Polya - Szego Type Inequality and Imbedding Theorems for Weighted Sobolev Spaces

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Abstract. In this paper we will establish a Polya - Szego type inequality for a weighted gradient of a function on $\mathbb{R}^2$ with respect to to a weighted area. In order to do that we need to study an isoperimetric problem for the weighted area. We then apply the inequality to prove embedding theorems for a weighted Sobolev space and to calculate the best constant in the Sobolev imbedding theorem. In our upcoming manuscript the obtained results in this note will be used to study boundary value problems for semilinear degenerate elliptic equations.

1 Introduction

The classical Polya - Szego inequality is a very important tool in many branches of mathematics such a analysis, partial differential equations, geometry, ... In this note, motived by studying degenerate elliptic equations, we will extend this inequality in the context of a weighted area in $\mathbb{R}^2$. To do this we need to solve an isoperimetric problem, namely to find a figure in $\bar{\mathbb{R}}^2_+$ with a fixed weighted area that has the least weighted perimeter. Here the weight of the area appears in the connection with the degenerate elliptic equations we have in mind. To solve the isoperimetric problem we use a result in [4]. The extended Polya - Szego inequality is then the result of the application of the co-area formula. Isoperimetric problems were studied by many authors, see for example [1], [2], [4], [7], [8], [9], and the references therein. As an application of the extended Polya - Szego inequality we present some imbedding theorems for weighted Sobolev spaces that come from the study of boundary value problems for semilinear degenerate elliptic equations. We also calculate the best constant and find the minimizers in the obtained Sobolev inequality. This can be seen as a generalization of the results in [2], [10], [3], [6]. We will use the obtained Sobolev embedding theorems in this paper to study semiliear degenerate equations in upcoming manuscripts.

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2 Main Results

In this section we will state our main results. For this purpose let us give some notations. Let $E \subset \mathbb{R}^2$ be a bounded open set with ∂E Lipschitz. Denote by $\nu = (\nu_1, \nu_2)$ the outward unit normal to ∂E. Let $k > 0, p > 1$ we denote $m = (k + 2 + pk)/(k + 1)$.

Definition 1. The (p, k)-area of E is defined by

$$|E|_{p,k} = \iint_E |x|^{pk} dxdy.$$

Definition 2. The (p, k)-perimeter of E is defined by

$$P_{p,k}(E) = \int_{\partial E} \sqrt{x^{2(p-1)k}\nu_1^2 + x^{2pk}\nu_2^2} d\mathcal{H}^1.$$

Note that the fraction $P_{p,k}(E)/|E|_{p,k}^{(m-1)/m}$ is invariant under the scaling $(x, y) \mapsto (\lambda x, \lambda^{k+1} y)$. Moreover we establish isoperimetric inequality as following.

Theorem 1. *Let $k > 0, p > 1, E \subset \mathbb{R}^2$ be a bounded open set with ∂E Lipschitz. Then we have*

$$\frac{P_{p,k}(E)}{|E|_{p,k}^{(m-1)/m}} \geq \frac{P_{p,k}(B_{k+})}{|B_{k+}|_{p,k}^{(m-1)/m}}$$

where $B_{k+} = \{(x, y) \in \mathbb{R}^2 : \; x^{2k+2} + (k + 1)^2 y^2 < 1, x > 0\}$.

Using the above isoperimetric inequality and co-area formular we get a new Polya-Szego type inequality. Firstly let us give a new arrangement.

Definition 3. Let $c \in \mathbb{R}, u \in C_0^\infty(\bar{\mathbb{R}}^2_{c+}; \mathbb{R}_+)$ where $\bar{\mathbb{R}}^2_{c+} = \{(x, y) \in \mathbb{R}^2 : \; x \geq c\}$. The arrangement $u^* : \bar{\mathbb{R}}^2_+ \to \mathbb{R}^+$, where $\bar{\mathbb{R}}^2_+ = \bar{\mathbb{R}}^2_{0+}$, is defined by

$$u^*(x, y) = \varphi(\rho_k) \text{ here } \rho_k = (x^{2k+2} + (k + 1)^2 y^2)^{\frac{1}{2k+2}}, \varphi : \mathbb{R}_+ \to \mathbb{R}_+,$$

such that $|\{u > t\}|_{p,k} = |\{u^* > t\}|_{p,k}$ for all $t > 0$.

Our Polya - Szego type inequality is stated as follows.

Theorem 2. *Let $k > 0, p > 1, c \in \mathbb{R}$ and $u \in C_0^\infty(\bar{\mathbb{R}}^2_{c+}; \mathbb{R}_+)$ Then*

$$\iint_{\mathbb{R}^2_+} |\nabla_G u^*|^p \, dxdy \leq \iint_{\mathbb{R}^2_{c+}} |\nabla_G u|^p \, dxdy,$$

where $|\nabla_G u| = (u_x^2 + x^{2k} u_y^2)^{1/2}$.

For giving our best Sobolev inequality, we need Sobolev type spaces as follows.

Definition 4. Let k, p as above and $q > 1$. We define $W^{1,p,q}_{0,k}(\mathbb{R}^2)$ is the completion of $C_0^\infty(\mathbb{R}^2)$ with respect to the norm

$$||u||_{W^{1,p,q}_{0,k}} = \left(\iint_{\mathbb{R}^2} |\nabla_G u|^p \, dxdy \right)^{1/p} + \left(\iint_{\mathbb{R}^2} |x|^{pk} \, |u|^q \, dxdy \right)^{1/q}.$$

For $u \in W^{1,p,q}_{0,k}(\mathbb{R}^2) \setminus \{0\}$ we consider the ratio

$$C_{p,q,k}(u) = \frac{\left(\iint_{\mathbb{R}^2} |\nabla_G u|^p \, dxdy \right)^{1/p}}{\left(\iint_{\mathbb{R}^2} x^{pk} \, |u|^q \, dxdy \right)^{1/q}}. \tag{1}$$

By rescaling $X = \lambda x, Y = \lambda^{k+1} y$, we have $U(x,y) = u(\lambda x, \lambda^{k+1} y)$ and

$$\iint_{\mathbb{R}^2} |\nabla_G U|^p \, dxdy = \lambda^{p-(k+2)} \iint_{\mathbb{R}^2} |\nabla_G u|^p \, dXdY,$$
$$\iint_{\mathbb{R}^2} x^{pk} \, |U|^q \, dxdy = \lambda^{-(p+1)k-2} \iint_{\mathbb{R}^2} X^{pk} \, |u|^q \, dXdY,$$

so $C_{p,q,k}(U) = \lambda^{\frac{p-(k+2)}{p} - \frac{pk+k+2}{q}} C_{p,q,k}(u)$. Hence, in order to have

$$\inf_{W^{1,p,q}_{0,k}(\mathbb{R}^2)} C_{p,q,k}(u) > 0$$

we need $1 < p < k+2$ and $q = \frac{p((p+1)k+2)}{k+2-p}$ is the critical exponent. For this case we obtain the best Sobolev inequality as follows:

Theorem 3. *Let $k > 0, 1 < p < k+2$ and $q = p((p+1)k+2)/(k+2-p)$. Then we have*

$$\left(\iint_{\mathbb{R}^2} x^{pk} \, |u|^q \, dxdy \right)^{1/q} \leq C^{-1}_{p,q,k} \left(\iint_{\mathbb{R}^2} |\nabla_G u|^p \, dxdy \right)^{1/p}, \forall u \in W^{1,p,q}_{0,k}(\mathbb{R}^2), \tag{2}$$

with the best constant obtained by

$$C_{p,q,k} = (k+1) m^{\frac{1}{p}} \left(\frac{p-1}{m-1} \right)^{-\frac{1}{p'}} \left[\frac{2}{(k+1)^2} B \left(\frac{2k+1}{2(k+1)}, \frac{1}{2} \right) \right]^{\frac{1}{p} - \frac{1}{q}} \left[\frac{1}{p'} B \left(\frac{m}{p}, \frac{m}{p'} \right) \right]^{\frac{1}{m}}$$

where $m = ((p+1)k+2)/(k+1), p' = p/(p-1)$. The equality sign holds in (2) if u has the form

$$u(x,y) = \left(a + b \rho_k^{p'(k+1)} \right)^{1-m/p}$$

where $\rho_k = (x^{2k+2} + (k+1)^2 y^2)^{\frac{1}{2(k+1)}}$ and a, b are positive constants.

3 Proofs

3.1 Isoperimetric Inequality

In order to proof Theorem 1 we consider isoperimetric inequality in half plane as follows.

Proposition 1. *Let $k > 0, p > 1, E \subset \bar{\mathbb{R}}^2_+$ be a bounded open set with ∂E Lipschitz. There is an $R > 0$ such that*

$$E^* = \left\{(x,y) \in \mathbb{R}^2_+ : \ \rho_k(x,y) < R\right\}, |E^*|_{p,k} = |E|_{p,k}$$

where $\rho_k(x,y) = \left(x^{2(k+1)} + (k+1)^2 y^2\right)^{\frac{1}{2(k+1)}}$. Then we have $P_{p,k}(E^*) \leq P_{p,k}(E)$.

Proof. Consider the transformation $\Psi : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ defined by

$$\Psi(x,y) = \left(\frac{x^{k+1}}{k+1}, y\right).$$

It is homeomorphism with the inverse $\Phi : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ definied by

$$\Phi(\xi,\eta) = (|(k+1)\xi|^{\frac{1}{k+1}}, \eta).$$

Let $F = \Psi(E), F^* = \Psi(E^*)$. It is not difficult to see that F^* is semi-disk in $\mathbb{R}^2_+$ with center at the origin. By calculating we get

$$|E|_{p,k} = \iint_F |(k+1)\xi|^{\frac{(p-1)k}{k+1}} d\xi d\eta, \ |E^*|_{p,k} = \iint_{F^*} |(k+1)\xi|^{\frac{(p-1)k}{k+1}} d\xi d\eta,$$
$$P_{p,k}(E) = \int_{\partial F} |(k+1)\xi|^{\frac{(p-1)k}{k+1}} d\mathcal{H}^1, \ P_{p,k}(E^*) = \int_{\partial F^*} |(k+1)\xi|^{\frac{(p-1)k}{k+1}} d\mathcal{H}^1.$$

Noting that $|E|_{p,k} = |E^*|_{p,k}$ and using Theorem 1.4 (in [4]) for F, F^* the proof is complete. $\square$

Remark 1. Let $C_0 = |P_{p,k}(B_{k+})|/|B_{k+}|^{(m-1)/m}_{p,k}$, from Proposition 1 it is not difficult to get

$$C_0 |E|^{(m-1)/m}_{p,k} \leq P_{p,k}(E) \tag{3}$$

for all open, bounded subsets E of $\bar{\mathbb{R}}^2_+$ with Lipschitz boundary ∂E. Because of symmetry isoperimetric inequality (3) still holds for open, bounded subsets E of $\bar{\mathbb{R}}^2_- = \mathbb{R}^2 \setminus \mathbb{R}^2_+$ with Lipschitz boundary.

Proof of Theorem 1. For $E \subset \mathbb{R}^2$ be open, bounded set with Lipschitz boundary, we participate

$$E = E_+ \cup E_- \text{ where } E_+ = E \cap \bar{\mathbb{R}}^2_+, E_- = E \cap \bar{\mathbb{R}}^2_-.$$

Then E_+, E_- is open, bounded subset in $\bar{\mathbb{R}}^2_+, \bar{\mathbb{R}}^2_-$ (respectively) and their boundary are Lipschitz. Hence we have isoperimetric inequality (3) for E_+, E_-. Because of $(m-1)/m \in (0,1)$ we get

$$|E|_{p,k}^{(m-1)/m} \leq |E_+|_{p,k}^{(m-1)/m} + |E_-|_{p,k}^{(m-1)/m}.$$

Using the above inequality and isoperimetric inequality (3) for E_+, E_- the proof is done. □

3.2 Polya-Szego Inequality

For $k > 0, p > 1, u \in C_0^\infty(\bar{\mathbb{R}}^2_{c+}, \mathbb{R}_+)$, let u^* be the rearrangement of u as in Definition 3. We need some technical lemmas as follows.

Lemma 1. *Let $M = \max_{\mathbb{R}^2_{c+}} u$ and $R_0 > 0$ such that $\operatorname{supp} u \subset \{(x,y) \in \bar{\mathbb{R}}^2_{c+} : \rho_k(x,y) < R_0\}$. Then the following statements hold.*

(i) *The map $t \mapsto |\{t < u \leq M\} \cap \{\nabla u = 0\}|_{p,k}$ is nonincreasing.*

(ii) *For $\rho_k > R_0, \varphi(\rho_k) = 0$. Moreover, $\varphi : [0, R_0] \to [0, M]$ is continuous, nonincreasing and $\{t \in \mathbb{R} : \exists s \in [0, R_0], \varphi(s) = t, \varphi'(s) = 0\}$ has Lebesgue measure 0 in $\mathbb{R}$.*

(iii) *The map $h : [0, M] \to [0, \infty)$ defined by*

$$h(t) = |\{t < u^* \leq M\} \cap \{\nabla u^* = 0\}|_{p,k}$$

is nonincreasing. Moreover $h'(t) = 0$ a.e. on $[0, M]$.

Proof. It is not difficult to prove (i)-(ii). For the proof of (iii) we refer to [5]. □

Lemma 2. *Assume that $u \not\equiv 0$. Denote $M = \max_{\mathbb{R}^2_+} u$. Then*

$$\iint_{\{u^*=t\}} \frac{|x|^{pk}}{|\nabla u^*|} d\mathcal{H}^1 \geq \iint_{\{u=t\}} \frac{|x|^{pk}}{|\nabla u|} d\mathcal{H}^1 \text{ for a.e. } t \in [0, M]. \tag{4}$$

Proof. For $0 < t < M$, we have

$$|\{u > t\}|_{p,k} = |\{\nabla u = 0\} \cap \{t < u \leq M\}|_k + \int_t^M d\tau \int_{\{u=\tau\} \cap \{\nabla u \neq 0\}} \frac{|x|^{pk} d\mathcal{H}^1}{|\nabla u|}.$$

By Sard's Theorem, for $u \in C_0^\infty(\bar{\mathbb{R}}^2_+)$,

$$\big\{t \in [0, M] : \exists (x, y) \in \bar{\mathbb{R}}^2_+, u(x, y) = t, \nabla u(x, y) = 0\big\}$$

has Lebesgue measure 0 in $\mathbb{R}$. Hence, using Lemma 1 (i) we have

$$-\frac{d}{dt}\left|\{u > t\}\right|_{p,k} \geq \int_{u^{-1}\{t\}} \frac{|x|^{pk} d\mathcal{H}^{-1}}{|\nabla u|} \text{ for } a.e.\ t \in [0, M]. \tag{5}$$

For u^*, using Lemma 1 (ii)-(iii) we get

$$-\frac{d}{dt}\left|\{u^* > t\}\right|_{p,k} = \int_{u^{*-1}\{t\}} \frac{|x|^{pk} d\mathcal{H}^{-1}}{|\nabla u^*|} \text{ for } a.e.\ t \in [0, M]. \tag{6}$$

Since $|\{u > t\}|_{p,k} = |\{u^* > t\}|_{p,k}$, from (5)-(6) we get (4). $\square$

We are able to prove Theorem 2.

Proof of Theorem 2. It is easy when $u \equiv 0$. So we assume that $u \not\equiv 0$. Let $M = \max_{\mathbb{R}^2_{c+}} u$. Since $u \in C_0^\infty(\bar{\mathbb{R}}^2_{c+})$ so by Sard's Theorem,

$$\big\{t \in [0, M] : \exists (x, y) \in \bar{\mathbb{R}}^2_{c+}, u(x, y) = t, \nabla u(x, y) = 0\big\}$$

has Lebesgue measure 0 in $\mathbb{R}$. Using Lemma 1 (ii),

$$\big\{t \in [0, M] : \exists (x, y) \in \bar{\mathbb{R}}^2_+, u^*(x, y) = t, \nabla u^*(x, y) = 0\big\}$$

has Lebesgue measure 0 in $\mathbb{R}$. Using co-area formula we have

$$\iint_{\mathbb{R}^2_{c+}} |\nabla_G u|^p \, dxdy = \int_0^M dt \int_{u^{-1}\{t\}} |\nabla_G u|^{p-1} \, d\mu_G,$$
$$\iint_{\mathbb{R}^2_+} |\nabla_G u^*|^p \, dxdy = \int_0^M dt \int_{u^{*-1}\{t\}} |\nabla_G u|^{p-1} \, d\mu_G,$$

where $d\mu_G = \frac{|\nabla_G u|}{|\nabla u|} d\mathcal{H}^1$. So in order to prove the Polya-Szego inequality we will prove that for $t \in [0, M]$ such that t is not a critical value of u and u^* then

$$\int_{u^{*-1}\{t\}} |\nabla_G u^*|^{p-1} \, d\mu_G \leq \int_{u^{-1}\{t\}} |\nabla_G u|^{p-1} \, d\mu_G. \tag{7}$$

Using Cauchy inequality we get

$$\begin{aligned}
(P_{p,k}(\{u > t\}))^p &= \left(\int_{u^{-1}\{t\}} |x|^{(p-1)k} \, d\mu_G\right)^p \\
&\leq \left(\int_{u^{-1}\{t\}} |\nabla_G u|^{p-1} \, d\mu_G\right)\left(\int_{u^{-1}\{t\}} \frac{|x|^{pk} \, d\mu_G}{|\nabla_G u|}\right)^{p-1}.
\end{aligned} \tag{8}$$

Note that $u^*(x,y) = \varphi(\rho_k)$ so

$$|\nabla_G u^*| = |x|^k \rho_k^{-k} |\varphi'| .$$

On the other hand $\frac{|\nabla_G u|}{|\nabla u|} = \sqrt{\nu_1^2 + x^{2k}\nu_2^2}$ doesn't depend on u, hence

$$\begin{aligned}
(P_{p,k}(\{u^* > t\}))^p &= \left(\int_{u^{*-1}\{t\}} |x|^{(p-1)k} \, d\mu_G\right)^p \\
&= \left(\int_{u^{*-1}\{t\}} |\nabla_G u^*|^{p-1} \, d\mu_G\right)\left(\int_{u^{*-1}\{t\}} \frac{|x|^{pk} \, d\mu_G}{|\nabla_G u^*|}\right)^{p-1} . \quad (9)
\end{aligned}$$

Recall that $|\{u > t\}|_{p,k} = |\{u^* > t\}|_{p,k}$, using the isoperimetric inequality in Theorem 1 we obtain

$$P_{p,k}(\{u^* > t\}) \leq P_{p,k}(\{u > t\}) . \quad (10)$$

From $\frac{d\mu_G}{|\nabla_G u|} = \frac{d\mathcal{H}^1}{|\nabla u|}$ and (4) in the proof of Lemma 2, (8) - (9) - (10) we get (7). We complete the proof.

□

Remark 2. Using the symmetry we can get the above Polya - Szego type inequality for $u \in C_0^\infty(\bar{\mathbb{R}}^2_{c-}; \mathbb{R}_+)$ and its rearrangement $u^* : \bar{\mathbb{R}}^2_+ \to \mathbb{R}_+$ defined by

$$u^*(x,y) = \varphi(\rho_k), |\{u^* > t\}|_{p,k} = |\{u > t\}|_{p,k} \; \forall t > 0.$$

3.3 Sobolev Inequality

In order to prove Theorem 3 we need some calculation. Considering the case $u(x,y) = \phi(r), r = \rho_k^{k+1} = (x^{2k+2} + (k+1)^2 y^2)^{1/2}$. Using the polar coordinates (as in [11]):

$$x = (r\sin\theta)^{1/(k+1)}, y = \frac{r\cos\theta}{k+1}$$

we have $dxdy = \frac{r^{1/(k+1}}{(k+1)^2}(\sin\theta)^{-k/(k+1)} dr d\theta$. Then we get

$$\iint_{\mathbb{R}^2_+} |\nabla_G u|^p \, dxdy = (k+1)^{p-2} B\left(\frac{2k+1}{2(k+1)}, \frac{1}{2}\right) \int_0^\infty r^{\frac{pk+1}{k+1}} |\phi'(r)|^p dr, \quad (11)$$

$$\iint_{\mathbb{R}^2_+} x^{pk} |u|^q \, dxdy = (k+1)^{-2} B\left(\frac{2k+1}{2(k+1)}, \frac{1}{2}\right) \int_0^\infty r^{\frac{pk+1}{k+1}} |\phi(r)|^q dr. \quad (12)$$

Then we need Lemma 2 (in [10]). For convinient, we recall it:

Lemma 3. *Let $1 < p < m$ and $q = mp/(m-p)$. Assume that $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is Lipschitz and satisifies*

$$\int_0^\infty r^{m-1}|\phi'(r)|^p < \infty, \phi(r) \to 0 \text{ when } r \to \infty. \tag{13}$$

Then

$$\frac{(\int_0^\infty r^{m-1}|\phi'(r)|^p)^{1/p}}{(\int_0^\infty r^{m-1}|\phi(r)|^q dr)^{1/q}} \geq D_{p,q,m}$$

with the best constant obtained by

$$D_{p,q,m} = m^{\frac{1}{p}} \left(\frac{p-1}{m-1}\right)^{-\frac{1}{p'}} \left[\frac{1}{p'} B\left(\frac{m}{p}, \frac{m}{p'}\right)\right]^{\frac{1}{m}}$$

where $p' = p/(p-1)$. The equality sign holds in (13) if ϕ has the form

$$\phi(r) = \left(a + br^{p'}\right)^{1-m/p}$$

where a, b are positive constants.

We now prove Theorem 3.

Proof of Theorem 3. Note that $C_0^\infty(\bar{\mathbb{R}}_+^2)$ is dense in $W_{0,k}^{1,p,q}(\mathbb{R}^2)$, we prove (2) for $u \in C_0^\infty(\mathbb{R}^2) \setminus \{0\}$. Then we know that $|\nabla_G |u|| \leq |\nabla_G u|$. Besides for every nonnegative function $w \in W_{0,k}^{1,p,q}(\mathbb{R}^2)$ there is a sequence of nonnegative functions $w_j \in C_0^\infty(\mathbb{R}^2)$ such that

$$\lim_{j\to\infty} ||w_j - w||_{W_{0,k}^{1,p,q}} = 0.$$

So we can assume that $u \in C_0^\infty(\mathbb{R}^2) \setminus \{0\}$ and $u \geq 0$. It is obvious that there is a $c \in \mathbb{R}$ such that

$$\iint_{\mathbb{R}_{c+}^2} |x|^{pk} |u|^q \, dxdy = \iint_{\mathbb{R}_{c-}^2} |x|^{pk} |u|^q \, dxdy = \frac{1}{2} \iint_{\mathbb{R}^2} |x|^{pk} |u|^q \, dxdy. \tag{14}$$

Put $u_+ = u\Big|_{\bar{\mathbb{R}}_{c+}^2}, u_- = u\Big|_{\bar{\mathbb{R}}_{c-}^2}$ and u_+^*, u_-^* are their rearrangement respectively. Using the property of the rearrangement and (14) we have

$$\iint_{\mathbb{R}_+^2} |x|^{pk} \left|u_+^*\right|^q dxdy = \iint_{\mathbb{R}_-^2} |x|^{pk} \left|u_-^*\right|^q dxdy = \frac{1}{2} \iint_{\mathbb{R}^2} |x|^{pk} |u|^q \, dxdy. \tag{15}$$

Note that $u \in C_0^\infty(\mathbb{R}^2), u \geq 0$, using Theorem 2 and Remark 2 we get

$$\iint_{\mathbb{R}^2_+} |\nabla_G u_+^*|^p dxdy \leq \iint_{\mathbb{R}^2_{c+}} |\nabla_G u_+|^p dxdy, \tag{16}$$

$$\iint_{\mathbb{R}^2_+} |\nabla_G u_-^*|^p dxdy \leq \iint_{\mathbb{R}^2_{c-}} |\nabla_G u_-|^p dxdy, \tag{17}$$

On the other hand we have

$$u_+^*(x,y) = \phi_+(r), u_-^*(x,y) = \phi_-(r), r = \rho_k^{k+1},$$

and ϕ_+, ϕ_- satisfy Lemma 3. Using (11)-(12) and Lemma 3 for $1 < p < k+2$, $m = (k+2+pk)/(k+1)$, $q = p(k+2+pk)/(k+2-p)$ we obtain

$$D_{p,q,k} \left(\iint_{\mathbb{R}^2_+} |x|^{pk} \left| u_+^* \right|^q dxdy \right)^{1/q} \leq \left(\iint_{\mathbb{R}^2_+} |\nabla_G u_+^*|^p dxdy \right)^{1/p}, \tag{18}$$

$$D_{p,q,k} \left(\iint_{\mathbb{R}^2_+} |x|^{pk} \left| u_-^* \right|^q dxdy \right)^{1/q} \leq \left(\iint_{\mathbb{R}^2_+} |\nabla_G u_-^*|^p dxdy \right)^{1/p}. \tag{19}$$

where

$$D_{p,q,k}(u) = (k+1) \left(\frac{1}{(k+1)^2} B\left(\frac{2k+1}{2(k+1)}, \frac{1}{2} \right) \right)^{1/p-1/q} D_{p,q,m}. \tag{20}$$

From (15)-(16)-(17)-(18)-(19) the proof is complete. □

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