EXISTENCE AND NONEXISTENCE OF NONTRIVIAL SOLUTIONS FOR DEGENERATE ELLIPTIC EQUATIONS ON A TORUS, I

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Abstract. In this paper, we study the existence and non existence of nontrivial solutions to the Dirichlet boundary value problem for the following degenerate elliptic equation

$$-div(s^{\alpha}\nabla u) = s^{\ell}\,|u|^{p-1}\,u \text{ in } T(R,a), \tag{1}$$
$$u = 0 \text{ on } \partial T(R,a) \tag{2}$$

where

$$T(R,a) = \{(x_1,x_2,x_3) \in \mathbb{R} : x_3^2 + (r-R)^2 < a^2\},$$
$$r = \sqrt{x_1^2 + x_2^2}, 0 < a < R$$

is a torus in $\mathbb{R}^3, s = \sqrt{x_3^2 + (r-R)^2}$ and $\alpha \geq 0, \ell \geq -2, 1 < p < \infty$. The main results show that when p is small then the problem has a nontrivial positive solution. On the other hand, when p is big there is not a nontrivial soltion. To obtain the existence of nontrivial solutions we use the variational method and the symmetric property of the torus. To obtain the nonexistence of nontrivial solutions we derive a Pohozaev's type identity and then apply it.

1 Introduction

Boundary value problems (BVP) for degenerate elliptic equations (DEE), especially nontrivial solutions to BVP for DEE, have been extensively studied recently. Many results concerning the existence, nonexistence, multiplicity of nontrivial solutions to BVP for DEE were obtained, see for example [9], [10], [14], [4], [6] and

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the references therein. In this paper we deal with a DEE on a torus. We essentially use the symmetric property of the torus to obtain the results. Recall that nontrivial solutions to BVP for elliptic equations were considered in [13], [3], [1], [2], [7], [12], [11] (see also the references therein). The plan of the paper is as follows: In §2 we introduce some notations and the formulations of main results. Next in §3 we present some auxilliary statements. Finally in §4 we give the proofs of the results.

2 Main Results

Let us first introduce the notations that will be used later on.
$L^p_\ell(T(R,a)), \ell \in \mathbb{R}, 1 \le p < \infty$ is the space of measurable functions f on $T(R,a)$ such that $s^\ell |f|^p \in L^1(T(R,a))$.
$L^p_\ell(T(R,a)), \ell \in \mathbb{R}, 1 \le p < \infty$ is a Banach space with the norm

$$\|f\|_{L^\ell_p} = \left(\iiint_{T(R,a)} s^\ell |f|^p \, dx \right)^{\frac{1}{p}}. \tag{3}$$

In this paper, we are only interested in measurable functions f on $T(R,a)$ depending only on $r = \sqrt{x_1^2 + x_2^2}$ and x_3, i.e.

$$f(x) = g(r, x_3). \tag{4}$$

In particular, we consider the subspace $L^p_{\ell,sym}(T(R,a))$ containing all functions $f \in L^p_\ell(T(R,a))$ which satisfy (4). Then the norm in (3) is rewritten as follows

$$\|f\|_{L^\ell_p} = \left(2\pi \iint_{B_a} s^\ell |g|^p \, r dr dx_3 \right)^{\frac{1}{p}}$$

with $B_a = \{(r, x_3) : (r-R)^2 + x_3^2 < a^2\}$.
If $(r, x_3) \in B_a$ then $R - a < r < R + a$ so we can consider $L^p_{l,sym}(T(R,a))$ as $L^p_l(B_a)$.
$H^{1,\alpha}_0(T(R,a)), \alpha \ge 0$, is the closure of $C^1_0(T(R,a))$ in the norm

$$\|u\|_{H^{1,\alpha}} = \left(\iiint_{T(R,a)} s^\alpha |\nabla u|^2 \, dx \right)^{\frac{1}{2}}. \tag{5}$$

As above, we are interested in the subspace $H^{1,\alpha}_{0,sym}(T(R,a))$ containing all functions $u \in H^{1,\alpha}_0(T(R,a))$ written as $u(x) = v(r, x_3)$. Then the norm in (5) is rewritten as

$$\|u\|_{H^{1,\alpha}} = \left(2\pi \iint_{T(R,a)} s^\alpha |\nabla v|^2 \, r dr dx_3 \right)^{\frac{1}{2}}$$

with $\nabla v = (v_r, v_{x_3})$. Then we can consider $H^{1,\alpha}_{0,sym}(T(R,a))$ as $H^{1,\alpha}_0(B_a)$.
Now we are in a position to state the main theorems.

Theorem 1. *The problem* (1)-(2) *has a positive solution* $u \in H^{1,\alpha}_{0,sym}(T(R,a))$ *when either* $\alpha = 0, \ell \geq -2$ *and* $1 < p < \infty$ *or* $\alpha > 0, \ell \geq \alpha - 2$ *and* $1 < p < \frac{2(\ell+2)}{\alpha} - 1$.

Theorem 2. *When* $\alpha > 0, \ell \geq \alpha - 2, p > \frac{2(\ell+2)}{\alpha} - 1$, *there exists* $\epsilon_0 > 0$ *such that for* $0 < \frac{\alpha}{R} < \epsilon_0$ *the problem* (1)-(2) *has only trivial solution in* $H^{1,\alpha}_0(T(R,a))$.

3 Some auxilliary statements

Proposition 1. *We have the following continuous embedding*

$$H^{1,\alpha}_{0,sym}(T(R,a)) \hookrightarrow L^p_{\ell,sym}(T(R,a)) \tag{6}$$

when either $\alpha = 0, \ell > -2$ *and* $1 \leq p < \infty$ *or* $\alpha > 0$, $\ell \geq \alpha - 2$ *and* $1 \leq p \leq \frac{2(\ell+2)}{\alpha}$. *The embedding* (6) *is compact when either* $\alpha = 0, \ell > -2$ *and* $1 \leq p < \infty$ *or* $\alpha > 0, l \geq \alpha - 2$ *and* $1 \leq p < \frac{2(\ell+2)}{\alpha}$.

In order to prove Proposition 1, we only need to prove the following lemma.

Lemma 1. *The embedding* $H^{1,\alpha}_0(B_a) \hookrightarrow L^p_\ell(B_a)$ *is continuous when* $\alpha > 0, \ell \geq \alpha - 2$ *and* $1 \leq p \leq \frac{2(\ell+2)}{\alpha}$. *It is compact when either* $\alpha = 0, \ell > -2$ *and* $1 \leq p < \infty$ *or* $\alpha > 0, \ell \geq \alpha - 2$ *and* $1 \leq p < \frac{2(\ell+2)}{\alpha}$.

Proof. We recall Caffarelli - Kohn - Nirenberg inequality

$$\left(\iint_{B_a} s^\ell \left|v\right|^{\frac{2(\ell+2)}{\alpha}} drdx_3 \right)^{\frac{\alpha}{\ell+2}} \leq C \iint_{B_a} s^\alpha \left|\nabla v\right|^2 drdx_3 \tag{7}$$

when $\alpha > 0, \ell \geq \alpha - 2$. Therefore, in order to prove Lemma 1, we only need to prove these embeddings are compact.

Case 1: $\alpha = 0, \ell > -2$, $1 \leq p < \infty$.

Since B_a is a disk in $\mathbb{R}^2$, the embedding $H^1_0(B_a) \hookrightarrow L^q(B_a), 1 \leq q < \infty$ is compact. Now we only need to prove the embedding $L^q(B_a) \hookrightarrow L^p_\ell(B_a)$ is continuous when $-2 < \ell < 0, 1 \leq p < \infty$, for some $q > p$. Choose $q > \frac{2p}{\ell+2} > p$. By using Holder's inequality we have

$$\iint_{B_a} s^\ell \left|v\right|^p drdx_3 \leq \left(\iint_{B_a} s^{\frac{\ell q}{q-p}} drdx_3 \right)^{1-\frac{p}{q}} \left(\iint_{B_a} \left|v\right|^q drdx_3 \right)^{\frac{p}{q}}.$$

Since

$$\iint_{B_a} s^{\frac{\ell q}{q-p}} dr dx_3 = 2\pi \int_0^a s^{1+\frac{\ell q}{q-p}} ds = \frac{2\pi (q-p)}{(\ell+2)\, q - 2p} a^{2+\frac{\ell q}{q-p}},$$

the embedding $L^q(B_a) \hookrightarrow L^p_\ell(B_a)$ is continuous. Thus case 1 is proved.

Case 2: $\alpha > 0, \ell \geq \alpha - 2, 1 \leq p < \frac{2(\ell+2)}{\alpha}$.

We can consider $u \in H_0^{1,\alpha}(B_a)$ as $u \in H_0^{1,\alpha}(\mathbb{R}^2)$ by setting $u = 0$ outside B_a. Similarly for $L^p_\ell(B_a)$. Let $\mathcal{F}$ be a bounded subset in $H_0^{1,\alpha}(B_a)$. In order to prove the embedding is compact for Case 2, we will show that $\mathcal{F}$ is relatively compact subset in $L^p_\ell(B_a)$. For $1 \leq p < \frac{2(\ell+2)}{\alpha}$, by using the Holder's inequality we have

$$\iint_{B_a} s^\ell |u|^p dr dx_3 \leq \left(\iint_{B_a} s^\ell |u|^{\frac{2(l+2)}{\alpha}} dr dx_3 \right)^{\frac{\alpha p}{2(l+2)}} \left(\iint_{B_a} s^\ell dr dx_3 \right)^{1-\frac{\alpha p}{2(l+2)}}.$$

Since $\iint_{B_a} s^\ell dr dx_3 = 2\pi \int_0^a s^{\ell+1} ds = \frac{2\pi}{\ell+2} a^{\ell+2}$ and Caffarelli - Kohn - Nirenberg inequality, $\mathcal{F}$ is bounded in $L^p_\ell(B_a)$ (or $L^p_\ell(\mathbb{R}^2)$). When $p = 1$, in order to prove $\mathcal{F}$ is relatively compact in $L^1_\ell(B_a)$ we show that $\mathcal{G} = \left\{ s^\ell u : u \in \mathcal{F} \right\}$ is relatively compact in $L^1(B_a)$ (or $L^1(\mathbb{R}^2)$). Because $\mathcal{F}$ is bounded in $L^1_\ell(\mathbb{R}^2)$, $\mathcal{G}$ is bounded in $L^1(\mathbb{R}^2)$. Thus, in order to prove $\mathcal{G}$ is relatively compact in $L^1(\mathbb{R}^2)$, according to Frechet-Kolmogorov, we only need to prove

$$\sup_{v \in \mathcal{G}} \iint_{\mathbb{R}^2} |v(y+h) - v(y)|\, dy \to 0$$

as $h \to 0$. Let $\epsilon > 0, |h| < \epsilon$, we have

$$\iint_{\mathbb{R}^2} |v(y+h) - v(y)|\, dy = \iint_{|y|>a} |v(y+h) - v(y)|\, dy +$$

$$+ \iint_{2\epsilon<|y|<a} |v(y+h) - v(y)|\, dy + \iint_{|y|<2\epsilon} |v(y+h) - v(y)|\, dy := I_1 + I_2 + I_3$$

where $y = (r - R, x_3)\,, h = (h_1, h_2)\,.$

Since $v = s^\ell u, u \in \mathcal{F}$, supp$v \subset B_a$. Therefore, for $0 \leq |h| < \epsilon < a$

$$\begin{aligned} I_1 &= \iint_{a<|y|<a+\epsilon} |y+h|^\ell |u(y+h)|\, dy \\ &\leq (2a)^{|\ell|} \iint_{a<|y|<a+\epsilon} |u(y+h)|\, dy \\ &\leq (2a)^{|\ell|} \left(\iint_{a-\epsilon<|y|<a} s^\ell |u(y)|^{\frac{2(l+2)}{\alpha}} dy \right)^{\frac{\alpha}{2(\ell+2)}} \left(\iint_{a-\epsilon<|y|<a} s^{-\frac{\alpha}{2(\ell+2)-\alpha}} \right)^{1-\frac{\alpha}{2(\ell+2)}}. \end{aligned}$$

Since $0 < \alpha \leq \ell + 2$ and

$$\iint_{a-\epsilon<|y|<a} s^{-\frac{\alpha}{2(\ell+2)-\alpha}} dy = 2\pi \int_{a-\epsilon}^a s^{1-\frac{\alpha}{2(\ell+2)-\alpha}} ds \leq C\epsilon,$$

we have

$$I_1 \leq C\epsilon^{1-\frac{\alpha}{2(\ell+2)}}, \forall u \in \mathcal{F}. \tag{8}$$

Because

$$\begin{aligned} I_3 &\leq \iint_{|y|<2\epsilon} \left(|v(y+h)| + |v(y)|\right) dy \leq 2\iint_{|y|<3\epsilon} |v(y)|\, dy = 2\iint_{|y|<3\epsilon} s^\ell |u(y)|\, dy \\ &\leq 2\left(\iint_{|y|<3\epsilon} s^\ell dy\right)^{1-\frac{\alpha}{2(l+2)}} \left(\iint_{|y|<3\epsilon} s^\ell |u(y)|^{2\frac{(\ell+2)}{\alpha}}\, dy\right)^{\frac{\alpha}{2(\ell+2)}} \end{aligned}$$

and

$$\iint_{|y|<3\epsilon} s^\ell dy = 2\pi \int_0^{3\epsilon} s^{\ell+1} ds = \frac{2\pi}{l+2}(3\epsilon)^{\ell+2}$$

we get

$$I_3 \leq C\epsilon^{\frac{2(\ell+2)-\alpha}{2}}, \forall u \in \mathcal{F}. \tag{9}$$

Note that $v(y+h) - v(y) = \int_0^1 h \cdot \nabla v(y+th)\, dt$ and $\nabla v = s^\ell \nabla u + \left(\ell s^{\ell-2} u\right) y$ so

$$\begin{aligned} I_2 &= \iint_{2\epsilon<|y|<a} \left| \int_0^1 h \cdot \nabla v(y+th)\, dt \right| dy \\ &\leq |h| \int_0^1 \left(\iint_{2\epsilon<|y|<a} |\nabla v(y+th)|\, dy \right) dt \\ &\leq \epsilon \iint_{\epsilon<|y|<a} |\nabla v(y)|\, dy \end{aligned}$$

$$\leq C\epsilon \left[\iint_{\epsilon<|y|<a} s^\ell |\nabla u|\, dy + \iint_{\epsilon<|y|<a} s^{\ell-1} |u|\, dy \right] := C\epsilon \left[J_1 + J_2\right]. \tag{10}$$

Again using Holder's inequality we have

$$J_1 \leq \left(\iint_{\epsilon<|y|<a} s^\alpha |\nabla u|^2 dy \right)^{\frac{1}{2}} \left(\iint_{\epsilon<|y|<a} s^{2\ell-\alpha} dy \right)^{\frac{1}{2}}, \tag{11}$$

$$J_2 \leq \left(\iint_{\epsilon<|y|<a} s^\alpha |u|^{\frac{2(2+\ell)}{\alpha}} dy \right)^{\frac{\alpha}{2(2+\ell)}} \left(\iint_{\epsilon<|y|<a} s^{\ell-\frac{2(2+\ell)}{2(2+\ell)-\alpha}} dy \right)^{1-\frac{\alpha}{2(2+\ell)}}. \tag{12}$$

Note that $\ell - \frac{2(\ell+2)}{2(\ell+2)-\alpha} + 2 = \frac{(\ell+2)}{2(\ell+2)-\alpha}(2\ell - \alpha + 2)$.
If $2\ell - \alpha = -2$ then $\ell - \frac{2(\ell+2)}{2(\ell+2)-\alpha} = -2$. Since

$$\iint_{\epsilon<|y|<a} s^{-2} dy = 2\pi \int_\epsilon^a s^{-1} ds = 2\pi \ln\left(\frac{\alpha}{\epsilon}\right)$$

and (10)-(11)-(12), we get

$$I_2 \leq C\epsilon \ln\left(\frac{\alpha}{\epsilon}\right), \forall u \in \mathcal{F}. \tag{13}$$

If $2\ell - \alpha < -2$ then

$$\iint_{\epsilon<|y|<a} s^{2\ell-2} dy = 2\pi \int_{\epsilon}^{a} s^{2\ell-\alpha+1} ds \leq \frac{2\pi}{\alpha - 2 - 2\ell} \epsilon^{2\ell-\alpha+2},$$

$$\begin{aligned}\iint_{\epsilon<|y|<a} s^{\ell-\frac{2(2+\ell)}{2(2+\ell)-\alpha}} dy &= 2\pi \int_{\epsilon}^{a} s^{\ell+1-\frac{2(2+\ell)}{2(2+\ell)-\alpha}} ds \\ &\leq \frac{2\pi\left(2\left(2+\ell\right)-\alpha\right)}{\left(\ell+2\right)\left(\alpha-2-2\ell\right)} \epsilon^{\frac{(\ell+2)(2\ell-\alpha+2)}{2(2+\ell)-\alpha}}.\end{aligned}$$

Thus, from (10)-(11)-(12) we get

$$I_2 \leq C\epsilon^{\frac{2(\ell-\alpha+2)+\alpha}{2}}, \forall u \in \mathcal{F}. \tag{14}$$

If $2\ell - \alpha > -2$ then from

$$\iint_{\epsilon<|y|<a} s^{2\ell-\alpha} dy \leq C_1, \iint_{\epsilon<|y|<a} s^{\ell-\frac{2(2+\ell)}{2(2+\ell)-\alpha}} dy \leq C_2$$

and (10)-(11)-(12) we get

$$I_2 \leq C\epsilon, \forall u \in \mathcal{F}. \tag{15}$$

From (8)-(9) and (13)-(14)-(15) we conclude that $\mathcal{F}$ is relatively compact in $L^1_\ell(B_a)$.
Consider the case $1 < p < \frac{2(\ell+2)}{\alpha}$ we have

$$\iint_{B_a} s^\ell |u|^p dy \leq \left(\iint_{B_a} s^\ell |u|^{\frac{2(\ell+2)}{\alpha}} dy\right)^\lambda \left(\iint_{B_a} s^\ell |u| dy\right)^{1-\lambda}$$

with $p = 1 - \lambda + \frac{2(2+\ell)}{\alpha}\lambda$. Therefore, $\mathcal{F}$ is relatively compact in $L^p_\ell(B_a)$. □

Proof of Proposition 1. We can consider $H^{1,\alpha}_{0,sym}(T(R,a))$ as $H^{1,\alpha}_0(B_a)$ so from Lemma 1 we obtain Proposition 1. □

Proposition 2. *The Nemytskii mapping $u \mapsto s^\ell |u|^p$ is continuous from $L^{pq}_\ell(T(R,a))$ to $L^q_{\ell(1-q)}(T(R,a))$, when $\ell \in \mathbb{R}, 1 \leq p < \infty, 1 < q < \infty$. Moreover, it is compact from $H^{1,\alpha}_{0,sym}(T(R,a))$ to $L^q_{\ell(1-q)}(T(R,a))$ when either $\alpha = 0, \ell > -2$ and $1 \leq p < \infty, 1 < q < \infty$ or $\alpha > 0, \ell \geq \alpha - 2, 1 \leq p < \frac{2(\ell+2)}{\alpha} - 1, q = \frac{2(\ell+2)}{2(\ell+2)-\alpha}$.*

Proof. Using Propopsition 1 and the continuity of Nemytskii mapping it is not difficult to get the compactness of this mapping. The proof of the continuity of Nemytskii is elementary. □

Proposition 3. *Let $u \in H^{1,\alpha}_{0,sym}(T(R,a))$. Then we have $u^- \in H^{1,\alpha}_{0,sym}(T(R,a))$, $u^- \geq 0$, $\nabla u^- = -\chi_{\{u<0\}}\nabla u$, where $u^- = \max\{0, -u\}$.*

Proof. Noting that C_0^1 is dense in $H^{1,\alpha}_{0,sym}$, the proof of Proposition 3 is similar to the proof of the same result for $u \in H_0^1$. □

4 The proofs of the main results

4.1 The proof of Theorem 1

To prove problem (1) - (2) has a positive solution $u \in H^{1,\alpha}_{0,sym}(T(R,a))$, we consider the following function

$$J : H^{1,\alpha}_{0,sym}(T(R,a)) \to \mathbb{R}$$

$$J(u) = \frac{1}{2}\iiint_{T(R,a)} s^\alpha |\nabla u|^2 dx - \frac{1}{p+1}\iiint_{T(R,a)} s^\ell u |u|^p dx. \tag{16}$$

By using Mountain Pass Lemma, we imply that J has a nontrivial critical point $u \in H^{1,\alpha}_{0,sym}(T(R,a))$, i. e.

$$J'(u)(\phi) = \iiint_{T(R,a)} s^\alpha \nabla u \cdot \nabla \phi dx - \iiint_{T(R,a)} s^\ell |u|^p \phi dx = 0, \forall \phi \in H^{1,\alpha}_{0,sym}(T(R,a)). \tag{17}$$

Next we show that (17) is valid for all $\phi \in H_0^{1,\alpha}(T(R,a))$, that means the nontrivial critical point u is a weak solution of the following problem

$$-div\left(s^\alpha \nabla u\right) = s^\ell |u|^p \text{ in } T(R,a), \tag{18}$$

$$u = 0 \text{ on } \partial T(R,a). \tag{19}$$

By using the Maximum Principle for solution of the problem (18) - (19) we have $u > 0$ in $T(R,a)$. Therefore u is a positive solution of the problem (1) - (2). In order to use Mountain Pass Lemma, we will show that J satisfies all conditions of Mountain Pass Lemma.

Lemma 2. *We have the following assertions:*
(i) $J(0) = 0$.
(ii) $\exists \rho > 0, \exists \gamma > 0$ *such that* $\forall u \in H^{1,\alpha}_{0,sym}(T(R,a)), \|u\| = \rho$ *then* $J(u) \geq \gamma$.
(iii) $\exists e \in H^{1,\alpha}_{0,sym}(T(R,a)), \|e\|_{H^{1,\alpha}} > \rho, J(e) = 0$.

Proof. (i) is obvious.

(ii) From the assumptions of α, ℓ, p we imply that the embedding $H^{1,\alpha}_{0,sym}(T(R,a)) \hookrightarrow L^{p+1}_{\ell,sym}(T(R,a))$ is continuous. Then we have

$$\left|\iiint_{T(R,a)} s^{\ell} u\,|u|^p\,dx\right| \leq \iiint_{T(R,a)} s^{\ell}\,|u|^{p+1}\,dx \leq C\left(\iiint_{T(R,a)} s^{\alpha}\,|\nabla u|^p\,dx\right)^{\frac{p+1}{2}}.$$

Thus,

$$J(u) \geq \frac{1}{2}\,\|u\|^2_{H^{1,\alpha}} - \frac{C}{p+1}\,\|u\|^{p+1}_{H^{1,\alpha}}. \tag{20}$$

Because $p > 1$, from (20), (ii) is obvious.

(iii) Fix $u_0 \in H^{1,\alpha}_{0,sym}(T(R,a))$ such that $u_0 \geq 0, \|u_0\|_{H^{1,\alpha}} > 0$. Let $\alpha > 0$, we have

$$J(\lambda u_0) = \frac{\lambda^2}{2}\iiint_{T(R,a)} s^{\alpha}\,|\nabla u_0|^2\,dx - \frac{\lambda^{p+1}}{p+1}\iiint_{T(R,a)} s^{\ell}\,|u_0|^{p+1}\,dx.$$

It is easy to see that, since $p > 1, \lim_{\lambda\to+\infty} J(\lambda u_0) = -\infty$. From here, it is obvious that (iii) holds. □

We only need to prove that J satisfies the Palais-Smale condition.

Lemma 3. *Suppose that $\{u_n\}_{n\in\mathbb{N}}$ in $H^{1,\alpha}_{0,sym}(T(R,a))$ satisfies*

$$|J(u_n)| \leq M, \forall n \in \mathbb{N}, \tag{21}$$

$$\lim_{n\to+\infty} J'(u_n) = 0. \tag{22}$$

Then there exists a subsequence of $\{u_n\}_{n\in\mathbb{N}}$ that converges in $H^{1,\alpha}_{0,sym}(T(R,a))$.

Proof. For $u \in H^{1,\alpha}_{0,sym}(T(R,a))$ we have $J'(u_n) \in \left(H^{1,\alpha}_{0,sym}(T(R,a))\right)'$. According to Riesz representation Theorem, there exists a unique element $v \in H^{1,\alpha}_{0,sym}(T(R,a))$ such that

$$J'(u)(\phi) = \iiint_{T(R,a)} s^{\alpha}\nabla v\cdot\nabla\phi dx, \forall\phi \in H^{1,\alpha}_{0,sym}(T(R,a)).$$

We will define v as follows.

Case 1. $\alpha = 0$. For $q > 1$, by Proposition 2, we have $f(x,u) = s^{\ell}\,|u|^p \in L^q_{\ell(1-q)}(T(R,a))$. Also we have

$$\left|\iiint_{T(R,a)} f(x,u)\phi(x)dx\right| \leq \left(\iiint_{T(R,a)} s^{\ell(1-q)}\,|f|^q\,dx\right)^{\frac{1}{q}}\left(\iiint_{T(R,a)} s^{\ell}\,|\phi|^{\frac{q}{q-1}}\,dx\right)^{1-\frac{1}{q}}$$

Since the embedding $H^{1,0}_{0,sym}(T(R,a)) \hookrightarrow L_{\ell}^{\frac{q}{q-1}}(T(R,a))$ is continuous, the functional $\phi \mapsto \iiint_{T(R,a)} f(x,u)\phi(x)dx$ is an element in $\left(H^{1,0}_{0,sym}(T(R,a))\right)'$.

Case 2. $\alpha > 0$. By Proposition 2, we have $f(x,u) = s^{\ell}|u|^p \in L^{\frac{2(\ell+2)}{2(\ell+2)-\alpha}}_{\frac{-\alpha\ell}{2(\ell+\alpha)-\alpha}}(T(R,a))$. Also we have

$$\left|\iiint_{T(R,a)} f(x,u)\phi(x)dx\right| \le \left(\iiint_{T(R,a)} s^{\frac{-\alpha l}{2(l+\alpha)-\alpha}}|f|^{\frac{2(\ell+2)}{2(\ell+2)-\alpha}}dx\right)^{1-\frac{\alpha}{2(\ell+2)}} \times$$

$$\times\left(\iiint_{T(R,a)} s^{\ell}|\phi|^{\frac{2(\ell+2)}{\alpha}}dx\right)^{\frac{\alpha}{2(\ell+2)}}.$$

Since the embedding $H^{1,\alpha}_{0,sym}(T(R,a)) \hookrightarrow L_{\ell}^{\frac{2(\ell+2)}{\alpha}}(T(R,a))$ is continuous, the functional $\phi \mapsto \iiint_{T(R,a)} f(x,u)\phi(x)dx$ is an element in $\left(H^{1,\alpha}_{0,sym}(T(R,a))\right)'$.
Then according to Riesz representation Theorem, there exists a unique element $w = Tu \in H^{1,\alpha}_{0,sym}(T(R,a))$ such that

$$\iiint_{T(R,a)} s^{\alpha}\nabla v\cdot\nabla\phi dx = \iiint_{T(R,a)} f(x,u)\phi(x)dx, \forall\phi \in H^{1,\alpha}_{0,sym}(T(R,a)).$$

Therefore, from (17) we can rewrite

$$v = J'(u) = u - w = u - Tu \in H^{1,\alpha}_{0,sym}(T(R,a)) \tag{23}$$

We look at the way to define Tu

$$T : H^{1,\alpha}_{0,sym}(T(R,a)) \to L^{\frac{2(\ell+2)}{2(\ell+2)-\alpha}}_{\frac{-\alpha\ell}{2(\ell+\alpha)-\alpha}}(T(R,a)) \to H^{1,\alpha}_{0,sym}(T(R,a)), u \mapsto s^{\ell}|u|^p \mapsto w.$$

By Proposition 2, $u \mapsto s^{\ell}|u|^p$ is compact. Since $s^{\ell}|u|^p \mapsto w$ is continuous, the map

$$T : H^{1,\alpha}_{0,sym}(T(R,a)) \to H^{1,\alpha}_{0,sym}(T(R,a))$$

is compact. From (23) and from the assumption that $J'(u_n) \to 0$ as $n \to \infty$ we only need to prove that $\{u_n\}$ is bounded in $H^{1,\alpha}_{0,sym}(T(R,a))$. We have

$$J'(u_n)(u_n) = \iiint_{T(R,a)} s^{\alpha}|\nabla u_n|^2 dx - \iiint_{T(R,a)} s^{\ell}|u_n|^p u_n dx.$$

Since $J'(u_n) \to 0$ as $n \to \infty$, there exists N_0 such that

$$\left|\iiint_{T(R,a)} s^{\alpha}|\nabla u_n|^2 dx - \iiint_{T(R,a)} s^{\ell}|u_n|^p u_n dx\right| \le \|u_n\|_{H^{1,\alpha}}, \forall n \ge N_0. \tag{24}$$

Since $|J(u_n)| \leq M$ or

$$\left| \frac{1}{2} \iiint_{T(R,a)} s^{\alpha} |\nabla u_n|^2 \, dx - \frac{1}{p+1} \iiint_{T(R,a)} s^{\ell} u_n |u_n|^p \, dx \right| \leq M, \forall n,$$

and (24) we have

$$\begin{aligned} \|u_n\|^2_{H^{1,\alpha}} &\leq 2M + \frac{2}{p+1} \iiint_{T(R,a)} s^{\ell} u_n |u_n|^p \, dx \\ &\leq 2M + \frac{2}{p+1} \left(\|u_n\|_{H^{1,\alpha}} + \|u_n\|^2_{H^{1,\alpha}} \right), \forall n \geq N_0. \end{aligned}$$

Therefore, $\frac{p-1}{p+1} \|u_n\|^2_{H^{1,\alpha}} - \frac{2}{p+1} \|u_n\|_{H^{1,\alpha}} \leq 2M, \forall n \geq N_0$. So $\{u_n\}$ is bounded in $H^{1,\alpha}_{0,sym}(T(R,a))$. □

In conclusion, J satisfies all conditions of Mountain Pass Lemma, so J has a nontrivial critical solution $u \in H^{1,\alpha}_{0,sym}(T(R,a))$, i.e. $u \neq 0$ and

$$J'(u)(\phi) = \iiint_{T(R,a)} s^{\alpha} \nabla u \cdot \nabla \phi dx - \iiint_{T(R,a)} s^{\ell} |u|^p \phi dx = 0, \forall \phi \in H^{1,\alpha}_{0,sym}(T(R,a)) \tag{25}$$

We consider the coordinate $x_1 = (R + s\cos\theta)\cos\phi, x_2 = (R + s\cos\theta)\sin\phi$, $x_3 = s\sin\theta$ we have $T(R,a) = \{0 \leq s < a, 0 \leq \theta, \phi < 2\pi\}$. Then from (25), $u \in C^{2,\beta}(T(R,a)\backslash S_a)$ for some $\beta \in (0,1)$, where $S_a = \{(x_1, x_2, 0) : x_1^2 + x_2^2 = a^2\}$, satisfies the following equation

$$\partial_s \left(s^{\alpha+1}(R + s\cos\theta)\partial_s u \right) + \partial_\theta (s^{\alpha-1}(R + s\cos\theta)\partial_\theta u) + s^{\ell+1}(R + s\cos\theta) |u|^p = 0 \tag{26}$$

in $T(R,a)\backslash S_a$.

Next we will show that the critical point $u \in H^{1,\alpha}_{0,sym}(T(R,a))$ is the weak solution of the following problem

$$-div(s^{\alpha} \nabla u) = s^{\ell} |u|^p \text{ in } T(R,a), u = 0 \text{ on } \partial T(R,a).$$

Specifically, we prove the following lemma.

Lemma 4. (25) *holds for all* $\Phi \in H^{1,\alpha}_0(T(R,a))$.

Proof. Because of density, it is enough to prove Lemma 4 for $\Phi \in C^1_0(T(R,a))$. Let $\epsilon \in (0,a)$. Note that $u \in C^{2,\beta}(T(R,a)\backslash S_a)$ satisfies

$$-div(s^{\alpha} \nabla u) = s^{\ell} |u|^p \text{ in } T(R,a)\backslash T(R,\epsilon), u = 0 \text{ on } \partial T(R,a).$$

Then by using Divergence Theorem we have

$$\iiint_{T(R,a)\backslash T(R,\epsilon)} s^{\alpha} \nabla u . \nabla \Phi dx = - \iiint_{T(R,a)\backslash T(R,\epsilon)} div(s^{\alpha} \nabla u) \Phi dx +$$

$$+\iint_{\partial(T(R,a)\setminus T(R,\epsilon))}(\nu\cdot s^{\alpha}\nabla u)\Phi dS=\iiint_{T(R,a)\setminus T(R,\epsilon)}s^{\ell}\left|u\right|^{p}dx+\iint_{\partial T(R,\epsilon)}(\nu\cdot s^{\alpha}\nabla u)\Phi dS.$$

We need to prove

$$\lim_{\epsilon\to 0^{+}}\iint_{\partial T(R,\epsilon)}(\nu\cdot s^{\alpha}\nabla u)\Phi dS=0. \tag{27}$$

In coordinate system (s,θ,φ), we rewrite

$$\iint_{\partial T(R,\epsilon)}(\nu\cdot s^{\alpha}\nabla u)\Phi dS=\int_{0}^{2\pi}\int_{0}^{2\pi}\epsilon^{\alpha+1}(R+\epsilon\cos\theta)u_{s}(\epsilon,\theta)\Phi(\epsilon,\theta,\varphi)d\theta d\varphi. \tag{28}$$

By integrating $\int_{\epsilon}^{a}ds$ both sides of (26) we get

$$a^{\alpha+1}(R+a\cos\theta)u_{s}(a,\theta)-\epsilon^{\alpha+1}(R+\epsilon\cos\theta)u_{s}(\epsilon,\theta)=$$

$$-\int_{\epsilon}^{a}\partial_{\theta}(s^{\alpha-1}(R+s\cos\theta)u_{\theta})ds-\int_{\epsilon}^{a}s^{\ell+1}(R+s\cos\theta)\left|u\right|^{p}ds.$$

By multiplying $\Phi(\epsilon,\theta,\varphi)$ and then integrating $\int_{0}^{2\pi}\int_{0}^{2\pi}d\theta d\varphi$ both sides of the above equation, we get

$$\begin{aligned}&\int_{0}^{2\pi}\int_{0}^{2\pi}a^{\alpha+1}(R+a\cos\theta)u_{s}(a,\theta)\Phi(\epsilon,\theta,\varphi)d\theta d\varphi-\\&-\epsilon^{\alpha+1}\int_{0}^{2\pi}\int_{0}^{2\pi}(R+\epsilon\cos\theta)u_{s}(\epsilon,\theta)\Phi(\epsilon,\theta,\varphi)d\theta d\varphi\\&=-\int_{\epsilon}^{a}\int_{0}^{2\pi}\int_{0}^{2\pi}s^{\alpha-1}(R+s\cos\theta)u_{\theta}(s,\theta)\Phi_{\theta}(\epsilon,\theta,\varphi)d\theta d\varphi ds\\&-\int_{\epsilon}^{a}\int_{0}^{2\pi}\int_{0}^{2\pi}s^{\ell+1}(R+s\cos\theta)\left|u\right|^{p}\Phi(\epsilon,\theta,\varphi)d\theta d\varphi ds.\end{aligned} \tag{29}$$

Since $u\in H^{1,\alpha}_{0,sym}(T(R,a))\hookrightarrow L^{p}_{\ell,sym}(T(R,a))$, we have

$$\int_{0}^{a}\int_{0}^{2\pi}\int_{0}^{2\pi}s^{\ell+1}(R+s\cos\theta)\left|u\right|^{p}d\theta d\varphi ds<+\infty,$$

$$\int_{0}^{a}\int_{0}^{2\pi}\int_{0}^{2\pi}s^{\alpha-1}(R+s\cos\theta)\left|u_{\theta}\right|^{2}d\theta d\varphi ds<+\infty.$$

Moreover, $\Phi\in C_{0}^{1}(T(R,a))$ so the right side of (29) converges as $\epsilon\to 0^{+}$. Thus, there exists a limit $A=\lim_{\epsilon\to 0^{+}}\int_{0}^{2\pi}\int_{0}^{2\pi}\epsilon^{\alpha+1}(R+\epsilon\cos\theta)u_{s}(\epsilon,\theta)\Phi(\epsilon,\theta,\varphi)d\theta d\varphi$.

To prove (27), from (28) we only need to prove $A = 0$. Suppose that $A \neq 0$. Then there exists $\epsilon_0 > 0$ such that

$$\left| \int_0^{2\pi} \int_0^{2\pi} \epsilon^{\alpha+1}(R + \epsilon \cos\theta) u_s(\epsilon, \theta) \Phi(\epsilon, \theta, \varphi) d\theta d\varphi \right| > \frac{|A|}{2}, 0 < \epsilon < \epsilon_0.$$

Since $\Phi \in C_0^1(T(R, a))$, there exists $M > 0$ such that

$$|\Phi(\epsilon, \theta, \varphi)| \leq M, \forall (\epsilon, \theta, \varphi) \in T(R, a).$$

Then

$$\int_0^{2\pi} \int_0^{2\pi} \epsilon^{\alpha+1}(R + \epsilon \cos\theta) |u_s(\epsilon, \theta)|^2 d\theta d\varphi \geq C\epsilon^{-\alpha-1}, 0 < \epsilon < \epsilon_0.$$

This contradicts to $\int_0^a \int_0^{2\pi} \epsilon^{\alpha+1}(R + \epsilon \cos\theta) |u_s(\epsilon, \theta)|^2 d\epsilon d\theta < +\infty$. Therefore, $A = 0$. $\square$

Next we will prove that $u > 0$ in $T(R, a)$. In fact, by Lemma 4 and Proposition 3 we have

$$\iiint_{T(R,a)} s^\alpha \nabla u . \nabla u^- dx = \iiint_{T(R,a)} s^\ell |u|^p u^- dx$$

or

$$-\iiint_{T(R,a)} s^\alpha \left|\nabla u^-\right|^2 dx = \iiint_{T(R,a)} s^\ell \left|u^-\right|^p dx.$$

So $u^- = 0$ a.e. in $T(R, a)$. In other words, $u \geq 0$ a.e. in $T(R, a)$. Since $-div(s^\alpha \nabla u) = s^\ell |u|^p \geq 0$ in $T(R, a) \backslash S_a$ and $u \neq 0$, by strong maximum principle $u > 0$ in $T(R, a) \backslash S_a$. Theorem 1 is proved.

4.2 The proof of Theorem 2

To prove Theorem 2 for nonexistence of nontrivial solution of problem (1) - (2), we need the following Pohozaev-type identity.

Lemma 5. *Suppose that u is a weak solution of problem* (1) - (2). *Choose a field $m = (\frac{r-R}{r}x_1, \frac{r-R}{r}x_2, \frac{R}{r}x_3)$. Then we have*

$$\iiint_{T(R,a)} (div\ m)\mathcal{K}F(u)dx + \iiint_{T(R,a)} (m \cdot \nabla \mathcal{K})F(u)dx =$$

$$= \frac{1}{2}\left[\iiint_{T(R,a)} (div\ m)J |\nabla u|^2 dx + \iiint_{T(R,a)} (m \cdot \nabla J) |\nabla u|^2 dx\right] -$$

$$- \sum_{i,j=1}^{3} \iiint_{T(R,a)} J(\partial_i m_j)\partial_i u \partial_j u dx + \frac{1}{2}\iint_{\partial T(R,a)} J(m \cdot \nu) |\partial_\nu u|^2 dS$$

where $\mathcal{K} = s^\ell, J = s^\alpha, f(u) = |u|^{p-1} u, F(u) = \int_0^u f(t)dt = \frac{1}{p+1} |u|^{p+1}$.

Proof. Because (2), we have $\nabla u = (\nabla u, \nu)\nu$ and

$$\partial_\nu u(m \cdot \nabla u) = (m \cdot \nu) \left|\partial_\nu u\right|^2 . \tag{30}$$

Since u is a solution of (1),

$$\iiint_{T(R,a)} div\ (J\nabla u).(m \cdot \nabla u) dx = -\iiint_{T(R,a)} \mathcal{K} f(u)(m \cdot \nabla u) dx. \tag{31}$$

Since $f(s) = F'(s)$,we have $\mathcal{K} f(u) m_j \partial_j u = m_j \partial_j(\mathcal{K} F(u)) - (m_j \partial_J \mathcal{K}) F(u)$. Then

$$\iiint_{T(R,a)} \mathcal{K} f(u)(m\cdot\nabla u)dx = \iiint_{T(R,a)} m\cdot\nabla(\mathcal{K}F(u))dx - \iiint_{T(R,a)} (m\cdot\nabla(\mathcal{K})F(u)dx.$$

By using integation by parts we get

$$\iiint_{T(R,a)} m\cdot\nabla(\mathcal{K}F(u))dx = -\iiint_{T(R,a)} (div\ m)\mathcal{K}F(u)dx + \iint_{\partial T(R,a)} (m\cdot\nu)\mathcal{K}F(u)dS.$$

From (2), we get $F(u)\Big|_{\partial T(R,a)} = 0$. So

$$\iiint_{T(R,a)} \mathcal{K} f(u)(m\cdot\nabla u)dx = -\iiint_{T(R,a)} div\ (m)\mathcal{K}f(u)dx - \iiint_{T(R,a)} (m\cdot\nabla(\mathcal{K})F(u)dx. \tag{32}$$

By using integation by parts again we get

$$\iiint_{T(R,a)} div\ (J\nabla u)(m \cdot \nabla u)dx = -\iiint_{T(R,a)} J\nabla u \cdot \nabla(m \cdot \nabla u)dx +$$

$$+\iint_{\partial T(R,a)} J\partial_\nu u(m \cdot \nabla u)dS. \tag{33}$$

We have

$$\begin{aligned}
J\nabla u \cdot \nabla(m \cdot \nabla u) &= \sum_{i=1}^{3} J\partial_i u.\partial_i(\sum_{j=1}^{3} m_j \partial_j u) \\
&= \sum_{i,j=1}^{3} J m_j \partial_i u \partial_{i,j}^2 u + \sum_{i,j=1}^{3} J(\partial_i m_j)\partial_i u.\partial_j u \\
&= \sum_{i,j=1}^{3} J m_j \frac{\partial_j \left|\partial_i u\right|^2}{2} + \sum_{i,j=1}^{3} J(\partial_i m_j)\partial_i u.\partial_j u \\
&= \sum_{j=1}^{3} J m_j \frac{\partial_j \mid \nabla u|^2}{2} + \sum_{i,j=1}^{3} J(\partial_i m_j)\partial_i u.\partial_j u,
\end{aligned}$$

$$\iiint_{T(R,a)} Jm_j\frac{\partial_j \mid \nabla u|^2}{2}dx = -\iiint_{T(R,a)} \partial_j(Jm_j)\frac{\mid \nabla u|^2}{2}dx + \iint_{\partial T(R,a)} Jm_j\nu_j\frac{\mid \nabla u|^2}{2}dS.$$

So

$$\iiint_{T(R,a)} J\nabla u \cdot \nabla(m\cdot\nabla u)dx = -\frac{1}{2}\iiint_{T(R,a)} (div\ m)J \mid \nabla u|^2\, dx -$$

$$-\frac{1}{2}\iiint_{T(R,a)} (m\cdot\nabla J)J \mid \nabla u|^2\, dx + \frac{1}{2}\iint_{\partial T(R,a)} J(m\cdot\nu) \mid \nabla u|^2\, dS +$$

$$+\sum_{i,j=1}^{3}\iiint_{T(R,a)} J(\partial_i m_j)\partial_i u.\partial_j u dx. \tag{34}$$

From (30), (33), (34) we have

$$\iiint_{T(R,a)} div(J\nabla u)(m\cdot\nabla u)dx = \frac{1}{2}\iiint_{T(R,a)} (div\ m)J \mid \nabla u|^2\, dx +$$

$$+\frac{1}{2}\iiint_{T(R,a)} (m\cdot\nabla J)J \mid \nabla u|^2\, dx + \frac{1}{2}\iint_{\partial T(R,a)} J(m\cdot\nu) \mid \nabla u|^2\, dS -$$

$$-\sum_{i,j=1}^{3}\iiint_{T(R,a)} J(\partial_i m_j)\partial_i u.\partial_j u dx. \tag{35}$$

From (31), (32), (35), we have the conclusion. □

For $m, \mathcal{K}, J$ as in Lemma 5, and $\lambda = a/R$ we have the following lemma.

Lemma 6. *(i)* $div(m) = 2$.
(ii) $(\nabla J\cdot m) = \alpha s^{\alpha-2}\left((r-R)^2 + \frac{R}{r}x_3^2\right) \geq \frac{\alpha}{\lambda+1}J.$
(iii) $(\nabla\mathcal{K}\cdot m) = \ell s^{\ell-1}\left((r-R)^2 + \frac{R}{r}x_3^2\right) \leq \frac{\ell}{1-\delta\lambda}\mathcal{K}$ *where* $\delta = \begin{cases} 1 & \textit{for } \ell \geq 0, \\ -1 & \textit{for } \ell < 0. \end{cases}$
(iv) $m\cdot\nu = \frac{(r-R)^2}{a} + \frac{Rx_3^2}{ra} > 0$ *on* $\partial T(R,a)$.
(v) $\sum_{i,j=1}^{3}\partial_i m_j\xi_i\xi_j = \frac{r^3-Rx_2^2}{r^3}\xi_1^2 + \frac{r^3-Rx_1^2}{r^3}\xi_2^2 + \frac{R}{r}\xi_3^2 + \frac{2Rx_1x_2}{r^3}\xi_1\xi_2 - \frac{Rx_1x_3}{r^3}\xi_1\xi_3 - \frac{Rx_2x_3}{r^3}\xi_2\xi_3.$

Assume that $u \in H_0^{1,\alpha}(T(R,a))$ is a nontrivial weak solution of (1) - (2), we have

$$\iiint_{T(R,a)} J|\nabla u|^2 dx = (p+1)\iiint_{T(R,a)} \mathcal{K}F(u)dx. \tag{36}$$

From Lemma 5 and Lemma 6 we obtain that

$$\left(2+\frac{\ell}{1-\delta\lambda}\right)\iiint_{T(R,a)}\mathcal{K}F(u)dx \geq \left(1+\frac{\alpha}{2(1+\lambda)}\right)\iiint_{T(R,a)}\mathcal{K}F(u)dx- \\ -\iiint_{T(R,a)} J\sum_{i,j=1}^{3}\partial_i m_j \partial_i u \partial_j u dx. \qquad (37)$$

For $\epsilon > 0$ we consider the following matrix

$$M = \begin{pmatrix} \epsilon + \frac{Rx_2^2}{r^3} & -\frac{Rx_1x_2}{r^3} & \frac{Rx_1x_3}{2r^3} \\ -\frac{Rx_1x_2}{r^3} & \epsilon + \frac{Rx_1^2}{r^3} & \frac{Rx_2x_3}{2r^3} \\ \frac{Rx_1x_3}{2r^3} & \frac{Rx_2x_3}{2r^3} & 1+\epsilon-\frac{R}{r} \end{pmatrix}.$$

In order to have $\sum_{i,j=1}^{3}\partial_i m_j \xi_i \xi_j \leq (1+\epsilon)|\xi|^2, \forall \xi \in \mathbb{R}^3$, the matrix M is positive semi-define. It is not difficult to see that if

$$\left(\epsilon^2+\frac{R}{r}\epsilon\right)\left(1+\epsilon-\frac{R}{r}\right)-\frac{R^2x_3^2}{4r^4}\epsilon-\frac{R^3x_3^2}{4r^5}\geq 0$$

for $(r/R-1)^2+(x_3/R)^2 < \lambda^2$ then M is positive semi-define. So we can choose $\epsilon = 2\sqrt{\lambda}/(1-\lambda)^2 > 0$ such that

$$\sum_{i,j=1}^{3}\partial_i m_j \xi_i \xi_j \leq (1+\epsilon)|\xi|^2, \forall \xi \in \mathbb{R}^3.$$

Hence, from (37) - (36) and noting that u is nontrivial solution we get

$$2+\frac{\ell}{1-\delta\lambda} \geq (p+1)\left(\frac{\alpha}{2(1+\lambda)}-\epsilon\right). \qquad (38)$$

So for α, ℓ, p as in Theorem 2 there is $\epsilon_0 > 0$ such that (38) does not hold if $0 < \lambda = a/R < \epsilon_0$. Therefore Theorem 2 is proved.

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