Subdiffusive concentration of the graph distance in Bernoulli percolation

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Abstract

Considering Bernoulli percolation on $\mathbb{Z}^d$, in the supercritical regime, Garet and Marchand [GM09] proved a diffusive concentration for the graph distance. In this paper, we sharpen this result by establishing the subdiffusive concentration inequality in sublinear scale for the graph distance. As consequence, we revisit a recent result by Dembin [Dem22] on the sublinear variance of distance.

1 Introduction

1.1 Model and main result

Bernoulli percolation is a simple but well-known probabilistic model for porous material introduced by Broadbent and Hammersley [BH57]. Let $d \geq 2$ and $\mathcal{E}(\mathbb{Z}^d)$ be the set of the edges $e = \langle x, y \rangle$ of endpoints $x = (x_1, \ldots, x_d), y = (y_1, \ldots, y_d) \in \mathbb{Z}_d$ such that $\|x - y\|_1 := \sum_{i=1}^d |x_i - y_i| = 1$. Given the parameter $p \in (0,1)$, we let each edge $e \in \mathcal{E}(\mathbb{Z}^d)$ be *open* with probability p and *closed* otherwise, independently of the state of other edges. The phase transition of model has been well-known since 1960s. There exists a critical parameter $p_c \in (0,1)$, such that there is almost surely a unique infinite open cluster $\mathcal{C}_\infty$ if $p > p_c$, whereas all open clusters are finite if $p < p_c$, see [Gri89]. Let $x \in \mathbb{Z}^d$, we denote by x^* the closest point to x in $\mathcal{C}_\infty$ (in $\|.\|_\infty$ distance), called regularized point of x. We define the graph distance as

$$\forall x, y \in \mathbb{Z}^d,\ D^*(x,y) = D(x^*, y^*) = \inf_{\gamma: x^* \to y^*} \#\gamma,$$

where infimum is taken over the set of lattice open paths

$$\gamma = (u_0 = x^*, u_1, \ldots, y^* = u_N), \quad \|u_{i+1} - u_i\|_1 = 1.$$

Notice that if these points x, y are not in $\mathcal{C}_\infty$ then $D(x,y)$ might be ∞. Hence, Garet and Marchand [GM09] introduced the definition of graph distance using regularized points, which guarantees that $D^*(x,y) = D(x^*, y^*) < \infty$ almost surely for all x and y. Let $\mathbf{e}_1 = (1, 0, \ldots, 0)$ be the first standard basis vector. We aim to study the graph distance from the original 0 to $n\mathbf{e}_1$:

$$D^*_n = D^*(0, n\mathbf{e}_1).$$

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First passage percolation. We consider the general model of the first passage percolation on $\mathbb{Z}^d$. Now each edge $e \in \mathcal{E}(\mathbb{Z}^d)$, we assign a random weight t_e taking values in $[0, \infty]$ such that the family $(t_e)_{e\in\mathcal{E}(\mathbb{Z}^d)}$ is independent and identically distributed with distribution ζ. We interpret t_e as the time needed to cross the edge e. Similarly, the quantity we are interested in is the passage time:

$$T(x, y) = \inf_{\gamma: x\to y} \sum_{e\in\gamma} t_e,$$

where infimum is taken over the set of lattice paths

$$\gamma = (u_0 = x, u_1, \dots, y = u_N), \quad \|u_{i+1} - u_i\|_1 = 1.$$

We will assume throughout that

$$\zeta([0, \infty)) > p_c, \quad \zeta(\{0\}) < p_c, \tag{1}$$

where p_c is the critical probability for Bernoulli percolation on $\mathbb{Z}^d$. Under the condition (1), we now look at the supercritical Bernoulli Percolation as a particular case of first passage percolation with the distribution

$$\zeta = \zeta_p = p\delta_1 + (1 - p)\delta_\infty, \quad p > p_c. \tag{2}$$

Here one mean that the edge $e \in \mathcal{E}(\mathbb{Z}^d)$ is open if $t_e = 1$ (with probability p) and closed if $t_e = \infty$ (with probability $1 - p$). The first passage time travel from the original 0 to ne_1 was denote by

$$T_n^* = T^*(0, n\mathbf{e}_1) = \inf_{\gamma: 0^*\to(n\mathbf{e}_1)^*} \sum_{e\in\gamma} t_e,$$

Time constant. The first order of growth of T_n^* was described by Cerfa and Théret [CT16]: under the assumption (1), there exists a constant $\mu(e_1) \in [0, \infty)$ such that,

$$\lim_{n\to\infty} \frac{T_n^*}{n} = \mu(\mathbf{e}_1) \quad \text{a.e and in } L^1.$$

The function μ is the so-called time constant. Moreover, we also obtain lower tail large deviations by Kesten [Kes86]: for any $\varepsilon > 0$ small enough,

$$\lim_{n\to\infty} \frac{\log \mathbb{P}[T_n^* \le (\mu(\mathbf{e}_1) - \varepsilon)n]}{n} = r(\varepsilon) < 0, \tag{3}$$

In [BSG21], Basu, Sly and Ganguly have just shown that for any bounded distribution $\zeta \in [0, b]$ with continuity densities and $d = 2$,

$$\forall\varepsilon \in (0, b - \mu(\mathbf{e}_1)), \quad \lim_{n\to\infty} \frac{\log \mathbb{P}[T_n^* \ge (\mu(\mathbf{e}_1) + \varepsilon)n]}{n^2} = r(\varepsilon, \zeta) < 0, \tag{4}$$

and some further results for unbounded distribution was done by Cosco and Nakajima [CN21] (the speed of large deviation and rate function now depend on the tail assumption of t_e). In the Bernoulli percolation case, Garet and Marchand [GM07] showed that:

$$\forall\varepsilon > 0 \quad \limsup_{n\to\infty} \frac{\log \mathbb{P}[\frac{D_n^*}{n\mu(\mathbf{e}_1)} \notin (1 - \varepsilon, 1 + \varepsilon)]}{n} < 0,$$

Fluctuation and concentration. It is expected in the physics literature that the variance T_n^* should have the order n^α for some $\alpha < 1$ depending on the dimension d for general distribution ζ. However, these predictions are far from being proved in the first-passage percolation model. In particular, the best known upper bound of variance obtained in [DHS15] by Damron, Hanson and Sosoe for the general distribution ζ: if $\zeta(0) < p_c$ and

$$\mathbb{E}[t_e^2 \log_+ t_e] < \infty, \tag{5}$$

then there exists a constant $C > 0$ such that

$$\mathrm{Var}[T_n^*] \le C \frac{n}{\log n}. \tag{6}$$

Recently, Dembin [Dem22] extend this result to supercritical Bernoulli percolation (note that the moment condition (5) is failed). The sublinearity of variance is also called the superconcentration, see e.g. (6). Chatterjee [Cha14] discover the deep connection among properties of superconcentration, chaos and multiplevaleys in, for example, the Gaussian polymer and mixed-p spin model. Under stronger assumptions on the moments, this phenomenon can be supplemented with concentration results by Damron, Hanson and Sosoe [DHS14]: if $\mathbb{E}[e^{2\alpha t_e}] < \infty$, then there exist $c_1, c_2 > 0$ such that

$$\mathbb{P}\left(|T_n^* - \mathbb{E}[T_n^*]| \ge \sqrt{\frac{n}{\log n}}\lambda\right) \le c_1 e^{-c_2\lambda} \text{ for } \lambda \ge 0.$$

either $\mathbb{E}[t_e^2 \log_+ t_e] < \infty$, there exist $c_1, c_2 > 0$ such that

$$\mathbb{P}\left(T_n^* - \mathbb{E}[T_n^*] \le -\sqrt{\frac{n}{\log n}}\lambda\right) \le c_1 e^{-c_2\lambda} \text{ for } \lambda \ge 0.$$

To our best knowledge, the moderate deviation of D_n^* was established in the supercritical Bernoulli percolation by Garet and Marchand [GM09]: for each $c_4 > 0$, there exist some constants c_1, c_2, c_3 such that for all $\lambda \in [c_3(1 + \log n), \sqrt{n}]$,

$$\mathbb{P}[|D_n^* - \mathbb{E}[D_n^*]| \ge \sqrt{n}\lambda] \le c_1 e^{-c_2\lambda}.$$

The aim of the present paper is to prove a sub-diffusive concentration of D_n^* for supercritical Bernoulli percolation as follows.

Theorem 1.1. *Let $p > p_c$. There exist some constant $c_1, c_2 > 0$ depending on p and d such that*

$$\mathbb{P}\left(|D_n^* - \mathbb{E}[D_n^*]| \ge \sqrt{\frac{n}{\log n}}\kappa\right) \le c_1 e^{-c_2\kappa} \text{ for all } \kappa \ge 0. \tag{7}$$

Consequently, we recover the sub-linear bound for the variance:

$$\mathrm{Var}[D_n^*] \le C_0 \frac{n}{\log n}, \tag{8}$$

where C_0 is a positive constant depending on p and d.

1.2 Setup for proof

We will use a strategy of Benjamini, Kalai and Schramm [BKS11] (called BKS trick) that allows to show the subdiffusive concentration of D_n^*. First of all, we define a partial average version of D_n^*,

$$F_m^* = \frac{1}{\#B(m)} \sum_{z \in B(m)} D^*(z, z+x) = \frac{1}{\#B(m)} \sum_{z \in B(m)} D_z^*, \tag{9}$$

where $D^*(z, z+x) = D_z^*$

$$B(m) = \{x : \|x\|_1 \leq m\}, \quad m = n^{1/4}.$$

We will show the concentration bounds for F_m^* are analogous to those for D_n^*. In particular, we will show that the following result can imply Theorem 1.1 (see more detail at Section 4).

Theorem 1.2. *Under the assumption* (2)*, there exist* $c_1, c_2 > 0$ *such that*

$$\mathbb{P}\left(|F_n^* - \mathbb{E}[F_n^*]| \geq \sqrt{\frac{n}{\log n}\kappa}\right) \leq c_1 e^{-c_2\kappa} \text{ for } \kappa \geq 0. \tag{10}$$

Based on the strategy introduced in [[BR08], Lemma 4.1] and [DHS14], to prove Theorem 1.2 we will derive appropriate bounds for $\mathrm{Var}[e^{\lambda F_m^*}]$.

Theorem 1.3. *There exist a constant* $c > 0$ *such that*

$$\mathrm{Var}[e^{\lambda F_m^*/2}] \leq K\lambda^2 \mathbb{E}[e^{\lambda F_m^*}] < \infty \quad \text{for } |\lambda| < \frac{1}{2\sqrt{K}}, \tag{11}$$

where $K = \dfrac{cn}{\log n}$.

1.3 Organization of this paper

In Section 2, we present some standard results of the supercritical percolation and recall the concentration inequalities. In Section 3, we prove two key components of the proof. Finally, we prove Theorem 1.1 in Section 4.

2 Preliminaries

2.1 Background on Percolation

Let R be a positive integer, and let $B_x(R) = x + [-R, R]^d$ be a box centering at $x \in \mathbb{Z}^d$ with radius R. We now suppose that $M_x(R)$ the largest (open) cluster in $B_x(R)$ (if there exist two or more largest clusters, we pick one according to some predetermined rule). We say that $M_x(R)$ crosses $B_x(R)$ in the i^{th} direction if $M_x(R)$ contains an open path $\gamma = (y^1, \ldots, y^n)$ satisfying $y_i^1 = x_i - R$ and $y_i^n = x_i + R$. In addition, we call $M_x(R)$ a crossing cluster of $B_x(R)$ if $M_x(R)$ crosses $B_x(R)$ in all directions. Furthermore, for $A \subset \mathbb{Z}^d$, let

$$L_i = \inf\{y_i : y \in A\}, \quad R_i = \inf\{y_i : y \in A\}, \tag{12}$$

and we define the diameter $\text{diam}(A)$ by

$$\text{diam}(A) = \max(R_i - L_i : 1 \le i \le d).$$

We define the following events:

$$L_R = \{\text{there exists a crossing cluster in } B_x(R)\},$$

$$T_R = \{B_x(R) \text{ has a crossing cluster and contains at least two open clusters having diameter at least } R\}.$$

Lemma 2.1. *(Lemma 7.104,[Gri89]) Let $p > p_c$ and $d \ge 2$. There exist two positive constants β_1 and β_2 depending on p, such that*

$$\mathbb{P}(T_R) \le \beta_1 e^{-\beta_2 R}. \tag{13}$$

Lemma 2.2. *(Theorem 8.97,[Gri89]) Let $p > p_c$. Then there exists a positive constant β_2 depending on p, such that*

$$\mathbb{P}[L(R) \ge 1 - e^{-\beta_2 R}. \tag{14}$$

Lemma 2.3. *(Lemma 2.3,[GM09]) Let $p > p_c$. There exist positive constants $\rho_1, \rho_2, \alpha, \beta > 0$, such that for any $x \in \mathbb{Z}^d$*

(i)

$$\forall t \ge \rho_1 \|x\|_1, \quad \mathbb{P}[D^*(0,x) \ge t] \le e^{-\rho_2 t}. \tag{15}$$

(ii)

$$\mathbb{E}[e^{\alpha D^*(0,x)}] \le e^{\beta \|x\|_1}. \tag{16}$$

2.2 Entropy inequalities

Fix $\lambda \in \mathbb{R}$, we define

$$G = G_\lambda = e^{\lambda F_m^*}.$$

We notice that $G = e^{\lambda F_m^*}$ is a function on $\{1, \infty\}^{\mathcal{E}(\mathbb{Z}^d)}$. Hence, sometimes we write

$$G = G(t_{e_i}, t_{e_i^c})$$

to emphasize the dependence of G on the random variables t_{e_i} and $t_{e_i^c} = (t_{e_j})_{j \ne i}$. Let us enumerate the edges of $\mathcal{E}(\mathbb{Z}^d)$ as $e_1, e_2, \dots$ and define a sequence of σ-algebra by

$$\mathcal{F}_0 = \varnothing, \quad \mathcal{F}_i = \sigma(X_1, \dots, X_i),$$

for $i \ge 0$. Now we consider the martingale increments

$$\Delta_i = \mathbb{E}[G \mid \mathcal{F}_i] - \mathbb{E}[G \mid \mathcal{F}_{i-1}] = \mathbb{E}[G(t'_{e_i}, t_{e_i^c}) - G(t_{e_i}, t_{e_i^c}) \mid \mathcal{F}_{i-1}],$$

where t'_{e_i} is an independent copy of t_{e_i} and $G(t'_{e_i}, t_{e_i^c})$ is obtained from $G = G(t_{e_i}, t_{e_i^c})$ by replacing the variable t_{e_i} by t'_{e_i}. It is clear that

$$G - \mathbb{E}[G] = \sum_{i=1}^{\infty} \Delta_i.$$

Combining this with the orthogonality of the $(\Delta_i)_{i=1}^{\infty}$, we have

$$\operatorname{Var}[G] = \sum_{i=1}^{\infty} \Delta_i^2. \tag{17}$$

To control the concentration of the averaged passage time F_m^*, we bound the variance of G based on an entropy inequality due to Falik and Samorodnitsky[FS07].

Lemma 2.4.

$$\sum_{i=1}^{\infty} \operatorname{Ent}[\Delta_i^2] \geq \operatorname{Var}[G] \log \frac{\operatorname{Var}[G]}{\sum_{i=1}^{\infty} (\mathbb{E}[|\Delta_i|])^2}, \tag{18}$$

where Ent *denotes the entropy operator:*

$$\operatorname{Ent}[f] = \mathbb{E}\Big[f \log \frac{f}{\mathbb{E}[f]}\Big].$$

We will need the following lemma to control the entropy:

Lemma 2.5. *There exists a constant $C > 0$ depending on p such that*

$$\sum_{i=1}^{\infty} \operatorname{Ent}[\Delta_i^2] \leq C \sum_{i=1}^{\infty} \mathbb{E}[(G(\infty, t_{e_i^c}) - G(1, t_{e_i^c}))^2]. \tag{19}$$

This lemma is a direct consquence of the two following results.

Lemma 2.6. *(Bernoulli log-Sobolev inequalities). Assume that $f : \{a, b\} \to \mathbb{R}$ and $\zeta = p\delta_a + (1-p)\delta_b$. There exist a constant $C > 0$ depending on p such that*

$$\operatorname{Ent}[f(\zeta)^2] \leq C|f(b) - f(a)|^2.$$

Proposition 1. *Let g be a non-negative function on a product probability space $(\prod_{i=1}^{\infty} \Omega_i, \mathcal{F} = \bigvee_{i=1}^{\infty} \mathcal{G}_i, \zeta = \prod_{i=1}^{\infty} \zeta_i)$ where $(\Omega_i, \mathcal{G}_i, \zeta_i)$ is a probability space for all i. Then*

$$\operatorname{Ent}[g] \leq \sum_{i=1}^{\infty} \mathbb{E}[\operatorname{Ent}_{\zeta_i}[g], \tag{20}$$

where $\operatorname{Ent}_{\zeta_i}$ *is the entropy of g with respect to ζ_i, all other coordinates remain fixed.*

3 Construction of detour and its application

3.1 The linearization of the graph distance

For any $z \in \mathbb{Z}^d$, we denote D_z^* by

$$D_z^* := D^*(z, z + n\mathbf{e}_1).$$

Proposition 2. *Let $p > p_c(d)$ and γ_z be a geodesic from z^* to $(z + n\mathbf{e}_1)^*$. Then there exists a collection of random variables $(R_e, R_z, R_{z+ne_1})_{e \in \mathcal{E}(\mathbb{Z}^d)}$, such that the following holds.*

(i) There exist a constant $C > 1$,

$$0 \leq \left| D_z^*(\infty, t_{\bar{e}}) - D_z^*(1, t_{\bar{e}}) \right| \leq C(R_e + R_z + R_{z+n\mathbf{e}_1}) \mathbb{I}(e \in \gamma_z(1, t_{\bar{e}})), \tag{21}$$

(ii) There exist constants α_1 and α_2 depending on p, such that for all $t \in \mathbb{N}$

$$\max\{\mathbb{P}[R_e \geq t], \mathbb{P}[R_z \geq t], \mathbb{P}[R_{z+n\mathbf{e}_1} \geq t]\} \leq \alpha_1 \exp(-\alpha_2 t) \tag{22}$$

(iii) For any $t \in \mathbb{N}$, the event $\{R_e \leq t\}$ depends only on the status of edges in $B_e(t)$, where $B_e(t)$ is the set of edges having distance at most t from e.

Proof. We first observe that $D_z^*(\infty, t_{e^c}) \geq D_z^*(1, t_{e^c})$. Now if $e \notin \gamma_z(1, t_{e^c})$ then by the definition of chemical distance,

$$D_z^*(\infty, t_{e^c}) \leq D_z^*(1, t_{e^c}).$$

Hence,

$$(D_z^*(\infty, t_{e^c}) - D_z^*(1, t_{e^c}))\mathbb{I}(e \notin \gamma_z(1, t_{e^c})) = 0. \tag{23}$$

On the other hand, it is more complicated if $e \in \gamma(1, t_{e^c})$. In the cases that distribution ζ of each edges is bounded, resampling an edge on geodesic γ_z cannot affect too much to the passage time T_z^*. We can bound this discrepancy by a constant that known for as the linearization of the passage time. However, in the context of Bernoulli percolation, closing one edge on the geodesic can have big impact on the graph distance D_n^*. To solve this issue, all we need to do now is to build a detour bypassing one closed edge. Additionally, the closest point of $z, z + n\mathbf{e}_1$ in infinite cluster can be changed when close one edge e in γ_z.

Construction of detour avoiding one closed edge. Let $\mathcal{C}_\infty^e$ be the infinite cluster of $\mathcal{C}_\infty \setminus \{e\}$ which is unique almost surely. For $x \in \mathbb{Z}^d$, we denote the regularized point x_e^* of x is the closest point of x in $\mathcal{C}_\infty^e$. First we built two box $B_e(R) \subset B_e(2R)$ having the same center and depending on e (see Figure 1). We also denote annulus $A_e(R)$ by

$$A_e(R) = B_e(2R) \setminus B_e(R).$$

For Γ_1, Γ_2 two crossing path joining $B_e(R)$ to $\partial B_e(2R)$, we define

$$D(\Gamma_1, \Gamma_2) = \inf\{\#\gamma : \gamma \text{ is an open path connecting } \Gamma_1 \text{ and } \Gamma_2 \text{ in } A_e(R)\}$$

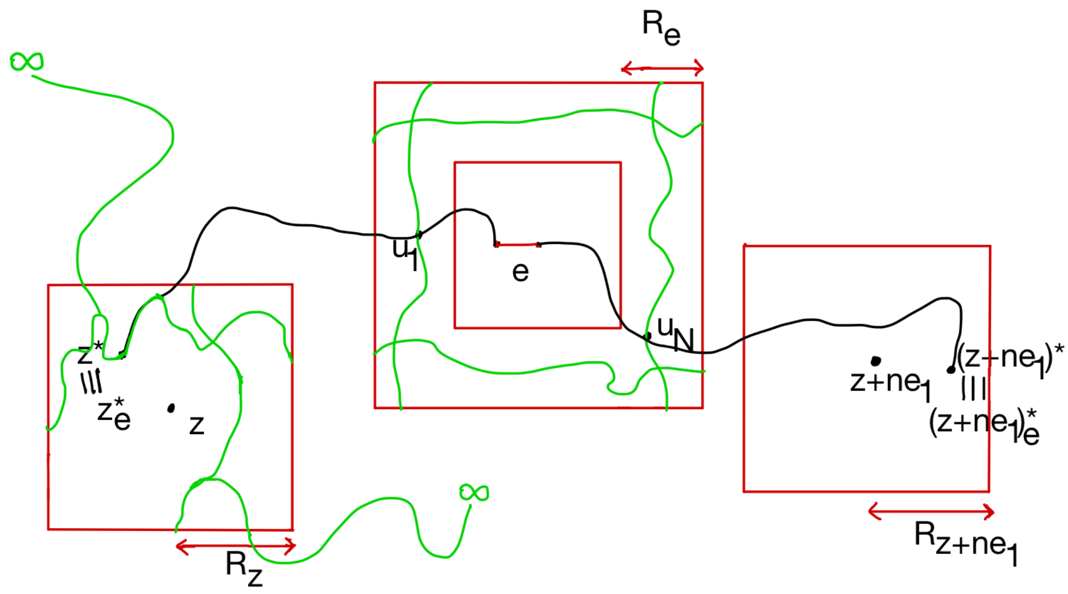

Figure 1: The open detour from z_e^* to $(z+n\mathbf{e}_1)_e^*$ bypass the closed edge e (when e is not in $B_z(R_z)$)

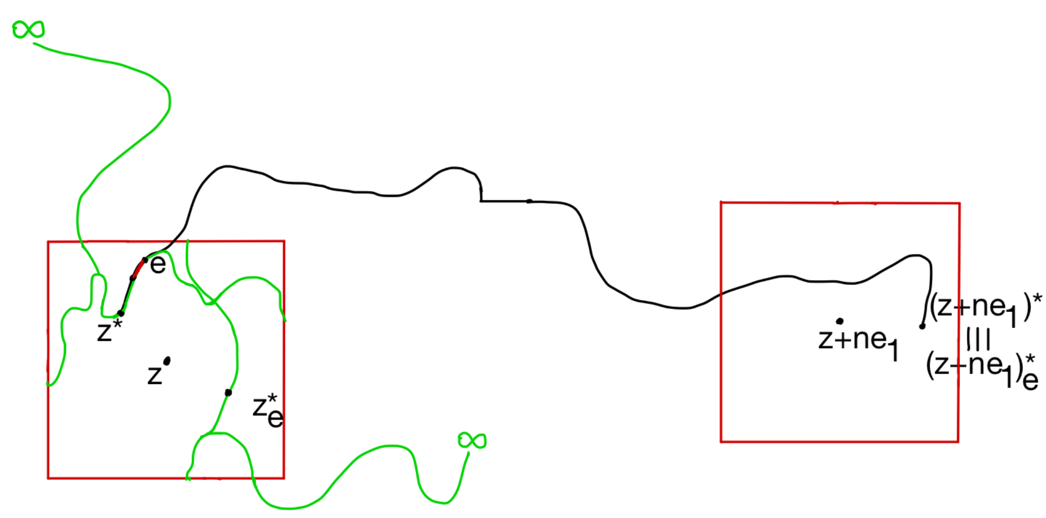

Figure 2: The open detour from z_e^* to $(z+n\mathbf{e}_1)_e^*$ bypass the closed edge e (when e is in $B_z(R_z)$)

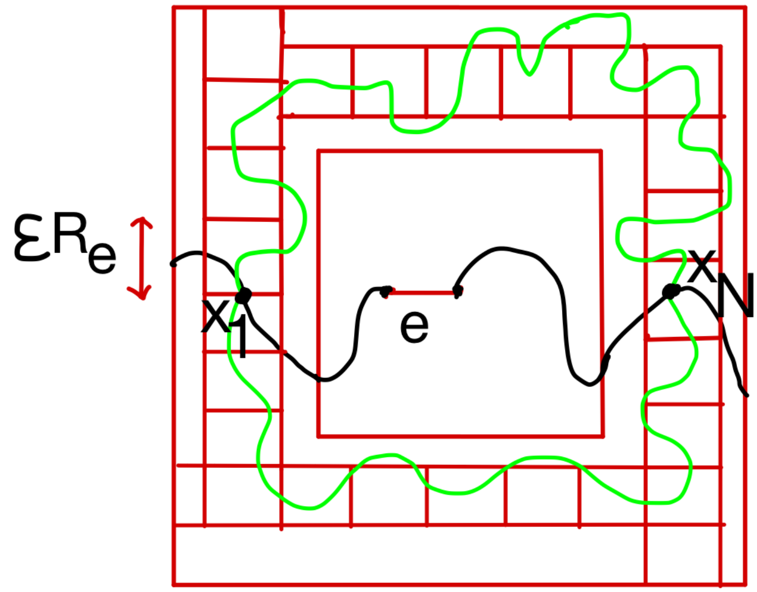

Figure 3: Illustration of the construction of consecutive small boxes control the length of bypass avoiding the closed edge e

For some $C_1 > 0$ and $R \geq 1$, let us define the event

$$U_e(R) = \{\forall \Gamma_1, \Gamma_2 : D(\Gamma_1, \Gamma_2) \leq C_1 R\}.$$

We now set

$$R_e = \inf\{R > 0 : U_e(R) \text{ occurs }\}.$$

To prove large deviation for R_e, we prove the key following property: there exists a constant $\beta = \beta(p)$ such that

$$\mathbb{P}[U_e(R)] \geq 1 - \exp(-\beta R), \tag{24}$$

Indeed, we built a family of consecutive small boxes with radius εR (see Figure 3). Let $\mathcal{C}_\infty(A_e(R))$ be the part of infinite cluster $\mathcal{C}_\infty$ in annulus $A_e(R)$. We denote N for the number of small boxes $B(\varepsilon R)$, thus $N \leq 16/\varepsilon$. Assuming that these boxes are enumerated by $B_0(\varepsilon R), \ldots, B_N(\varepsilon R)$ and

$$\mathcal{C}_\infty(A_e(R)) \bigcap \bigcup_{i=0}^{N} B_i(\varepsilon R) = \{x_0, \ldots, x_N\}, \quad x_0 \in \Gamma_1, x_N \in \Gamma_2.$$

Notice that there exist some constant c_1, c_2 such that for all $i \geq 1$

$$\mathbb{P}[\mathcal{C}_\infty(A_e(R)) \cap B_i(\varepsilon R)] \geq 1 - c_1 e^{-c_2 R}.$$

Using Lemma 2.3 (ii), we deduce that

$$\max_{N\geq i\geq 1}\max_{x_{i-1},x_i\in B_{i-1}(\varepsilon R)\cup B_i(\varepsilon R)}\mathbb{P}\left[D(x_{i-1},x_i)\leq R\right]\geq 1-\alpha_1 e^{-\alpha_2 R}.$$

As a result, we obtain that

$$\begin{aligned}\mathbb{P}\left[\forall\Gamma_1,\Gamma_2: D(\Gamma_1,\Gamma_2)\leq C_1R\right] &\geq \mathbb{P}\left[D(x_0,x_N)\leq C_1R\right]\\ &\geq \prod_{i=1}^{16/\varepsilon}\mathbb{P}[D(x_{i-1},x_i)\leq R]\geq 1-\alpha_1' e^{-\alpha_2 R}.\end{aligned}$$

Combining this with the definition of the event $U_e(R)$, we have

$$\mathbb{P}[U_e(R)]=\mathbb{P}\left[\forall\Gamma_1,\Gamma_2: D(\Gamma_1,\Gamma_2)\leq C_1R\right]\geq 1-\exp(-\beta R). \tag{25}$$

Furthermore, for each $0\leq i\leq N$, we denote γ_e^i the geodesic of $D(x_i,x_{i+1})$. By taking $\gamma_e=\bigcup_{i=0}^N\gamma_e^i$, we obtain

$$\mathbb{P}\left[L(A_e(R)\cap\gamma_z)\xleftrightarrow{\gamma_e\subset A_e(R)}R(A_e(R)\cap\gamma_z)\bigcap\#\gamma_e\leq C_1R\right]\geq\mathbb{P}[U_e(R)]\geq 1-\exp(-\beta R). \tag{26}$$

Next, since using (25) we obtain the large deviation for R_e,

$$\mathbb{P}[R_e\geq t]\leq\mathbb{P}\left[\bigcap_{R\leq t-1}(U_e(R))^c\right]\leq\mathbb{P}\left[(U_e(R))^c_{t-1}\right]\leq\alpha\exp(-\beta t).$$

Finally, we observe that event $\{R_e\leq t\}$ occures if and only if there exists $R\leq t$ satisfies $U_e(R)$. Therefore, this event depends only on the status of edges in $B_e(t)$. We prove (iii).

Construction of open path linking between z^* and z_e^*. First we built a box $B_z(R)$ centering at z with radius R. We now consider some events

$$V_z^1(R)=\{\forall\, x,y\in B_z(R):\ \text{if } x\xleftrightarrow{B_z(R)}y\text{ then }D(x,y)\leq C_2R\},$$

and

$$V_z^2(R)=\{\text{there exist two disjoint path connecting from }B_z(R)\text{ go to infinite}\}.$$

We thus define event $U_z(R)$ by

$$U_z(R)=V_z^1(R)\cap V_z^2(R).$$

By using Lemma 2.3 (i), it is easy to see that

$$\mathbb{P}[V_z^1(R)]\geq 1-e^{-\rho_2 R}, \tag{27}$$

for some constant ρ_2. Next, we built a sequence of boxes with radius $2R,4R,6R,\ldots$, namely $B(z):=B_z(R),B(2R),B(4R),B(6R),\ldots$ such that for all $i\geq 0$

$$B(2i(R)\cap B(2(i+1)R)=\hat{B}(2(i+1)/4R), \tag{28}$$

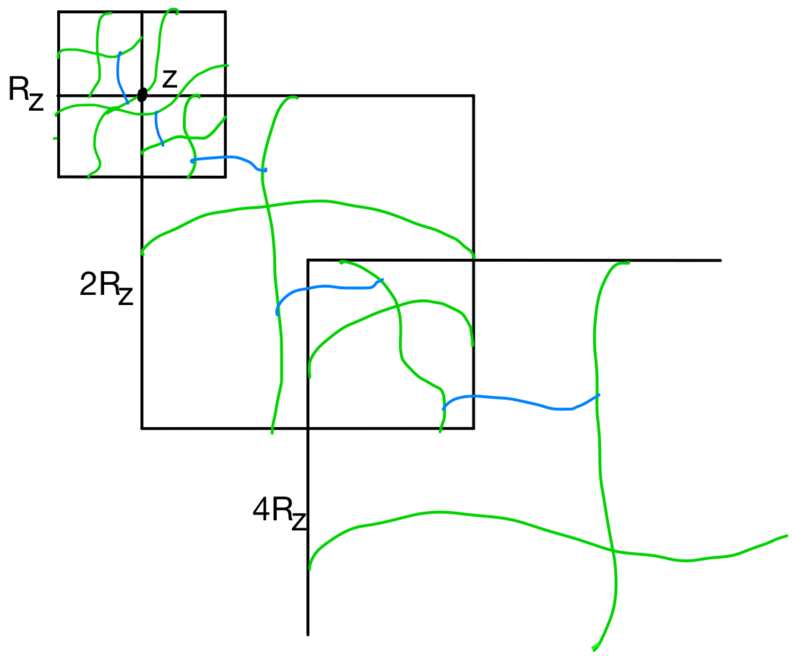

Figure 4: Illustration of the construction of a open path γ^1_∞ connect from $\hat{B}(2R)$ to ∞.

where $\hat{B}(2(i+1)/4R)$ is the box with radius $2(i+1)/4R$. Using Lemma 2.2, we get that for all $i \geq 0$,

$$\mathbb{P}[\text{ there exist a crossing cluster in } B(2iR)] \geq 1 - e^{-\beta_2 R}$$

and

$$\mathbb{P}[\text{ there exist a crossing cluster in } \hat{B}(2iR)] \geq 1 - e^{-\beta_2 R}.$$

By using these inequality and union bound we have

$$\begin{aligned}
&\mathbb{P}[\text{there exist a open path } \gamma^1_\infty \text{ connect from } \hat{B}(R) \text{ to } \infty] \\
&\quad \geq \mathbb{P}[\text{ for all } i \geq 0\text{: there exist a crossing cluster in both } B(2iR) \text{ and } \hat{B}(2iR)] \\
&\quad \geq 1 - \sum_{i=0}^{\infty} e^{-2i\beta_2 R} \geq 1 - \beta_1 e^{-\beta_2 R}
\end{aligned}$$

By using similar construction, we also indicate that there exits a open path γ^2_∞ (disjoint with γ^1_∞) with probability at least $1 - \beta_1 e^{-\beta_2 R}$. As a consequence, we have

$$\mathbb{P}[V^2_z(R)] \geq 1 - \beta_1 e^{-\beta_2 R}, \tag{29}$$

which implies that

$$\mathbb{P}[U_z(R)] \geq 1 - \beta_1 e^{-\beta_2 R}. \tag{30}$$

If $U_z(R)$ occurs then both z^* and z_e^* are in infinite cluster, moreover, $D(z^*, z_e^*) \le C_1 R$. Therefore,

$$\mathbb{P}[z^* \xleftrightarrow{\gamma^* z} z_e^*, \#\gamma_z^* \le C_1 R] \ge \mathbb{P}[U_z(R)] \ge 1 - \beta_1 e^{-\beta_2 R}.$$

Finally, we set

$$R_z = \inf\{R : U_z(R) \text{ occurs }\}. \tag{31}$$

By the definition of R_z and using union bound, we get that

$$\mathbb{P}[R_z \ge t] \le \mathbb{P}\left[\bigcap_{R \le t-1} (U_z(R))^c\right] \le \mathbb{P}\left[(U_z(R))_{t-1}^c\right] \le \alpha \exp(-\beta t).$$

Similarly, we have

$$\mathbb{P}[(z + n\mathbf{e}_1)^* \xleftrightarrow{\gamma^*_{z+n\mathbf{e}_1}} (z + n\mathbf{e}_1)_e^*, \#\gamma^*_{z+n\mathbf{e}_1} \le C_1 R] \ge 1 - \beta_1 e^{-\beta_2 R},$$

and

$$\mathbb{P}[R_{z+n\mathbf{e}_1} \ge t] \le \alpha \exp(-\beta t).$$

Notice that by construction of $R_z, R_{z+n\mathbf{e}_1}, R_e$, we claim (i).
We finish the proof of Proposition 2. □

3.2 The weighted average of dependent edge-weights in geodesic

We first recall a result on controlling maximal weight of paths (Lemma 2.6, [CN19]) whose the proof is based on the theory of greedy lattice animals. Given $M \ge 1$, let $\{B_e, e \in \mathcal{E}(\mathbb{Z}^d)\}$ be a collection of Bernoulli random variables satisfying

- (E1) $\{B_e, e \in \mathcal{E}(\mathbb{Z}^d)\}$ is $M-$dependent, i.e., for all $e \in \mathcal{E}(\mathbb{Z}^d)$, the variable B_e is independent of all variables $\{e' : e' \notin B_e(M)\}$.
- (E2) There exist a function $\phi : \mathbb{R} \to \mathbb{R}$ such that $\phi(M) = \mathcal{O}((3M + 1)^{-d})$ and

$$q_M = \sup_{e \in \mathcal{E}(\mathbb{Z}^d)} \mathbb{E}[B_e] \le \phi(M).$$

For any self-avoiding path γ, we define

$$N(\gamma) = \sum_{e \in \gamma} B_e, \quad N_{L,M} = \max_{\gamma \in \Xi_L} N(\gamma),$$

where

$$\Xi_L = \{\gamma : \gamma \subset B(L); \#\gamma \le L\}.$$

Lemma 3.1. *[CN19, Lemma 2.6] Let $M \ge 1$ and $\{B_e : e \in \mathcal{E}(\mathbb{Z}^d)\}$ be a collection of random variables satisfying (E1) and (E2). Then there is a positive constant $C = C(d)$ depending only on the dimension d such that* *(i) For all $L \in \mathbb{N}$*

$$\frac{\mathbb{E}[N_{L,M}]}{Lq_M^{1/d}} \le CM^{d+1}.$$

(ii) if $t \ge CM^d \max\left(1, MLq_M^{1/d}\right)$, then

$$\mathbb{P}[N_{L,M} \ge t] < 2^d \exp(-t/(16M)^d).$$

Let $f : \mathbb{R} \to \mathbb{R}$ and consider the following condition:

$$(f(M))^2 M^{d+1} (\phi(M))^{1/d} = o(1/M). \tag{H}$$

The following lemma aim to control the weighted average of edges-weight in geodesic.

Lemma 3.2. *There exist some positive constant $C_1 = C_1(d)$ and $C_2 = C_2(d)$ such that the following holds for all $L \ge 1$,*

(i) If the functions f satisfies (H) then

$$\mathbb{E}\Big[\Big(\max_{\gamma\in\Xi_L} \sum_{e\in\gamma} f(R_e)\Big)^2\Big] \le C_1 L^2.$$

(ii) Suppose that γ is a random nearest-neighbor path starting from 0. If the functions f^2 satisfies (H) then

$$\mathbb{E}\Big[\Big(\sum_{e\in\gamma} f(R_e)\Big)^2\Big] \le C_1 L^2 + C_2 \sum_{\ell\ge L} \ell^2 (\mathbb{P}[\#\gamma = \ell])^{1/2}.$$

Assume in addition that $\mathbb{P}(\#\gamma = \ell) = \mathcal{O}(\ell^{-6-\varepsilon})$, for some $\varepsilon > 0$. Then we have

$$\mathbb{E}\Big[\Big(\sum_{e\in\gamma} f(R_e)\Big)^2\Big] = \mathcal{O}(L^2).$$

(iii) Suppose that γ is a random path such that $\gamma \subset B(m)$ almost surely for some $m \ge 1$. If the functions f^2 satisfies (H) then

$$\mathbb{E}\Big[\Big(\sum_{e\in\gamma} f(R_e)\Big)^2\Big] \le C_1 (m+L)^2 + C_2 \sum_{\ell\ge L} (m+\ell)^2 (\mathbb{P}[\#\gamma = \ell])^{1/2}.$$

Assume in addition that $\mathbb{P}(\#\gamma = \ell) = \mathcal{O}(\ell^{-6-\varepsilon})$, for some $\varepsilon > 0$. Then we have

$$\mathbb{E}\Big[\Big(\sum_{e\in\gamma} f(R_e)\Big)^2\Big] = \mathcal{O}(m^2).$$

Proof. We first prove (i). By Cauchy-Schwarz inequality,

$$\mathbb{E}\Big[\Big(\max_{\gamma\in\Xi_L} \sum_{e\in\gamma} f(R_e)\Big)^2\Big] \le \mathbb{E}\Big[\max_{\gamma\in\Xi_L} (\#\gamma) \sum_{e\in\gamma} f^2(R_e)\Big] \le L\mathbb{E}\Big[\max_{\gamma\in\Xi_L} \sum_{e\in\gamma} f^2(R_e)\Big], \tag{32}$$

since $\#\gamma \leq L$ for all $\gamma \in \Xi_L$. For any self-avoiding path γ, we define

$$A_M^\gamma = \{e \in \gamma : R_e = M\}.$$

Thus we can express

$$\sum_{e\in\gamma} f^2(R_e) = \sum_{M\geq 1} f^2(M)(\#A_M^\gamma). \tag{33}$$

Notice that it follows from the definition of A_M^γ,

$$\#A_M^\gamma = \sum_{e\in\gamma} \mathbb{I}(R_e = M) = \sum_{e\in\gamma} B_{e,M}, \tag{34}$$

where

$$B_{e,M} = \mathbb{I}(R_e = M).$$

Plugging this into (33), we obtain

$$\begin{aligned}\mathbb{E}\left[\max_{\gamma\in\Xi_L}\sum_{e\in\gamma} f^2(R_e)\right] &= \mathbb{E}\left[\sum_{M\geq 1} f^2(M)\max_{\gamma\in\Xi_L}\sum_{e\in\gamma} B_{e,M}\right] \\ &= \sum_{M\geq 1} f^2(M)\mathbb{E}\left[N_{L,M}\right]. \end{aligned} \tag{35}$$

By Proposition 2 (iii), $\{B_e, e \in \mathcal{E}(\mathbb{Z}^d)\}$ is a collection of M-dependent Bernoulli random variables. Moreover,

$$q_M = \sup_{e\in\mathcal{E}(\mathbb{Z}^d)} \mathbb{E}[B_e] \leq \phi(M) = \alpha_1 \exp(-\alpha_2 M),$$

since Proposition 2 (ii). Therefore, the conditions (E1) and (E2) are satisfied for all $M \geq M_0$, with $M_0 = M_0(d)$ is a large enough constant. Now using Lemma 3.1, we obtain that for all $M \geq M_0$,

$$\mathbb{E}[N_{L,M}] \leq CLM^{d+1}\phi(M)^{1/d}. \tag{36}$$

In contrast, it yields that for $M \leq M_0$,

$$N_{L,M} \leq \max_{\gamma\in\Xi_L} \#A_M^\gamma \leq L. \tag{37}$$

By using (35) with (36) and (37), we get that

$$\mathbb{E}\left[\max_{\gamma\in\Xi_L}\sum_{e\in\gamma} f^2(R_e)\right] \leq L\left[\sum_{M=1}^{M_0} f^2(M) + C\sum_{M\geq M_0} f^2(M)M^{d+1}(\phi(M))^{1/d}\right] \leq C_1 L.$$

Here for the last inequality we used the condition (H1). Finally, combining this with (32), we conclude that

$$\mathbb{E}\Big[\Big(\max_{\gamma\in\Xi_L}\sum_{e\in\gamma} f(R_e)\Big)^2\Big] \leq C_1 L^2.$$

Next, we will prove (ii). We first obverse that for all $L \geq 1$,

$$\mathbb{E}\Big[\Big(\sum_{e\in\gamma} f(R_e)\Big)^2\Big] = \mathbb{E}\Big[\Big(\sum_{e\in\gamma} f(R_e)\Big)^2 \mathbb{I}(\gamma \leq L)\Big] + \mathbb{E}\Big[\Big(\sum_{e\in\gamma} f(R_e)\Big)^2 \mathbb{I}(\#\gamma \geq L)\Big]$$
$$\leq \mathbb{E}\Big[\Big(\max_{\gamma\in\Xi_L}\sum_{e\in\gamma} f(R_e)\Big)^2 + \sum_{l=L}^{\infty}\mathbb{E}\Big[\#\gamma\sum_{e\in\gamma} f^2(R_e)\mathbb{I}(\#\gamma = l)\Big]$$
$$\leq C_1 L^2 + \sum_{l=L}^{\infty} l\mathbb{E}\Big[\max_{\gamma\in\Xi_l}\sum_{e\in\gamma} f^2(R_e)\mathbb{I}(\#\gamma = l)\Big]. \tag{38}$$

Here the last line we used (i) for the function $f(R_e)$. Besides, thanks to (i) for the function $(f(R_e))^2$ again,

$$\mathbb{E}\Big[\max_{\gamma\in\Xi_l}\sum_{e\in\gamma} f^2(R_e)\mathbb{I}(\#\gamma = l)\Big] \leq \mathbb{E}\left[\left(\max_{\gamma\in\Xi_l}\sum_{e\in\gamma} f^2(R_e)\right)^2\right]^{1/2} \mathbb{E}\left[\mathbb{I}(\#\gamma = l)\right]^{1/2}$$
$$\leq C_2 l(\mathbb{P}[(\#\gamma = l)])^{1/2}.$$

Combining this with (38), we obtain

$$\mathbb{E}\Big[\Big(\sum_{e\in\gamma} f(R_e)\Big)^2\Big] \leq C_1 L^2 + C_2\sum_{l=L}^{\infty} l^2(\mathbb{P}[(\#\gamma = l)])^{1/2}. \tag{39}$$

Finally, we prove (iii). We now separate the expectation into two parts,

$$\mathbb{E}\Big[\Big(\sum_{e\in\gamma} f(R_e)\Big)^2\Big] = \mathbb{E}\Big[\Big(\sum_{e\in\gamma} f(R_e)\Big)^2 \mathbb{I}(\#\gamma \leq L)\Big] + \mathbb{E}\Big[\Big(\sum_{e\in\gamma} f(R_e)\Big)^2 \mathbb{I}(\#\gamma > L)\Big]. \tag{40}$$

By the hypothesis that γ is a random path such that $\gamma \subset B(m)$ almost surely, moreover if $\#\gamma \leq L$ then $\gamma \in \Xi_{L+m}$. As a result, using (i) for the function $f(R_e)$, we obtain that

$$\mathbb{E}\Big[\Big(\sum_{e\in\gamma} f(R_e)\Big)^2 \mathbb{I}(\#\gamma \leq L)\Big] \leq \mathbb{E}\Big[\max_{\gamma\in\Xi_{L+m}}\Big(\sum_{e\in\gamma} f(R_e)\Big)^2\Big] = \mathbb{E}\Big[\Big(\max_{\gamma\in\Xi_{L+m}}\sum_{e\in\gamma} f(R_e)\Big)^2\Big]$$
$$\leq C_1(L+m)^2.$$

For the second part of (40) we have

$$\mathbb{E}\Big[\Big(\sum_{e\in\gamma} f(R_e)\Big)^2 \mathbb{I}(\#\gamma > L)\Big] = \sum_{l=L}^{\infty}\mathbb{E}\Big[\Big(\sum_{e\in\gamma} f(R_e)\Big)^2 \mathbb{I}(\#\gamma = l)\Big]$$
$$\leq \sum_{l=L}^{\infty}\mathbb{E}\Big[\#\gamma\sum_{e\in\gamma} f^2(R_e)\mathbb{I}(\#\gamma = l)\Big]$$
$$\leq \sum_{l=L}^{\infty} l\mathbb{E}\Big[\max_{\gamma\in\Xi_{l+m}}\sum_{e\in\gamma} f^2(R_e)\mathbb{I}(\#\gamma = l)\Big]$$
$$\leq \sum_{l=L}^{\infty} l\Big(\mathbb{E}\Big[\Big(\max_{\gamma\in\Xi_{l+m}}\sum_{e\in\gamma} f^2(R_e)\Big)^2\Big]\Big)^{1/2}(\mathbb{P}[(\#\gamma = l)])^{1/2}$$
$$\leq C_2\sum_{l=L}^{\infty} l(l+m)(\mathbb{P}[(\#\gamma = l)])^{1/2}. \tag{41}$$

Here the last line thanks to (i) for the function $(f(R_e))^2$. Combining this with (40), it follows that

$$\mathbb{E}\Big[\Big(\sum_{e\in\gamma} f(R_e)\Big)^2\Big] \le C_1(L+m)^2 + C_2\sum_{l=L}^{\infty} l(l+m)(\mathbb{P}[(\#\gamma = l)])^{1/2}.$$

We complete the proof. □

4 Proof of Theorem 1.1

We have known that Theorem 1.2 could be deduced from Theorem 1.3. Now we clarify how Theorem 1.2 implies Theorem 1.1. Assume that the average passage time F_m^* such that subdiffusive concentration,

$$\mathbb{P}\Big[|F_m^* - \mathbb{E}[F_m^*]| \ge \sqrt{\frac{n}{\log n}}\kappa\Big] \le c_1 e^{-c_2\kappa}, \quad \kappa \ge 0, \tag{42}$$

where $c_1, c_2 > 0$. Our task is to give the corresponding estimate for the first passage time D_n^* from (42):

$$\mathbb{P}\Big[|D_n^* - \mathbb{E}[D_n^*]| \ge \sqrt{\frac{n}{\log n}}\kappa\Big] \le c_1' e^{-c_2'\kappa}, \quad \kappa \ge 0, \tag{43}$$

We first write

$$\begin{aligned} D_n^* - \mathbb{E}[D_n^*] &= F_m^* - \mathbb{E}[D_n^*] + D_n^* - F_m^* \\ &= F_m^* - \mathbb{E}[F_m^*] + D_n^* - F_m^* \end{aligned}$$

here we used $\mathbb{E}[F_m^*] = \mathbb{E}[D_n^*]$. Let $M > 0$ we will choose later. Using triangle inequality we observe that if event $\{|D_n^* - \mathbb{E}[D_n^*]| \ge 4M\}$ occur then $\{|F_m^* - \mathbb{E}[F_m^*]| \ge 2M\}$ or $\{|D_n^* - \mathbb{E}[F_m^*]| \ge 2M\}$. By union bound we estimate,

$$\mathbb{P}[|D_n^* - \mathbb{E}[D_n^*]| \ge 4M] \le \mathbb{P}[|F_m^* - \mathbb{E}[F_m^*]| \ge 2M] + \mathbb{P}[|D_n^* - \mathbb{E}[F_m^*]| \ge 2M. \tag{44}$$

Using subadditivity property, we can write,

$$\begin{aligned} |D_n^* - \mathbb{E}[F_m^*]| = \Big|D_n^* - \frac{1}{\#B(m)}\sum_{z\in B(m)} D_z^*\Big| &\le \frac{1}{\#B(m)}\sum_{z\in B(m)} |D^*(0,z) - D^*(z, z+ne_1)| \\ &\le \frac{1}{\#B(m)}\sum_{z\in B(m)} (D^*(0,z) + D^*(ne_1, ne_1+z)). \end{aligned}$$

Now we see that if event $\Big\{\frac{1}{\#B(m)}\sum_{z\in B(m)}(D^*(0,z) + D^*(ne_1, ne_1+z)) \ge 2M\Big\}$ occurs then

$$\max_{z\in B(m)} D^*(0,z) \ge M \text{ or } \max_{z\in B(m)} D^*(ne_1, ne_1+z) \ge M. \tag{45}$$

Combining this with union bound we obtain that

$$\begin{aligned} &\mathbb{P}\Big[\frac{1}{\#B(m)}\sum_{z\in B(m)} (D^*(0,z) + D^*(ne_1, ne_1+z)) \ge 2M\Big] \\ &\le \mathbb{P}[\max_{z\in B(m)} D^*(0,z) \ge M] + \mathbb{P}[\max_{z\in B(m)} D^*(ne_1, ne_1+z) \ge M] \\ &= 2\mathbb{P}[\max_{z\in B(m)} D^*(0,z) \ge M], \end{aligned}$$

where the last line we used the translation invariant. By Lemma 2.3, the right hand side was bounded by

$$2(\#B(m)) \max_{z\in B(m)} \mathbb{P}[D^*(0,z) \geq M] \leq 2(\#B(m)) e^{-\alpha M} e^{\beta m}$$
$$\leq 2m^d e^{\beta m} e^{-\alpha M}. \tag{46}$$

Taking $4M = \sqrt{\frac{n}{\log n}}\kappa$, we have

$$\mathbb{P}\Big[|D_n^* - F_m^*| \geq \frac{\kappa}{2}\sqrt{\frac{n}{\log n}}\Big] \leq 2n^{d/4} \exp\Big(-\alpha\sqrt{\frac{n}{4\log n}}\kappa + \beta n^{1/4}\Big) = \mathcal{O}(1)e^{-\mathcal{O}(1)\kappa}.$$

Combining this with (42) and (44) we claim (43).

4.1 Bound on influence

Theorem 4.1. *Assume that α_2 and C be two constants in Lemma 2.3 and Proposition 2, respectively. There exists a constant $C_1 > 0$,*

$$\sum_{i=1}^{\infty} (\mathbb{E}[|\Delta_i|])^2 \leq C_1 \lambda^2 \mathbb{E}\big[e^{2\lambda F_m^*}\big] n^{1-d}, \quad \forall \lambda \in \mathbb{R}.$$

The above theorem is a direct consequence of the following propositions:

Proposition 3. *There exists a constant C_2 such that for all $i \geq 1$, we have*

$$\mathbb{E}[|\Delta_i|] \leq C_2 m^{\frac{1-d}{2}}, \quad \forall \lambda \in \mathbb{R}. \tag{47}$$

Proposition 4. *There exists a constant C_3 such that*

$$\sum_{i=1}^{\infty} \mathbb{E}[|\Delta_i|] \leq C_3 n, \quad \forall \lambda \in \mathbb{R}. \tag{48}$$

4.1.1 Proof of Proposition 3

We first note that

$$\Delta_i = \mathbb{E}[G|\mathcal{F}_i] - \mathbb{E}[G|\mathcal{F}_{i-1}] = \mathbb{E}[G(t'_{e_i}) - G(t_{e_i})|\mathcal{F}_{i-1}]. \tag{49}$$

Thus,

$$\mathbb{E}[|\Delta_i|] \leq \mathbb{E}[|G(t'_{e_i}) - G(t_{e_i})|] = 2\mathbb{E}\big[\big(e^{\lambda F_m^*(t'_{e_i})} - e^{\lambda F_m^*(t_{e_i})}\big)_+\big].$$

Hence, using the inequality that $(e^{\lambda a} - e^{\lambda b})_+ \leq |\lambda|(e^{\lambda a} + e^{\lambda b})(a-b)_+$, we get

$$\mathbb{E}[|\Delta_i|] \leq 2|\lambda|\mathbb{E}\big[\big(e^{\lambda F_m^*(t'_{e_i})} + e^{\lambda F_m^*(t_{e_i})}\big)(F_m^*(t'_{e_i}) - F_m^*(t_{e_i}))_+\big]$$
$$= 4|\lambda|\mathbb{E}\big[e^{\lambda F_m^*(t_{e_i})}(F_m^*(t'_{e_i}) - F_m^*(t_{e_i}))_+\big]. \tag{50}$$

By Proposition 2 (i), there exists a positive constant C and random variables R_z, R_{z+ne_1} and R_{e_i}, such that

$$D_z^*(t'_{e_i}) - D_z^*(t_{e_i}) \leq \big(D_z^*(t'_{e_i}) - D_z^*(t_{e_i})\big)_+ = D_z^*(\infty, t_{e_i^c}) - D_z^*(1, t_{e_i^c})$$
$$\leq CR_{z,e_i}\mathbb{I}(e_i \in \gamma_z), \tag{51}$$

where

$$R_{z,e_i} = R_{e_i} + R_z + R_{z+ne_1}. \tag{52}$$

Therefore,

$$\begin{aligned} F_m^*(t'_{e_i}) - F_m^*(t_{e_i}) &\le \big(F_m^*(t'_{e_i}) - F_m^*(t_{e_i})\big)_+ \\ &\le \frac{1}{\#B(m)} \sum_{z\in B(m)} (D_z^*(t'_{e_i}) - D_z^*(t_{e_i}))_+ \le A_i, \end{aligned} \tag{53}$$

where

$$A_i = \frac{C}{\#B(m)} \sum_{z\in B(m)} R_{z,e_i} \mathbb{I}(e_i \in \gamma_z).$$

Combining (50) and (53), we obtain

$$\begin{aligned} \mathbb{E}[|\Delta_i|] &\le 4|\lambda| \mathbb{E}\big[e^{\lambda F_m^*} A_i\big] &(54)\\ &\le 4|\lambda| \mathbb{E}\big[e^{2\lambda F_m^*}\big]^{1/2} \mathbb{E}\big[\mathbb{E}\big[A_i^2\big]^{1/2}, &(55) \end{aligned}$$

here for the equation in the first line, we remark that $F_m^*(t_{e_i}) = F_m^*$ and for the last inequality, we used Hölder's inequality.

We now consider $\mathbb{E}[A_i^4]$. Using Cauchy-Schwarz inequality,

$$\begin{aligned} A_i^2 &\le \frac{C^2}{\#B(m)} \sum_{z\in B(m)} R_{z,e_i}^2 \mathbb{I}(e_i \in \gamma_z) \\ &\le \frac{3C^4}{\#B(m)} \sum_{z\in B(m)} (R_z^2 + R_{e_i}^2 + R_{z+ne_1}^2) \mathbb{I}(e_i \in \gamma_z). \end{aligned} \tag{56}$$

Therefore, thanks to translation invariant we have

$$\begin{aligned} \mathbb{E}[A_i^2] \le \frac{3C^2}{\#B(m)} \Big(&\mathbb{E}\Big[\sum_{z\in B(m)} R_{e_i-z}^2 \mathbb{I}(e_i - z \in \gamma_n)\Big] + \mathbb{E}\Big[R_0^2 \sum_{z\in B(m)} \mathbb{I}(e_i - z \in \gamma_n)\Big] \\ &+ \mathbb{E}\Big[R_{ne_1}^2 \sum_{z\in B(m)} \mathbb{I}(e_i - z \in \gamma_n)\Big]\Big) \\ = \frac{3C^2}{\#B(m)} \Big(&\mathbb{E}\Big[\sum_{e\in\gamma} R_e^2\Big] + \mathbb{E}\left[R_0^2 \#\gamma\right] + \mathbb{E}\left[R_{ne_1}^2 \#\gamma\right] \Big), \end{aligned} \tag{57}$$

where

$$\gamma = \gamma_n \cap \{e_i - B(m)\},$$

and

$$\{e_i - B(m)\} = \{(x_{e_i} - z, y_{e_i} - z) : z \in B(m)\}.$$

We will focus on our effort to compute the upper bound for each terms of the right-hand side of (57). By using Cauchy-Schwarz inequality, the second term was bounded by

$$\left(\mathbb{E}\left[R_0^4\right]\right)^{1/2} (\mathbb{E}[(\#\gamma)^2])^{1/2}. \tag{58}$$

Since Proposition (2) (ii), we obtain

$$(\mathbb{E}[R_0^4])^{1/2} = \mathcal{O}(1). \tag{59}$$

It is clear that if event $\{\#\{(e_i - B(m)) \cap \gamma_n\} = l\}$ occurs, then we may find the first and last intersections (that we call x and y respectively) of γ_n with V, the set of vertices in $\{e_i - B(m)\}$. Moreover, the portion of γ_n from x to y denoted by $\gamma_{x,y}$ is then a geodesic with at least l edges. Using the union bound, we have for any $l \geq \rho_1 m$,

$$\begin{aligned}\mathbb{P}[\#\gamma \geq l] \leq \mathbb{P}\left[\exists x, y \in V : \#\gamma_{x,y} \geq l\right] &\leq (2m+1)^{2d} \max_{x,y\in V} \mathbb{P}[\#\gamma_{x,y} \geq l] \\ &= (2m+1)^{2d} \max_{x,y\in V} \mathbb{P}[T(x,y) \geq l] \\ &\leq (2m+1)^{2d} e^{-\rho_2 l},\end{aligned} \tag{60}$$

where the last line we used Lemma 2.3 (i). As a result, we get

$$\mathbb{E}[(\#\gamma)^2] \leq (\rho_1 m)^2 + \sum_{l \geq \rho_1 m} l(2m+1)^{2d} e^{-\rho_2 l} \leq \mathcal{O}(1) m^2. \tag{61}$$

Combining (58) with (59) and (61), it follows that

$$\mathbb{E}\Big[R_0^2 \#\gamma\Big] \leq \mathcal{O}(1) m. \tag{62}$$

Similarly, we also have

$$\mathbb{E}\Big[R_{ne_1}^2 \#\gamma\Big] \leq \mathcal{O}(1) m. \tag{63}$$

We next control the first term of (57). Note that the function $f(x) = x^2$ satisfies (H1) and (H2). Therefore, applying Lemma 3.2 (iii) to $f(x) = x^2$, $\gamma = \gamma_n \cap \{e_i - B(m), L = \rho_1 m$ we obtain that

$$\begin{aligned}\mathbb{E}\Big[\sum_{e\in\gamma} R_e^2\Big] &\leq \Big(\mathbb{E}\Big[\Big(\sum_{e\in\gamma} R_e^2\Big)^2\Big]\Big)^{1/2} \\ &\leq C_1(m+\rho_1 m)^2 + C_2 \sum_{l\geq\rho_1 m} m(m+l)\sqrt{\mathbb{P}[\#\gamma = l]} \\ &\leq 4C_1\rho_1^2 m^2 + C_2 \sum_{l\geq\rho_1 m} 2l^2 \exp(-\rho_2 l) = \mathcal{O}(1) m^2.\end{aligned} \tag{64}$$

Here for the last line we used (60). By combining (57) with (62), (63) and (64) we have

$$\mathbb{E}[A_i^2] \leq \mathcal{O}(1) m^{1-d}. \tag{65}$$

Finally, we summarize (55) with (65) to conclude that

$$\mathbb{E}[|\Delta_i|] \leq \mathcal{O}(1)|\lambda|\big(\mathbb{E}\big[e^{2\lambda F_m^*}\big]\big)^{1/2} m^{(1-d)/2}. \tag{66}$$

4.1.2 Proof of Proposition 4

By using (55), we have the following bound,

$$\sum_{i=1}^{\infty}\mathbb{E}[|\Delta_i|] \leq 4|\lambda|\mathbb{E}\left[e^{\lambda F_m^*(t_{e_i})}\sum_{i=1}^{\infty}A_i\right]$$
$$\leq 4|\lambda|\big(\mathbb{E}\big[e^{2\lambda F_m^*(t_{e_i})}\big]\big)^{1/2}\Big(\mathbb{E}\Big[\Big(\sum_{i=1}^{\infty}A_i\Big)^2\Big]\Big)^{1/2}, \tag{67}$$

where we recall that

$$A_i = \frac{C}{\#B(m)}\sum_{z\in B(m)} R_{z,e_i}\mathbb{I}(e_i\in\gamma_z).$$

Now we can write

$$\mathbb{E}\Big[\Big(\sum_{i=1}^{\infty}A_i\Big)^2\Big] \leq \frac{C^2}{(\#B(m))^2}\mathbb{E}\Big[\Big(\sum_{z\in B(m)}\sum_{i=1}^{\infty}R_{z,e_i}\Big)^2\Big]$$
$$\leq \frac{C^2}{(\#B(m))}\sum_{z\in B(m)}\mathbb{E}\Big[\Big(\sum_{i=1}^{\infty}R_{z,e_i}\Big)^2\Big]. \tag{68}$$

Since the definition of R_{z,e_i} and Cauchy-Schwarz inequality, we get

$$\mathbb{E}\Big[\Big(\sum_{i=1}^{\infty}R_{z,e_i}\Big)^2\Big] \leq \mathbb{E}\Big[\Big(\sum_{i=1}^{\infty}R_{e_i}\mathbb{I}(e_i\in\gamma_z)\Big)^2 + \Big(\sum_{i=1}^{\infty}R_{z}\mathbb{I}(e_i\in\gamma_z)\Big)^2$$
$$+\Big(\sum_{i=1}^{\infty}R_{z+ne_1}\mathbb{I}(e_i\in\gamma_z)\Big)^2\Big]. \tag{69}$$

Thanks to Cauchy-Schwarz inequality and Proposition 2.3 (i),

$$\mathbb{E}\Big[\Big(\sum_{i=1}^{\infty}R_{z}\mathbb{I}(e_i\in\gamma_z)\Big)^2\Big] \leq \Big(\mathbb{E}\Big[R_z^2\Big]\Big)^{1/2}\Big(\mathbb{E}\Big[\Big(\sum_{i=1}^{\infty}\mathbb{I}(e_i\in\gamma_{z_2})\Big)^4\Big]\Big)^{1/2}$$
$$\leq \mathcal{O}(1)(\mathbb{E}[(\#\gamma_{z_2})^4])^{1/2} \leq \mathcal{O}(1)n^2, \tag{70}$$

Similarly, we have

$$\mathbb{E}\Big[\Big(\sum_{i=1}^{\infty}\ R_{z+ne_1}\mathbb{I}(e_i\in\gamma_z)\Big)^2\Big] \leq \mathcal{O}(1)n^2. \tag{71}$$

We next get the bound for the first term of (69). We observe that

$$\mathbb{E}\Big[\Big(\sum_{i=1}^{\infty}R_{e_i}\mathbb{I}(e_i\in\gamma_z)\Big)^2\Big] = \mathbb{E}\Big[\Big(\sum_{e\in\gamma_z}R_{e_i}\Big)^2\Big]. \tag{72}$$

It clears that the function $f(x) = x$ satisfies (H1) and (H2). Thus, applying Lemma 3.2 (ii) to $f(x) = x, \gamma = \gamma_z, L = \rho_1 n$, we obtain

$$\begin{aligned} \mathbb{E}\Big[\Big(\sum_{e\in\gamma_z} R_{e_i}\Big)^2\Big] &\leq C_1(\rho_1 n)^2 + C_2 \sum_{l\geq\rho_1 n} l^2 \sqrt{\mathbb{P}[\#\gamma_{z_2} = l]} \\ &\leq C_1(\rho_1 n)^2 + C_2 \sum_{l\geq\rho_1 n} l^2 e^{-\rho_2 l/2} \leq \mathcal{O}(1)n^2. \end{aligned} \tag{73}$$

Here for the last line we used Lemma 2.3 (i). Combining (69) with (70), (71) and (73), it follows that

$$\mathbb{E}\Big[\Big(\sum_{i=1}^{\infty} A_i\Big)^2\Big] \leq \mathcal{O}(1)n^2.$$

Combining this with (67), it yields that

$$\sum_{i=1}^{\infty} \mathbb{E}[|\Delta_i|] \leq \mathcal{O}(1)|\lambda| n \big(\mathbb{E}\big[e^{2\lambda F_m^*}\big]\big)^{1/2}.$$

The result follows.

4.2 Entropy bound

We obtain the upper bound of the entropy thank to Lemma 2.5,

$$\begin{aligned} \sum_{i=1}^{\infty} \mathrm{Ent}_\zeta[\Delta_i^2] &\leq C \sum_{i=1}^{\infty} \mathbb{E}\Big[(G(\infty, t_{e_i^c}) - G(1, t_{e_i^c}))^2\Big] \\ &\leq |\lambda|^2 \sum_{i=1}^{\infty} \mathbb{E}\Big[\Big(e^{2\lambda F_m^*(\infty, t_{e_i^c})} + e^{2\lambda F_m^*(1, t_{e_i^c})}\Big)(F_m^*(\infty, t_{e_i^c}) - F_m^*(1, t_{e_i^c}))^2\Big]. \end{aligned} \tag{74}$$

Notice that

$$\begin{aligned} &\mathbb{E}\Big[e^{2\lambda F_m^*(\infty, t_{e_i^c})}(F_m^*(\infty, t_{e_i^c}) - F_m^*(1, t_{e_i^c}))^2\Big] \\ &\qquad = \frac{1}{1-p}\mathbb{E}\Big[e^{2\lambda F_m^*(\infty, t_{e_i^c})}(F_m^*(\infty, t_{e_i^c}) - F_m^*(1, t_{e_i^c}))^2 \mathbb{I}(t_e = \infty)\Big) \\ &\qquad \leq \frac{1}{1-p}\mathbb{E}\Big[e^{2\lambda F_m^*}(F_m^*(\infty, t_{e_i^c}) - F_m^*(1, t_{e_i^c}))^2\Big]. \end{aligned}$$

and

$$\begin{aligned} &\mathbb{E}\Big[e^{2\lambda F_m^*(1, t_{e_i^c})}(F_m^*(\infty, t_{e_i^c}) - F_m^*(1, t_{e_i^c}))^2\Big] \\ &\qquad = \frac{1}{p}\mathbb{E}\Big[e^{2\lambda F_m^*(\infty, t_{e_i^c})}(F_m^*(\infty, t_{e_i^c}) - F_m^*(1, t_{e_i^c}))^2 \mathbb{I}(t_e = 1)\Big) \\ &\qquad \leq \frac{1}{p}\mathbb{E}\Big[e^{2\lambda F_m^*}(F_m^*(\infty, t_{e_i^c}) - F_m^*(1, t_{e_i^c}))^2\Big]. \end{aligned}$$

Combining these inequalities with (74) we obtain

$$\begin{aligned} \sum_{i=1}^{\infty} \mathrm{Ent}_\zeta[\Delta_i^2] &\leq C\lambda^2 \sum_{i=1}^{\infty} \mathbb{E}\Big[e^{2\lambda F_m^*}(F_m^*(\infty, t_{e_i^c}) - F_m^*(1, t_{e_i^c}))^2\Big] \\ &\leq C\lambda^2 \sum_{i=1}^{\infty} \mathbb{E}\Big[e^{2\lambda F_m^*} A_i^2\big], \end{aligned}$$

where we recall that

$$A_i = \frac{C}{\#B(m)} \sum_{z \in B(m)} R_{z,e_i} \mathbb{I}(e_i \in \gamma_z).$$

Now by similar argument as in the proof of Proposition 4 we have the following bound for entropy,

Proposition 5. *There exists a constant $C > 0$ such that*

$$\sum_{i=1}^{\infty} \mathrm{Ent}_\zeta[\Delta_i^2] \leq C\lambda^2 n \mathbb{E}\big[e^{2\lambda F_m^*}\big], \quad \forall \lambda \in \mathbb{R}. \tag{75}$$

4.3 Proof of Theorem 1.3

We have already proved the final result via Falik-Samorondonesky inequality. By using Lemma 2.4, Theorem 4.1, Proposition 5, we have

$$\mathrm{Var}\left[e^{\lambda F_m^*}\right] \leq \mathcal{O}(1)\left(\log \frac{\mathrm{Var}[e^{\lambda F_m^*}]}{\mathcal{O}(1)n^{(1-d)/8}}\right)^{-1} |\lambda|^2 n \mathbb{E}\left[e^{2\lambda F_m^*}\right]. \tag{76}$$

From this bound, we may assume that

$$\mathrm{Var}[e^{\lambda F_m^*}] \geq \mathcal{O}(1)|\lambda|^2 n^{15/16} \mathbb{E}\left[e^{2\lambda F_m^*}\right], \tag{77}$$

otherwise there is nothing to do. By both (76) and (77), for any $\lambda \in \mathbb{R}$,

$$\mathrm{Var}[e^{\lambda F_m^*}] \geq \mathcal{O}(1)|\lambda|^2 \frac{n}{\log n} \mathbb{E}\left[e^{2\lambda F_m^*}\right].$$

The result follows.

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