A.7. A rigid abelian tensor category $\mathscr{C}$ equipped with an exact faithful k-linear tensor functor $\omega : \mathscr{C} \to \mathrm{mod} R$, R is a k-algebra, is called a *(general) Tannakian category* over k if. The functor ω is called a *fibre functor* for $\mathscr{C}$ with values in R-modules.

Theorem A.8. *Let $(\mathscr{C}, \omega)$ be a general Tannakian category with values in R-modules. Then, there exists a k-groupoid scheme $\mathscr{C}$, acting transitively upon* $\mathrm{Spec} R \times_k \mathrm{Spec} R$ *such that ω induces an equivalence between $\mathscr{C}$ and* $\mathrm{Rep}(R : \mathscr{G})$.

The groupoid $\mathscr{G}$ is called the Tannakian groupoid of $(\mathscr{C}, \omega)$. Conversely, if we start from a groupoid scheme $\mathscr{G}$ acting transitively upon a ring R, then $\mathrm{Rep}(R : \mathscr{G})$ equipped with the forgetful functor is a Tannakian category. The corresponding Tannakian groupoid is isomorphic to $\mathscr{G}$.

A.4. **Tannakian duality over a Dedekind ring.** Reference for this subsection is [Sa72], see also [DH18].

Assume that $\mathscr{C}$ is an R-linear abelian tensor category. Denote by $\mathscr{C}^o$ the full subcategory of $\mathscr{C}$ consisting of rigid objects. We say that $\mathscr{C}$ is *dominated* by $\mathscr{C}^o$ if each object of $\mathscr{C}$ is a quotient of a rigid object.

Definition A.9. A (neutral) Tannakian category over a Dedekind ring R is an R-linear abelian tensor category $\mathscr{C}$, dominated by $\mathscr{C}^o$, together with an exact faithful tensor functor $\omega : \mathscr{C} \to \mathsf{Mod}(R)$.

Theorem A.10 ([Sa72, Thm. II.4.1.1])**.** *Let $(\mathscr{C},\omega)$ be a neutral Tannakian category over a Dedekind ring R. Then the group functor $A \mapsto \mathsf{Aut}^{\otimes}_A(\omega\otimes A)$ is representable by a flat group scheme G and ω factors through an equivalence between $\mathscr{C}$ and* $\mathrm{Rep}_R(G)$.

APPENDIX B. MORPHISMS BETWEEN AFFINE FLAT GROUP SCHEMES

Let R be a Dedekind domain. We denote $\mathbf{FGSch}/R$ be the full subcategory of the category of affine group scheme over R whose objects are R-flat. Let $\Pi \in \mathbf{FGSch}/R$. We recall some concepts:

(1) A subcomodule M of an $R[\Pi]$-comodule N is said to be *special* if N/M is flat over R; a *special subquotient* M of an $R[\Pi]$-comodule N is a special submodule of a quotient of N, or, equivalently, a quotient of a special submodule of N.
(2) Let N be an $R[\Pi]-$comodule. Then N_{tor}, the R-torsion submodule of N is an $R[\Pi]$-subcomodule. Hence for any $R[\Pi]$-subcomodule M, the preimage of $(N/M)_{\mathrm{tor}}$ in N, denoted M^{sat}, is an $R[\Pi]$-comodule. Since R is a Dedekind ring, the quotient N/M^{sat} is flat, being torsion-free. Thus M^{sat} is the smallest special subcomodule of N, containing M. It is called the *saturation* of M in N.

Definition B.1. Let H′ be a flat Hopf algebra over R. A Hopf subalgebra H of H′ is an R-submodule equipped with a Hopf algebra structure such that the inclusion $\mathrm{H} \to \mathrm{H}'$ is a homomorphism of Hopf algebras. We say that H is *saturated* in H′ if H'/H is flat as an R-module.

Let $\rho : \Pi \to \mathrm{G}$ be a morphism in $\mathbf{FGSch}/R$. We describe the "images" of ρ in two ways.

Definition B.2 (The diptych)**.** Define Ψ_ρ as the group scheme whose Hopf algebra is the image of $R[G]$ in $R[\Pi]$. Define $R[\Psi'_\rho]$ as the saturation of the latter inside $R[\Pi]$. The obvious commutative diagram

$$\begin{array}{ccc} \Psi'_\rho & \longrightarrow & \Psi_\rho \\ \uparrow & & \downarrow \\ \Pi & \xrightarrow[\rho]{} & G \end{array}$$

is called the diptych of ρ.

Remark B.3. We have

(1) The image of a Hopf algebra homomorphism is a Hopf subalgebra (of the target).
(2) Implicit in the above definition is the fact that $\mathrm{R}[\Psi'_\rho]$ is a Hopf algebra. Indeed, we have the filtration

$$R[\Psi'_\rho]^{\mathrm{sat}} \otimes R[\Psi'_\rho]^{\mathrm{sat}} \subset R[\Psi'_\rho]^{\mathrm{sat}} \otimes R[\Pi] \subset R[\Pi] \otimes R[\Pi],$$

the successive quotients of which are flat, hence $R[\Pi]\otimes R[\Pi]/R[\Psi'_\rho]^{\mathrm{sat}} \otimes R[\Psi'_\rho]^{\mathrm{sat}}$ is also flat. Thus

$$(R[\Psi'_\rho]\otimes R[\Psi'_\rho])^{\mathrm{sat}} \subset R[\Psi'_\rho]^{\mathrm{sat}} \otimes R[\Psi'_\rho]^{\mathrm{sat}}.$$

Hence, by the definition of $R[\Psi'_\rho]^{\mathrm{sat}}$, we have

$$\Delta\left(R[\Psi'_\rho]^{\mathrm{sat}}\right) \subset (R[\Psi'_\rho] \otimes R[\Psi'_\rho])^{\mathrm{sat}} \subset R[\Psi'_\rho]^{\mathrm{sat}} \otimes R[\Psi'_\rho]^{\mathrm{sat}}.$$

Lemma B.4. *The morphism $\Pi \to \Psi'_\rho$ is faithfully flat.*

Proof. See [DH18, Theorem 4.1.1]. □

Proposition B.5. *If $\Psi'_{\rho,\mathrm{k}} \to \Psi_{\rho,\mathrm{k}}$ is faithfully flat for every residue field k, then $\Psi'_\rho \to \Psi_\rho$ is an isomorphism.*

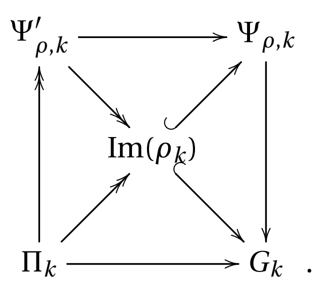
Proof. By construction, $R\left[\Psi_\rho\right] \to \mathrm{R}[\Psi'_\rho]$ is injective and $K[\Psi_\rho] = K[\Psi'_\rho]$. Since $\Psi'_{\rho,\mathrm{k}}$ is faithfully flat over $\Psi_{\rho,\mathrm{k}}$ for k being either the fraction field or any residue field of R, then Ψ'_ρ is faithfully flat over Ψ_ρ by Proposition 3.2 in [DHH17]. The faithful flatness of $R[\Psi'_\rho]$ over $R[\Psi_\rho]$ implies that $R[\Psi_\rho]$ is saturated in $R[\Psi'_\rho]$. Indeed, tensoring the exact sequence

$$0 \to R[\Psi_\rho] \to R[\Psi'_\rho] \to R[\Psi'_\rho]/R[\Psi_\rho] \to 0$$

on the right with $R[\Psi'_\rho]$ over $R[\Psi_\rho]$ we obtain a split exact sequence (the splitting is given by the map $R[\Psi'_\rho] \otimes_{R[\Psi_\rho]} R[\Psi'_\rho] \to R[\Psi'_\rho], m \otimes n \mapsto mn$). By assumption $R[\Psi'_\rho]$ is R-flat, hence so is $R[\Psi'_\rho] \otimes_{R[\Psi_\rho]} R[\Psi'_\rho]$. Consequently $R[\Psi'_\rho]/R[\Psi_\rho] \otimes_{R[\Psi_\rho]} R[\Psi'_\rho]$ is R-flat. Now the faithful flatness of $R[\Psi'_\rho]$ over $R[\Psi_\rho]$ implies that $R[\Psi'_\rho]/R[\Psi_\rho]$ is R-flat, that is, $R[\Psi_\rho]$ is saturated in $R[\Psi'_\rho]$ as an R-module. □

Over the residue field k, there is yet another interesting group scheme in sight: the image of ρ_k. We then have the *triptych* of ρ, which is the commutative diagram

(35)



Together with Proposition B.5, diagram (35) proves the following:

Corollary B.6. *The following claims are true.*

i) *If $\mathrm{Im}\left(\rho_{\mathrm{k}}\right) \to \Psi_{\rho,\mathrm{k}}$ is an isomorphism for every residue field k, then $\Psi'_\rho \to \Psi_\rho$ is an isomorphism.*
ii) *If $\Psi'_\rho \to \Psi_\rho$ is an isomorphism, then $\mathrm{Im}\left(\rho_{\mathrm{k}}\right) \to \Psi_{\rho,\mathrm{k}}$ is an isomorphism.*
iii) *The image of $\Psi'_{\rho,\mathrm{k}}$ in $\Psi_{\rho,\mathrm{k}}$ is none other than $\mathrm{Im}\left(\rho_{\mathrm{k}}\right)$.*

We have some notations, conventions and standard terminology.

(1) If V is a finite R-projective module, we write $\mathrm{GL}(V)$ for the general linear group scheme representing $A \longrightarrow \mathrm{Aut}_A(V \otimes A)$. If $V = R^n$, then $\mathrm{GL}(V) = \mathrm{GL}_n$.
(2) An object V of $\mathrm{Rep}^\circ_R(G)$ is said to be a *faithful representation* if the resulting morphism $G \longrightarrow \mathrm{GL}(V)$ is a closed immersion. Similar conventions are in force for group schemes over k. We admonish the reader that this is not the terminology of the authoritative [SGA3], where a faithful action is decreed to be one having no kernel (see Definition 2.3.6.2 of exposé I).

Let $\Pi \in \mathbf{FGSch}/R$. This group scheme admits a closed embedding into some $\mathrm{GL}_{n,R}$ as J.S. Milne mention in Aside 9.4 of his book [Mi12], or, according to the above notions, Π possesses a faithful representation. We remark on the concept of special subquotient which we introduced in Subsection A.1.1.

Definition B.7. Let $\Pi \in \mathbf{FGSch}/\mathrm{R}$ and $V \in \mathrm{Rep}^\circ_R(\Pi)$. Call an object $V'' \in \mathrm{Rep}^\circ_R(\Pi)$ a *special sub-quotient* of V if there exists a special monomorphism $V' \to V$ and an epimorphism $V' \to V''$. The category of all special sub-quotients of various $T^{a_1,b_1}(\mathrm{V}) \oplus \cdots \oplus T^{a_m,b_m}(\mathrm{V})$ is denoted by $\langle V\rangle^s_\otimes$.

Let V be a finite R- projective module and assume that our G (in the diptych) equals GL(V). We now interpret Ψ_ρ and Ψ'_ρ in terms of their representation categories.

Proposition B.8. *Let* V *be an object of* $\mathrm{Rep}^\circ_\mathrm{R}(\Pi)$ *and* ρ *be the natural homomorphism* $\Pi \to \mathrm{G} := \mathrm{GL(V)}$.

(1) *The obvious functor* $\mathrm{Rep}_R\left(\Psi_\rho\right) \to \mathrm{Rep}_R(\Pi)$ *defines an equivalence of categories between* $\mathrm{Rep}^\circ_R\left(\Psi_\rho\right)$ *and* $\langle V\rangle^s_\otimes$.
(2) *The obvious functor* $\mathrm{Rep}_R\left(\Psi'_\rho\right) \to \mathrm{Rep}_R(\Pi)$ *defines an equivalence between* $\mathrm{Rep}_R\left(\Psi'_\rho\right)$ *and* $\langle \mathrm{V}\rangle_\otimes$.

Proof. We prove by adapting the proofs in [dS09, Proposition 12]. We begin by remark some notions in [dS09, Sections 3.1 and 3.2] which will be modified slightly for our setting. We denote δ as a determinant representation of V : it is a locally R-module of rank one where Ψ_ρ acts via the group-like element δ of Ψ_ρ (see determinant of a finite projective module in [Stack, 0FJ9]). Since V is projective then V is direct summand of finite free module M with rank d, and we define

$$\Theta_M(a) = \left(1 \oplus (M^d)^{\otimes 1} \oplus \ldots (M^d)^{\otimes a}\right) \ (a \in \mathbb{N}).$$

There are natural arrows in $\mathrm{Rep}_R(\Psi_\rho)$

$$\theta_a : \Theta_M(a) \longrightarrow (R[\Psi_\rho], \rho_r),$$

where $(R[\Psi_\rho], \rho_r)$ is regular representation of Ψ_ρ. Indeed, we will construct the map θ_a steps by steps. Take m_i is a basis of M and let m_{ij} be a basis of M^d. The comodule map for M^d is

$$m_{ij} \mapsto \sum_\ell v_{i\ell} \otimes \rho_{\ell j},$$

by mapping m_{ij} to ρ_{ij} gives the Ψ_ρ-equivariant map $V^d \longrightarrow (R[\Psi_\rho], \rho_r)$. We can extend this to any tensor power $(V^d)^{\otimes a} \longrightarrow (R[\Psi_\rho], \rho_r)$, that is, the map θ_a sends a pure tensor

$$m_{I_1} \otimes \cdots \otimes m_{I_a} \in \left(M^d\right)^{\otimes a}, \quad I \in \{1, \ldots, d\} \times \{1, \ldots, d\}$$

to the element

$$\rho_{I_1} \cdots \rho_{I_a}.$$

We now define

$$\Theta_V(a,b,1) = \left(1 \oplus (V^d)^{\otimes 1} \oplus \ldots (V^d)^{\otimes a}\right) \otimes (\delta^{-1})^{\otimes b};$$
$$\Theta_V(a,b,c) = \Theta_V(a,b,1)^{\otimes c} \ \ (a,b,c \in \mathbb{N}).$$

There are natural arrows in $\mathrm{Rep}_R(\Psi_\rho)$

$$\theta_{a,b,c} : \Theta_V(a,b,c) \longrightarrow (R[\Psi_\rho], \rho_r)$$

defined by arrows

$$\theta_{a,b,1} : (V^d)^{\otimes a} \otimes (\delta^{-1})^{\otimes b} \longrightarrow (R[\Psi_\rho], \rho_r)$$

as follows. We restriction the map θ_a to $(V^d)^{\otimes a}$ and then multiplication by group like element δ, that is, we map the pure tensor $v_{I_1} \otimes \cdots \otimes v_{I_a} \in \left(V^d\right)^{\otimes a}$ to $\delta^{-b} \cdot \rho_{I_1} \cdots \rho_{I_a}$ instead.

Let V be a finite projective $R[\Psi_\rho]$-comodule. The coaction $\rho : \mathrm{V} \longrightarrow \mathrm{V} \otimes R[\Psi_\rho]$ induces a map

$$\mathrm{Cf} : V^\vee \otimes V \longrightarrow R[\Psi_\rho], \quad \varphi \otimes m \mapsto \sum \varphi(m_i)\, m_i', \quad \varphi \in V^\vee, m \in V, \Delta(m) = \sum_i m_i \otimes m_i'.$$

The image of this map, denoted by Cf(V), is called the *coefficient space* of V. Before giving the proof of (1), we denote

$$S_V := \bigcup_{a \geq 1} \mathrm{Im}\,\theta_{a,0,1}, \quad S_V' = \bigcup_{a,b \geq 1} \mathrm{Im}\,\theta_{a,b,1}.$$

These are Ψ_ρ-submodules of $(R[\Psi_\rho], \rho_r)$. The proof of (1) is a consequence of the following claims.

Claim 1. Assume that the projective module V is faithful. Then any finite projective representation W is belongs to $\langle V \rangle_\otimes^s$.

Verification. Assume that Ψ_ρ-module W is free module of rank r. We embed W equivariantly in $(R[\Psi_\rho], \rho)^{\oplus r}$. As the representation V is faithful, we have $R[\Psi_\rho] = S_V'$ and by tensoring with some $\delta^{\otimes b}$ we have $W \otimes \delta^{\otimes b} \subseteq S_V^{\oplus r}$. Hence $W \otimes \delta^{\otimes b}$ is a special sub-object of some $\mathrm{Im}\,\theta_{a,0,r}$ [dS09, Lemma 11]. By the Snake Lemma

$$\mathrm{Im}\,\theta_{a,0,r} / W \otimes \delta^{\otimes b} \cong \Theta(a,0,r) / \theta_{a,0,r}^{-1}\left(W \otimes \delta^{\otimes b}\right)$$

and we see that W is a special sub-quotient of $\Theta(a,b,r)$. It is clear that $\Theta_V(a,b,c)$ belongs to $\langle V \rangle_\otimes^s$, so W also belongs to $\langle V \rangle_\otimes^s$. Finally, since every finite projective is a direct summand of the finite free module, the result as follows.

Claim 2. If any finite projective representation of Ψ_ρ belongs to $\langle V \rangle_\otimes^s$, then V is faithful.

Verification. Let $W \subseteq \left(R[\Psi_\rho], \rho_r\right)$ be a G-submodule of finite rank. Since V is projective, then there exists projective module I such that $V \oplus I = M$ where M is R-free module. We consider M as G-module with the action of G on I is trivial. Choose $U \subseteq \sum_1^s T^{a_i,b_i}(M)$ special such that W is a quotient and U/W is R-free module. Then

$$W \subseteq \mathrm{Cf}(W) \subseteq \mathrm{Cf}(U) \subseteq \mathrm{Cf}\left(\sum_1^s T^{a_i,b_i}(M)\right) \subseteq S'_M$$

via [dS09, Claim in Proposition 12] and [DH18, 1.1.5]. This shows that $R[\Psi_\rho] = S'_M$ i.e. M is faithful. We conclude that V is also faithful since the action of G on I is trivial.

Proof of (2). The main idea of this proof is Theorem A.1. We have the morphism $\Pi \longrightarrow \Psi'_\rho$ is faithfully flat (Proposition B.4) since $\mathrm{Rep}^\circ_R(\Psi'_\rho) \longrightarrow \mathrm{Rep}^\circ_R(\Pi)$ is fully faithful and its image closed under taking subobject. On the other hand, since $\langle V\rangle_\otimes$ is the Tannakian category (we can check that it satisfies the Definition A.9), we have

$$\langle V\rangle_\otimes \cong \mathrm{Rep}_R(H),$$

where $H \in \mathbf{FGSch}/R$ (see Theorem A.10). According to (1), we have the following composition map

$$\mathrm{Rep}^\circ_R(\Psi_\rho) \longrightarrow \mathrm{Rep}^\circ_R(\Psi'_\rho) \longrightarrow \langle V\rangle^s_\otimes$$

is isomorphism. Combine with $\mathrm{Rep}^\circ_R(H) = \langle V\rangle^\circ_\otimes$ (the full subcategory of subobjects of finite direct sums of copies of tensor generated of V) and Claim 1, we see that the natural map $H \to \Psi'_\rho$ is closed immersion via Theorem A.1-(2). ☐

Note that, in general, $\mathrm{Rep}_R\left(\Psi_\rho\right)$ is not a full subcategory of $\mathrm{Rep}_R(\Pi)$. This means that we have the following interpretation of the diptych (Definition B.2) in terms of representation categories:

$$\begin{array}{ccc} \langle V\rangle_\otimes & \longleftarrow & \langle V\rangle^s_\otimes \\ \downarrow & & \downarrow \\ \mathrm{Rep}_R(\Pi) & \longleftarrow & \mathrm{Rep}^\circ_R(\mathrm{GL}(V)). \end{array}$$

We now deal with the representation theoretic interpretation of the triptych (Diagram (35)) of ρ. For that, given an R-linear category C, we define the special fiber C_s at a closed point s of $S := \mathrm{Spec}(R)$ to be the full subcategory whose objects W are annihilated by $\mathfrak{m}_s$, i.e. $m_s.id_W = 0$ in $\mathrm{Hom}_C(W,W) = 0$, where $\mathfrak{m}_s$ is the maximum ideal of R, which determines s. One can show that C_s is equivalent to the scalar extension C_{k_s} where $k_s = R/m_s$ is the residue field of R at s. We then have a commutative diagram of solid arrows between k-linear abelian categories:

$$
\begin{array}{ccc}
\mathrm{Rep}_R(\Psi'_\rho)_{(k_s)} & \longleftarrow & \mathrm{Rep}_R(\Psi_\rho)_{(k_s)} \\
\big\downarrow & \nwarrow \quad \swarrow\cdots & \\
 & \mathrm{Im}(\rho_{k_s}) & \\
 & \swarrow & \\
\mathrm{Rep}_R(\Pi)_{(k_s)}. & &
\end{array}
\tag{36}
$$

From [Ja87, Part I, 10.1, 162] the categories $\mathrm{Rep}_R(-)_{(k_s)}$ are simply the corresponding representations categories of the group schemes obtained by base change $R \to k_s$. Since V is a faithful representation of Ψ_ρ (recall that $\Psi_\rho \to \mathrm{GL(V)}$ is a closed immersion by construction), $V \otimes k_s$ is a faithful representation of $\Psi_\rho \otimes k_s$, so that each object of $\mathrm{Rep}_R\left(\Psi_\rho\right)_{(k_s)}$ is a sub-quotient of some $\bigoplus T^{a_{\mathrm{i}},\mathrm{b}_{\mathrm{i}}}(\mathrm{V} \otimes k_s)$. This means that the upper horizontal arrow in Diagram (36) factors through $\langle \mathrm{V} \otimes k_s \rangle_\otimes$, i.e. the dotted arrow exists and still produces a commutative diagram. We conclude that Diagram (36) captures the essence of Diagram (35) as the former can easily be completed by introducing the representation category of the general linear group in the lower right corner.

Theorem B.9 (Lemma 3.6)**.** *Let $\mathscr{M}$ be an absolute connection on X/k. Then each locally free relative connection in $\langle \inf(\mathscr{M})\rangle_\otimes$ is indeed a* special *subobject of a tensor generated object from* $\inf(\mathscr{M})$.

Proof. The R-point η induces an equivalence of abelian tensor categories

$$\eta^* : \langle \inf(\mathscr{M})\rangle_\otimes \longrightarrow \mathrm{Rep}_R(G'), \tag{37}$$

where G' is a flat group scheme over R (see Theorem A.10). Let $p : G' \to \mathrm{GL}(\eta^* \inf(\mathscr{M}))$ be the associated representation of the group G. We consider the diptych (Definition B.2) of p :

$$
\begin{array}{ccc}
\Psi'_\rho & \longrightarrow & G \\
\big\uparrow & & \big\downarrow \\
G' & \xrightarrow[p]{} & \mathrm{GL}(\eta^* \inf(\mathscr{M}))
\end{array}
$$

where G is the group scheme Ψ_ρ of Definition B.2. Theorem A.1 and Proposition B.8-(2) show that the leftmost arrow above is an isomorphism. Moreover, according to Proposition B.8-(1), the functor η^* induces an equivalence

$$\langle \inf(\mathscr{M})\rangle_\otimes^s \xrightarrow{\sim} \mathrm{Rep}_R^\circ(G). \tag{38}$$

Thus, the statement of this lemma can be proved if we can show that G isomorphic to G'.

To prove isomorphic between G and G', we only need to treat the local property, that is, we only need to prove that G'_{k_s} is fully faithful over G_{k_s} thanks to Proposition B.5. Moreover, we still prove that the functor

$$\mathrm{Rep}_k(G_{k_s}) \longrightarrow \langle \inf(\mathscr{M})|_{X_{k_s}}\rangle_\otimes$$

is an equivalence based on the Corollary B.6 (i) and Diagram (36). The proof is a consequence of the following claims which we use the notion of k instead of k_s for convenience.

Claim 1. Let $\mathcal{N}$ and $\mathcal{N}'$ be objects in $\mathscr{C}(X/k)$ and let

$$\mathcal{N}|_{X_k} = \inf(\mathcal{N})|_{X_k} \xrightarrow{\theta} \inf(\mathcal{N}')|_{X_k} = \mathcal{N}'_{X_k}$$

be an arrow of $\mathscr{C}(X_k/k)$. Then there exists a morphism of $\mathscr{C}(X/R)$

$$\tilde{\theta} : \inf(\mathcal{N}) \longrightarrow \inf(\mathcal{N}')$$

lifting θ.

Verification. We have an arrow of $\mathscr{C}\,(X_k/k)$

$$\theta :\ \mathcal{N}|_{X_k} \longrightarrow \mathcal{N}'\Big|_{X_k}$$

which gives us an arrow of $\mathscr{C}\,(X_k/k)$

$$\sigma :\ \mathscr{O}_{X_k} \longrightarrow \mathcal{N}\Big|_{X_k}^{\vee} \otimes \mathcal{N}'\Big|_{X_k} =: \mathscr{E}|_{X_k}$$

We will show that σ is the restriction of an arrow $\mathscr{O}_X \to \mathscr{E}$ of $\mathscr{C}(X/R)$. Now, let

$$\iota : \mathrm{T} \longrightarrow \mathscr{E}|_{\mathrm{X_k}}$$

be the maximal trivial subobject; the arrow σ can therefore be written as a composition in $\mathscr{C}\,(X_k/\mathrm{k})$

(39) $$\mathscr{O}_{X_k} \xrightarrow{\tau} \mathrm{T} \xrightarrow{\iota} \mathscr{E}\Big|_{X_k}.$$

According to Lemma 3.3-(1), it is possible to find $\mathscr{T} \in \mathscr{C}(S/k)$ and a morphism of $\mathscr{C}(X/k)$

$$\tilde{\iota} : f^*(\mathscr{T}) \longrightarrow \mathscr{E}$$

such that ι is the restriction to X_k of $\tilde{\iota}$. As f is proper, flat and geometrically integral, we have $f_*\mathscr{O}_\mathrm{X} = \mathscr{O}_\mathrm{S}$ [SGA1, X, Proposition 1.2, p.202]; it then follows that the functor f^* from vector bundles on S to vector bundles on X is full and faithful. As S is affine, we conclude that the morphism of $\mathscr{O}_{X_k}$-modules

$$f^*\,(\mathscr{O}_S)\Big|X_k = \mathscr{O}_{X_k} \xrightarrow{\tau} \mathrm{T} = f^*(\mathscr{T})\Big|\,X_k$$

appearing in (39) is the restriction of a morphism

$$\tilde{\tau} = f^*(\delta) : f^*\,(\mathscr{O}_S) \longrightarrow f^*(\mathscr{T}).$$

Of course, δ need not be a morphism of $\mathscr{C}(S/k)$, but $f^*(\delta)$ is surely a morphism of $\mathscr{C}(X/R)$, that is, an arrow between inflations

$$\mathrm{Inf}(\mathscr{O}_X) \longrightarrow \mathrm{Inf}\big(f^*\mathscr{T}\big).$$

In conclusion, we have proved that σ is the restriction of $\tilde{\iota} \circ \tilde{\tau}$.

Claim 2. For each $\mathcal{V} \in \langle \inf(\mathscr{M})|_{X_k}\rangle_\otimes$, there exist $\mathscr{E}$ and $\mathscr{E}'$ in $\langle \inf(\mathscr{M})\rangle_\otimes^s$ and an arrow of $\mathscr{C}(X/R)$

$$\tilde{\theta} : \mathscr{E} \longrightarrow \mathscr{E}'$$

such that

$$\mathcal{V} \cong \mathrm{Coker}\big(\tilde{\theta}|_{X_k} : \mathscr{E}|_{X_k} \longrightarrow \mathscr{E}'|_{X_k}\big).$$

Verification. According to Lemma 3.3-(2) (applied to the dual of $\mathcal{V}$), we can find $\mathcal{N}$ and $\mathcal{N}'$ in $\langle \inf(\mathcal{M})\rangle_\otimes$ fitting into an exact sequence of $\mathscr{C}(X/k)$-modules:

$$\mathcal{N}|_{\mathrm{X_k}} \xrightarrow{\theta} \mathcal{N}'\Big|_{\mathrm{X_k}} \longrightarrow \mathrm{V} \longrightarrow 0.$$

Using Claim 1, θ is the restriction to X_k of an arrow in $\mathscr{C}(X/R)$

$$\tilde{\theta}: \mathrm{Inf_X}(\mathcal{N}) \longrightarrow \mathrm{Inf_X}\left(\mathcal{N}'\right).$$

We then take $\mathscr{E} = \mathrm{Inf}_X(\mathcal{N})$ and $\mathscr{E}' = \mathrm{Inf_X}\left(\mathcal{N}'\right)$, and the proof is finished since these do belong to $\langle \mathcal{M}\rangle^{\mathrm{s}}_\otimes$.

Claim 3. Denote by η the composition of functors

$$\mathrm{Rep}_R(G) \longrightarrow \mathrm{Rep}_R(G') \xrightarrow{\sim} \langle \inf(\mathcal{M})\rangle_\otimes.$$

For each $V \in \mathrm{Rep}_k(G_k)$, there exists $N \in \mathrm{Rep}^\circ_R(G)$ such that

(1) V is a quotient of N_k and
(2) there exists some $\mathcal{N} \in \langle \inf(\mathcal{M})\rangle_\otimes$ such that $\eta(N) = \inf(\mathcal{N})$.

Verification. According to [Se68, Proposition 3, p.41] we can "almost lift" V. Precisely, there exists $E \in \mathrm{Rep}^\circ_R(G)$ and a surjection $E_k \to V$. By means of the equivalence

$$\eta : \mathrm{Rep}^\circ_{\mathrm{R}}(\mathrm{G}) \xrightarrow{\sim} \langle \mathcal{M}\rangle^{\mathrm{s}}_\otimes$$

of (38), we can find a diagram in $\mathscr{C}(X/R)$:

$$\begin{array}{ccc} \mathscr{F} & \longrightarrow & \mathscr{T} \\ \downarrow & & \\ \eta(E), & & \end{array}$$

where $\mathscr{T}$ is some tensor power of Inf($\mathcal{M}$), the vertical arrow is an epimorphism (in $\mathscr{C}(X/R)$), and the horizontal arrow is special. In particular, $\mathscr{T} = \mathrm{Inf}(\mathscr{T}')$ for some tensor power $\mathscr{T}'$ of $\mathcal{M}$. According to Lemma 3.4, there exists $\mathcal{N} \in \mathscr{C}(X/k)$ and an epimorphism

$$\mathrm{Inf}(\mathcal{N}) \longrightarrow \mathscr{F}.$$

Since Inf($\mathcal{N}$) in fact belongs to $\langle \mathcal{M}\rangle^{\mathrm{s}}_\otimes$; the above equivalence then produces the desired N, viz. any object of $\mathrm{Rep}^\circ_R(G)$ which is taken by η to Inf($\mathcal{N}$). Indeed, (2) is verified by construction, and (1) follows from the fact that if $\eta(\theta) : \eta(N) \to \eta(E)$ is an epimorphism of $\mathscr{C}(X/R)$, then θ is an epimorphism in $\mathrm{Rep}_R(G)$ (between objects of $\mathrm{Rep}^\circ_R(G)$).

Claim 4. The functor

$$\eta_k : \mathrm{Rep}_k(G_k) = \mathrm{Rep}_R(G)_{(k)} \longrightarrow \langle \inf(\mathcal{M})\rangle_{\otimes,(k)}$$

is full.

Verification. Let $\varphi : \eta_k(\mathrm{V}) \to \eta_k\left(V'\right)$ be a morphism in $\mathscr{C}\,(X_k/k)$. It then fits into a commutative diagram

$$\begin{array}{ccc} \eta(N)\otimes k & \xrightarrow{\ \theta\ } & \eta(N')\otimes k \\ \downarrow & & \downarrow \\ \eta_k(V) & \xrightarrow[\ \phi\ ]{} & \eta_k(V'), \end{array}$$

where $\eta(N) = \mathrm{Inf}(\mathscr{N})$ and $\eta\left(N'\right) = \mathrm{Inf}\left(\mathscr{N}'\right)$ are constructed from Claim 3. Claim 1 gives us a lift

$$\tilde{\theta} : \eta(\mathrm{N}) \longrightarrow \eta\left(\mathrm{N}'\right)$$

of θ. Since $\mathscr{N}$ and $\mathscr{N}'$ belong to $\langle\mathscr{M}\rangle_{\otimes}$, both $\mathrm{Inf}(\mathscr{N})$ and $\mathrm{Inf}\left(\mathscr{N}'\right)$ lie in $\langle\mathscr{M}\rangle_{\otimes}^{s}$. Since η is an equivalence between $\mathrm{Rep}_R^{\circ}(G)$ and $\langle\mathscr{M}\rangle_{\otimes}^{s}$, there exists $\sigma : \mathrm{N} \to \mathrm{N}'$ such that $\eta(\sigma) = \tilde{\theta}$. Since the vertical arrows in the above diagram also belong to the image of η_k, the proof of the claim is finished. □

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