Noname manuscript No.
(will be inserted by the editor)

Nonuniform Berry-Esseen bound for self-normalized series

Nguyen Chi Dzung · **Pham Viet Hung**

Abstract In this paper, we shall obtain nonuniform Berry-Esseen bounds in the central limit theorem for self-normalized series. We establish the exponential Berry-Esseen bounds for the probability of the self-normalized series under the condition that the third moment is finite.

Keywords Nonuniform bound, Berry-Esseen inequality, Random power series, Self-normalized series

Mathematics Subject Classification (2010) 60E15 · 60F15

1 Introduction

Let $X_1, X_2, ...$ be a sequence of independent random variables with $\mathbb{E}X_i = 0$ and $0 < \mathbb{E}X_i^2 < \infty$ for $i \geq 1$. Let $b \in (0, 1)$ be the discount factor. The random power series S_b can be defined as

$$S_b = X_0 + bX_1 + b^2X_2 + ...$$

From the financial point of view, X_i stands for the (random) money that we will get at i-th year of a contract, for example a coupon bond and S_b is the present value of the cash flow. In the literature, S_b is also called the perpetuities (see [2], [9], [10]) for more detail.

The study of this quantity has drawn much of interest and it has a long history for more than 50 years. Let us mention some remarkable results for the simplest case that the random variables X_i's are independent, identically distribution (i.i.d). In 1971, Gerber [8] provided a Berry-Esseen bound for the following central limit theorem as $b \to 1^-$,

$$\sqrt{1-b^2}S_b \xrightarrow{d} \mathcal{N}(0, \sigma^2),$$

where $\sigma^2 = \mathbb{E}X_1^2$.

Nguyen Chi Dzung
Institute of Mathematics, Vietnam Academy of Science and Technology, 18 Hoang Quoc Viet, Hanoi, Vietnam
E-mail: nczung@gmail.com

Pham Viet Hung
Institute of Mathematics, Vietnam Academy of Science and Technology, 18 Hoang Quoc Viet, Hanoi, Vietnam
E-mail: pvhung@math.ac.vn

In 1974, Lai [13] proved the strong law of large number

$$\frac{X_b}{1-b} \xrightarrow{a.s} 0.$$

The law of iterated logarithm

$$\limsup_{b\to 1^{-1}} \sqrt{\frac{1-b^2}{\log\log(1/(1-b^2))}} S_b = \sqrt{2}\sigma$$

was proved by Gaposhkin [7] in 1965.

Recently, Iksanov consider a generalization with stochastic discount rates and provide the analogue versions of the above results.

In this paper, we are interested in the self-normalized series (denote by S_b/V_b)

$$\frac{X_0 + bX_1 + b^2X_2 + ...}{\sqrt{X_0^2 + b^2X_1^2 + b^4X_2^2 + ...}}$$

with $V_b^2 = X_0^2 + b^2X_1^2 + b^4X_2^2 + ...$

Self-normalized series can be seen as an extension of self-normalized sum, defined as

$$\frac{S_n}{V_n} = \frac{X_1 + ... + X_n}{\sqrt{X_1^2 + ... + X_n^2}},$$

where again the random variables X_i's are i.i.d with mean zero and finite variance.

The self-normalized sum is also an attractive research direction both in Probability and Statistics, (see [15, Self-normalized limit theorem: A survey, Probability Surveys]) for more detail.

From the distribution of S_n/V_n, one can make a suitable change of variable to deduce the distribution of the classical Student t- statistics and also the studentized t-statistics. This research direction has been studied extensively with many interesting and nice results: Nonuniform Berry-Esseen bound [12, Wang and Jing], Cramér type large (moderate) deviation [11, Jing, Shao and Wang], the law of iterated logarithm [11, Jing, Shao and Wang], Donsker type functional central limit theorem [3, Csörgő, Szyszkowicz and Wang]. It is also interesting to consider some questions for the self-normalized series model. In 2006, Fu and Huang [6] confirmed the self-normalized law of iterated logarithm.

The purpose of this paper is to establish a nonuniform Berry-Esseen bound for the self-normalized series S_b/V_b. In other words, we wish to obtain a bound for

$$\delta_b(x) := |\mathbb{P}(S_b/V_b \le x) - \Phi(x)|.$$

Our main result is the following theorem.

Theorem 1 *Let $X_1, X_2, ...$ be a sequence of independent, symmetric random variables with $\mathbb{E}\left(|X_j|^3\right) < \infty$ for all $j = 1, 2, ...$ Set $B_b^2 = \sum_{j=1}^{\infty} b^{2j}\mathbb{E}X_j^2, L_{3b} = B_b^{-3}\sum_{j=1}^{\infty} b^{3j}\mathbb{E}|X_j|^3$.*

(i) If $|x| \le (5L_{3b}^{1/3})^{-1}$, we have

$$\delta_b(x) \le A\left((1+x^2)L_{3b} + \sum_{j=1}^{\infty}\mathbb{P}(|b^jX_j| > B_b/(6|x|))\right)\exp\left(-\frac{x^2}{2}\right). \tag{1.1}$$

(ii) If $|x| > (5L_{3b}^{1/3})^{-1}$, we have

$$\delta_b(x) \le \left(1 + \frac{1}{\sqrt{2\pi}|x|}\right) \exp\left(-\frac{x^2}{2}\right). \tag{1.2}$$

Under the assumption $\mathbb{E}|X_j|^3 < \infty$, by applying Markov's inequality for $|x| \le (5L_{3b}^{1/3})^{-1}$, we obtain the following corollary.

Corollary 1 *Let $X_1, X_2, \dots$ be a sequence of independent, symmetric random variables with $\mathbb{E}\left(|X_j|^3\right) < \infty$, for all $j = 1, 2, \dots$ Then for all $x \in \mathbb{R}$, we have*

$$\delta_b(x) \le A \min\{(1 + |x|^3)L_{3b}, 1\} \exp\left(-\frac{x^2}{2}\right).$$

Before proving the main result, we need some the following technical lemmas.

2 Some technical lemmas

Lemma 1 *Let $X_1, X_2, \dots$ be a sequence of independent, symmetric random variables with $\mathbb{E}\left(|X_n|^3\right) < \infty$ for all $n = 1, 2, \dots$*

(i) For all $n \ge 1$ and $x > 0$ such that $(1 + x^3)L_{3b} \le \dfrac{1}{125}$, we have

$$\mathbb{P}\left(S_b > x(V_b^2 + B_b^2)/(2B_b)\right) = (1 - \Phi(x)) \exp(r_{1b}(x)) + \exp\left(\frac{-x^2}{2}\right) r_{2b}(x), \tag{2.1}$$

where $|r_{1b}(x)| \le 14x^3 L_{3b}$ and $|r_{2b}(x)| \le A(1 + x^2)L_{3b} \exp(14x^3 L_{3b})$.

(ii) For $n \ge 1$ and $x \ge 1$ satisfying $x^3 L_{3b} \le \dfrac{1}{125}$, we have

$$\mathbb{P}(S_b > x(V_b^2 + B_b^2)/(2B_b)) = (1 - \Phi(x))(1 + r_{3b}(x)), \tag{2.2}$$

where $|r_{3b}(x)| \le Ax(1 + x^2)L_{3b} \exp(14x^3 L_{3b})$.

Proof Set

$$h = \frac{x}{B_b}, \quad \eta_j = b^j X_j - \frac{h}{2}(b^{2j} X_j^2 - b^{2j}\sigma_j^2).$$

Then the left-hand side of (2.1) is equivalent to

$$\mathbb{P}(S_b > x(V_b^2 + B_b^2)/(2B_b)) = \mathbb{P}\left(\sum_{j=1}^{\infty} \eta_j > xB_b\right). \tag{2.3}$$

Next, we apply the conjugate method which was first introduced by Esscher [4] and improved by Feller [5]. Let $\xi_1, \xi_2, \dots$ be independent random variables with ξ_i having distribution function defined by

$$V_j(u) = \mathbb{E}(\exp(h\eta_j)\mathbf{1}(\eta_j \le u))/\mathbb{E}(\exp(h\eta_j)) \quad \text{for } j = 1, 2, \dots$$

We also define

$$M_b^2(h) = \sum_{j=1}^{\infty} \mathrm{Var}(\xi_j)$$

and

$$G_b(t) = \mathbb{P}\left(\frac{\sum_{j=1}^{\infty}(\xi_j - \mathbb{E}\xi_j)}{M_b(h)} \le t\right), \quad R_b(h) = \frac{xB_b - \sum_{j=1}^{\infty}\mathbb{E}\xi_j}{M_b(h)}.$$

By the well-known equation $\int_0^\infty \exp(-sx)\,\mathrm{d}\,\Phi(x) = \exp\left(-\frac{x^2}{2}\right)(1-\Phi(s))$ and using inverse Laplace transform, we have

$$\begin{aligned}\mathbb{P}\left(\sum_{j=1}^{\infty}\eta_j > xB_b\right) &= \left(\prod_{j=1}^{\infty}\mathbb{E}\exp(h\eta_j)\right)\int_{xB_b}^{\infty}\exp(-hu)\mathrm{d}\mathbb{P}(\sum_{j=1}^{\infty}\xi_j \le u)\\ &= \left(\prod_{j=1}^{\infty}\mathbb{E}\exp(h\eta_j)\right)\int_0^{\infty}\exp(-hxB_b - hM_b(h)v)\mathrm{d}G_b(v+R_b(h))\\ &= \left(\prod_{j=1}^{\infty}\mathbb{E}\exp(h\eta_j)\right)e^{-x^2}\left(\int_0^{\infty}\exp(-hM_b(h)v)\mathrm{d}(G_b(v+R_b(h)) - \Phi(v))\right.\\ &\qquad \left. + \int_0^{\infty}\exp(-hM_b(h)v)\mathrm{d}\Phi(v)\right)\\ &= I_0(h)\exp(-x^2)\left(\exp\left(\frac{x^2}{2}\right)(1-\Phi(x)) + I_1(h) + I_2(h) + I_3(h)\right),\end{aligned} \tag{2.4}$$

where

$I_0(h) = \prod_{j=1}^{\infty}\mathbb{E}\exp(h\eta_j)$,
$I_1(h) = \int_0^\infty \exp(-hM_b(h)v)\mathrm{d}(G_b(v+R_b(h)) - \Phi(v+R_b(h)))$,
$I_2(h) = \int_0^\infty \exp(-hM_b(h)v)\mathrm{d}(\Phi(v+R_b(h)) - \Phi(v))$,
$I_3(h) = \int_0^\infty \exp(-hM_b(h)v - \exp(-xv))\mathrm{d}\Phi(v)$.

We will establish some inequalities before estimating $I_j(h)$ for $j = 1, 2, 3$.
It follows from Jensen's inequality that $\sigma_j^3 \le \mathbb{E}|X_j|^3$.

Combining this and the assumption $(1+x^3)L_{3b} \le \frac{1}{125}$, we have

$$b^j\sigma_j h = \frac{b^j\sigma_j x}{B_b} \le \left(x^3 B_b^{-3} b^{3j}\mathbb{E}|X_j|^3\right)^{1/3} \le \frac{1}{5}. \tag{2.5}$$

Thus

$$h\eta_j = -\frac{1}{2}h^2(b^jX_j - h^{-1})^2 + \frac{1}{2} + \frac{1}{2}b^{2j}\sigma_j^2h^2 \le \frac{13}{25}. \tag{2.6}$$

From (2.5), the symmetry assumption and $\mathbb{E}|X_j|^3 < \infty$, we have

$$\begin{aligned}
&|\mathbb{E}(\eta_j \mathbf{1}(|b^j X_j| \le h^{-1}))| = \left|\mathbb{E}\left(\left(b^j X_j - \frac{h}{2}(b^{2j}X_j^2 - b^{2j}\sigma_j^2)\right)\mathbf{1}(|b^j X_j| > h^{-1})\right)\right| \\
&\le \mathbb{E}(|b^j X_j|\mathbf{1}(|b^j X_j| > h^{-1})) + \frac{h}{2}\mathbb{E}((b^j X_j)^2\mathbf{1}(|b^j X_j| > h^{-1})) \\
&\quad + \frac{h}{2}\mathbb{E}((b^j \sigma_j)^2\mathbf{1}(|b^j X_j| > h^{-1})) \\
&\le h^2\mathbb{E}(|b^j X_j|^3\mathbf{1}(|b^j X_j| > h^{-1})) + \frac{h^2}{2}\mathbb{E}(|b^j X_j|^3\mathbf{1}(|b^j X_j| > h^{-1})) \\
&\quad + \frac{h^2}{2}\mathbb{E}(|b^j \sigma_j|^3 b^{2j}\sigma_j^2 h^2\mathbf{1}(|b^j X_j| > h^{-1})) \\
&\le 2h^2\mathbb{E}(|b^j X_j|^3\mathbf{1}(|b^j X_j| > h^{-1})).
\end{aligned} \tag{2.7}$$

Similarly, we also have

$$|\mathbb{E}(\eta_j^2\mathbf{1}(|b^j X_j| \le h^{-1}) - b^{2j}\sigma_j^2)| \le \frac{3}{2}h\left(\mathbb{E}|b^j X_j|^3 + hb^{4j}\sigma_j^4\right), \tag{2.8}$$

$$\mathbb{E}(|\eta_j|^3\mathbf{1}(|b^j X_j| \le h^{-1})) \le 6\mathbb{E}\left(|b^j X_j|^3\mathbf{1}(|b^j X_j| \le h^{-1})\right) + 2h^3 b^{6j}\sigma_j^6. \tag{2.9}$$

We have

$$\begin{aligned}
\mathbb{E}\exp(n\eta_j) &= \mathbb{E}\left(\exp(h\eta_j)\mathbf{1}(|b^j X_j| \le h^{-1})\right) + \mathbb{E}\left(\exp(h\eta_j)\mathbf{1}(|b^j X_j| > h^{-1})\right) \\
&= \mathbb{E}\left(\left(1 + h\eta_j + \frac{1}{2}(h\eta_j)^2\right)\mathbf{1}(|b^j X_j| \le h^{-1})\right) + \mathbb{E}\left(\exp(h\eta_j)\mathbf{1}(|b^j X_j| > h^{-1})\right) \\
&+ \mathbb{E}\left(\left(\exp(h\eta_j) - 1 - h\eta_j - \frac{1}{2}(h\eta_j)^2\right)\mathbf{1}(|b^j X_j| \le h^{-1})\right) \\
&= 1 + \frac{1}{2}h^2 b^{2j}\sigma_j^2 + l_{1j}(h) \\
&= \exp\left(\frac{1}{2}h^2 b^{2j}\sigma_j^2 + l_{2j}(h)\right),
\end{aligned} \tag{2.10}$$

$$(\mathbb{E}\exp(h\eta_j))^{-1} = 1 - \frac{1}{2}h^2 b^{2j}\sigma_j^2 + l_{3j}(h), \tag{2.11}$$

where

$$\begin{aligned}
l_{1j}(h) &= -\mathbb{P}(|b^j X_j| > h^{-1}) + h\mathbb{E}(\eta_j\mathbf{1}(|b^j X_j| \le h^{-1})) + \frac{1}{2}h^2\mathbb{E}(\eta_j^2\mathbf{1}(|b^j X_j| \le h^{-1}) - b^{2j}\sigma_j^2) \\
&+ \mathbb{E}(\exp(h\eta_j)\mathbf{1}(|b^j X_j| > h^{-1})) + \mathbb{E}\left((\exp(h\eta_j) - 1 - h\eta_j - \frac{1}{2}(h\eta_n)^2)\mathbf{1}(|b^j X_j| \le h^{-1})\right).
\end{aligned}$$

Applying the elementary inequality $\left|e^x - 1 - x - \frac{x^2}{2}\right| \le \frac{|x|^3 e^{|x|}}{6}$ for all $x \in \mathbb{R}$, and noting that $\exp(xh\eta_j \le 2)$ for $0 \le x \le 1$, we get

$$\begin{aligned}|l_{1j}(h)| &\le h|\mathbb{E}\left(\eta_j \mathbf{1}(|b^j X_j| \le h^{-1})\right) + \frac{1}{2}h^2|\mathbb{E}(\eta_j^2 \mathbf{1}(|b^j X_j| \le h^{-1}) - b^{2j}\sigma_j^2)| \\ &+ \frac{1}{3}h^3 \mathbb{E}\left(|\eta_j|^3 \mathbf{1}(|b^j X_j| \le h^{-1})\right) + 3\mathbb{P}(|b^j X_j| > h^{-1}) \\ &\le 2h^3 \mathbb{E}\left(|\eta_j|^3 \mathbf{1}(|b^j X_j| > h^{-1})\right) + \frac{3}{4}h^3\left(\mathbb{E}|b^j X_j|^3 + hb^{4j}\sigma_j^4\right) \\ &+ \frac{1}{3}h^3 6\mathbb{E}\left(|\eta_j|^3 \mathbf{1}(|b^j X_j| \le h^{-1})\right) + 2h^3 b^{6j}\sigma_j^6 \\ &\le 7h^3 \mathbb{E}|b^j X_j|^3,\end{aligned}$$

$$|l_{2j}(h)| \le 2|l_{1j}(h)| \le 14h^3 \mathbb{E}|b^j X_j|^3,$$
$$|l_{3j}(h)| \le 2|l_{1j}(h)| \le 14h^3 \mathbb{E}|b^j X_j|^3.$$

It is proved by Wang and Jing [12] that

$$|\mathbb{E}(\eta_j \exp(h\eta_j)) - hb^{2j}\sigma_j^2| \le 16h^2 \mathbb{E}|b^j X_j|^3, \tag{2.12}$$

$$|\mathbb{E}(\eta_j^2 \exp(h\eta_j)) - b^{2j}\sigma_j^2| \le 30h \mathbb{E}|b^j X_j|^3, \tag{2.13}$$

$$\mathbb{E}(|\eta_j|^3 \exp(h\eta_j)) \le 30\mathbb{E}|b^j X_j|^3. \tag{2.14}$$

It follows from (2.5)–(2.14) that

$$\mathbb{E}\xi_j = \frac{\mathbb{E}(\eta_j \exp(h\eta_j))}{\mathbb{E}(\exp(h\eta_j))} = hb^{2j}\sigma_j^2 + l_{4j}(h), \tag{2.15}$$

where

$$l_{4j}(h) = \left(\frac{1}{\mathbb{E}(\exp(h\eta_j))} - 1\right)\mathbb{E}(\eta_j \exp(h\eta_j)) + \mathbb{E}(\eta_j \exp(h\eta_j)) - hb^{2j}\sigma_j^2.$$

Thus, by (2.5),(2.11), (2.12), we get

$$\begin{aligned}|l_{4j}(h)| &\le \left|\left(\frac{1}{\mathbb{E}(\exp(h\eta_j))} - 1\right)\mathbb{E}(\eta_j \exp(h\eta_j))\right| + |\mathbb{E}(\eta_j \exp(h\eta_j)) - hb^{2j}\sigma_j^2| \\ &\le |(l_{3j}(h) - \frac{1}{2}h^2 b^{2j}\sigma_j^2)(hb^{2j}\sigma_j^2 + 16h^2 b^{3j}\mathbb{E}|X_j|^3)| + 16h^2 b^{3j}\mathbb{E}|X_j|^3 \\ &\le 22h^4 b^{5j}\sigma_j^2 \mathbb{E}|X_j|^3 + 224h^5 b^{6j}(\mathbb{E}|X_j|^3)^2 + 16h^2 b^{3j}\mathbb{E}|X_j|^3 + \frac{1}{2}h^3 b^{4j}\sigma_j^4 \\ &\le 20h^2 b^{3j}\mathbb{E}|X_j|^3.\end{aligned}$$

Similarly, we also have

$$\mathrm{Var}(\xi_j) = \frac{\mathbb{E}(\eta_j^2 \exp(h\eta_j))}{(\mathbb{E}\exp(h\eta_j))^2} - (\mathbb{E}\xi_j)^2 = b^{2j}\sigma_j^2 + l_{5j}(h), \tag{2.16}$$

and

$$\mathbb{E}|\xi_j|^3 = \mathbb{E}(|\eta|^3 \exp(h\eta_j))/\mathbb{E}\exp(h\eta_j) \le 34b^{3j}\mathbb{E}|X_j|^3, \tag{2.17}$$

where $|l_{5j}(h)| \le 41hb^{3j}\mathbb{E}|X_j|^3$.

It follows $|l_{5j}(h)| \le 41hb^{3j}\mathbb{E}|X_j|^3$ and the assumption $(1+x^3)L_{3b} \le 1/125$, it is easy to obtain that

$$M_b^2(h) = B_b^2 + \sum_{j=1}^{\infty} l_{5j}(h) > \frac{2}{3}B_b^2. \tag{2.18}$$

We are now estimate $I_j(h), 0 \le j \le 3$. For $I_0(h)$, using (2.10) we have

$$I_0(h) = \exp\left(\frac{1}{2}h^2B_b^2 + \sum_{j=1}^{\infty} l_{2j}(h)\right) = \exp\left(\frac{x^2}{2}\right)\exp\left(\sum_{j=1}^{\infty} l_{2j}(h)\right). \tag{2.19}$$

By (2.15)–(2.19), the Berry-Esseen bound and Taylor expansion, we have

$$I_1(h) \le \sup_x |G_n(v) - \Phi(v)| \le \frac{A}{M_b^3(h)}\sum_{j=1}^{\infty}\mathbb{E}|\xi_j - \mathbb{E}\xi_j|^3 \le AL_{3b}, \tag{2.20}$$

$$I_2(h) \le \sup_x |\Phi(v + R_n(h)) - \Phi(v)| \le \frac{A}{M_b(h)}\sum_{j=1}^{\infty}|l_{4j}(h)| \le Ax^2L_{3b}. \tag{2.21}$$

By applying the mean value estimate to $I_3(h)$ [see Petrov [14], page 227], we have

$$\begin{aligned} I_3(h) &\le \frac{1}{x}\left|\frac{M_b(h)}{B_b} - 1\right| \max\left\{1, \frac{B_b^2}{M_b^2}\right\} \\ &\le \frac{3}{2x}\left|\frac{M_b^2(h) - B_n^2}{B_b(M_b(h) + B_n)}\right| \\ &\le AL_{3b}. \end{aligned} \tag{2.22}$$

Combining (2.3), (2.4) and (2.19)–(2.22), we get

$$\begin{aligned} \mathbb{P}(S_b > x(V_b^2 + B_b^2)/(2B_b)) &= \exp\left(-\frac{x^2}{2}\right)\exp(r_{1b}(x))\left(\exp\left(\frac{x^2}{2}\right)(1 - \Phi(x)) + I_1(h) + I_2(h) + I_3(h)\right) \\ &= (1 - \Phi(x))\exp(r_{1b}(x)) + \exp\left(-\frac{x^2}{2}\right) r_{2b}(x), \end{aligned}$$

where $r_{1b}(x) = \sum_{j=1}^{\infty} l_{2j}(h)$ and $r_{2b}(x) = \exp(r_{1b}(x))(I_1(h) + I_2(h) + I_3(h))$.

Thus

$$|r_{1b}(x)| = \sum_{j=1}^{\infty}|l_{2j}(h)| \le 14x^3L_{3b},$$

and

$$|r_{2b}(x)| = \exp(|r_{1b}(x)|)(|I_1(h)| + |I_2(h)| + |I_3(h)|) \le A(1+x^2)L_{3b}\exp(14x^3\text{Ł}_{3b}).$$

We thus have completed the proof of Lemma 2.1(i).

(ii) By the same proof as part (i), we also see that in the case $n \ge 1$ and $x \ge 1$ satisfying $x^3L_{3b} \le 1/125$, (2.1) holds. Hence, the proof of (2.2) obtains by using the inequalities $e^t \le 1 + te^t$ and $1 - \Phi(t) \le \frac{1}{\sqrt{2\pi}t}\exp\left(-\frac{t^2}{2}\right)$ for $t > 0$.

Lemma 2 *Let $\varepsilon_1, \varepsilon_2, \ldots$ be i.i.d. Rademacher random variables, that is, $\mathbb{P}(\varepsilon_j = \pm 1) = 1/2$. Then for any $x \geq 1$ and any sequence $a_1, a_2, \ldots$ satisfying $|a_j| \leq B_b/(6x)$ and $\sum_{n=1}^{\infty} a_j^2 > \frac{4}{9}B_b^2$, we have*

$$\mathbb{P}\left(\sum_{j=1}^{\infty} a_j\varepsilon_j > x\left(\sum_{j=1}^{\infty} a_j^2\right)^{1/2}\right) \leq (1-\Phi(x))\left(1 + Ax(1+x^2)L_{3b}^* \exp(2x^3 L_{3b}^*)\right), \tag{2.23}$$

where $L_{3b}^* = B_b^{-3}\sum_{j=1}^{\infty}|a_j|^3$.

The proof of Lemma 2 follows very similar lines to those of Lemma 1, so we omit it.

Lemma 3 *Let $\{a_i, i \geq 1\}$ be any sequence of real numbers. Set $A_\infty^2 = \sum_{i=1}^{\infty} a_i^2$ and $T_\infty = \sum_{i=1}^{\infty}\varepsilon_i a_i$. Then, for all $x > 0$*

$$\mathbb{P}(T_\infty > xA_\infty) \leq \exp\left(-\frac{x^2}{2}\right).$$

where $\{\varepsilon_i, i \geq 1\}$ is i.i.d Rademacher random variables.

Proof For any $u > 0$, by Markov's inequality, we have

$$\begin{aligned}\mathbb{P}(T_\infty > xA_\infty) &\leq \exp(-uxA_\infty)\mathbb{E}\exp(uT_\infty) = \exp(-uxA_\infty)\prod_{i=1}^{\infty}\cosh(ua_i)\\ &\leq \exp(-uxA_\infty)\prod_{i=1}^{\infty}\exp\left(\frac{u^2a_i^2}{2}\right) = \exp(-uxA_\infty + u^2A_\infty^2/2).\end{aligned}$$

The proof of Lemma 3 is complete by choosing $u = -xA_\infty^2$.

Lemma 4 *Let $X_1, X_2, \ldots$ be independent, symmetric random variables. Then for any $x \geq 0$ and $n \geq 1$, we have*

$$\mathbb{P}(S_b > xV_b) \leq \exp\left(-\frac{x^2}{2}\right).$$

Proof Similarly, in [12, Lemma 43], we assume that $\{X_j, j \geq 1\}$ are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which also supports a sequence of independent Rademacher random variables $\{\varepsilon_j, j \geq 1\}$ independent of $\{X_j, j \geq 1\}$. By the symmetry of X_j and independence X_j and ε_j, we have that

$$\begin{aligned}\mathbb{P}(S_b > xV_b) &= \mathbb{P}\left(\sum_{j=1}^{\infty} X_j\varepsilon_j > xV_b\right)\\ &= \int\int \ldots \mathbb{P}\left(\sum_{j=1}^{\infty} x_j\varepsilon_j > x\left(\sum_{j=1}^{\infty} x_j^2\right)^{1/2}\right) \mathrm{d}F_1(x_1)\mathrm{d}F_2(x_2)\ldots\\ &\leq \exp\left(-\frac{x^2}{2}\right) \text{(by Lemma 3)}.\end{aligned}$$

This ends the proof of the lemma.

Lemma 5 *Let $X_1, X_2, ...$ be independent random variables. Then for $x \geq 1, y \geq 0$ and $k \geq 1$, we have*

$$\mathbb{P}(S_b > xV_b, |b^k X_k| > y) \leq \mathbb{P}(|b^k X_k| > y)\mathbb{P}\left(\sum_{j\neq k, j=1}^{\infty} b^j X_j > (x^2-1)^{1/2}\left(\sum_{j\neq k, j=1}^{\infty} b^{2j}X_j^2\right)^{1/2}\right).$$

Proof We observe that, for any real number a, then

$$\begin{aligned} ab^k X_k - \frac{x}{2}(b^{2k}X_k^2 + a^2) &= -\frac{x}{2}\left(b^k X_k - \frac{a}{x}\right)^2 - \frac{xa^2}{2} + \frac{a^2}{2x} \\ &\leq \frac{a^2}{2}\left(\frac{1}{x} - x\right). \end{aligned}$$

Thus, we get that

$$\begin{aligned} \mathbb{P}(S_b > xV_b, |b^k X_k| > y) &= \mathbb{P}\left(S_b > \inf_{a>0}\frac{x}{2a}(V_b^2 + a^2), |b^k X_k| > y\right) \\ &= \mathbb{P}\left(\sup_{a>0}\left(\sum_{j=1}^{\infty}(ab^j X_j - \frac{x}{2}b^{2j}X_j^2) - \frac{x}{2}a^2\right) > 0, |b^k X_k| > y\right) \\ &= \mathbb{P}\left(\sup_{a>0}\left(\sum_{j\neq k, j=1}^{\infty}(ab^j X_j - \frac{x}{2}b^{2j}X_j^2) + ab^k X_k - \frac{x}{2}b^{2k}X_k^2 - \frac{x}{2}a^2\right) > 0, |b^k X_k| > y\right) \\ &\leq \mathbb{P}\left(\sup_{a>0}\left(\sum_{j\neq k, j=1}^{\infty}(ab^j X_j - \frac{x}{2}b^{2j}X_j^2) + \frac{a^2}{2}(\frac{1}{x} - x)\right) > 0, |b^k X_k| > y\right) \\ &= \mathbb{P}\left(\sum_{j\neq k, j=1}^{\infty} b^j X_j > \inf_{a>0}\frac{x}{2a}\left(\sum_{j\neq k, j=1}^{\infty} b^{2j}X_j^2 + a^2(1 - \frac{1}{x^2})\right), |b^k X_k| > y\right) \\ &= \mathbb{P}\left(\sum_{j\neq k, j=1}^{\infty} b^j X_j > (x^2-1)^{1/2}\left(\sum_{j=1, j\neq k}^{\infty} b^{2j}X_j^2\right)^{1/2}, |b^k X_k| > y\right) \\ &= \mathbb{P}(|b^k X_k| > y)\mathbb{P}\left(\sum_{j\neq k, j=1}^{\infty} b^j X_j > (x^2-1)^{1/2}\left(\sum_{j\neq k, j=1}^{\infty} b^{2j}X_j^2\right)^{1/2}\right). \end{aligned} \tag{2.24}$$

The lemma is proved.

We are now ready to prove Theorem 1. To bound $\delta_b(x)$, it suffices to consider $x > 0$ since we can simply apply result to $-X_j$ when $x < 0$. For $0 < x \leq 1$, (1.1) was proved by Bentkus, Bloznelis and Götze [1]. We consider the case where $1 \leq x \leq (5L_{3b}^{1/3})^{-1}$. By applying the elementary inequality $2B_bV_b \leq B_b^2 + V_b^2$ and Lemma 1(ii), we have

$$\begin{aligned} \mathbb{P}(S_b > xV_b) &\geq \mathbb{P}(2B_bS_b > x(V_b^2 + S_b^2)) \\ &\geq (1 - \Phi(x))(1 - Ax(1+x^2)L_{3b}). \end{aligned} \tag{2.25}$$

Combining this and the inequality $1-\Phi(x) \le \frac{1}{\sqrt{2\pi}x}\exp\left(-\frac{x^2}{2}\right)$ for $x \ge 0$ implies that

$$\mathbb{P}(S_b \le xV_b) - \Phi(x) \le (1-\Phi(x))Ax(1+x^2)L_{3b} \le A(1+x^2)L_{3b}\exp\left(-\frac{x^2}{2}\right).$$

Hence, to prove (1.1), it suffices to show that

$$\mathbb{P}(S_b > xV_b) \le (1-\Phi(x)) + A\left((1+x^2)L_{3b} + \sum_{j=1}^{\infty}\mathbb{P}(|b^jX_j| > B_b/(6x))\exp\left(-\frac{x^2}{2}\right)\right). \tag{2.26}$$

Set

$$Y_j = b^jX_j\mathbf{1}(|b^jX_j| \le B_b/(6x)),\quad S_b^* = \sum_{j=1}^{\infty}Y_j,\quad V_b^{*2} = \sum_{j=1}^{\infty}Y_j^2.$$

For $x \ge 1$, it follows from Lemma 4 and Lemma 5 that

$$\begin{aligned}
\mathbb{P}(S_b > xV_b) &\le \sum_{k=1}^{\infty}\mathbb{P}\left(S_b > xV_b, |b^kX_k| > \frac{B_b}{6x}\right) + \mathbb{P}(S_b^* > xV_b^*)\\
&\le \sum_{k=1}^{\infty}\mathbb{P}\left(|b^kX_k| > \frac{B_b}{6x}\right)\mathbb{P}\left(\sum_{j\neq k, j=1}^{\infty} b^jX_j > (x^2-1)^{1/2}\left(\sum_{j\neq k,j=1}^{\infty} b^{2j}X_j^2\right)^{1/2}\right) + \mathbb{P}(S_b^* > xV_b^*)\\
&\le \sum_{k=1}^{\infty}\mathbb{P}\left(|b^kX_k| > \frac{B_b}{6x}\right)\exp\left(-\frac{x^2-1}{2}\right) + \mathbb{P}(S_b^* > xV_b^*)\\
&\le e\sum_{k=1}^{\infty}\mathbb{P}\left(|b^kX_k| > \frac{B_b}{6x}\right)\exp\left(-\frac{x^2}{2}\right) + \mathbb{P}(S_b^* > xV_b^*).
\end{aligned} \tag{2.27}$$

Denote $F_j(x)$ to be distribution function of X_j for $j \ge 1$. Using the assumption $|x| \le (5L_{3b}^{1/3})^{-1}$, the inequalities $e^t \le 1 + te^t$ and $e^t \ge 1+t$ for $t \ge 0$, we have for $i \ge 1$,

$$\begin{aligned}
\prod_{j\neq i,j=1}^{\infty}\mathbb{E}(\exp(2x^3B_b^{-3}|Y_j|^3)) &\le \prod_{j\neq i,j=1}^{\infty}\left(1 + 2x^3B_b^{-3}\mathbb{E}|Y_j|^3e^{2/125}\right)\\
&\le \prod_{j\neq i,j=1}^{\infty}\exp(4x^3B_b^{-3}\mathbb{E}|Y_j|^3)\\
&\le \exp(4x^3L_{3b})\\
&\le 2.
\end{aligned} \tag{2.28}$$

We will now bound $\mathbb{P}(S_b^* > xV_b^*)$. Similarly in the proof of Lemma 3, we assume that $\{Y_j, j \ge 1\}$ are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which also supports a sequence of independent Rademacher random variables $\{\varepsilon_j, j \geq 1\}$ independent of the sequence $\{Y_j, j \geq 1\}$. By symmetry of X_j, we have that

$$\begin{aligned}\mathbb{P}(S_b^* > xV_b^*) &= \mathbb{P}\left(\sum_{j=1}^{\infty} Y_j\varepsilon_j > xV_b^*\right) \\ &\leq \mathbb{P}\left(\sum_{j=1}^{\infty} Y_j\varepsilon_j > xV_b^*, V_b^{*2} > \frac{4}{9}B_b^2\right) + \mathbb{P}\left(V_b^{*2} \leq \frac{4}{9}B_b^2\right).\end{aligned} \tag{2.29}$$

It follows from (2.28) and Lemma 2 that

$$\begin{aligned}&\mathbb{P}\left(\sum_{j=1}^{\infty} Y_j\varepsilon_j > xV_b^*, V_b^{*2} > \frac{4}{9}B_b^2\right) \\ &= \int\int \cdots_{\sum_{j=1}^{\infty} y_j^2 > \frac{4}{9}B_b^2, |y_j| \leq B_b/(6x), j=1,2,..} \mathbb{P}\left(\sum_{j=1}^{\infty} y_j\varepsilon_j > x\left(\sum_{j=1}^{\infty} y_j^2\right)^{1/2}\right) \mathrm{d}F_1(y_1)\mathrm{d}F_2(y_2)... \\ &\leq (1-\Phi(x)) \int\int \cdots_{|y_j| \leq B_b/(6x), j=1,2,..} \\ &\times \left(1 + Ax(1+x^2)B_b^{-3}\sum_{i=1}^{\infty}|y_j|^3 \exp(2x^3B_b^{-3}\sum_{j=1}^{\infty}|y_j|^3)\right) \mathrm{d}F_1(y_1)\mathrm{d}F_2(y_2)... \\ &\leq (1-\Phi(x))\left(1 + Ax(1+x^2)B_b^{-3}\sum_{i=1}^{\infty}\mathbb{E}\left(|Y_j|^3\exp(2x^3B_b^{-3}\sum_{j=1}^{\infty}|Y_j|^3)\right)\right) \\ &\leq (1-\Phi(x))\left(1 + Ax(1+x^2)B_b^{-3}\sum_{i=1}^{\infty}\left(\mathbb{E}|Y_j|^3\left(\prod_{j\neq i, j=1}^{\infty}\mathbb{E}\exp(2x^3B_b^{-3}\sum_{j=1}^{\infty}|Y_j|^3)\right)\right)\right) \\ &\leq (1-\Phi(x)) + A(1+x^2)L_{3b}\exp\left(\frac{-x^2}{2}\right).\end{aligned} \tag{2.30}$$

We will next bound $\mathbb{P}\left(V_b^{*2} \leq \frac{4}{9}B_b^2\right)$. Firstly, we note that

$$\sum_{j=1}^{\infty}\mathbb{E}\left(b^{2j}X_j^2\mathbf{1}\left(|b^jX_j| > \frac{B_b}{6x}\right)\right) \leq \frac{6x}{B_b}\sum_{j=1}^{\infty}\mathbb{E}|b^jX_j|^3 \leq \frac{6}{125}B_b^2. \tag{2.31}$$

$$\sum_{j=1}^{\infty}\mathbb{E}Y_j^2 = B_b^2 - \sum_{j=1}^{\infty}\mathbb{E}(b^{2j}X_j^2\mathbf{1}(|b^jX_j| > B_b/(6x))). \tag{2.32}$$

Then for any $t > 0$, by using (2.31), (2.32), the inequalities $1 + |x| \le e^{|x|}$, $e^x \le 1 + x + 1/2x^2 e^{|x|}$, $\mathrm{Var}(Y_j^2) \le \mathbb{E}Y_j^4 \le b^{3j} E|X_j|^3 B_b/(6x)$ and Markov's inequality, we have

$$\begin{aligned}
\mathbb{P}\left(V_b^{*2} \le \frac{4}{9}B_b^2\right) &= \mathbb{P}\left(\sum_{j=1}^{\infty}(\mathbb{E}Y_j^2 - Y_j^2) > \frac{5}{9}B_b^2 - \sum_{j=1}^{\infty}\mathbb{E}(b^{2j}X_j^2\mathbf{1}(|b^jX_j| > B_b/(6x)))\right) \\
&\le \mathbb{P}\left(\sum_{j=1}^{\infty}(\mathbb{E}Y_j^2 - Y_j^2) > \frac{1}{2}B_b^2\right) \\
&\le e^{-t/2}\prod_{j=1}^{\infty}\mathbb{E}\exp(tB_b^{-2}(\mathbb{E}Y_j^2 - Y_j^2)) \\
&\le e^{-t/2}\prod_{j=1}^{\infty}\left(1 + \frac{1}{2}t^2B_b^{-4}\,\mathrm{Var}(Y_j^2)\exp\left(\frac{tx^{-2}}{36}\right)\right) \\
&\le e^{-t/2}\prod_{j=1}^{\infty}\exp\left(\frac{t^2b^{3j}\mathbb{E}|X_j|^3}{6xB_b^3}\left(\frac{tx^{-2}}{36}\right)\right) \\
&\le e^{-t/2}\prod_{j=1}^{\infty}\exp\left(\frac{t^2}{6x}L_{3b}\left(\frac{tx^{-2}}{36}\right)\right).
\end{aligned}$$

Choosing $t = 4x^2(1 + x^{-2}\log L_{3b}^{-1/2})$, we obtain

$$\mathbb{P}\left(V_b^{*2} \le \frac{4}{9}B_b^2\right) \le AL_{3b}\exp\left(\frac{-x^2}{2}\right). \tag{2.33}$$

Combining (2.29), (2.30) and (2.33), we get

$$\mathbb{P}(S_b^* > xV_b^*) \le (1 - \Phi(x)) + A(1 + x^2)L_{3b}\exp\left(\frac{-x^2}{2}\right). \tag{2.34}$$

Then (2.26) implies from (2.27) and (2.34). The proof of Theorem 1.1 is thus completed.

(ii) The proof of (1.2) implies from the inequality $1 - \Phi(x) \le \frac{1}{\sqrt{2\pi}x}\exp\left(-\frac{x^2}{2}\right)$ for $x > 0$ and Lemma 4.

Acknowledgments The research is supported under grant no. ICRTM03-2020.04.

References

1. Bentkus, V., Bloznelis, M., Götze, F.: A berry-esséen bound for student's statistic in the non-iid case. Journal of Theoretical Probability **9**(3), 765–796 (1996)
2. Buraczewski, D., Damek, E., Mikosch, T., et al.: Stochastic models with power-law tails. The equation X= AX+ B. Cham: Springer (2016)
3. Csörgő, M., Szyszkowicz, B., Wu, Q.: Donsker's theorem for self-normalized partial sums processes. The Annals of Probability **31**(3), 1228–1240 (2003)
4. Escher, F.: On the probability function in the collective theory of risk. Skand. Aktuarie Tidskr. **15**, 175–195 (1932)
5. Feller, W.: An introduction to probability theory and its applications. 1, 2nd

6. Fu, K.A., Huang, W.: A self-normalized law of the iterated logarithm for the geometrically weighted random series. Acta Mathematica Sinica, English Series **32**(3), 384–392 (2016)
7. Gaposhkin, V.F.: The law of the iterated logarithm for cesaros and abels methods of summation. Theory of Probability & Its Applications **10**(3), 411–420 (1965)
8. Gerber, H.U.: The discounted central limit theorem and its berry-esseen analogue. The Annals of Mathematical Statistics pp. 389–392 (1971)
9. Goldie, C.M., Maller, R.A.: Stability of perpetuities. The Annals of Probability **28**(3), 1195–1218 (2000)
10. Iksanov, A.: Renewal theory for perturbed random walks and similar processes. Springer (2016)
11. Jing, B.Y., Shao, Q.M., Wang, Q.: Self-normalized cramér-type large deviations for independent random variables. The Annals of probability **31**(4), 2167–2215 (2003)
12. Jing, B.Y., Wang, Q.: An exponential nonuniform berry-esseen bound for self-normalized sums. The Annals of Probability **27**(4), 2068–2088 (1999)
13. Lai, T.L.: Summability methods for independent identically distributed random variables. Proceedings of the American Mathematical Society **45**(2), 253–261 (1974)
14. Petrov, V.V.: Sums of independent random variables. In: Sums of Independent Random Variables. De Gruyter (2022)
15. Shao, Q.M., Wang, Q.: Self-normalized limit theorems: A survey. Probability Surveys **10**, 69–93 (2013)