A characterization of delay independent stability for linear off-diagonal delay difference equations

Duc Thuan Do[1], Thai Son Doan[2], Viet Cuong Le[3]

Abstract

In this paper, we consider linear off-diagonal difference equations of the form

$$x_i(k+1) = \sum_{j=1}^{n} a_{ij} x_j(k-\tau_{ij}), \qquad \text{for } i = 1, \dots, n, \tag{1}$$

where $A = (a_{ij})_{1\le i,j\le n} \in \mathbb{R}^{n\times n}$ and $\tau = (\tau_{ij})_{1\le i,j\le n}$ is the delay. Our main result is to establish an explicit characterization in terms of A for delay independent stability of (1). As an application, we establish a criterion for delay independent stability for an equilibrium of a discrete-time Lotka-Voltera equation.

Keywords: Delay Difference Equations, Stability, Lotka-Voltera equations

1. Introduction

Consider linear off-diagonal delay difference equations in $\mathbb{R}^n$ with constant coefficients of the form

$$x_i(k+1) = \sum_{j=1}^{n} a_{ij} x_j(k-\tau_{ij}), \qquad \text{for } i = 1, \dots, n, \tag{2}$$

where $A = (a_{ij})_{1\le i,j\le n} \in \mathbb{R}^{n\times n}$ and $\tau = (\tau_{ij})_{1\le i,j\le n}$ is the delay time satisfying that

$$\tau_{ij} \in \mathbb{Z}_{\ge 0} \text{ for } 1 \le i \ne j \le n \text{ and } \tau_{ii} = 0 \text{ for } i = 1, \dots, n. \tag{3}$$

Email addresses: thuan.doduc@hust.edu.vn, ducthuank7@gmail.com (Duc Thuan Do), dtson@math.ac.vn (Thai Son Doan), cuonglv2@huce.edu.vn (Viet Cuong Le)

[1]School of Applied Mathematics and Informatics, Hanoi University of Science and Technology, 01 Dai Co Viet str., Hai Ba Trung District, Hanoi, Vietnam

[2]Institute of Mathematics, Vietnam Academy of Science and Technology, 18 Hoang Quoc Viet Road, Cau Giay District, Hanoi, Vietnam

[3]Department of Information Technology, Hanoi University of Civil Engineering , 55 Giai Phong str., Hanoi, Vietnam

Our aim in this paper is to find a characterization for asymptotic stability of (2) when the delays $\tau_{ij}, i \neq j$, can take arbitrary non-negative integer values. This work lies in the research topic known as "absolute stability" or "harmless delay" which has been gained a lot of interest due to its vast applications in biological models [19, 13], neural models [20, 23]. Another motivation of this research topic is from the fact that stability of a delay system might be sensitive to the delays of the system, see e.g. [1, 5, 9, 10] and the references therein.

One of the central task in this research topic is to find a necessary and sufficient condition of the coefficients of systems for delay independent stability, i.e. the systems are stable regardless of the choice of the delays. Several techniques have been developed to deal with this problem, for examples two-variable criterion, see [16, 17, 11], frequency sweeping tests, see e.g. [3, 18, 22] and the references therein. Based on these techniques, systems are delay independent stable if and only if their coefficients must satisfy a family of algebraic inequalities. For off-diagonal linear delay systems, a special class of delay equations but many practical cases of constrained delay uncertainties can be formulated in this form, a remarkable result in [14] provided an explicit condition for their delay independent stability. Note that the condition of delay independent stability in [14] is formulated in terms of a finite number of algebraic inequalities. Then, several extensions of [14] to delay independent stability for neural networks were done in [2, 20].

In this paper, we establish an explicit characterization of delay independent stability for linear off-diagonal delay difference equations (2). This result can be considered as a natural discrete-time counterpart of the results for continuous-time delay systems presented in [14] and as far as we aware, this characterization has not been developed elsewhere. A work relating to our result is given in [15] in which the authors provided a necessary condition for delay independent stability of (2) when $a_{ii} < 0$.

The paper is organized as follows: In Section 2, we give the setting and state the main result of this paper about a characterization of delay independent stability for linear off-diagonal delay difference equations. Section 3 is devoted to proving the main result and the section consists of three subsections. In Subsection 3.1, we recall some fundamental properties of weakly diagonally dominant matrices. A proof of the necessary and sufficient part of the main theorem is given, respectively, in Subsection 3.2 and Subsection 3.3. An example about delay independent stability for an equilibrium of a discrete-time Lotka-Voltera equation is given in Section 4 to illustrate the theoretical result of the paper.

Notations: Let $\mathbb{Z}_{\geq 0}$ denote the set of non-negative integer numbers. Let $\mathbb{R}^n_{>0} := \{(c_1, \dots, c_n)^{\mathrm{T}} \in \mathbb{R}^n : c_1, \dots, c_n > 0\}$. Define the operator $\widehat{\cdot} : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ as follows: for each matrix $M = (m_{ij})_{1\leq i,j\leq n} \in \mathbb{R}^{n\times n}$ the corresponding matrix $\widehat{M} = (\widehat{m}_{ij})_{1\le i,j\le n}$ is given by

$$\widehat{m}_{ij} = \begin{cases} m_{ij}, & \text{if } i = j, \\ |m_{ij}|, & \text{if } i \neq j. \end{cases} \tag{4}$$

Finally, let I_n denote the identity matrix in $\mathbb{R}^{n\times n}$.

2. Preliminaries and the statement of the main result

Note that for a linear time-invariant difference equation, the notions of attractivity and asymptotic stability coincide, see e.g. [6]. Furthermore, system (2) can be rewritten as a higher order difference equation with time-invariant coefficient (see also the Remark 2 below). Then, system (2) is said to be *asymptotically stable* (equivalently *attractive*) if all solutions of (2) tend to 0 as the time tends to infinity.

It is also known that asymptotic stability of a linear difference equation can be characterized by the location of the root of its characteristic polynomial, see e.g. [6]. In what follows, we establish this type of result for (2). For this purpose, let

$$F_\tau(\lambda) := \det \begin{pmatrix} a_{11} - \lambda & a_{12}\lambda^{-\tau_{12}} & \dots & a_{1n}\lambda^{-\tau_{1n}} \\ a_{21}\lambda^{-\tau_{21}} & a_{22} - \lambda & \dots & a_{1n}\lambda^{-\tau_{1n}} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}\lambda^{-\tau_{n1}} & a_{n2}\lambda^{-\tau_{n2}} & \dots & a_{nn} - \lambda \end{pmatrix}. \tag{5}$$

The *characteristic equation* of (2) is defined by $F_\tau(\lambda) = 0$. Denote by $\mathcal{R}F_\tau$ the set of all roots of the characteristic equation, i.e. $\mathcal{R}F_\tau := \{\lambda \in \mathbb{C} : F_\tau(\lambda) = 0\}$.

Lemma 1. System (2) is asymptotically stable if and only if

$$|\lambda| < 1 \qquad \text{for all } \lambda \in \mathcal{R}F_\tau, \tag{6}$$

Proof. Let $A_{ij} \in \mathbb{R}^{n\times n}$ be the matrix having the element in row i and column j equal to a_{ij} and the other elements equal to 0. Put $\tau = \max \tau_{ij}$. Then, by shifting $x(k - \tau_{ij})$ by τ, system (2) is equivalent to a higher order difference equation

$$x(k + \tau + 1) = \sum_{1\le i,j\le n} A_{ij}x(k + \tau - \tau_{ij}). \tag{7}$$

By [21, Theorem 2.4], equation (7) is asymptotically stable if and only if all roots λ of the equation

$$P(\lambda) = \sum_{1\le i,j\le n} \lambda^{\tau-\tau_{ij}} A_{ij} - \lambda^{\tau+1} I = 0$$

satisfy $|\lambda| < 1$. It is easy to see that $P(\lambda) = \lambda^\tau F_\tau(\lambda)$. Thus, (6) is a necessary and sufficient condition for asymptotic stability of system (2). □

Remark 2. By transforming the higher order difference equation (7) to the first order difference equation in the companion form (see, e.g., [21, Theorem 2.4]), it is easy to see that system (2) is asymptotically stable if and only if it is *exponentially stable* (i.e., there exist $M, \alpha > 0$ such that $\|x(t)\| \leq Me^{-\alpha t}$ for all $t \geq 0$). Therefore, (6) is also a necessary and sufficient condition for exponential stability of system (2).

To formulate the main result of the paper, we recall that a matrix $M \in \mathbb{R}^{n\times n}$ is said to be *weakly diagonally dominant* if all the principle minors of $-\widehat{M}$ are non-negative (see (4) for the definition of the matrix $\widehat{M}$). A further discussion about weakly diagonally dominant matrices is given in Subsection 3.1.

Theorem 3 (Characterization of delay independent stability for linear off-diagonal delay difference equations)**.** Let

$$A_1 := -I_n - A, \qquad A_2 := -I_n + A. \tag{8}$$

System (2) is asymptotically stable for all choices of delay $\tau_{ij}, i \neq j$ if and only if the following conditions simultaneously hold:

(C1) $a_{ii} \in (-1, 1)$ for all $i = 1, \ldots, n$,

(C2) The matrices A_1 and A_2 are weakly diagonally dominant,

(C3) $\det -\widehat{A}_1 \neq 0, \det -\widehat{A}_2 \neq 0$.

3. Proof of the main results

3.1. Weakly diagonally dominant matrices

In this subsection, we state and prove a property of irreducible weakly diagonally dominant matrices. This result is an important ingredient in the proof of Theorem 3. Recall that a matrix $M = (m_{ij})_{1\leq i,j\leq n} \in \mathbb{R}^{n\times n}$ is called *reducible* if it can be transformed to a matrix of the form $\begin{pmatrix} M^{(1)} & J^{(12)} \\ 0 & M^{(2)} \end{pmatrix}$ by a simultaneous permutation of rows and columns, where $M^{(1)}$ and $M^{(2)}$ are square matrices of dimension at least 1. A matrix M is called *irreducible* if it is not reducible.

Proposition 4. Let $M \in \mathbb{R}^{n\times n}$ be a weakly diagonally dominant matrix. Suppose additionally that M is irreducible and $\det \widehat{M} \neq 0$. Then there exists $c \in \mathbb{R}^n_{>0}$ such that

$$m_{ii}c_i + \sum_{j\neq i} |m_{ij}|c_j < 0 \qquad \text{for all } i = 1, 2, \ldots, n. \tag{9}$$

Proof. By [8, Theorem 5.9], since M is irreducible and weakly diagonally dominant, it follows that there is a $x \in \mathbb{R}^n_{>0}$ such that $y := \widehat{M}x \leq 0$, i.e.

$$y_i = m_{ii}x_i + \sum_{j \neq i} |m_{ij}|x_j \leq 0 \qquad \text{for all } i = 1, 2, \ldots, n.$$

We now find nearby y a desired c satisfying (9). Let $\varepsilon := \min_{1 \leq i \leq n} \frac{x_i}{2}$. Then,

$$B_\varepsilon(x) := \left\{ (u_1, \ldots, u_n)^{\mathrm{T}} : \max_{1 \leq i \leq n} |u_i - x_i| < \frac{\varepsilon}{2} \right\}$$

is an open subset of $\mathbb{R}^n_{>0}$. Since $\widehat{M}$ is invertible, the set $\widehat{M}B_\varepsilon(x)$ is open and contains y. Thus, there exists $\delta > 0$ such that

$$(y_1 - \delta, y_2 - \delta, \ldots, y_n - \delta)^{\mathrm{T}} \in \widehat{M}B_\varepsilon(x).$$

Then, $c := \widehat{M}^{-1}(y_1 - \delta, y_2 - \delta, \ldots, y_n - \delta)^T$ satisfies (9) and the proof is complete. □

We now have a corollary of the above proposition that is useful in verifying condition (C1) of the main result stated in Theorem 3.

Corollary 5. Suppose that $M \in \mathbb{R}^{n \times n}$ is weakly diagonally dominant and satisfies that $\det \widehat{M} \neq 0$. Then,

$$m_{ii} \neq 0 \qquad \text{for all } i = 1, \ldots, n.$$

Proof. Obviously, by Proposition 4 the statement follows when M is irreducible. In the general case, thanks to [8, Theorem 3.6] by a simultaneous permutation of rows and columns P, the matrix M is transformed to an upper block triangular matrix whose diagonal blocks are irreducible

$$PMP^{\mathrm{T}} = \begin{pmatrix} M^{(1)} & J^{(12)} & \ldots & J^{(1r)} \\ 0 & M^{(2)} & \ldots & J^{(2r)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & M^{(r)} \end{pmatrix}.$$

Since M is weakly diagonally dominant, all sub-matrices $M^{(1)}, \ldots, M^{(r)}$ are too. From $\det \widehat{M} \neq 0$, we also obtain that $\det \widehat{M^{(1)}} \neq 0, \ldots, \det \widehat{M^{(r)}} \neq 0$. Furthermore, $M^{(1)}, \ldots, M^{(r)}$ are irreducible and hence by Proposition 4 all diagonal entries of $M^{(1)}, \ldots, M^{(r)}$ are non-zero. Consequently, all diagonal entries of M are non-zero. The proof is complete. □

3.2. Necessary part

Before going the the proof of the necessary part of Theorem 3, we need the following preparatory results.

Lemma 6. Let A_1 be given as in (8). Then, the following statements hold:

(i) Suppose that $\det(-\widehat{A_1}) = 0$. Then, there exist delays τ^*_{ij} such that $F_{\tau^*}(-1) = 0$.

(ii) Suppose that $\det(-\widehat{A_1}) < 0$. Then, there exist delays τ^*_{ij} and $\lambda^* \in (-\infty, -1)$ such that

$$F_{\tau^*}(-1) < 0 \quad \text{and} \quad F_{\tau^*}(\lambda^*) > 0.$$

Proof. We first construct off-diagonal delays for both (i) and (ii). By definition of A_1 and the operator $\widehat{\cdot}$, we have

$$\det(-\widehat{A}_1) = \det \begin{pmatrix} 1 + a_{11} & -|a_{12}| & \dots & -|a_{1n}| \\ -|a_{21}| & 1 + a_{22} & \dots & -|a_{2n}| \\ \vdots & \vdots & \ddots & \vdots \\ -|a_{n1}| & -|a_{n2}| & \dots & 1 + a_{nn} \end{pmatrix}. \tag{10}$$

We now choose an off-diagonal delay $\tau^* = (\tau^*_{ij})_{i \neq j}$ as

$$\tau^*_{ij} = \begin{cases} 2 & \text{if } a_{ij} < 0, \\ 1 & \text{if } a_{ij} \geq 0. \end{cases} \tag{11}$$

It means that $(-1)^{\tau^*_{ij}} a_{ij} = -|a_{ij}|$. Then, by definition of $F_{\tau^*}(\lambda)$ as in (5) we arrive at

$$F_{\tau^*}(\lambda) = \det \begin{pmatrix} a_{11} - \lambda & -|a_{12}|(-\lambda)^{-\tau^*_{12}} & \dots & -|a_{1n}|(-\lambda)^{-\tau^*_{1n}} \\ -|a_{21}|(-\lambda)^{-\tau^*_{21}} & a_{22} - \lambda & \dots & -|a_{2n}|(-\lambda)^{-\tau^*_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ -|a_{n1}|(-\lambda)^{-\tau^*_{n1}} & -|a_{n2}|(-\lambda)^{-\tau^*_{n2}} & \dots & a_{nn} - \lambda \end{pmatrix}.$$

Hence, from (10) we have $F_{\tau^*}(-1) = \det(-\widehat{A}_1)$. Then, (i) is proved. Now we consider the case that $F_{\tau^*}(-1) = \det(-\widehat{A}_1) < 0$. Then,

$$\frac{F_{\tau^*}(\lambda)}{(-\lambda)^n} = \det \begin{pmatrix} 1 - \frac{a_{11}}{\lambda} & \frac{|a_{12}|(-\lambda)^{-\tau^*_{12}}}{\lambda} & \dots & \frac{|a_{1n}|(-\lambda)^{-\tau^*_{1n}}}{\lambda} \\ \frac{|a_{21}|(-\lambda)^{-\tau^*_{21}}}{\lambda} & 1 - \frac{a_{22}}{\lambda} & \dots & -|a_{2n}|(-\lambda)^{-\tau^*_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{|a_{n1}|(-\lambda)^{-\tau^*_{n1}}}{\lambda} & \frac{|a_{n2}|(-\lambda)^{-\tau^*_{n2}}}{\lambda} & \dots & 1 - \frac{a_{nn}}{\lambda} \end{pmatrix}.$$

Consequently,

$$\lim_{\lambda\to-\infty}\frac{F_{\tau^*}(\lambda)}{(-\lambda)^n}=1.$$

This implies that there exists $\lambda^*\in(-\infty,-1)$ such that $F_{\tau^*}(\lambda^*)>0$ and (ii) is proved. The proof is complete. □

Next, we formulate a dual version of Lemma 6. In fact, we now consider the adjoint equation to (2) of the form

$$x_i(k+1)=\sum_{j=1}^{n}\bar{a}_{ij}x_j(k-\tau_{ij}),\qquad \text{for } i=1,\dots,n, \tag{12}$$

where $\bar{a}_{ii}=-a_{ii}$ and $\bar{a}_{ij}=|a_{ij}|$ for $i\neq j$ and τ_{ij} is given as in (11). The characteristic equation of (12) is given as $F^{\mathrm{ad}}_{\tau^*}(\lambda)=0$, where

$$F^{\mathrm{ad}}_{\tau^*}(\lambda):=\det\begin{pmatrix}\bar{a}_{11}-\lambda & \bar{a}_{12}\lambda^{-\tau_{12}} & \dots & \bar{a}_{1n}\lambda^{-\tau_{1n}}\\ \bar{a}_{21}\lambda^{-\tau_{21}} & \bar{a}_{22}-\lambda & \dots & \bar{a}_{1n}\lambda^{-\tau_{1n}}\\ \vdots & \vdots & \ddots & \vdots\\ \bar{a}_{n1}\lambda^{-\tau_{n1}} & \bar{a}_{n2}\lambda^{-\tau_{n2}} & \dots & \bar{a}_{nn}-\lambda\end{pmatrix}. \tag{13}$$

Obviously, $F^{\mathrm{ad}}_{\tau^*}(\lambda)=(-1)^nF_{\tau^*}(-\lambda)$. Thus, by Lemma 1, (2) is asymptotically stable if and only if (12) is asymptotically stable.

Lemma 7. Let A_2 be given as in (8). Then, the following statements hold:

(i) Suppose that $\det(-\widehat{A_2})=0$. Then, there exist delays τ^*_{ij} such that $F^{\mathrm{ad}}_{\tau^*}(-1)=0$.

(ii) Suppose that $\det(-\widehat{A_2})<0$ holds. Then, there exist delays τ^*_{ij} and $\lambda^*\in(-\infty,-1)$ such that

$$F^{\mathrm{ad}}_{\tau^*}(-1)<0 \quad\text{and}\quad F^{\mathrm{ad}}_{\tau^*}(\lambda^*)>0.$$

Proof. The proof follows by applying the result in Lemma 6 to the adjoint equation (12). □

Now we are in a position to prove the necessity part of the main theorem.

Proof of the necessity part of Theorem 3. Suppose that system (2) (and hence also (12)) is asymptotically stable for all choice of off-diagonal delays τ. We now verify that all conditions (C1), (C2) and (C3) hold:

Verification of (C1): By virtue of Corollary 5, $a_{ii}\notin\{-1,1\}$ for $i=1,\dots,n$. Then, it remains to show that $a_{ii}\notin(-\infty,-1)\cup(1,\infty)$ for $i=1,\dots,n$. We divide the proof of this fact into two steps:

Step 1: In this step, we show that $a_{ii} \notin (1, \infty)$ for $i = 1, \ldots, n$. Suppose the contrary, i.e. $M > 1$, where $M := \max_{1 \le i \le n} a_{ii}$. Let $\mathcal{I} := \{i : a_{ii} = M\}$. Then, there exists $\varepsilon \in (0, 1)$ satisfying that

$$M - \frac{\varepsilon}{2} > 1 \quad \text{and} \quad M - \frac{\varepsilon}{2} \ge a_{ii} \qquad \text{for all } i \in \{1, \ldots, n\} \setminus \mathcal{I}. \tag{14}$$

Let $K := \{\lambda \in \mathbb{C} : |\lambda - M| \le \frac{\varepsilon}{2}\}$. To complete the proof, we show that there are off-diagonal delays τ such that the characteristic equation $F_\tau(\lambda) = 0$ has a root on K. We decompose $F_\tau(\lambda)$ as

$$F_\tau(\lambda) = \prod_{i=1}^{n} (a_{ii} - \lambda) + R_\tau(\lambda) \tag{15}$$

where $R_\tau(\lambda) := F_\tau(\lambda) - \prod_{i=1}^{n} (a_{ii} - \lambda)$. By definition of $F_\tau(\lambda)$ as in (5), there is a constant C (independence of τ) such that the remainder term $R_\tau(\lambda)$ satisfies

$$|R_\tau(\lambda)| \le C|\lambda|^{-\min_{1\le i,j\le n} \tau_{ij}} \le C(M - \frac{\varepsilon}{2})^{-\min_{1\le i,j\le n} \tau_{ij}} \qquad \text{for } \lambda \in \partial K. \tag{16}$$

On the other hand, by definition of ε as in (14) we have

$$\begin{aligned} |\prod_{i=1}^{n} (a_{ii} - \lambda)| &= \prod_{i \in \mathcal{I}} |M - \lambda| \prod_{i \notin \mathcal{I}} |a_{ii} - \lambda| \\ &\ge (\frac{\varepsilon}{2})^n \quad \text{for } \lambda \in \partial K, \end{aligned}$$

which together with (16) and the fact that $M - \frac{\varepsilon}{2} > 1$ implies that there exist off-diagonal delays τ_{ij} such that

$$|\prod_{i=1}^{n} (a_{ii} - \lambda)| > |R_\tau(\lambda)| \quad \text{for all } \lambda \in \partial K.$$

Thus, by using Rouché's Theorem (see e.g. [4]) and (15), the functions $F_\tau(\lambda)$ and $\prod_{i=1}^{n} (a_{ii} - \lambda)$ have the same number of zeros inside in K. Therefore, $F_\tau(\cdot)$ has a root λ in K and this leads to a contradiction.

Step 2: In this step, we show that $a_{ii} \notin (-\infty, -1)$ for $i = 1, \ldots, n$. Suppose the contrary, i.e. $m < -1$, where $m := \min_{1 \le i \le n} a_{ii}$. Similarly to Step 1, there exist off-diagonal delays τ_{ij} such that the adjoint characteristic equation $F_\tau^{\mathrm{ad}}(\lambda) = 0$ has a root λ with $|\lambda| > 1$ and this is a contradiction.

Verification of (C2): We will use Lemma 6 to show that A_1 is weakly diagonally dominant. By analogous statements, we use Lemma 7 to deduce that A_2 is weakly diagonally dominant. Suppose the contrary that A_1 is not weakly diagonally dominant, i.e. there is a negative principle minor of $-\widehat{A}_1$. Exchanging the columns and rows of $-\widehat{A}_1$ (equivalently, re-indexing the variables $x_1, \ldots, x_n$

of (2)), if necessary, we can assume that there exists $\ell \in \{1, \dots, n\}$ such that

$$\det -\widehat{A^{(\ell)}} < 0, \quad \text{where} \quad A^{(\ell)} := -I_\ell - \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1\ell} \\ a_{21} & a_{22} & \dots & a_{2\ell} \\ \vdots & \vdots & \ddots & \vdots \\ a_{\ell 1} & a_{\ell 2} & \dots & a_{\ell\ell} \end{pmatrix}.$$

Denote by $F_\tau^{(\ell)}(\lambda)$ the characteristic function of the following "sub-system" of (2)

$$x_i(k+1) = \sum_{j=1}^{\ell} a_{ij} x_j(k - \tau_{ij}), \qquad \text{for } i = 1, \dots, \ell.$$

It means that $F_\tau^{(\ell)}(\lambda)$ depends only on $\tau^\ell := (\tau_{ij})_{1 \le i,j \le \ell}$ and

$$F_{\tau^\ell}^{(\ell)}(\lambda) = \det \begin{pmatrix} a_{11} - \lambda & a_{12}\lambda^{-\tau_{12}} & \dots & a_{1\ell}\lambda^{-\tau_{1\ell}} \\ a_{21}\lambda^{-\tau_{21}} & a_{22} - \lambda & \dots & a_{2\ell}\lambda^{-\tau_{1\ell}} \\ \vdots & \vdots & \ddots & \vdots \\ a_{\ell 1}\lambda^{-\tau_{\ell 1}} & a_{\ell 2}\lambda^{-\tau_{\ell 2}} & \dots & a_{\ell\ell} - \lambda \end{pmatrix}. \tag{17}$$

By virtue of Lemma 6, there exist $\tau^{*\ell}$ and $\lambda^* \in (-1, -\infty)$ such that

$$F_{\tau^{*\ell}}^{(\ell)}(-1) < 0 \quad \text{and} \quad F_{\tau^{*\ell}}^{(\ell)}(\lambda^*) > 0. \tag{18}$$

For each $\tau = (\tau_{ij})_{1 \le i,j \le n}$ satifying that $\tau_{ij} = \tau_{ij}^{*(\ell)}$ for $1 \le i, j \le \ell$, we let

$$m(\tau) = \min\{\tau_{ij} : i > \ell \text{ or } j > \ell\}.$$

From (5) and (17), we derive that for all $\lambda \in (-\infty, -1]$

$$\lim_{m(\tau) \to \infty} F_\tau(\lambda) = F_{\tau^{*\ell}}^{(\ell)}(\lambda) \prod_{k=\ell+1}^{n} (a_{kk} - \lambda).$$

By (C1), $\prod_{k=\ell+1}^n (a_{kk} - \lambda) > 0$ for all $\lambda \in (-\infty, -1]$. Then, from (18) we can extend $\tau^{*\ell}$ to $\tau^* = (\tau_{ij}^*)_{1 \le i,j \le n}$ such that

$$F_{\tau^*}(-1) < 0 \quad \text{and} \quad F_{\tau^*}(\lambda^*) > 0.$$

Then, for such a choice of τ_{ij}^* the characteristic equation $F_\tau(\lambda) = 0$ has a root λ with $|\lambda| > 1$ and the system (2) is not asymptotically stable. This completes the verification of (C2).

Verification of (C3): This can be deduced directly from Lemma 6(i) and Lemma 7(i). □

3.3. Sufficiency part

To proof the sufficiency part of Theorem 3, we recall the Gershgorin circle theorem which is used to bound the spectrum of a square matrix, see e.g. [12, Theorem 7.2.1].

Theorem 8 (Gershgorin circle theorem)**.** Let $M \in \mathbb{C}^{n\times n}$. Then every eigenvalue of M lays within at least one of the Gershgorin discs

$$\left\{ z \in \mathbb{C} : |z - m_{ii}| < \sum_{j\neq i} |m_{ij}| \right\}.$$

Proof of the sufficiency part of Theorem 3. Assume that (C1), (C2) and (C3) hold. Suppose the contrary that there exist off-diagonal delays τ^* such that the characteristic function $F_{\tau^*}(\lambda)$ has a root λ^* with $|\lambda^*| \geq 1$. Let

$$M := \begin{pmatrix} a_{11} & a_{12}(\lambda^*)^{-\tau_{12}} & \dots & a_{1n}(\lambda^*)^{-\tau_{1n}} \\ a_{21}(\lambda^*)^{-\tau_{21}} & a_{22} & \dots & a_{2n}(\lambda^*)^{-\tau_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(\lambda^*)^{-\tau_{n1}} & a_{n2}(\lambda^*)^{-\tau_{n2}} & \dots & a_{nn} \end{pmatrix}.$$

Then, λ^* is an eigenvalue of M. We now derive a contradiction by considering the following two separated cases:

Case 1: A is irreducible. Then, $A_1 = -I_n - A$ and $A_2 = -I_n + A$ are also irreducible matrices. By (C2) and (C3), A_2 is weakly diagonally dominant and $\det \widehat{A}_2 \neq 0$. Hence, using Proposition 4 and the fact that $A_2 = -I_n + A$, there exists $c \in \mathbb{R}^n_{>0}$ such that

$$(a_{ii} - 1)c_i + \sum_{j\neq i} |a_{ij}|c_j < 0 \quad \text{for all } i = 1, 2, \dots, n,$$

which together with the fact that $|\lambda^*| \geq 1$ implies that

$$(a_{ii} - 1)c_i + \sum_{j\neq i} |a_{ij}(\lambda^*)^{-\tau_{ij}}|c_j < 0 \quad \text{for all } i = 1, 2, \dots, n.$$

Equivalently,

$$\sum_{j\neq i} c_i^{-1}|a_{ij}(\lambda^*)^{-\tau_{ij}}|c_j < 1 - a_{ii} \quad \text{for all } i = 1, 2, \dots, n. \tag{19}$$

Now, let $N = \mathrm{diag}(c_1^{-1}, \dots, c_n^{-1})M\,\mathrm{diag}(c_1, \dots, c_n)$. Then, λ^* is also an eigenvalue of N and by definition of M, we have

$$N = \begin{pmatrix} a_{11} & c_1^{-1}a_{12}(\lambda^*)^{-\tau_{12}}c_2 & \dots & c_1^{-1}a_{1n}(\lambda^*)^{-\tau_{1n}}c_n \\ a_{21}(\lambda^*)^{-\tau_{21}} & a_{22} & \dots & a_{2n}(\lambda^*)^{-\tau_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(\lambda^*)^{-\tau_{n1}} & a_{n2}(\lambda^*)^{-\tau_{n2}} & \dots & a_{nn} \end{pmatrix}.$$

Then, by Theorem 8 and (19) we arrive at

$$\lambda^* \in \bigcup_{i=1}^{n} (2a_{ii} - 1, 1). \tag{20}$$

Similarly, using (C2), (C3) and the fact that A_1 is weakly diagonally dominant and $\det \widehat{A}_2 \neq 0$, there exists $d \in \mathbb{R}^n_{>0}$ such that

$$\sum_{j \neq i} d_i^{-1} |a_{ij} (\lambda^*)^{-\tau_{ij}}| d_j < 1 + a_{ii} \quad \text{for all } i = 1, 2, \ldots, n.$$

Then, applying Theorem 8 to the matrix $\operatorname{diag}(d_1^{-1}, \ldots, d_n^{-1}) M \operatorname{diag}(d_1, \ldots, d_n)$ yields that

$$\lambda^* \in \bigcup_{i=1}^{n} (-1, 2a_{ii} + 1),$$

which together with (20) leads to a contradiction that $\lambda^* \in (-1, 1)$. The proof is complete in this case.

Case 2: A is reducible. Then, according to [8, Theorem 3.6] by relabeling the indices of variables $x_1, \ldots, x_n$, if necessary, we assume that

$$A = \begin{pmatrix} A^{(1)} & J^{(12)} & \ldots & J^{(1r)} \\ 0 & A^{(2)} & \ldots & J^{(2r)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A^{(r)} \end{pmatrix},$$

where $A^{(1)} \in \mathbb{R}^{n_1 \times n_1}, \ldots, A^{(r)} \in \mathbb{R}^{n_r \times n_r}$ are irreducible, weakly diagonally dominant and $\det I_{n_i} \pm \widehat{A^{(i)}} \neq 0$. Now, for $i = 1, \ldots, r$ we define $\tau^{*(i)}$ by

$$\tau^{*(i)}_{k\ell} = \tau_{n_1 + \cdots + n_{i-1} + k, n_1 + \cdots + n_{i-1} + \ell} \quad \text{for } 1 \leq k, \ell \leq n_i.$$

Then, by definition of $F_\tau(\lambda)$ as in (5) we have

$$F_{\tau^*}(\lambda^*) = \prod_{i=1}^{r} F^{(i)}_{\tau^*}(\lambda^*),$$

where

$$F^{(i)}_{\tau^*}(\lambda) = \det \begin{pmatrix} a^{(i)}_{11} - \lambda & a^{(i)}_{12} \lambda^{-\tau^*_{12}} & \ldots & a^{(i)}_{1n_i} \lambda^{-\tau^*_{1n_i}} \\ a^{(i)}_{21} \lambda^{-\tau^*_{21}} & a^{(i)}_{22} - \lambda & \ldots & a^{(i)}_{2n_i} \lambda^{-\tau^*_{2n_i}} \\ \vdots & \vdots & \ddots & \vdots \\ a^{(i)}_{n_i 1} \lambda^{-\tau^*_{n_i 1}} & a^{(i)}_{n_i 2} \lambda^{-\tau^*_{n_i 2}} & \ldots & a^{(i)}_{n_i n_i} - \lambda \end{pmatrix}.$$

Then, there exists $i \in \{1, \ldots, r\}$ such that $F^{(i)}_{\tau^*}(\lambda^*) = 0$. Applying the result in Case 1 to the matrix $A^{(i)}$ and the characteristic function $F^{(i)}_{\tau^*}(\lambda^*)$, we obtain that $\lambda^* \in (-1, 1)$ which leads to a contradiction. The proof is complete. $\square$

4. Example

Consider a discrete-time Lotka-Volterra system

$$y_i(k+1) = y_i(k)\left(1 + hr_i + \sum_{j=1}^{n} ha_{ij}y(k-\tau_{ij})\right), \;\; i = 1, 2\ldots, n, \qquad (21)$$

which can be obtained by applying the numerical Euler method with step size h to the continuous-time Lotka-Volterra system

$$\dot{y}_i(t) = y_i(t)\left(r_i + \sum_{j=1}^{n} a_{ij}y_j(t-\tau_{ij})\right), \;\; i = 1, 2\ldots, n.$$

Assume that there exists a positive vector $\hat{y} \in \mathbb{R}^n$ such that $r + A\hat{y} = 0$. Then $\hat{y}$ is an equilibrium of system (21), i.e., $y(k) = \hat{y}$ is a solution of (21). To study stability of the equilibrium $\hat{y}$, for each solution $y(k)$ of (21) we let $x(k) := y(k) - \hat{y}$. By (21), $x(k)$ satisfies the following equation

$$x_i(k+1) = \sum_{j=1}^{n} b_{ij}x(k-\tau_{ij}) + x_i(k)\sum_{j=1}^{n} ha_{ij}x(k-\tau_{ij}), \;\; i = 1, 2\ldots, n, \qquad (22)$$

where

$$b_{ij} = \begin{cases} 1 + h\hat{y}_i a_{ij} & \text{if } i = j, \\ h\hat{y}_i a_{ij} & \text{if } i \neq j \end{cases}. \qquad (23)$$

Furthermore, the equilibrium $\widehat{y}$ of system (21) is locally asymptotically stable if and only if the trivial solution of system (22) is locally asymptotically equivalent. Thus, the equilibrium $\widehat{y}$ of system (21) is locally asymptotically stable if and only if the linearization of system (22) along the trivial solution

$$x_i(k+1) = \sum_{j=1}^{n} b_{ij}x(k-\tau_{ij}), \qquad (24)$$

is asymptotically stable. Now, let $B_1 = -I_n - B, B_2 = -I_n + B$. By Theorem 3, the equilibrium $\hat{y}$ of system (21) is locally asymptotically stable for all delays τ_{ij} if and only if

(L1) $b_{ii} \in (-1, 1)$ for all $i = 1, \ldots, n$,

(L2) The matrices B_1 and B_2 are weakly diagonally dominant,

(L3) $\det -\widehat{B}_1 \neq 0, \det -\widehat{B}_2 \neq 0$.

Acknowledgement

The authors would like to thank the reviewers and the associated editors for many constructive comments that lead to a significant improvement of the paper. This research is funded by Hanoi University of Civil Engineering (HUCE) under grant number 25-2021/KHXD-TD.

References

[1] A.Y. Aleksandrov, O. Mason, Diagonal Lyapunov–Krasovskii functionals for discrete-time positive systems with delay, *Systems Control Lett.* **63** (2014), 63–67.

[2] S.A. Campbell, Delay independent stability for additive neural networks. New millennium special issue on neural networks and neurocomputing theory, models, and applications, Part I, *Differential Equations Dynam. Systems* **9** (2001), no. 3-4, 115–138.

[3] J. Chen and H.A. Latchman, Frequency sweeping tests for stability independent of delay, *IEEE Trans. Automat. Control* **40** (1995), no. 9, 1640–1645.

[4] J.B. Conway. *Functions of One Complex Variable I*, Springer-Verlag New York, 1978.

[5] A. Domoshnitsky, E. Fridman, A positivity-based approach to delay dependent stability of systems with large time-varying delays, *Systems Control Lett.* **97** (2016), 139–148.

[6] S. Elaydi, *An Introduction to Difference Equations*, 3rd edition, Springer, New York, 2005.

[7] L. E. Elsgolts and S.B. Norkin, *Introduction to the Theory and Application of Differential Equations with Deviating Arguments*, Academic Press, New York, 1973.

[8] M. Fiedler, *Special Matrices and Their Applications in Numerical Mathematics*, Martinus Nijhoff Publ. (Kluwer), Dordrecht, 1986.

[9] J.K. Hale and S.M. Verduyn Lunelm, Strong stabilization of neutral functional differential equations, *IMA J. Math. Control Inform.* **19** (2002), no. 1-2, 5–23.

[10] J.K. Hale and S.M. Verduyn Lunelm, Stability and control of feedback systems with time delays. Time delay systems: theory and control, *Internat. J. Systems Sci.* **34** (2003), no. 8-9, 497–504.

[11] D. Hertz, E.I. Jury and E. Zeheb, Stability independent and dependent of delay for delay differential systems, *J. Franklin Inst.* **318** (1984), no. 3, 143–150.

[12] G.H. Golub and C.F. Van Loan, *Matrix Computations*, Johns Hopkins University Press, 1996.

[13] J. Hofbauer and K. Sigmund, *The Theory of Evolution and Dynamical Systems*, Cambridge University Press, Cambridge, 1988.

[14] J. Hofbauer and J.W.-H. So, Diagonal dominance and harmless off-diagonal delays, *Proc. Amer. Math. Soc.* **128** (2000), no. 9, 2675-2682.

[15] Y. Hong and W. Ma, Sufficient and necessary conditions for global attractivity and stability of a class of discrete Hopfield-type neural networks with time delays, *Math. Biosci. Eng.* **16** (2019), no. 5, 4936–4946.

[16] E.W. Kamen, Linear systems with commensurate time delays: stability and stabilization independent of delay, *IEEE Trans. Automat. Control* **27** (1982), no. 2, 367–375.

[17] E.W. Kamen, Correction to "Linear systems with commensurate time delays: stability and stabilization independent of delay", *IEEE Trans. Automat. Control* **28** (1983), no. 2, 248–249.

[18] X-G. Li, S-I. Niculescu, A. Çela, L. Zhang, X. Li, A frequency-sweeping framework for stability analysis of time-delay systems, *IEEE Trans. Automat. Control* **62** (2017), no. 8, 3701–3716.

[19] Z. Lu and W. Wang, Global stability for two–species Lotka–Volterra systems with delay, *J. Math. Anal. Appl.* **208** (1997), 277–280.

[20] W. Ma, Y. Saito and Y. Takeuchi, M-matrix structure and harmless delays in a Hopfield-type neural network, *Appl. Math. Lett.* **22** (2009), no. 7, 1066–1070.

[21] V. Mehrmann and Duc Thuan Do, Stability analysis of implicit difference equations under restricted perturbations, *SIAM J. Matrix Anal. Appl.* **36** (2015), no. 1, pp. 178–202.

[22] S-I. Niculescu, X-G. Li, A. Çela, Counting characteristic roots of linear delay differential equations. Part I: frequency-sweeping stability tests and applications, In: Breda, D. (eds) Controlling Delayed Dynamics, *CISM International Centre for Mechanical Sciences*, vol **604** (2023), Springer, Cham..

[23] S. Zhang, W. Ma and Y. Kuang, Necessary and sufficient conditions for global attractivity of Hopfield-type neural networks with time delays, *Rocky Mountain J. Math.* **38** (2008), no. 5, 1829–1840.