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Differential Stability of Discrete Optimal Control Problems with Possibly Nondifferentiable Costs*

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Abstract In this paper, a family of discrete optimal control problems that depend on parameters is considered. The control problems are reformulated as parametric optimization problems. By establishing/exploiting abstract results on subdifferentials of optimal value functions of parametric optimization problems, we derive formulas for estimating/computing subdifferentials of optimal value functions of parametric discrete optimal control problems in both nonconvex and convex cases. Namely, for control problems with nonconvex costs, upper-evaluations on the regular subdifferential and the limiting (Mordukhovich) subdifferential of the optimal value function are obtained without using the (strict) differentiability of the costs. Meanwhile, for control problems with convex costs, besides results on estimating/computing the subdifferential (in the sense of convex analysis) of the optimal value function, it is worth pointing out that some properties of the optimal value function are first discussed in this paper.

* Dedicated to Professor Nguyen Dong Yen on the occasion of his 65th birthday

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1 Introduction

Discrete optimal control problems arise in investigations of controlled systems where changes of the control and current state can take place only at strictly defined, isolated instants of time. Problems of this type often appear in applications. For example, many control problems in economics (see, [7, Chap. 1], [8, Chap. 1], [9, Chap. 1], [30], [38, Chap. 9]) can be stated in a natural way as discrete optimal control problems.

A major part in the literature of discrete optimal control problems is on characterizations of the solutions. The necessary condition for an extremum in discrete optimal control problems, which is known as the *discrete maximum principle*, was discussed in the paper [28] and in the book [14, Chap. 6] for general cases. The readers can find more discussions and counterexamples about the discrete maximum principle and related necessary optimality conditions for discrete control systems in [21, Section 6.4]. Besides, the *dynamic programming method*, which, on the one hand, gives a convenient computational formalism for solving problems of this type, and, on the other hand, contains a possible approach toward obtaining sufficient conditions in optimal control theory, was studied in the books [6, Chap. 3], [14, Chap. 6] and papers cited therein. It is well-known that optimal control problems for discrete-time systems can be reduced to optimization problems, which are also called mathematical programming problems. Therefore, one can use tools and techniques from mathematical programming to study necessary/sufficient optimality conditions of optimal control problems. The interested reader is referred to the papers [12, 13, 19, 35, 36] and the references therein where the mathematical programming approach plays a key role in deriving optimality conditions for discrete optimal control problems.

Investigations on stability and solution sensitivity of *parametric optimization problems* are vital in optimization and variational analysis. They allow us to understand behaviors of the optimal value function and of the solution map when parameters appearing in the problem under investigation witness some perturbation. Optimal value functions of parametric optimization problems are usually nonsmooth, even when the problems are given by smooth data. Thus, in order to study differential stability of optimal value functions, one may need to use different types of subdifferential–generalized concepts of the classical derivative. For nonconvex optimization problems, Mordukhovich and his co-authors in [20, Chap. 1], [22, Chap. 4], [27], Penot in [29, Chap. 4] have derived formulas for computing or estimating the regular subdifferential, the *limiting (Mordukhovich) subdifferential* of optimal value functions. Meanwhile, by using different versions of the Moreau-Rockafellar theorem and appropriate regularity conditions, the papers [3], [5] and the works [23, Chap. 2], [24], [25, Chap. 4], [26] have provided formulas for computing the *subdifferential in the sense of convex analysis* of optimal value functions in convex optimization problems. Such results and related developments in optimization and variational analysis have severed as main tools for studying differential stability of optimal value functions of *parametric optimal control problems*. This direction has attracted attention of many researchers, see [2,4,11,15,16,17,18,31,33,34, 37] and the references therein.

Motivated by the work of Kien et al. [15], Chieu and Yao [11], Toan and Yao [37], and An and Toan [2], this paper presents new results on differential stability of optimal value functions of parametric discrete optimal control problems with possibly nondifferentiable costs. Formulas for estimating or computing the regular subdifferential, the limiting subdifferential, and the subdifferential in the sense of convex analysis of the optimal value function via corresponding subdifferentials of functions describing the cost and differential information of the constraint system are obtained. Two regularity conditions (see, conditions (A1) and (A2) in Section 5 below) are needed, among other assumptions. To our best knowledge, the first one putting on data of the constraint is weaker than regularity conditions ever used in the literature for problems with linear and convex constraints. Meanwhile, the second one involving an interaction between the cost and the constraint is introduced to work for nonconvex optimal control problems with nondifferentiable costs, which none of the previous works have investigated. Besides, some properties of optimal value functions in convex optimal control problems are first examined in this paper. To be able to achieve such results by considering the parametric optimal control problem as a parametric optimal problem, we have exploited state-of-the-art tools in optimization and variational analysis (including [22] and [24]) as well as our new abstract results built up for optimization problems with nonsmooth objectives (Theorems 1, 2, 4, 5).

The remaining sections are as follows. Section 2 is for the problem formulation and some auxiliary concepts in variational analysis from the books [20], [22], and [23]. Results on differential stability of parametric optimization problems with convex constraint maps are established/represented in Section 3. In Section 4, we specify the results in the previous section for the case where the constraint map is defined by linear operators and convex sets. Section 5 is devoted to the study of differential stability of the parametric discrete optimal control problem in both nonconvex and convex cases. Several illustration examples are given in Section 6. The last section provides some concluding remarks.

Throughout the paper, the considered spaces are finite-dimensional Euclidean with the inner product and the norm being denoted by $\langle \cdot, \cdot \rangle$ and by $|| \cdot ||$, respectively. For a set Ω in X, the interior, the closure, and the relative interior of Ω are denoted by $\operatorname{int} \Omega$, $\operatorname{cl} \Omega$, and $\operatorname{ri} \Omega$, respectively. For a linear operator A, $\ker A$ (resp., $\operatorname{rge} A$) stands for the kernel (resp., the range) of A.

2 Preliminaries

This section is divided into two subsections. In the first subsection, the parametric discrete optimal control problem, which we are interested in, is introduced and then is transformed into a parametric optimization problem under an inclusion constraint. In the second one, basic concepts from variational analysis are recalled.

2.1 Problem Formulation

Let X_k, U_k, W_k, for $k = 0, 1, \ldots, N-1$, and X_N be finite-dimensional spaces, where N is a positive natural number. Let there be given
- nonempty, convex sets $C \subset X_0$ and $\Omega_0 \subset U_0, \ldots, \Omega_{N-1} \subset U_{N-1}$;
- linear operators $A_k : X_k \to X_{k+1}$, $B_k : U_k \to X_{k+1}$, and $T_k : W_k \to X_{k+1}$ for $k = 0, 1, \ldots, N-1$;
- functions $h_k : W_k \times X_k \times U_k \to \mathbb{R}$ for $k = 0, 1, \ldots, N-1$, and $h_N : X_N \to \mathbb{R}$.

In what follows, we will deal with a control system where the *state variable* (resp., the *control variable*) at time k is $x_k \in X_k$ (resp., $u_k \in U_k$). The control system contains the *parameter* $w_k \in W_k$ at each stage k. We call X_k, U_k, and W_k the space of state variables, the space of control variables, and the space of parameters at stage k, respectively.

Put $X = X_0 \times X_1 \times \cdots \times X_N$, $U = U_0 \times U_1 \times \cdots \times U_{N-1}$, and $W = W_0 \times W_1 \times \cdots \times W_{N-1}$. For a given parameter $w = (w_0, w_1, \ldots, w_{N-1}) \in W$, consider the following *parametric discrete optimal control problem*: Find a pair (x, u) of state $x = (x_0, x_1, \ldots, x_N) \in X$ and control $u = (u_0, u_1, \ldots, u_{N-1}) \in U$, which minimizes the *cost/objective function*

$$\sum_{k=0}^{N-1} h_k(w_k, x_k, u_k) + h_N(x_N) \tag{2.1}$$

among those pairs satisfying the *linear state equations*

$$x_{k+1} = A_k x_k + B_k u_k + T_k w_k, \ k = 0, 1, \ldots, N-1, \tag{2.2}$$

the *initial condition*

$$x_0 \in C, \tag{2.3}$$

and the *control constraints*

$$u_k \in \Omega_k \subset U_k, \quad k = 0, 1, \ldots, N-1. \tag{2.4}$$

Problems of this type have been considered in many papers (see [2,11,15, 30,34,35,36,37]). A classical example for the model (2.1)–(2.4) is the *inventory control problem* in economics (see [9, pp. 2-6, 13-14, 162-168]).

We are interested in studying stability properties of the discrete control problem (2.1)–(2.4) w.r.t. the change of parameter w; in particular, differential stability of the optimal value function $V(\cdot)$. In this paper we consider parametric discrete optimal control problem (2.1)–(2.4) as a mathematical programming problems. Because then one can use effective tools and techniques from mathematical programming to study differential stability of $V(\cdot)$. Let us show how the parametric discrete optimal control problem (2.1)–(2.4) is transformed into a parametric optimization problem under an inclusion constraint.

For each parameter $w = (w_0, w_1, \ldots, w_{N-1}) \in W$, denote by $V(w)$ the *optimal value* of problem (2.1)–(2.4) and $S(w)$ the corresponding *solution set*. Thus, we have an extended real-valued function $V : W \to \overline{\mathbb{R}} := [-\infty, +\infty]$ depending on $w \in W$ with the values $V(w)$, which is called the *optimal value function* of problem (2.1)–(2.4). Let

$$f(w, x, u) := \sum_{k=0}^{N-1} h_k(w_k, x_k, u_k) + h_N(x_N), \quad (w, x, u) \in W \times X \times U, \quad (2.5)$$

$$G(w) := \{(x, u) \in X \times U | x_{k+1} = A_k x_k + B_k u_k + T_k w_k, k = 0, 1, ..., N-1\} \quad (2.6)$$

for each $w \in W$, and $\Omega := \Omega_0 \times \Omega_1 \times \cdots \times \Omega_{N-1}$, $\widetilde{X} := X_1 \times X_2 \times \cdots \times X_N$. Then the function $V : W \to \overline{\mathbb{R}}$ can be rewritten as the optimal value function of the following *parametric optimization problem under an inclusion constraint*

$$V(w) = \inf_{(x,u) \in G(w) \cap (C \times \widetilde{X} \times \Omega)} f(w, x, u), \quad w \in W. \quad (2.7)$$

Since the optimal value function $V(\cdot)$ is generally nondifferentiable, we will focus on estimating/computing its subdifferentials which are generalized concepts of derivatives. Those are the regular, limiting/Mordukhovich subdifferentials when $V(\cdot)$ is not necessarily convex (the *nonconvex case*) and the subdifferential in the sense of convex analysis when $V(\cdot)$ is convex (the *convex case*).

2.2 Auxiliary Concepts

Let us recall some notions related to generalized differentiation from [20, 22, 23]. Along with single-valued maps usually denoted by $f : \mathbb{R}^n \to \mathbb{R}^m$, we consider set-valued maps (or multifunctions) $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ with values $F(x)$ in the collection of all the subsets of $\mathbb{R}^m$. The limiting construction

$$\operatorname*{Lim\,sup}_{x \to \bar{x}} F(x) := \Big\{ y \in \mathbb{R}^m \mid \exists x_k \to \bar{x}, y_k \to y \text{ with } y_k \in F(x_k), \forall k = 1, 2, ... \Big\}$$

is known as the *Painlevé-Kuratowski outer/upper limit* of F at $\bar{x}$. All the maps considered below are proper, i.e., $F(x) \neq \emptyset$ for some $x \in \mathbb{R}^n$.

Let Ω be a nonempty subset of $\mathbb{R}^n$ and $\bar{x} \in \Omega$. The regular normal cone to Ω at $\bar{x}$ is defined by

$$\widehat{N}(\bar{x}, \Omega) = \Big\{ v \in \mathbb{R}^n \mid \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle v, x - \bar{x} \rangle}{\|x - \bar{x}\|} \leq 0 \Big\},$$

where $x \xrightarrow{\Omega} \bar{x}$ means that $x \to \bar{x}$ and $x \in \Omega$. The *limiting (Mordukhovich) normal cone* to Ω at $\bar{x}$ is given by

$$N(\bar{x}, \Omega) = \mathop{\mathrm{Lim\,sup}}_{x \xrightarrow{\Omega} \bar{x}} \widehat{N}(x, \Omega).$$

We put $\widehat{N}(\bar{x}, \Omega) = N(\bar{x}, \Omega) = \emptyset$ if $\bar{x} \notin \Omega$.

Clearly, one always has

$$\widehat{N}(\bar{x}, \Omega) \subset N(\bar{x}, \Omega), \ \forall \Omega \subset \mathbb{R}^n, \forall \bar{x} \in \Omega. \tag{2.8}$$

When the reverse inclusion holds, one says that the set Ω is *normally regular* at $\bar{x}$. It is well-known (see, e.g., [22, Prop. 1.7]) that when Ω is a convex subset of $\mathbb{R}^n$, the regular normal cone to Ω at $\bar{x}$ coincides with the limiting normal cone and both constructions reduce to the *normal cone in the sense of convex analysis*, i.e.,

$$\widehat{N}(\bar{x}, \Omega) = N(\bar{x}, \Omega) = \big\{v \in \mathbb{R}^n \mid \langle v, x - \bar{x}\rangle \le 0, \ \forall x \in \Omega\big\}, \quad \forall \bar{x} \in \Omega. \tag{2.9}$$

Thus, a convex set is normally regular at any point in it. However, the class of normally regular sets is really bigger than the class of convex sets. Indeed, it is easy to show that the nonconvex set

$$\Omega := \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_2 \ge 0\} \setminus \{x = (0, x_2) \in \mathbb{R}^2 \mid x_2 > 0\}$$

is normally regular at $\bar{x} := (0, 0)$ and one has

$$\widehat{N}(\bar{x}, \Omega) = N(\bar{x}, \Omega) = \{v = (0, v_2) \in \mathbb{R}^2 \mid v_2 \le 0\}.$$

Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set-valued map with the *domain*

$$\mathrm{dom}\, F := \{x \in \mathbb{R}^n \mid F(x) \neq \emptyset\}$$

and the *graph*

$$\mathrm{gph}\, F := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y \in F(x)\}.$$

The set-valued map F is called *closed* (resp., *convex*) if $\mathrm{gph}\, F$ is closed (resp., convex) in the product space $\mathbb{R}^n \times \mathbb{R}^m$, which is endowed with the norm $\|(x, y)\| = \|x\| + \|y\|$ for any $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$.

The regular coderivative $\widehat{D}^*F(\bar{x}, \bar{y}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ and the *limiting (Mordukhovich) coderivative* $D^*F(\bar{x}, \bar{y}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ of F at $(\bar{x}, \bar{y})$ in $\mathrm{gph}\, F$ are given respectively by

$$\widehat{D}^*F(\bar{x}, \bar{y})(v) := \Big\{u \in \mathbb{R}^n \mid (u, -v) \in \widehat{N}\left((\bar{x}, \bar{y}), \mathrm{gph}\ F\right)\Big\}, \quad v \in \mathbb{R}^m$$

and

$$D^*F(\bar{x}, \bar{y})(v) := \left\{u \in \mathbb{R}^n \mid (u, -v) \in N\left((\bar{x}, \bar{y}), \mathrm{gph}\ F\right)\right\}, \quad v \in \mathbb{R}^m.$$

If $(\bar{x}, \bar{y}) \notin \mathrm{gph}\, F$, one puts $\widehat{D}^*F(\bar{x}, \bar{y})(v) = D^*F(\bar{x}, \bar{y})(v) = \emptyset$ for any $v \in \mathbb{R}^m$.

Note that, by the relation (2.8), one has $\widehat{D}^*F(\bar{x},\bar{y})(v) \subset D^*F(\bar{x},\bar{y})(v)$, for all $v \in \mathbb{R}^m$. Meanwhile, it follows from (2.9) that the above inclusion becomes equality when F is a convex set-valued map. Thus, in this case, the concepts of the regular coderivative and limiting coderivative of the map F at $(\bar{x},\bar{y}) \in \operatorname{gph} F$ are coincided (and therefore will be referred shortly as *coderivative*) and the normal cones appearing in their definitions are understood as the normal cone in the sense of convex analysis.

Consider a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ with the *effective domain*

$$\operatorname{dom} f := \{x \in \mathbb{R}^n \mid f(x) < +\infty\}$$

and the *epigraph*

$$\operatorname{epi} f := \{(x,\alpha) \in \mathbb{R}^n \times \mathbb{R} \mid \alpha \geq f(x)\}.$$

Suppose that $\bar{x} \in \mathbb{R}^n$ and $|f(\bar{x})| < \infty$. One calls the sets

$$\widehat{\partial} f(\bar{x}) := \left\{ v \in \mathbb{R}^n \mid (v,-1) \in \widehat{N}((\bar{x}, f(\bar{x})), \operatorname{epi} f) \right\},$$

$$\partial f(\bar{x}) := \{ v \in \mathbb{R}^n \mid (v,-1) \in N((\bar{x}, f(\bar{x})), \operatorname{epi} f) \},$$

and

$$\partial^\infty f(\bar{x}) := \{ v \in \mathbb{R}^n \mid (v,0) \in N((\bar{x}, f(\bar{x})), \operatorname{epi} f) \}$$

the regular subdifferential, *limiting (Mordukhovich) subdifferential*, and *singular subdifferential* of f at $\bar{x}$, respectively. If $|f(\bar{x})| = \infty$, one lets $\widehat{\partial} f(\bar{x})$, $\partial f(\bar{x})$, and $\partial^\infty f(\bar{x})$ to be empty sets.

Notice that the singular subdifferential $\partial^\infty f(\bar{x})$ reduces to $\{0\}$ if f is locally Lipschitzian around $\bar{x}$ (see [22, Thm. 1.22]). Meanwhile, the regular subdifferential of f at $\bar{x}$ can be express as (see [20, p. 90])

$$\widehat{\partial} f(\bar{x}) = \left\{ v \in \mathbb{R}^n \mid \liminf_{x \to \bar{x}} \frac{f(x) - f(\bar{x}) - \langle v, x - \bar{x}\rangle}{||x - \bar{x}||} \geq 0 \right\}.$$

From this, it is not difficult to prove that

$$\widehat{\partial} f(\bar{x}) \subset \widehat{\partial}_{x_1} f(\bar{x}) \times \widehat{\partial}_{x_2} f(\bar{x}) \times ... \times \widehat{\partial}_{x_n} f(\bar{x}),\ \forall \bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n) \in \mathbb{R}^n \quad (2.10)$$

with $\widehat{\partial}_{x_1} f(\bar{x})$, $\widehat{\partial}_{x_2} f(\bar{x}), \ldots, \widehat{\partial}_{x_n} f(\bar{x})$ respectively being the regular subdifferentials of $f(\cdot, \bar{x}_2, \ldots, \bar{x}_n)$ at $\bar{x}_1$, $f(\bar{x}_1, \cdot, \bar{x}_3, \ldots, \bar{x}_n)$ at $\bar{x}_2$, ..., $f(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n-1}, \cdot)$ at $\bar{x}_n$.

Due to (2.8), it always holds that $\widehat{\partial} f(\bar{x}) \subset \partial f(\bar{x})$. When the reverse inclusion is valid, one says that the function f is *lower regular* at $\bar{x}$. By (2.9), if f is a *convex function*, i.e., $\operatorname{epi} f$ is convex, then f is lower regular at any point $\bar{x}$ with $|f(\bar{x})| < \infty$. In this case, the regular subdifferential and limiting subdifferential of f at $\bar{x}$ coincide with the *subdifferential in the sense of convex analysis*

$$\widehat{\partial} f(\bar{x}) = \partial f(\bar{x}) = \{ v \in \mathbb{R}^n \mid f(x) - f(\bar{x}) \geq \langle v, x - \bar{x}\rangle, \quad \forall x \in \mathbb{R}^n \}.$$

In the case, where f is a $\mathcal{C}^1$ function around $\bar{x}$ (i.e., f is continuously Fréchet differentiable and the gradient map is continuous on a neighborhood of $\bar{x}$), the limiting subdifferential contains only the gradient $\{\nabla f(\bar{x})\}$ (see [22, Cor. 1.24]). Thus, a part from convex functions, $\mathcal{C}^1$ functions (in particular strictly differentiable functions) are also lower regular. However, there do exist functions which are lower regular but neither convex nor differentiable (see Example 1).

We end this section by recalling the relationship between normal cones to a convex set $\Omega \subset \mathbb{R}^n$ and subdifferentials of its *indicator function* $\delta(\cdot, \Omega) : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ defined by $\delta(x, \Omega) = 0$ if $x \in \Omega$ and $\delta(x, \Omega) = +\infty$ if $x \notin \Omega$ as follows

$$N(\bar{x}, \Omega) = \partial\delta(\bar{x}, \Omega) = \partial^{\infty}\delta(\bar{x}, \Omega), \quad \forall \bar{x} \in \mathbb{R}^n.$$

3 Parametric Optimization Problems with Convex Constraint Maps

Let $\varphi : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$ be an extended-real-valued function, $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a convex multifunction. Consider the *parametric optimization problem under an inclusion constraint*

$$\min\{\varphi(x, y) \mid y \in G(x)\} \tag{3.1}$$

depending on the parameter $x \in \mathbb{R}^n$. Define the *optimal value function* $\mu : \mathbb{R}^n \to \overline{\mathbb{R}}$ of (3.1) by

$$\mu(x) := \inf\{\varphi(x, y) \mid y \in G(x)\}, \quad x \in \mathbb{R}^n, \tag{3.2}$$

and the *solution map* $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ of (3.1) by

$$M(x) := \{y \in G(x) \mid \mu(x) = \varphi(x, y)\}, \quad x \in \mathbb{R}^n. \tag{3.3}$$

The following theorem is about an upper estimate for the regular subdifferential of the optimal value function $\mu(\cdot)$ in (3.2) via the regular subdifferential of the objective function φ and the coderivative of the constraint map G. This result complements the one in [27, Thm. 1] where another upper estimate for the regular subdifferential of the optimal value function $\mu(\cdot)$ is provided under the assumption on the nonemptiness of the so-called *Fréchet upper subdifferential* of the objective function φ.

Theorem 1 *Suppose that the optimal value function $\mu(\cdot)$ in (3.2) is finite at $\bar{x}$ and $\bar{y} \in M(\bar{x})$. If φ is lower semicontinuous and lower regular at $(\bar{x}, \bar{y})$, G has a closed graph, and the qualification condition*

$$\partial^{\infty}\varphi(\bar{x}, \bar{y}) \cap [-N((\bar{x}, \bar{y}), \operatorname{gph} G)] = \{(0_{\mathbb{R}^n}, 0_{\mathbb{R}^m})\} \tag{3.4}$$

is satisfied, then we have

$$\widehat{\partial}\mu(\bar{x}) \subset \bigcup_{(x^*, y^*) \in \widehat{\partial}\varphi(\bar{x}, \bar{y})} \big\{x^* + D^*G(\bar{x}, \bar{y})(y^*)\big\}. \tag{3.5}$$

Proof It is well-known that the constrained minimization problem (3.1) is equivalent to the unconstrained one

$$\min\{f(x,y) := \varphi(x,y) + \delta((x,y), \operatorname{gph} G) \mid y \in \mathbb{R}^m\}.$$

Thus, for an arbitrarily given $x^* \in \widehat{\partial}\mu(\bar{x})$, Theorem 4.47 in [29] yields

$$(x^*, 0) \in \widehat{\partial} f(\bar{x}, \bar{y}). \tag{3.6}$$

Since $\operatorname{gph} G$ is a closed and convex set, it follows that $\delta(\cdot, \operatorname{gph} G)$ is lower semicontinuous and lower regular at $(\bar{x}, \bar{y})$. Combining this with the lower semicontinuity and the lower regularity of φ at $(\bar{x}, \bar{y})$, the qualification condition (3.4) and applying Theorem 2.19 in [22], we get

$$\widehat{\partial} f(\bar{x}, \bar{y}) = \widehat{\partial}\varphi(\bar{x}, \bar{y}) + \widehat{\partial}\delta((\bar{x}, \bar{y}), \operatorname{gph} G).$$

Thus, we have

$$\widehat{\partial} f(\bar{x}, \bar{y}) = \widehat{\partial}\varphi(\bar{x}, \bar{y}) + N((\bar{x}, \bar{y}), \operatorname{gph} G) \tag{3.7}$$

due to $\widehat{\partial}\delta((\bar{x}, \bar{y}), \operatorname{gph} G) = \partial\delta((\bar{x}, \bar{y}), \operatorname{gph} G) = N((\bar{x}, \bar{y}), \operatorname{gph} G)$. It follows from (3.6) and (3.7) that $(x^*, 0) \in \widehat{\partial}\varphi(\bar{x}, \bar{y}) + N((\bar{x}, \bar{y}), \operatorname{gph} G)$. Therefore, there exist $(x_1, y_1) \in \widehat{\partial}\varphi(\bar{x}, \bar{y})$ and $(x_2, y_2) \in N((\bar{x}, \bar{y}), \operatorname{gph} G)$ satisfying $(x^*, 0) = (x_1, y_1) + (x_2, y_2)$. In other words, $x^* \in x_1 + D^*G(\bar{x}, \bar{y})(y_1)$, which completes the proof. □

The next theorem on an upper estimate for the limiting subdifferential of the optimal value function (3.2) is obtained by slight modifications from [22, Thm. 4.1 (i)]. Namely, by adding the requirement on the lower regularity of the objective function φ to the assumptions of [22, Thm. 4.1 (i)], we can obtain the inclusion (3.8) of which the union in the right-hand-side is taken via $(x^*, y^*) \in \widehat{\partial}\varphi(\bar{x}, \bar{y})$, instead of via $(x^*, y^*) \in \partial\varphi(\bar{x}, \bar{y})$ as in [22, Thm. 4.1 (i)]. Thus, we will skip the proof of Theorem 2.

Recall that the solution map $M(\cdot)$ in (3.3) is said to be *μ-inner semicontinuous* at $(\bar{x}, \bar{y}) \in \operatorname{gph} M$ if for every sequence $x_k \xrightarrow{\mu} \bar{x}$ there exists a sequence $y_k \in M(x_k)$ that contains a subsequence converging to $\bar{y}$. This definition extends the corresponding notion in [20, Def. 1.63]. The difference is that the condition $x_k \to \bar{x}$ in [20] is replaced by the weaker condition $x_k \xrightarrow{\mu} \bar{x}$ and this change does not affect the proof of [22, Thm. 4.1 (i)].

Theorem 2 *Suppose that the optimal value function $\mu(\cdot)$ in (3.2) is finite at $\bar{x}$ and $\bar{y} \in M(\bar{x})$. Suppose in addition to the assumptions of Theorem 1 that the solution map M in (3.3) is μ-inner semicontinuous at $(\bar{x}, \bar{y}) \in \operatorname{gph} M$. Then, one has*

$$\partial\mu(\bar{x}) \subset \bigcup_{(x^*, y^*) \in \widehat{\partial}\varphi(\bar{x}, \bar{y})} \left\{x^* + D^*G(\bar{x}, \bar{y})(y^*)\right\}. \tag{3.8}$$

A nature question arises: "*When do the estimates in Theorems 1 and 2 become equalities?* There is one situation, that is when the objective function is (strictly) differentiable and the solution map admits a local Lipschitzian selection at the referencing point (see [27]). Another situation is when the objective function φ is convex, a part from the convexity of the constraint map G. Note that, in this case the optimal value function $\mu(\cdot)$ is convex (see [23, Prop. 1.50]). Let us now recall a result from [24, Thm. 9.1] where the inclusions (3.5) in Theorem 1 and (3.8) in Theorem 2 become equalities in the second situation. Interestingly, the lower semicontinuity of φ, the closeness of G as well as the μ-inner semicontinuity of the solution map $M(\cdot)$ are not required anymore herein.

Theorem 3 (See [23, Prop. 1.50], [24, Thm. 9.1] and [25, Thm. 4.56(c)].) *If the objective function φ of the problem* (3.1) *is convex, then the optimal value function $\mu(\cdot)$ in* (3.2) *is convex too. Suppose that the optimal value function $\mu(\cdot)$ is finite at $\bar{x}$ and $\bar{y} \in M(\bar{x})$. In addition, if the qualification condition*

$$\mathrm{ri}\,(\mathrm{dom}\,\varphi) \cap \mathrm{ri}\,(\mathrm{gph}\,G) \neq \emptyset \tag{3.9}$$

is satisfied, then

$$\partial\mu(\bar{x}) = \bigcup_{(x^*,y^*)\in\partial\varphi(\bar{x},\bar{y})} \big\{x^* + D^*G(\bar{x},\bar{y})(y^*)\big\}. \tag{3.10}$$

Remark 1 In a convex setting, the authors in [5] obtained formulas for computing the subdifferential of the optimal value function $\mu(\cdot)$ in (3.2) under the assumption on the continuity of φ. Recently, in [1, Sect. 5], the authors showed the assumption on the continuity of φ is stronger than the following condition

$$N((\bar{x},\bar{y}),\mathrm{dom}\,\varphi) \cap [-N((\bar{x},\bar{y}),\mathrm{gph}\,G)] = \{(0_{\mathbb{R}^n},0_{\mathbb{R}^n})\}. \tag{3.11}$$

Meanwhile, according to [24, Cor. 5.3], the condition (3.11) implies (3.9). Thus, the qualification condition (3.9) is the weakest condition guaranteeing the validity of (3.10) under the convex setting.

4 Parametric Optimization Problems with Linear–Convex Constraint Maps

Let $f : \mathbb{R}^m \times \mathbb{R}^p \to \overline{\mathbb{R}}$ be an extended real-valued function, $M : \mathbb{R}^p \to \mathbb{R}^n$ and $T : \mathbb{R}^m \to \mathbb{R}^n$ be linear operators, and $\Omega \subset \mathbb{R}^p$ be a nonempty convex set. For each $w \in \mathbb{R}^m$, put $H(w) = \big\{z \in \mathbb{R}^p \mid Mz = Tw\big\}$. Consider the parametric optimization problem

$$\min\{f(w,z) \mid z \in H(w) \cap \Omega\} \tag{4.1}$$

depending on parameter $w \in \mathbb{R}^m$. The optimal value function $h : \mathbb{R}^m \to \overline{\mathbb{R}}$ and the solution map $\widetilde{S} : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ of (4.1) are given by

$$h(w) := \inf\{f(w,z) \mid z \in H(w) \cap \Omega\} \tag{4.2}$$

and $\widetilde{S}(w) := \{z \in H(w) \cap \Omega \mid h(w) = f(w, z)\}$ for each $w \in \mathbb{R}^m$.

Let $\widetilde{H} : \mathbb{R}^m \rightrightarrows \mathbb{R}^p$ with $\widetilde{H}(w) := H(w) \cap \Omega$ for all $w \in \mathbb{R}^m$. Clearly,

$$\operatorname{gph} \widetilde{H} = (\operatorname{gph} H) \cap (\mathbb{R}^m \times \Omega). \tag{4.3}$$

So, (4.1) is a special form of the parametric optimization problem with convex constraint map (3.1). Since the graph of constraint map of the former can be represented as an intersection of a linear subspace and a convex set, we name it a *parametric optimization problem with linear–convex constraint map*.

This section aims at giving formulas for estimating/computing subdifferentials of the optimal value function $h(\cdot)$ of the problem (4.1). The purpose is twofold. Firstly, it concretizes abstract results in the previous section for the subclass of parametric optimization problems with linear–convex constraint maps. Secondly, it serves as a key tool in deriving results in the next section on differential stability of the parametric discrete control problem (2.1)–(2.4), the target of this paper. It is worthy to note that (4.1) generally is not a convex parameter optimization problem, despising to the fact that the constraint map $\widetilde{H}(\cdot)$ is convex. Thus, the regular subdifferential and the limiting subdifferential of $h(\cdot)$ are needed to be involved when studying the differential stability of the problem. However, if the objective function is convex, then so is the optimal value function $h(\cdot)$. In this case, we consider the subdifferential in the sense of convex analysis of $h(\cdot)$.

We need several auxiliary results to compute the coderivative of the constraint map. The first lemma is a finite-dimensional version of [2, Lem. 1]. Since every finite-dimensional linear subspace of $\mathbb{R}^n$ is closed (see, e.g. [10, Prop. 2.18]), the linear operator $\Phi(z, w) := -Tw + Mz$, $(w, z) \in \mathbb{R}^m \times \mathbb{R}^p$, has closed range. Thus, we skip that requirement on the formulation of the lemma.

Lemma 1 *For each $(\bar{w}, \bar{z}) \in \operatorname{gph} H$, one has*

$$N\big((\bar{w}, \bar{z}), \operatorname{gph} H\big) = \big\{(-T^*x^*, M^*x^*) \mid x^* \in \mathbb{R}^n\big\}, \tag{4.4}$$

where $M^* : \mathbb{R}^n \to \mathbb{R}^p$ and $T^* : \mathbb{R}^n \to \mathbb{R}^m$ are adjoint operators of $M : \mathbb{R}^p \to \mathbb{R}^n$ and $T : \mathbb{R}^m \to \mathbb{R}^n$, respectively.

The next lemma plays a decisive role to obtain upcoming results. It allows us to compute the normal cone to the intersection $\operatorname{gph} H \cap (\mathbb{R}^m \times \Omega)$ under a very mild regularity condition put on three elements describing the constraint of the problem (4.1): the linear operators M and T, and the set Ω.

Lemma 2 *If the regularity condition*

$$M(\operatorname{ri} \Omega) \cap \operatorname{rge} T \neq \emptyset \tag{4.5}$$

holds, then one has

$$N\big((\bar{w}, \bar{z}), \operatorname{gph} H \cap (\mathbb{R}^m \times \Omega)\big) = N\big((\bar{w}, \bar{z}), \operatorname{gph} H\big) + \{0_{\mathbb{R}^m}\} \times N(\bar{z}, \Omega) \tag{4.6}$$

for each $(\bar{w}, \bar{z}) \in \operatorname{gph} H \cap (\mathbb{R}^m \times \Omega)$.

Proof Observe that $M(\mathrm{ri}\,\Omega) = \{x \in \mathbb{R}^n \mid x = Mz,\ z \in \mathrm{ri}\,\Omega\}$ and

$$\mathrm{rge}\,T = \{x \in \mathbb{R}^n \mid x = Tw,\ w \in \mathbb{R}^m\}.$$

Let us show that the regularity condition (4.5) is equivalent to

$$(\mathbb{R}^m \times \mathrm{ri}\,\Omega) \cap \mathrm{gph}\ H \neq \emptyset. \tag{4.7}$$

Indeed, suppose that (4.5) holds, then there exists $x \in M(\mathrm{ri}\,\Omega) \cap \mathrm{rge}\,T$. So, we can find $(w, z) \in \mathbb{R}^m \times \mathrm{ri}\,\Omega$ such that $x = Mz$ and $x = Tw$. It means that $Mz = Tw$, or, $(w, z) \in \mathrm{gph}\,H$. Therefore, $(w, z) \in (\mathbb{R}^m \times \mathrm{ri}\,\Omega) \cap \mathrm{gph}\ H$. Conversely, suppose that (4.7) holds. Take $(w, z) \in (\mathbb{R}^m \times \mathrm{ri}\,\Omega) \cap \mathrm{gph}\ H$. Then $z \in \mathrm{ri}\,\Omega$ and $Mz = Tw$. Set $x := Mz = Tw$. Obviously, $x \in \mathrm{rge}\,T$ and $x \in M(\mathrm{ri}\,\Omega)$. Thus, $M(\mathrm{ri}\,\Omega) \cap \mathrm{rge}\,T \neq \emptyset$.

Let $A_0 := \mathrm{gph}\,H$ and $A_1 := \mathbb{R}^m \times \Omega$. Since $\mathrm{gph}\,H$ is a linear subspace of $\mathbb{R}^m \times \mathbb{R}^p$, we have $\mathrm{ri}\,(\mathrm{gph}\,H) = \mathrm{gph}\,H$. Moreover, $\mathrm{ri}\,(\mathbb{R}^m \times \Omega) = \mathbb{R}^m \times \mathrm{ri}\,\Omega$ (see, e.g., [32, p. 67]). So, from (4.7) one has $\mathrm{ri}\,(\mathbb{R}^m \times \Omega) \cap \mathrm{ri}\,(\mathrm{gph}\,H) \neq \emptyset$. Hence, applying Theorem 5.3 in [24] for the sets A_0, A_1 and the point $(\bar{w}, \bar{z}) \in A_0 \cap A_1$, we have

$$N((\bar{w}, \bar{z}), A_0 \cap A_1) = N((\bar{w}, \bar{z}), A_0) + N((\bar{w}, \bar{z}), A_1). \tag{4.8}$$

As $N((\bar{w}, \bar{z}), A_0) = N\big((\bar{w}, \bar{z}), \mathrm{gph}\,H\big)$ and $N((\bar{w}, \bar{z}), A_1) = \{0_{\mathbb{R}^m}\} \times N(\bar{z}, \Omega)$, the equality (4.6) follows from (4.8). □

Remark 2 If $\mathrm{ri}\,\Omega \cap \ker M \neq \emptyset$, in particular, $0 \in \mathrm{ri}\,\Omega$, then (4.5) is satisfied. Indeed, if $\mathrm{ri}\,\Omega \cap \ker M \neq \emptyset$, then there exists $\bar{z} \in \mathrm{ri}\,\Omega \cap \ker M$. On the one hand, since $\bar{z} \in \ker M$, it follows that $M\bar{z} = 0$. On the other hand, choose $\bar{w} = 0$, one has $H(\bar{w}) = \{z \in \mathbb{R}^p \mid Mz = T\bar{w} = 0\}$.Thus $\bar{z} \in H(\bar{w})$, or, $(\bar{w}, \bar{z}) \in \mathrm{gph}\,H$. In addition, it is clear that $(\bar{w}, \bar{z}) \in \mathbb{R}^m \times \mathrm{ri}\,\Omega$. So $(\bar{w}, \bar{z}) \in (\mathbb{R}^m \times \mathrm{ri}\,\Omega) \cap \mathrm{gph}\ H$. It means that (4.7) holds. Therefore (4.5) is also fulfilled.

Remark 3 In [15, Lem. 2], Kien et al. used the assumption $\mathrm{rge}\,M \subset \mathrm{rge}\,T$, which is true in particular if T is surjective, to obtain (4.6) in the case where Ω can be nonconvex but normally regular at $\bar{z}$. In our setting, this condition implies (4.5). Indeed, we first see that $\mathrm{ri}\,\Omega$ is always nonempty (see, [24, Thm. 2.7 (i)]). Take $z \in \mathrm{ri}\,\Omega$ and put $x := Mz$. So, $x \in M(\mathrm{ri}\,\Omega)$; hence $x \in \mathrm{rge}\,M$. Thus, if $\mathrm{rge}\,M \subset \mathrm{rge}\,T$, then $x \in \mathrm{rge}\,T$. We have shown that $x \in M(\mathrm{ri}\,\Omega) \cap \mathrm{rge}\,T$, i.e., (4.5) is valid under the condition $\mathrm{rge}\,M \subset \mathrm{rge}\,T$.

Remark 4 Chieu and Yao [11] also studied the problem (4.1) under the normal regularity of Ω. In our notation, to get (4.6), the authors assumed further that

$$[-N(\bar{z}, \Omega)] \cap M^*(\ker T^*) = \{0_{\mathbb{R}^p}\}. \tag{4.9}$$

In the proof of Theorem 2.1 (Step 2) in [11], the authors showed that (4.9) yields

$$N((\bar{w}, \bar{z}), \mathrm{gph}\,H) \cap [-N((\bar{w}, \bar{z}), \mathbb{R}^m \times \Omega)] = \{(0_{\mathbb{R}^m}, 0_{\mathbb{R}^p})\}. \tag{4.10}$$

Note that $\mathrm{gph}\,H$ is convex. When Ω is convex, the authors in [24, Cor. 5.5] pointed out that condition (4.10) is stronger than $\mathrm{ri}\,(\mathrm{gph}\ H) \cap \mathrm{ri}\,(\mathbb{R}^m \times \Omega) \neq \emptyset$. Clearly, the latter coincides with (4.7). While, in the proof of Lemma 2, we have shown that (4.7) is equivalent to (4.5). Thus, (4.9) implies our condition (4.5).

Remark 5 Under a Banach space setting, the authors in [2, Lem. 1 and 2] employed the condition that the linear operator $\Phi(w,z) = -Tw + Mz$ has closed range and $\ker T^* \subset \ker M^*$ in order to derive (4.6). In our current paper with the finite dimensional setting, it is clear that $\Phi(\cdot)$ has closed range. We will show that the condition $\ker T^* \subset \ker M^*$ is also stronger than (4.5). First, observe that $\ker T^* \subset \ker M^*$ implies $\operatorname{rge} M \subset \operatorname{rge} T$. Indeed, by [10, Prop. 2.173 (iv)], one has $(\ker T^*)^\perp = \operatorname{cl}(\operatorname{rge} T)$ and $(\ker M^*)^\perp = \operatorname{cl}(\operatorname{rge} M)$, where $A^\perp$ is the orthogonal complement of A. Moreover, as $\operatorname{rge} T \subset \mathbb{R}^m$ and $\operatorname{rge} M \subset \mathbb{R}^p$, one gets $\operatorname{cl}(\operatorname{rge} T) = \operatorname{rge} T$ and $\operatorname{cl}(\operatorname{rge} M) = \operatorname{rge} M$. Thus $(\ker T^*)^\perp = \operatorname{rge} T$, and $(\ker M^*)^\perp = \operatorname{rge} M$. In addition, as $\ker T^* \subset \ker M^*$, it yields that $(\ker M^*)^\perp \subset (\ker T^*)^\perp$. Consequently, we obtain $\operatorname{rge} M \subset \operatorname{rge} T$. Now by Remark 3 the latter implies (4.5). Thus, (4.5) is weaker than $\ker T^* \subset \ker M^*$.

With the preparations in the above lemmas, we are now able to compute the coderivative of the constraint map $\widetilde{H}(\cdot)$ of (4.1).

Lemma 3 *Suppose that the regularity condition* (4.5) *is fulfilled. Then one has*

$$D^*\widetilde{H}(\bar{w},\bar{z})(z^*) = \bigcup_{v^* \in N(\bar{z},\Omega)} \big\{T^*[(M^*)^{-1}(z^*+v^*)]\big\},\ \forall (\bar{w},\bar{z}) \in \operatorname{gph}\widetilde{H}. \quad (4.11)$$

Proof Using definition of coderivative and formula (4.3), we have

$$\begin{aligned} D^*\widetilde{H}(\bar{w},\bar{z})(z^*) &= \big\{\widetilde{w}^* \in \mathbb{R}^m \mid (\widetilde{w}^*, -z^*) \in N((\bar{w},\bar{z}), \operatorname{gph}\widetilde{H})\big\} \\ &= \big\{\widetilde{w}^* \in \mathbb{R}^m \mid (\widetilde{w}^*, -z^*) \in N\big((\bar{w},\bar{z}), \operatorname{gph} H \cap (\mathbb{R}^m \times \Omega)\big)\big\}. \end{aligned}$$

Since the condition (4.5) is fulfilled, we can apply formula (4.6) in Lemma 2 to get

$$\begin{aligned} D^*\widetilde{H}(\bar{w},\bar{z})(z^*) &= \big\{\widetilde{w}^* \in \mathbb{R}^m \mid (\widetilde{w}^*, -z^*) \in N((\bar{w},\bar{z}), \operatorname{gph} H) + \{0\} \times N(\bar{z},\Omega)\big\} \\ &= \bigcup_{v^* \in N(\bar{z},\Omega)} \big\{\widetilde{w}^* \in \mathbb{R}^m \mid (\widetilde{w}^*, -z^* - v^*) \in N((\bar{w},\bar{z}), \operatorname{gph} H)\big\}. \end{aligned}$$

Moreover, from (4.4), $\widetilde{w}^* \in D^*\widetilde{H}(\bar{w},\bar{z})(z^*)$ if and only if there exist $v^* \in N(\bar{z},\Omega)$ and $x^* \in X^*$ such that $(\widetilde{w}^*, -z^* - v^*) = (-T^*x^*, M^*x^*)$. Consequently, $x^* \in (M^*)^{-1}(-z^* - v^*)$ and $\widetilde{w}^* = -T^*x^*$. So, $\widetilde{w}^* \in D^*\widetilde{H}(\bar{w},\bar{z})(z^*)$ if and only if $\widetilde{w}^* \in T^*[(M^*)^{-1}(z^*+v^*)]$ for some $z^* \in N(\bar{z},\Omega)$. In other words, the equality (4.11) has been proved. □

The regular subdifferential of the function $h(\cdot)$ can be estimated as follows.

Theorem 4 *Let the optimal value function $h(\cdot)$ in* (4.2) *be finite at $\bar{w}$ and $\bar{z} \in \widetilde{S}(\bar{w})$. Suppose that the regularity condition* (4.5) *is satisfied. If f is lower semicontinuous and lower regular at $(\bar{w},\bar{z})$, Ω is closed, and the qualification condition*

$$\partial^\infty f(\bar{w},\bar{z}) \cap [-N((\bar{w},\bar{z}), \operatorname{gph} H \cap (\mathbb{R}^m \times \Omega))] = \{(0_{\mathbb{R}^m}, 0_{\mathbb{R}^p})\} \quad (4.12)$$

is satisfied, then

$$\widehat{\partial} h(\bar{w}) \subset \bigcup_{(w^*,z^*)\in\widehat{\partial} f(\bar{w},\bar{z})} \bigcup_{v^*\in N(\bar{z},\Omega)} \big[w^* + T^*\big((M^*)^{-1}(z^*+v^*)\big)\big]. \tag{4.13}$$

Proof We will apply Theorem 1 for the case where w, z, $f(w,z)$, $\widetilde{H}(w)$ and $h(w)$ play the roles of x, y, $\varphi(x,y)$, $G(x)$ and $\mu(x)$, respectively. By (4.3), the constraint map $\widetilde{H}(\cdot)$ is closed if Ω is closed. Under the assumptions, applying Theorem 1 yields

$$\widehat{\partial} h(\bar{w}) \subset \bigcup_{(w^*,z^*)\in\widehat{\partial} f(\bar{w},\bar{z})} \big\{w^* + D^*\widetilde{H}(\bar{w},\bar{z})(z^*)\big\}. \tag{4.14}$$

Combining (4.14) with (4.11), we obtain (4.13). The proof is completed. □

The next theorem equips us an formula for estimating the limiting subdifferential of the optimal value function $h(\cdot)$.

Theorem 5 *Let the optimal value function $h(\cdot)$ in* (4.2) *be finite at $\bar{w} \in \operatorname{dom} \widetilde{S}$ and $\bar{z} \in \widetilde{S}(\bar{w})$. Assume that the condition* (4.5) *is satisfied. If, in addition to the assumptions of Theorem 4, the map $\widetilde{S}(\cdot)$ is h-inner semicontinuous at $(\bar{w},\bar{z})$, then*

$$\partial h(\bar{w}) \subset \bigcup_{(w^*,z^*)\in\widehat{\partial} f(\bar{w},\bar{z})} \bigcup_{v^*\in N(\bar{z},\Omega)} \big[w^* + T^*\big((M^*)^{-1}(z^*+v^*)\big)\big]. \tag{4.15}$$

Proof We will follow the scheme in the proof of Theorem 4. Under the assumptions made, applying Theorem 2 with w, z, $f(w,z)$, $H(w)\cap\Omega$, $h(w)$, and $\widetilde{S}(w)$ respectively playing the roles of x, y, $\varphi(x,y)$, $G(x)$, $\mu(x)$, and $M(x)$, we obtain

$$\partial h(\bar{w}) \subset \bigcup_{(w^*,z^*)\in\widehat{\partial} f(\bar{w},\bar{z})} \big\{w^* + D^*\widetilde{H}(\bar{w},\bar{z})(z^*)\big\}. \tag{4.16}$$

Combining (4.16) with (4.11) implies (4.15), which completes the proof. □

Remark 6 If f is locally Lipschitzian around $(\bar{w},\bar{z})$, then $\partial^\infty f(\bar{w},\bar{z})=\{(0,0)\}$. In addition, $N((\bar{w},\bar{z}), \operatorname{gph} H \cap (\mathbb{R}^m \times \Omega))$ always contains the origin due to the convexity of $\operatorname{gph} H \cap (\mathbb{R}^m \times \Omega)$. It follows immediately that (4.12) is satisfied.

We end this subsection by a result for the case when assuming further that the objective function f of the problem (4.1) is convex. With this additional assumption, the optimal value function $h(\cdot)$ is convex. Thus, its subdifferential appearing in the next theorem is understood in the sense of convex analysis. The most difference from Theorems 4 and 5 is that herein we have exact formulas for computing subdifferentials of the optimal value function $h(\cdot)$, instead of its upper estimates only.

Theorem 6 *Suppose that the objective function f is convex. Then so is the optimal value function $h(\cdot)$ in* (4.2). *Let the optimal value function $h(\cdot)$ be finite at a point $\bar{w}$ and $\bar{z} \in \widetilde{S}(\bar{w})$. Suppose, in addition, that the regularity condition* (4.5) *and the qualification condition*

$$\mathrm{ri}\,(\mathrm{dom}\, f) \cap [\mathrm{gph}\, H \cap (\mathbb{R}^m \times \mathrm{ri}\, \Omega)] \neq \emptyset \tag{4.17}$$

are satisfied. Then,

$$\partial h(\bar{w}) = \bigcup_{(w^*, z^*) \in \partial f(\bar{w}, \bar{z})} \; \bigcup_{v^* \in N(\bar{z}, \Omega)} \left[w^* + T^*\big((M^*)^{-1}(z^* + v^*)\big) \right]. \tag{4.18}$$

Proof We will apply Theorem 3 to the case where w, z, $f(w, z)$, $\widetilde{H}(w)$ and $h(w)$ play, respectively, the roles of x, y, $\varphi(x, y)$, $G(x)$ and $\mu(x)$. The fact that $h(\cdot)$ is a convex function is obtained directly from the convexity of f and $\widetilde{H}(\cdot)$ and the first assertion of Theorem 3. By (4.3) and the regularity condition (4.5), we have $\mathrm{ri}\,\mathrm{gph}\, \widetilde{H} = \mathrm{gph}\, H \cap (\mathbb{R}^m \times \mathrm{ri}\, \Omega)$. So if the qualification condition (4.17) is satisfied, then (3.9) holds, and hence, we have

$$\partial h(\bar{w}) = \bigcup_{(w^*, z^*) \in \partial f(\bar{w}, \bar{z})} \left\{ w^* + D^* \widetilde{H}(\bar{w}, \bar{z})(z^*) \right\}. \tag{4.19}$$

Thus, the formula (4.18) follows from (4.19) and (4.11). This completes the proof. □

5 Differential Stability of Parametric Discrete Optimal Control Problems

In the notation of Subsection 2.1, put $Z = X \times U$ and $K = C \times \widetilde{X} \times \Omega$ and consider two linear operators $M : Z \to \widetilde{X}$ and $T : W \to \widetilde{X}$ defined respectively by

$$Mz = \begin{pmatrix} -A_0 & I & 0 & 0 & \dots & 0 & 0 & -B_0 & 0 & 0 & \dots & 0 \\ 0 & -A_1 & I & 0 & \dots & 0 & 0 & 0 & -B_1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -A_{N-1} & I & 0 & 0 & 0 & \dots & -B_{N-1} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \\ u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{pmatrix}$$

and

$$Tw = \begin{pmatrix} T_0 w_0 \\ T_1 w_1 \\ \vdots \\ T_{N-1} w_{N-1} \end{pmatrix}.$$

Consider also functions $\tilde{h}_k : W \times Z \to \mathbb{R}$ given by $\tilde{h}_k(w,z) = h_k(w_k, x_k, u_k)$ with $k = 0, 1, ..., N-1$, and function $\tilde{h}_N : W \times Z \to \mathbb{R}$ defined by $\tilde{h}_N(w,z) = h_N(x_N)$. Then the objective function $f(\cdot)$ in (2.5), the constraint map $G(\cdot)$ in (2.6), and the optimal value function $V(\cdot)$ in (2.7) of problem (2.1)–(2.4) can be expressed as

$$f(w,z) = \sum_{k=0}^{N} \tilde{h}_k(w,z), \quad (w,z) \in W \times Z, \tag{5.1}$$

$$G(w) = \big\{ z = (x,u) \in Z \,|\, Mz = Tw \big\}, \quad w \in W, \tag{5.2}$$

and

$$V(w) = \inf_{z \in G(w) \cap K} f(w,z), \quad w \in W. \tag{5.3}$$

Due to (5.3), the optimal value function $V(\cdot)$ of the problem (2.1)–(2.4) has the form of (4.2) (with the map $G(w)$ in (5.2) playing the role of $H(w)$ and the set K the role of Ω). Based on theoretical tools built for the parametric optimization problem with linear–convex constraint map (4.1) in the previous section, we are now ready to give formulas for estimating/computing subdifferentials of the optimal value function $V(\cdot)$. We will first investigate the regular, limiting subdifferentials of $V(\cdot)$ in the *nonconvex case* where $V(\cdot)$ is not neccessarily convex. Then, we will present formulas for estimating/computing the subdifferential in the sense of convex analysis of $V(\cdot)$ when it is a convex function, by which this is named the *convex case*.

Note that for every $\tilde{x}^* = (\tilde{x}_1^*, \tilde{x}_2^*, ..., \tilde{x}_n^*) \in \widetilde{X}$, one has

$$M^* \tilde{x}^* = \begin{pmatrix} -A_0^* & 0 & 0 \ldots 0 \\ I & -A_1^* & 0 \ldots 0 \\ 0 & I & \ldots 0 \\ \vdots & \vdots & \vdots \; \vdots \; \vdots \\ 0 & 0 & 0 \ldots -A_{N-1}^* \\ 0 & 0 & 0 \ldots I \\ -B_0^* & 0 & 0 \ldots 0 \\ 0 & -B_1^* & 0 \ldots 0 \\ \vdots & \vdots & \vdots \; \vdots \; \vdots \\ 0 & 0 & 0 \ldots -B_{N-1}^* \end{pmatrix} \begin{pmatrix} \tilde{x}_1^* \\ \tilde{x}_2^* \\ \vdots \\ \tilde{x}_N^* \end{pmatrix} \in Z \tag{5.4}$$

and

$$T^* \tilde{x}^* = \big(T_0^* \tilde{x}_1^*, T_1^* \tilde{x}_2^*, \cdots, T_{N-1}^* \tilde{x}_N^* \big) \in W \tag{5.5}$$

with M^*, T^*, A_i^*, and B_i^* being the adjoint operators of T, M, A_i, and B_i, respectively.

5.1 Nonconvex Case

The next theorem provides us an upper estimate for the regular subdifferential of $V(\cdot)$. It is worthy to note that the Fréchet differentiability of the functions h_k, $k = 0, 1, ..., N$ is not required herein.

Theorem 7 *Let the optimal value function* $V(\cdot)$ *of the problem* (2.1)–(2.4) *be finite at a point* $\bar{w}$ *and let* $\bar{z} = (\bar{x}, \bar{u}) \in S(\bar{w})$ *with* $\bar{w} = (\bar{w}_0, \bar{w}_1, \dots, \bar{w}_{N-1})$, $\bar{x} = (\bar{x}_0, \bar{x}_1, \dots, \bar{x}_N)$ *and* $\bar{u} = (\bar{u}_0, \bar{u}_1, \dots, \bar{u}_{N-1})$. *Let further the functions* h_k, $k = 0, 1, \dots, N-1$ *and* h_N *be lower semicontinuous and lower regular at* $(\bar{w}_k, \bar{x}_k, \bar{u}_k)$ *and* $\bar{x}_N$, *respectively. Suppose that the sets* C *and* Ω_i, $i = 0, 1, \dots, N-1$, *are closed and the assumptions*

$$M(\operatorname{ri} K) \cap \operatorname{rge} T \neq \emptyset \tag{A1}$$

and

$$\begin{cases} v_k^* \in \partial^\infty \tilde{h}_k(\bar{w}, \bar{z}), \ k = 0, 1, .., N \\ \tilde{v}^* \in N((\bar{w}, \bar{z}), \operatorname{gph} G \cap (W \times K)) \\ v_0^* + v_1^* + ... + v_N^* + \tilde{v}^* = 0 \end{cases} \Longrightarrow v_0^* = v_1^* = ... = v_N^* = \tilde{v}^* = 0 \tag{A2}$$

are fulfilled. If $w^* = (w_0^*, w_1^*, \dots, w_{N-1}^*) \in W$ *is a regular subgradient of the optimal value function* $V(\cdot)$ *at* $\bar{w}$, *it is necessary that there exist* $x_0^* \in N(\bar{x}_0, C)$, $\tilde{x}^* = (\tilde{x}_1^*, \tilde{x}_2^*, ..., \tilde{x}_N^*) \in \widetilde{X}$ *and* $u^* = (u_0^*, u_1^*, ..., u_{N-1}^*) \in N(\bar{u}, \Omega)$ *such that*

$$\begin{cases} \tilde{x}_N^* \in \widehat{\partial} h_N(\bar{x}_N) \\ \tilde{x}_k^* \in \widehat{\partial}_{x_k} h_k(\bar{w}_k, \bar{x}_k, \bar{u}_k) + A_k^* \tilde{x}_{k+1}^*, \ k = 1, 2, ..., N-1 \\ x_0^* \in -\widehat{\partial}_{x_0} h_0(\bar{w}_0, \bar{x}_0, \bar{u}_0) - A_0^* \tilde{x}_1^* \\ u_k^* \in -\widehat{\partial}_{u_k} h_k(\bar{w}_k, \bar{x}_k, \bar{u}_k) - B_k^* \tilde{x}_{k+1}^*, \ k = 0, 1, ..., N-1 \\ w_k^* \in \widehat{\partial}_{w_k} h_k(\bar{w}_k, \bar{x}_k, \bar{u}_k) + T_k^* \tilde{x}_{k+1}^*, \ k = 0, 1, ..., N-1, \end{cases} \tag{5.6}$$

with $\widehat{\partial}_{w_k} h_k(\bar{w}_k, \bar{x}_k, \bar{u}_k), \widehat{\partial}_{x_k} h_k(\bar{w}_k, \bar{x}_k, \bar{u}_k), \widehat{\partial}_{u_k} h_k(\bar{w}_k, \bar{x}_k, \bar{u}_k)$ *being the regular subdifferentials of* $h_k(., \bar{x}_k, \bar{u}_k)$ *at* $\bar{w}_k$, $h_k(\bar{w}_k, ., \bar{u}_k)$ *at* $\bar{x}_k$, *and* $h_k(\bar{w}_k, \bar{x}_k, .)$ *at* $\bar{u}_k$, *respectively.*

Proof First, since the functions h_k, $k = 0, 1, \dots, N$ are lower semicontinuous at $(\bar{w}_k, \bar{x}_k, \bar{u}_k)$, so are the functions $\tilde{h}_k(\cdot)$, $k = 0, 1, \dots, N$ at $(\bar{w}, \bar{z})$. This implies that the objective function $f(\cdot)$ in (5.1) is lower semicontinuous at $(\bar{w}, \bar{z})$. Next, $\tilde{h}_k(\cdot)$, $k = 0, 1, \dots, N$ are lower regular at $(\bar{w}, \bar{z})$ because of the lower regularity of h_k, $k = 0, 1, \dots N$ and of [29, Prop. 4.36 and 6.17 (e)]; hence $f(\cdot)$ is lower regular at $(\bar{w}, \bar{z})$ due to [22, Cor. 2.21]. Furthermore, the closedness of K follows from the closedness of Ω_i, $i = 0, 1, \dots, N-1$, and C, while the condition (4.5) is satisfied due to the assumption (A1). Thus, it remains to show the validity of the qualification condition (4.12) from the assumption (A2).

Let $v_k^* \in \partial^\infty \tilde{h}_k(\bar{w}, \bar{z})$, $k = 0, 1, .., N$, be such that $v_0^* + ... + v_N^* = 0$. Then by considering $\tilde{v}^* := 0 \in N((\bar{w}, \bar{z}), \operatorname{gph} G \cap (W \times K))$, we have that

$v_0^* + ... + v_N^* + \tilde{v}^* = 0$. Thus, it follows from the assumption (A2) that $v_0^* = ... = v_N^* = 0$. Therefore, we can apply the sum rules in [22, Cor. 2.21] for the function $f(\cdot)$ to get

$$\widehat{\partial} f(\bar{w},\bar{z}) = \widehat{\partial}\left(\sum_{k=0}^{N} \tilde{h}_k(\bar{w},\bar{z})\right) = \sum_{k=0}^{N} \widehat{\partial}\tilde{h}_k(\bar{w},\bar{z}) \tag{5.7}$$

and

$$\partial^\infty f(\bar{w},\bar{z}) = \partial^\infty\left(\sum_{k=0}^{N} \tilde{h}_k(\bar{w},\bar{z})\right) = \sum_{k=0}^{N} \partial^\infty\tilde{h}_k(\bar{w},\bar{z}). \tag{5.8}$$

Due to (5.8), the assumption (A2) becomes

$$\partial^\infty f(\bar{w},\bar{z}) \cap [-N((\bar{w},\bar{z}), \operatorname{gph} G \cap (W \times K))] = \{(0_W, 0_Z)\},$$

which means that condition (4.12) is satisfied.

Applying Theorem 4, we get

$$\widehat{\partial} V(\bar{w}) \subset \bigcup_{(w^*,z^*)\in\widehat{\partial} f(\bar{w},\bar{z})} \ \bigcup_{v^*\in N(\bar{z},K)} \big[w^* + T^*\big((M^*)^{-1}(z^*+v^*)\big)\big]. \tag{5.9}$$

Let $w^* \in \widehat{\partial} V(\bar{w})$. It follows from (5.9) that there exist $(w_1^*, z_1^*) \in \widehat{\partial} f(\bar{w},\bar{z})$ and $v_1^* \in N(\bar{z},K)$ satisfying $w^* \in w_1^* + T^*((M^*)^{-1}(z_1^* + v_1^*))$. The latter means that there is $\tilde{x}^* = (\tilde{x}_1^*, \tilde{x}_2^*, ..., \tilde{x}_N^*) \in \widetilde{X}^*$ with

$$M^*\tilde{x}^* = z_1^* + v_1^* \tag{5.10}$$

and

$$w^* \in w_1^* + T^*\tilde{x}^*. \tag{5.11}$$

Let us explore the inclusion $(w_1^*, z_1^*) \in \widehat{\partial} f(\bar{w},\bar{z})$. Using the product rule (2.10) for the functions $\tilde{h}_k(\cdot)$ at $(\bar{w},\bar{z})$, we obtain

$$\begin{aligned}\sum_{k=0}^{N} \widehat{\partial}\tilde{h}_k(\bar{w},\bar{z}) &\subset \sum_{k=0}^{N} \Big(\widehat{\partial}_w \tilde{h}_k(\bar{w},\bar{z}) \times \widehat{\partial}_z \tilde{h}_k(\bar{w},\bar{z})\Big) \\ &= \sum_{k=0}^{N} \widehat{\partial}_w \tilde{h}_k(\bar{w},\bar{z}) \times \sum_{k=0}^{N} \widehat{\partial}_z \tilde{h}_k(\bar{w},\bar{z}). \end{aligned} \tag{5.12}$$

By the relationship between $\tilde{h}_k(\cdot)$ and h_k and by using again the product rule (2.10) for the regular subdifferentials of functions h_k at $(\bar{w},\bar{x}_k,\bar{u}_k)$, we get

$$\begin{aligned}
\sum_{k=0}^{N}\widehat{\partial}_w\tilde{h}_k(\bar{w},\bar{z}) &= \widehat{\partial}_w\tilde{h}_0(\bar{w},\bar{z})+\widehat{\partial}_w\tilde{h}_1(\bar{w},\bar{z})+\cdots+\widehat{\partial}_w\tilde{h}_{N-1}(\bar{w},\bar{z})\\
&\subset \Big[\widehat{\partial}_{w_0}\tilde{h}_0(\bar{w},\bar{z})\times\widehat{\partial}_{w_1}\tilde{h}_0(\bar{w},\bar{z})\times\cdots\times\widehat{\partial}_{w_{N-1}}\tilde{h}_0(\bar{w},\bar{z})\Big]\\
&\quad+\Big[\widehat{\partial}_{w_0}\tilde{h}_1(\bar{w},\bar{z})\times\widehat{\partial}_{w_1}\tilde{h}_1(\bar{w},\bar{z})\times\cdots\times\widehat{\partial}_{w_{N-1}}\tilde{h}_1(\bar{w},\bar{z})\Big]\\
&\quad+\cdots\\
&\quad+\Big[\widehat{\partial}_{w_0}\tilde{h}_{N-1}(\bar{w},\bar{z})\times\widehat{\partial}_{w_1}\tilde{h}_{N-1}(\bar{w},\bar{z})\times\cdots\times\widehat{\partial}_{w_{N-1}}\tilde{h}_{N-1}(\bar{w},\bar{z})\Big]\\
&= \Big[\widehat{\partial}_{w_0}h_0(\bar{w}_0,\bar{x}_0,\bar{u}_0)\times\{0\}\times\cdots\times\{0\}\Big]\\
&\quad+\Big[\{0\}\times\widehat{\partial}_{w_1}h_1(\bar{w}_1,\bar{x}_1,\bar{u}_1)\times\{0\}\times\cdots\times\{0\}\Big]\\
&\quad+\cdots\\
&\quad+\Big[\{0\}\times\cdots\times\{0\}\times\widehat{\partial}_{w_{N-1}}h_{N-1}(\bar{w}_{N-1},\bar{x}_{N-1},\bar{u}_{N-1})\Big].
\end{aligned}$$

Consequently,

$$\sum_{k=0}^{N}\widehat{\partial}_w\tilde{h}_k(\bar{w},\bar{z})\subset\widehat{\partial}_{w_0}h_0(\bar{w}_0,\bar{x}_0,\bar{u}_0)\times\ldots\times\widehat{\partial}_{w_{N-1}}h_{N-1}(\bar{w}_{N-1},\bar{x}_{N-1},\bar{u}_{N-1}). \tag{5.13}$$

Similarly,

$$\begin{aligned}
\sum_{k=0}^{N}\widehat{\partial}_z\tilde{h}_k(\bar{w},\bar{z}) &\subset \sum_{k=0}^{N}\widehat{\partial}_x\tilde{h}_k(\bar{w},\bar{z})\times\sum_{k=0}^{N}\widehat{\partial}_u\tilde{h}_k(\bar{w},\bar{z})\\
&\subset \widehat{\partial}_{x_0}h_0(\bar{w}_0,\bar{x}_0,\bar{u}_0)\times\ldots\times\widehat{\partial}_{x_{N-1}}h_{N-1}(\bar{w}_{N-1},\bar{x}_{N-1},\bar{u}_{N-1}) \qquad (5.14)\\
&\quad\times\widehat{\partial}h_N(\bar{x}_N)\times\widehat{\partial}_{u_0}h_0(\bar{w}_0,\bar{x}_0,\bar{u}_0)\times\ldots\\
&\quad\times\widehat{\partial}_{u_{N-1}}h_{N-1}(\bar{w}_{N-1},\bar{x}_{N-1},\bar{u}_{N-1}).
\end{aligned}$$

Therefore, it follows from (5.7), (5.12), (5.13), and (5.14) that

$$\begin{aligned}
\widehat{\partial}f(\bar{w},\bar{z}) &\subset \widehat{\partial}_{w_0}h_0(\bar{w}_0,\bar{x}_0,\bar{u}_0)\times\ldots\times\widehat{\partial}_{w_{N-1}}h_{N-1}(\bar{w}_{N-1},\bar{x}_{N-1},\bar{u}_{N-1})\\
&\quad\times\widehat{\partial}_{x_0}h_0(\bar{w}_0,\bar{x}_0,\bar{u}_0)\times\ldots\times\widehat{\partial}_{x_{N-1}}h_{N-1}(\bar{w}_{N-1},\bar{x}_{N-1},\bar{u}_{N-1})\times\widehat{\partial}h_N(\bar{x}_N)\\
&\quad\times\widehat{\partial}_{u_0}h_0(\bar{w}_0,\bar{x}_0,\bar{u}_0)\times\ldots\times\widehat{\partial}_{u_{N-1}}h_{N-1}(\bar{w}_{N-1},\bar{x}_{N-1},\bar{u}_{N-1}).
\end{aligned}$$

Thus, the inclusion $(w_1^*,z_1^*)\in\widehat{\partial}f(\bar{w},\bar{z})$ yields

$$w_1^*\in\widehat{\partial}_{w_0}h_0(\bar{w}_0,\bar{x}_0,\bar{u}_0)\times\ldots\times\widehat{\partial}_{w_{N-1}}h_{N-1}(\bar{w}_{N-1},\bar{x}_{N-1},\bar{u}_{N-1}) \tag{5.15}$$

and

$$z_1^* \in \widehat{\partial}_{x_0} h_0(\bar{w}_0,\bar{x}_0,\bar{u}_0)\times ...\times \widehat{\partial}_{x_{N-1}} h_{N-1}(\bar{w}_{N-1},\bar{x}_{N-1},\bar{u}_{N-1})\times \widehat{\partial} h_N(\bar{x}_N) \qquad (5.16)$$
$$\times\, \widehat{\partial}_{u_0} h_0(\bar{w}_0,\bar{x}_0,\bar{u}_0)\times ...\times \widehat{\partial}_{u_{N-1}} h_{N-1}(\bar{w}_{N-1},\bar{x}_{N-1},\bar{u}_{N-1}).$$

Now we compute the normal cone $N(\bar{z},K)$ by using the product formula for normal cones. Because $K = C\times \widetilde{X}\times \Omega$ and $\Omega = \Omega_0\times\Omega_1\cdots\times\Omega_{N-1}$, we have

$$N(\bar{z},K) = N(\bar{x}_0,C)\times\{0_{\tilde{X}^*}\}\times N(\bar{u}_0,\Omega_0)\times ...\times N(\bar{u}_{N-1},\Omega_{N-1}). \qquad (5.17)$$

Because $v_1^* \in N(\bar{z},K)$, it follows from (5.17) that there exist $x_0^* \in N(\bar{x}_0,C)$ and $u^* = (u_0^*,u_1^*,\ldots,u_{N-1}^*)$ with $u_k^* \in N(\bar{u}_k,\Omega_k)$ for all $k = 0,1,\ldots,N-1$ such that $v_1^* = (x_0^*,0,u^*)$. Therefore, using (5.4), (5.10) and (5.16), we get

$$\begin{pmatrix} -A_0^* & 0 & 0\ldots 0 \\ I & -A_1^* & 0\ldots 0 \\ 0 & I & 0\ldots 0 \\ \vdots & \vdots & \vdots\;\vdots\;\;\vdots \\ 0 & 0 & 0\ldots -A_{N-1}^* \\ 0 & 0 & 0\ldots I \end{pmatrix}\begin{pmatrix}\tilde{x}_1^* \\ \tilde{x}_2^* \\ \vdots \\ \tilde{x}_N^*\end{pmatrix}$$
$$\in \left(\widehat{\partial}_{x_0} h_0(\bar{w}_0,\bar{x}_0,\bar{u}_0)+x_0^*\right)\times \widehat{\partial}_{x_1} h_1(\bar{w}_1,\bar{x}_1,\bar{u}_1)\times ...\times$$
$$\widehat{\partial}_{x_{N-1}} h_{N-1}(\bar{w}_{N-1},\bar{x}_{N-1},\bar{u}_{N-1})\times \widehat{\partial} h_N(\bar{x}_N)$$

and

$$\begin{pmatrix} -B_0^* & 0 & 0\ldots 0 \\ 0 & -B_1^* & 0\ldots 0 \\ \vdots & \vdots & \vdots\;\vdots\;\;\vdots \\ 0 & 0 & 0\ldots -B_{N-1}^* \end{pmatrix}\begin{pmatrix}\tilde{x}_1^* \\ \tilde{x}_2^* \\ \vdots \\ \tilde{x}_N^*\end{pmatrix}$$
$$\in \left(\widehat{\partial}_{u_0} h_0(\bar{w}_0,\bar{x}_0,\bar{u}_0)+u_0^*\right)\times ...\times \left(\widehat{\partial}_{u_{N-1}} h_{N-1}(\bar{w}_{N-1},\bar{x}_{N-1},\bar{u}_{N-1})+u_{N-1}^*\right).$$

This means that

$$\begin{cases} -x_0^* \in A_0^*\tilde{x}_1^* + \widehat{\partial}_{x_0} h_0(\bar{w}_0,\bar{x}_0,\bar{u}_0), \\ \tilde{x}_1^* \in A_1^*\tilde{x}_2^* + \widehat{\partial}_{x_1} h_1(\bar{w}_1,\bar{x}_1,\bar{u}_1), \\ ... \\ \tilde{x}_{N-1}^* \in A_{N-1}^*\tilde{x}_N^* + \widehat{\partial}_{x_{N-1}} h_{N-1}(\bar{w}_{N-1},\bar{x}_{N-1},\bar{u}_{N-1}), \\ \tilde{x}_N^* \in \widehat{\partial} h_N(\bar{x}_N), \end{cases} \qquad (5.18)$$

and

$$-B_k^*\tilde{x}_{k+1}^* \in \widehat{\partial}_{u_k} h_k(\bar{w}_k,\bar{x}_k,\bar{u}_k) + u_k^*,\;\; k = 0,1,...,N-1. \qquad (5.19)$$

Similarly, the last formula of (5.6) can be derived from (5.5), (5.11), and (5.15)

$$w_k^* \in \widehat{\partial}_{w_k} h_k(\bar{w}_k,\bar{x}_k,\bar{u}_k) + T_k^*\tilde{x}_{k+1}^*,\;\; k = 0,1,...,N-1. \qquad (5.20)$$

Finally, we obtain (5.6) from (5.18), (5.19), and (5.20). The proof is completed. □

The following theorem allows us to estimate the limiting subdifferential of $V(\cdot)$.

Theorem 8 *Under the assumptions of Theorem 7, suppose further that the solution map $S(\cdot)$ is V- inner semicontinuous at $(\bar{w}, \bar{z})$. If $w^* \in W^*$ is a limiting subgradient of the optimal value function $V(\cdot)$ of problem* (2.1)–(2.4) *at $\bar{w}$, then there exist $x_0^* \in N(\bar{x}_0, C)$, $\tilde{x}^* = (\tilde{x}_1^*, \tilde{x}_2^*, ..., \tilde{x}_N^*) \in \widetilde{X}$ and $u^* = (u_0^*, u_1^*, ..., u_{N-1}^*) \in N(\bar{u}, \Omega)$ such that* (5.6) *is satisfied.*

Proof We will follow the scheme in the proof of Theorem 7. Under the additional assumption on the solution map $S(\cdot)$, we are able to apply Theorem 5, instead of Theorem 4, to obtain

$$\partial V(\bar{w}) \subset \bigcup_{(w^*, z^*) \in \widehat{\partial} f(\bar{w}, \bar{z})} \ \bigcup_{v^* \in N(\bar{z}, K)} \big[w^* + T^*\big((M^*)^{-1}(z^* + v^*)\big)\big]. \quad (5.21)$$

We see that the right hand side of (5.21) is also the one in (5.9). Therefore, by the same manner as in the proof of Theorem 7, we get (5.6). □

Remark 7 In [15], Kien et al. obtained the upper estimate (5.6) for the regular subdifferential of $V(\cdot)$ under the following assumptions:

(i) the functions $h_k, k = 0, ..., N-1$ and h_N are Fréchet differentiable at $(\bar{w}_k,\bar{x}_k,\bar{u}_k)$ and $\bar{x}_N$, respectively;

(ii) $T_k, k = 0, 1, ..., N-1$ are surjective;

(iii) Ω_k are normally regular at $\bar{u}_k$ for all $k = 0, 1, ..., N-1$.

In this paper, since $\Omega_k, k = 0, ..., N-1$ are convex, it follows that they are normally regular. Besides, if $T_k, k = 0, 1, ..., N-1$ are surjective, then $T : W \to \tilde{X}$ is surjective too and the latter implies condition (A1) by Remark 3. Meanwhile, the functions $h_k, k = 0, ..., N-1$ and h_N in Theorem 7 can be nondifferentiable, provided that they are lower semicontinuous, lower regular and satisfy the qualification condition (A2). Therefore, our result in Theorem 7 is quite different from the one in [15, Thm 1.1].

Remark 8 In the paper [11], Chieu and Yao studied the first-order behavior of the optimal value function of the parametric discrete optimal control problem (2.1)–(2.4). More precisely, in our notation the authors derived the upper estimate (5.6) for the regular subdifferential of $V(\cdot)$ under the conditions (i), (iii) as in Remark 7 and the validity of the condition

$$[-\widehat{N}(\bar{z}, K)] \cap M^*(\ker T^*) = \{0\}. \quad (5.22)$$

In addition, the authors showed that the condition (ii) in Remark 7 implies condition (5.22). Meanwhile, in Remark 4, we have asserted that (5.22) is stronger than our assumption (A1).

Remark 9 An estimate for the limiting subdifferential of $V(\cdot)$ was shown in [37, Thm. 2.1]. Namely, the authors employed the assumptions that: the functions h_k, $k = 0, 1, ..., N$ are strictly differentiable (hence lower regular), Ω_k are normally regular with $\operatorname{int} \Omega_k \neq \emptyset$, for $k = 0, 1, ..., N-1$ and an assumption which implies (5.22). Here, in our setting, $\Omega_k, k = 0, 1, ..., N-1$ are convex. It means that Ω_k are normally regular. However, we do not need the assumption $\operatorname{int} \Omega_k \neq \emptyset$. In addition, if h_k, $k = 0, 1, ..., N$ are strictly differentiable, then $\tilde{h}_k(\cdot)$, $k = 0, 1, ..., N$ are also strictly differentiable. Note that every mapping strictly differentiable is locally Lipschitzian around the point in question (see [20, p. 19]). Consequently, $\tilde{h}_k(\cdot)$, $k = 0, 1, ..., N$ are locally Lipschitzian around $(\bar{w}, \bar{z})$ and our qualification condition (A2) is satisfied. Meanwhile, as mentioned in Remark 8, our assumption (A1) is weaker than (5.22).

Remark 10 To obtain sufficient conditions for $w^* \in \widehat{\partial} V(\bar{w})$ (resp., $w^* \in \partial V(\bar{w})$), the authors in [15] and [11] (resp., in [37]) assumed furthermore that the solution map $S(\cdot)$ has a *local upper Lipschitzian selection* at $(\bar{w}, \bar{z})$. In general, this condition is quite difficult to check. In the forthcoming section, we will discuss in detail another type of sufficient condition without using any special requirement on the solution map.

5.2 Convex Case

When the data of the problem (2.1)–(2.4) are all convex, there are more to tell about properties of the optimal value function $V(\cdot)$.

Theorem 9 *Let h_k, $k = 0, 1, \ldots, N$ be convex functions and C, Ω_i, $i = 0, 1, \ldots, N-1$ be nonempty convex sets. Then the optimal value function $V : W \to \overline{\mathbb{R}}$ of the problem* (2.1)–(2.4) *is convex and possesses the following properties.*

(i) $V(\cdot) \equiv -\infty$ *or* $V(\cdot)$ *takes finite value on the whole space* W.

(ii) $V(\cdot)$ *is locally Lipschitzian on* W.

(iii) $\partial^\infty V(w) = \{0\}$ *for every* $w \in W$.

(iv) $\partial V(w)$ *is a nonempty compact set for every* $w \in W$.

(v) *If* $V(\cdot)$ *is finite on the whole space* W, *then* $V(\cdot)$ *is Gâteaux differentiable at some* $w \in W$ *if and only if* $\partial V(w)$ *is a singleton.*

Proof The fact that $V(\cdot)$ is convex is due to the assumptions on the convexity of input data, the representation (5.3), and the first assertion of Theorem 6.

(i) We will first show that $\operatorname{dom} V = W$. Indeed, since h_k, $k = 0, 1, \ldots, N$ are finite-valued functions, it follows that $f(\cdot)$ is finite. Moreover, for each $w \in W$,

$$V(w) = \inf_{(x,u) \in G(w) \cap (C \times \tilde{X} \times \Omega)} f(w, x, u) \leq f(\bar{w}, \bar{x}, \bar{u}) < +\infty$$

with $(\bar{w}, \bar{x}, \bar{u})$ being in the nonempty set $W \times (C \times \tilde{X}) \times \Omega$. So, $\operatorname{dom} V = W$. Besides, according to [39, Prop. 2.1.4], if there exists $\bar{w} \in W$ with $V(\bar{w}) = -\infty$, then the convex function $V(\cdot)$ will take the value $-\infty$ on $\operatorname{dom} V$. Therefore, we obtain (i).

(ii) Since $V(\cdot)$ is a convex function defined on the finite-dimensional space W, it follows from Corollary 2.27 in [23] that $V(\cdot)$ is locally Lipschitzian on $\mathrm{int}\,(\mathrm{dom}\,V)$. Combining this with the fact that $\mathrm{dom}\,V = W$, we get (ii).

(iii) This property is a consequence of (ii).

(iv) This property follows directly from Proposition 2.47 in [23] and the fact that $V(\cdot)$ is a convex function on the finite-dimensional space W.

(v) This property is obtained by using Corollary 2.4.10 in [39]. □

The main results on differential stability of the problem (2.1)–(2.4) in the convex case are presented in the following theorem. Especially, when the Fréchet differentiability of the functions $h_k, k = 0, 1, \ldots, N,$ describing the objective of the problem (2.1)–(2.4) is available, the theorem provide us a procedure for finding elements in the subdifferential of $V(\cdot)$ at the reference paramater.

Theorem 10 *Suppose that $h_k,\ k = 0, 1, \ldots, N$ are convex functions and C, Ω_i, $i = 0, 1, \ldots, N-1$ are nonempty convex sets. Let the optimal value function $V(\cdot)$ of the problem (2.1)–(2.4) be finite at $\bar{w}$ and let $\bar{z} = (\bar{x}, \bar{u}) \in S(\bar{w})$ with $\bar{w} = (\bar{w}_0, \bar{w}_1, \ldots, \bar{w}_{N-1})$, $\bar{x} = (\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_N)$ and $\bar{u} = (\bar{u}_0, \bar{u}_1, \ldots, \bar{u}_{N-1})$. Assume further that the condition (A1) holds. If $w^* = (w_0^*, w_1^*, \ldots, w_{N-1}^*) \in W$ is a subgradient (in the sense of convex analysis) of the optimal value function $V(\cdot)$ at $\bar{w}$, then there exist $x_0^* \in N(\bar{x}_0, C)$, $\tilde{x}^* = (\tilde{x}_1^*, \tilde{x}_2^*, ..., \tilde{x}_N^*) \in \widetilde{X}$ and $u^* = (u_0^*, u_1^*, ..., u_{N-1}^*) \in N(\bar{u}, \Omega)$ such that*

$$\begin{cases} \tilde{x}_N^* \in \partial h_N(\bar{x}_N), \\ \tilde{x}_k^* \in \partial_{x_k} h_k(\bar{w}_k, \bar{x}_k, \bar{u}_k) + A_k^* \tilde{x}_{k+1}^*,\ k = 1, 2, ..., N-1, \\ x_0^* \in -\partial_{x_0} h_0(\bar{w}_0, \bar{x}_0, \bar{u}_0) - A_0^* \tilde{x}_1^*, \\ u_k^* \in -\partial_{u_k} h_k(\bar{w}_k, \bar{x}_k, \bar{u}_k) - B_k^* \tilde{x}_{k+1}^*,\ k = 0, 1, ..., N-1, \\ w_k^* \in \partial_{w_k} h_k(\bar{w}_k, \bar{x}_k, \bar{u}_k) + T_k^* \tilde{x}_{k+1}^*,\ k = 0, 1, ..., N-1, \end{cases} \tag{5.23}$$

where $\partial_{w_k} h_k(\bar{w}_k, \bar{x}_k, \bar{u}_k), \partial_{u_k} h_k(\bar{w}_k, \bar{x}_k, \bar{u}_k), \partial_{x_k} h_k(\bar{w}_k, \bar{x}_k, \bar{u}_k)$ are the subdifferentials (in the sense of convex analysis) of $h_k(., \bar{x}_k, \bar{u}_k)$ at $\bar{w}_k$, $h_k(\bar{w}_k, ., \bar{u}_k)$ at $\bar{x}_k$, and $h_k(\bar{w}_k, \bar{x}_k, .)$ at $\bar{u}_k$ respectively.

In particular, if for every $k = 0, 1, \ldots, N-1$, h_k are Fréchet differentiable at $(\bar{w}_k, \bar{x}_k, \bar{u}_k)$ and h_N is Fréchet differentiable at $\bar{x}_N$, then $w^* \in W^*$ is a subgradient (in the sense of convex analysis) of $V(\cdot)$ at $\bar{w}$ if and only if there exist $x_0^* \in N(\bar{x}_0, C)$, $\tilde{x}^* = (\tilde{x}_1^*, \tilde{x}_2^*, ..., \tilde{x}_N^*) \in \widetilde{X}$ and $u^* = (u_0^*, u_1^*, ..., u_{N-1}^*) \in N(\bar{u}, \Omega)$ such that

$$\begin{cases} \tilde{x}_N^* = \nabla h_N(\bar{x}_N), \\ \tilde{x}_k^* = \nabla_{x_k} h_k(\bar{w}_k, \bar{x}_k, \bar{u}_k) + A_k^* \tilde{x}_{k+1}^*, \quad k = 1, 2, \ldots, N-1, \\ x_0^* = -\nabla_{x_0} h_0(\bar{w}_0, \bar{x}_0, \bar{u}_0) - A_0^* \tilde{x}_1^*, \\ u_k^* = -\nabla_{u_k} h_k(\bar{w}_k, \bar{x}_k, \bar{u}_k) - B_k^* \tilde{x}_{k+1}^*, \quad k = 0, 1, \ldots, N-1, \\ w_k^* = \nabla_{w_k} h_k(\bar{w}_k, \bar{x}_k, \bar{u}_k) + T_k^* \tilde{x}_{k+1}^*, \quad k = 0, 1, \ldots, N-1, \end{cases} \tag{5.24}$$

where $\nabla_{w_k} h_k(\bar{w}_k, \bar{x}_k, \bar{u}_k)$, $\nabla_{x_k} h_k(\bar{w}_k, \bar{x}_k, \bar{u}_k)$, $\nabla_{u_k} h_k(\bar{w}_k, \bar{x}_k, \bar{u}_k)$ stand for the Fréchet derivatives of the functions $h_k(., \bar{x}_k, \bar{u}_k)$, $h_k(\bar{w}_k, ., \bar{u}_k)$, $h_k(\bar{w}_k, \bar{x}_k, .)$ at $\bar{w}_k$, $\bar{x}_k$, and $\bar{u}_k$, respectively.

Proof Since h_k are convex functions taking finite values, so are $\tilde{h}_k(\cdot)$, $k = 0, 1, \dots, N$. It follows that $f(\cdot)$ is convex with $\operatorname{dom} f = \bigcap_{k=0}^{N} \operatorname{dom} \tilde{h}_k = W \times Z$ (see, e.g., [39, Thm. 2.1.3]). Hence, $\operatorname{ri}(\operatorname{dom} f) = W \times Z$. Besides, the condition (A1) guarantees $\operatorname{gph} G \cap (W \times \operatorname{ri} K) \neq \emptyset$ (see the proof of Lemma 2). This implies that

$$\begin{aligned}\operatorname{ri}(\operatorname{dom} f) \cap [\operatorname{gph} G \cap (W \times \operatorname{ri} K)] &= [W \times Z] \cap [\operatorname{gph} G \cap (W \times \operatorname{ri} K)] \\ &= \operatorname{gph} G \cap (W \times \operatorname{ri} K) \neq \emptyset,\end{aligned}$$

which shows the validity of (4.17). On account of Theorem 6, one has

$$\partial V(\bar{w}) = \bigcup_{(w^*,z^*) \in \partial f(\bar{w},\bar{z})} \ \bigcup_{v^* \in N(\bar{z},K)} \left[w^* + T^*\big((M^*)^{-1}(z^* + v^*)\big)\right]. \tag{5.25}$$

Let $w^* = (w_0^*, w_1^*, \dots, w_{N-1}^*)$ belong to $\partial V(\bar{w})$. Due to (5.25), there exist $(w_1^*, z_1^*) \in \partial f(\bar{w}, \bar{z})$ and $v_1^* \in N(\bar{z}, K)$ with $w^* \in w_1^* + T^*((M^*)^{-1})(z_1^* + v_1^*)$. The last inclusion means that there exists $\tilde{x}^* = (\tilde{x}_1^*, \tilde{x}_2^*, ..., \tilde{x}_N^*) \in \widetilde{X}^*$ satisfying

$$M^* \tilde{x}^* = z_1^* + v_1^* \quad \text{and} \quad \tilde{w}^* \in w_1^* + T^* \tilde{x}^*. \tag{5.26}$$

Similarly as in the proof of Theorem 7, it remains to expand the two inclusions $(w_1^*, z_1^*) \in \partial f(\bar{w}, \bar{z})$ and $v_1^* \in N(\bar{z}, K)$. As shown in the proof of Theorem 7, the latter means there exist $x_0^* \in N(\bar{x}_0, C)$ and $u^* = (u_0^*, u_1^*, \dots, u_{N-1}^*)$ with $u_k^* \in N(\bar{u}_k, \Omega_k)$ for all $k = 0, 1, \dots, N-1$ such that $v_1^* = (x_0^*, 0, u^*)$. The former will be explored in the same way as follows.

Since $\tilde{h}_k(\cdot)$, $k = 0, 1, \dots, N$ are convex functions taking finite values, applying the Moreau-Rockafellar theorem [23, Cor. 2.46], we obtain

$$\partial f(\bar{w}, \bar{z}) = \partial \left(\sum_{k=0}^{N} \tilde{h}_k(\bar{w}, \bar{z}) \right) = \sum_{k=0}^{N} \partial \tilde{h}_k(\bar{w}, \bar{z}). \tag{5.27}$$

Besides, using the product rule (2.10) for subdifferentials of convex functions $\tilde{h}_k(\cdot)$ at $(\bar{w}, \bar{z})$, we get

$$\sum_{k=0}^{N} \partial \tilde{h}_k(\bar{w}, \bar{z}) \subset \sum_{k=0}^{N} \partial_w \tilde{h}_k(\bar{w}, \bar{z}) \times \sum_{k=0}^{N} \partial_z \tilde{h}_k(\bar{w}, \bar{z}) \tag{5.28}$$

with $\partial_w \tilde{h}_k(\bar{w}, \bar{z}), \partial_z \tilde{h}_k(\bar{w}, \bar{z})$ being the subdifferentials of $\tilde{h}_k(., \bar{z})$ at $\bar{w}$ and $\tilde{h}_k(\bar{w}, .)$ at $\bar{z}$, respectively. Now, using the relationship between $\tilde{h}_k(\cdot)$ and h_k and applying again the product rule (2.10) for convex functions h_k at $(\bar{w}, \bar{x}, \bar{u})$, we have

$$\begin{aligned}\sum_{k=0}^{N}\partial_w\tilde{h}_k(\bar{w},\bar{z}) &= \partial_w\tilde{h}_0(\bar{w},\bar{z})+\partial_w\tilde{h}_1(\bar{w},\bar{z})+\cdots+\partial_w\tilde{h}_{N-1}(\bar{w},\bar{z})\\ &\subset \Big[\partial_{w_0}\tilde{h}_0(\bar{w},\bar{z})\times\partial_{w_1}\tilde{h}_0(\bar{w},\bar{z})\times\cdots\times\partial_{w_{N-1}}\tilde{h}_0(\bar{w},\bar{z})\Big]\\ &\quad+\Big[\partial_{w_0}\tilde{h}_1(\bar{w},\bar{z})\times\partial_{w_1}\tilde{h}_1(\bar{w},\bar{z})\times\cdots\times\partial_{w_{N-1}}\tilde{h}_1(\bar{w},\bar{z})\Big]\\ &\quad+\cdots\\ &\quad+\Big[\partial_{w_0}\tilde{h}_{N-1}(\bar{w},\bar{z})\times\partial_{w_1}\tilde{h}_{N-1}(\bar{w},\bar{z})\times\cdots\times\partial_{w_{N-1}}\tilde{h}_{N-1}(\bar{w},\bar{z})\Big]\\ &= [\partial_{w_0}h_0(\bar{w}_0,\bar{x}_0,\bar{u}_0)\times\{0\}\times\cdots\times\{0\}]\\ &\quad+[\{0\}\times\partial_{w_1}h_1(\bar{w}_1,\bar{x}_1,\bar{u}_1)\times\{0\}\times\cdots\times\{0\}]\\ &\quad+\cdots\\ &\quad+\big[\{0\}\times\cdots\times\{0\}\times\partial_{w_{N-1}}h_{N-1}(\bar{w}_{N-1},\bar{x}_{N-1},\bar{u}_{N-1})\big].\end{aligned}$$

As a result, we get

$$\sum_{k=0}^{N}\partial_w\tilde{h}_k(\bar{w},\bar{z})\subset\partial_{w_0}h_0(\bar{w}_0,\bar{x}_0,\bar{u}_0)\times\ldots\times\partial_{w_{N-1}}h_{N-1}(\bar{w}_{N-1},\bar{x}_{N-1},\bar{u}_{N-1}). \quad (5.29)$$

In a same manner, we have

$$\begin{aligned}\sum_{k=0}^{N}\partial_z\tilde{h}_k(\bar{w},\bar{z}) &\subset\sum_{k=0}^{N}\partial_x\tilde{h}_k(\bar{w},\bar{z})\times\sum_{k=0}^{N}\partial_u\tilde{h}_k(\bar{w},\bar{z})\\ &\subset\partial_{x_0}h_0(\bar{w}_0,\bar{x}_0,\bar{u}_0)\times\ldots\times\partial_{x_{N-1}}h_{N-1}(\bar{w}_{N-1},\bar{x}_{N-1},\bar{u}_{N-1})\\ &\quad\times\partial h_N(\bar{x}_N)\times\partial_{u_0}h_0(\bar{w}_0,\bar{x}_0,\bar{u}_0)\times\ldots\\ &\quad\times\partial_{u_{N-1}}h_{N-1}(\bar{w}_{N-1},\bar{x}_{N-1},\bar{u}_{N-1}).\end{aligned} \quad (5.30)$$

So, it follows from (5.27), (5.28), (5.29), and (5.30) that

$$\begin{aligned}\partial f(\bar{w},\bar{z})\subset{}&\partial_{w_0}h_0(\bar{w}_0,\bar{x}_0,\bar{u}_0)\times\ldots\times\partial_{w_{N-1}}h_{N-1}(\bar{w}_{N-1},\bar{x}_{N-1},\bar{u}_{N-1})\\ &\times\partial_{x_0}h_0(\bar{w}_0,\bar{x}_0,\bar{u}_0)\times\ldots\times\partial_{x_{N-1}}h_{N-1}(\bar{w}_{N-1},\bar{x}_{N-1},\bar{u}_{N-1})\times\partial h_N(\bar{x}_N)\\ &\times\partial_{u_0}h_0(\bar{w}_0,\bar{x}_0,\bar{u}_0)\times\ldots\times\partial_{u_{N-1}}h_{N-1}(\bar{w}_{N-1},\bar{x}_{N-1},\bar{u}_{N-1}).\end{aligned}$$

Thus, the inclusion $(w_1^*,z_1^*)\in\partial f(\bar{w},\bar{z})$ implies

$$w_1^*\in\partial_{w_0}h_0(\bar{w}_0,\bar{x}_0,\bar{u}_0)\times\ldots\times\partial_{w_{N-1}}h_{N-1}(\bar{w}_{N-1},\bar{x}_{N-1},\bar{u}_{N-1})$$

and

$$\begin{aligned}z_1^*\in{}&\partial_{x_0}h_0(\bar{w}_0,\bar{x}_0,\bar{u}_0)\times\ldots\times\partial_{x_{N-1}}h_{N-1}(\bar{w}_{N-1},\bar{x}_{N-1},\bar{u}_{N-1})\times\partial h_N(\bar{x}_N)\\ &\times\partial_{u_0}h_0(\bar{w}_0,\bar{x}_0,\bar{u}_0)\times\ldots\times\partial_{u_{N-1}}h_{N-1}(\bar{w}_{N-1},\bar{x}_{N-1},\bar{u}_{N-1}).\end{aligned}$$

Combining the last inclusions with (5.26) yields

$$M^*\tilde{x}^*\in\partial_{w_0}h_0(\bar{w}_0,\bar{x}_0,\bar{u}_0)\times\ldots\times\partial_{w_{N-1}}h_{N-1}(\bar{w}_{N-1},\bar{x}_{N-1},\bar{u}_{N-1})+v_1^*, \quad (5.31)$$

and

$$\begin{aligned} w^* \in & \big[\partial_{x_0} h_0(\bar{w}_0, \bar{x}_0, \bar{u}_0) \times ... \times \partial_{x_{N-1}} h_{N-1}(\bar{w}_{N-1}, \bar{x}_{N-1}, \bar{u}_{N-1}) \\ & \times\ \partial h_N(\bar{x}_N) \times \partial_{u_0} h_0(\bar{w}_0, \bar{x}_0, \bar{u}_0) \times ... \\ & \times \partial_{u_{N-1}} h_{N-1}(\bar{w}_{N-1}, \bar{x}_{N-1}, \bar{u}_{N-1})\big] + T^* \tilde{x}^*. \end{aligned} \tag{5.32}$$

By same arguments as the last part of the proof in Theorem 7, we obtain (5.23).

It is well-known that if $\phi : \mathbb{R}^n \to \overline{\mathbb{R}}$ is convex and Fréchet differentiable at $\bar{x} \in \operatorname{int}(\operatorname{dom}\phi)$, then $\partial\phi(\bar{x}) = \{\nabla\phi(\bar{x})\}$ (see, e.g. [23, Prop. 2.36]). So, as h_k, $k = 0, 1, ..., N$, are Fréchet differentiable by our assumptions, it follows that the inclusions in (5.31)–(5.32) become equalities, i.e.,

$$\begin{aligned} M^* \tilde{x}^* &= \nabla_{w_0} h_0(\bar{w}_0, \bar{x}_0, \bar{u}_0) \times ... \times \nabla_{w_{N-1}} h_{N-1}(\bar{w}_{N-1}, \bar{x}_{N-1}, \bar{u}_{N-1}) + v_1^*, \\ w^* &= \big[\nabla_{x_0} h_0(\bar{w}_0, \bar{x}_0, \bar{u}_0) \times ... \times \nabla_{x_{N-1}} h_{N-1}(\bar{w}_{N-1}, \bar{x}_{N-1}, \bar{u}_{N-1}) \times \nabla h_N(\bar{x}_N) \\ & \quad \times \nabla_{u_0} h_0(\bar{w}_0, \bar{x}_0, \bar{u}_0) \times ... \times \nabla_{u_{N-1}} h_{N-1}(\bar{w}_{N-1}, \bar{x}_{N-1}, \bar{u}_{N-1})\big] + T^* \tilde{x}^*. \end{aligned}$$

Therefore, $\tilde{w}^* \in \partial V(\bar{w})$ if and only if we can find $x_0^* \in N(\bar{x}_0, C)$, $\tilde{x}^* \in \widetilde{X}^*$ and $u^* \in N(\bar{u}, \Omega)$ such that (5.24) is satisfied. The proof is complete. □

6 Illustrative Examples

This section presents some illustrative examples for the obtained results. The first one is designed to show how Theorems 7 and 8 work for parametric optimal control problems with neither convex nor differentiable costs.

Example 1 Let $X_0 = X_1 = X_2 := \mathbb{R}, U_0 = U_1 = \mathbb{R}$, $W_0 = W_1 := \mathbb{R}$, $C := [-2, 0]$, $\Omega_0 := [1, +\infty)$, and $\Omega_1 := \mathbb{R}$. Let $A_0 : X_0 \to X_1, B_0 : U_0 \to X_1$, $T_0 : W_0 \to X_1, A_1 : X_1 \to X_2, B_1 : U_1 \to X_2$ and $T_1 : W_1 \to X_2$ be linear operators given by $A_0x_0 = 0, B_0u_0 = 0, T_0w_0 = w_0, A_1x_1 = 0, B_1u_1 = -u_1$, and $T_1w_1 = 0$, respectively. Furthermore, let $h_0 : W_0 \times X_0 \times U_0 \to \mathbb{R}, h_1 : W_1 \times X_1 \times U_1 \to \mathbb{R}$ and $h_2 : X_2 \to \mathbb{R}$ be functions defined by

$$\begin{aligned} & h_0(w_0, x_0, u_0) = (x_0 + u_0)^3 + (x_0 + u_0)^2, \\ & h_1(w_1, x_1, u_1) = |x_1 - w_1|, \\ & h_2(x_2) = 0. \end{aligned}$$

Then, for each $w = (w_0, w_1) \in \mathbb{R}^2$, the optimal control problem (2.1)–(2.4) is as follow

$$\begin{cases} (x_0 + u_0)^3 + (x_0 + u_0)^2 + |x_1 - w_1| \to \inf, \\ x_2 = -u_1, \\ x_1 = w_0, \\ x_0 \in [-2, 0], \\ u_0 \in [1, +\infty), \\ u_1 \in \mathbb{R}. \end{cases} \tag{6.1}$$

Let $\bar{w} = (\bar{w}_0, \bar{w}_1) = (0,0)$. It is not hard to show that $z = (-u_0, 0, x_2, u_0, -x_2)$ and $z = (-2, 0, x_2, 1, -x_2)$ for some $u_0 \in [1,2]$ and $x_2 \in \mathbb{R}$ belong to $S(\bar{w})$.

Choose $\bar{z} = (\bar{x}, \bar{u})$ with $\bar{x} = (\bar{x}_0, \bar{x}_1, \bar{x}_2) = (-1,0,0)$ and $\bar{u} = (\bar{u}_0, \bar{u}_1) = (1,0)$. Then $\bar{z} \in S(\bar{w})$. We are going to show that the assumptions in Theorem 7 are satisfied. Indeed, it is clear that the functions $h_0(\cdot)$, $h_1(\cdot)$, and $h_2(\cdot)$ are respectively continuous and lower regular at $(\bar{w}_0, \bar{x}_0, \bar{u}_0)$, $(\bar{w}_1, \bar{x}_1, \bar{u}_1)$, and $\bar{x}_2$. Also, the sets C, Ω_0, and Ω_1 are closed. It remains to verify conditions (A1) and (A2).

Invoking notations at the beginning of Section 5 and input data of (6.1), we have $K = [-2,0] \times \mathbb{R} \times \mathbb{R} \times [1,+\infty) \times \mathbb{R}$, the linear operators $M : \mathbb{R}^5 \to \mathbb{R}^2$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$ are defined by

$$Mz = \begin{pmatrix} 0\,1\,0\,0\,0 \\ 0\,0\,1\,0\,1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ u_0 \\ u_1 \end{pmatrix} \quad \text{and} \quad Tw = \begin{pmatrix} 1\,0 \\ 0\,0 \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \end{pmatrix}.$$

Clearly, $\tilde{x} = (0,0) \in \operatorname{rge} T$ and $\tilde{x} = M\hat{z}$ with $\hat{z} = (1,0,0,2,0) \in \operatorname{ri} K$. So, $\tilde{x} \in M(\operatorname{ri} K) \cap \operatorname{rge} T$, which means condition (A1) is satisfied. To verify (A2), we note that functions $\tilde{h}_0, \tilde{h}_1, \tilde{h}_2 : \mathbb{R}^2 \times \mathbb{R}^5 \to \mathbb{R}$ mapping $(w,z) = (w_0, w_1, x_0, x_1, x_2, u_0, u_1)$ to $\tilde{h}_0(w,z) = h_0(w_0, x_0, u_0)$, $\tilde{h}_1(w,z) = h_1(w_1, x_1, u_1)$ and $\tilde{h}_2(w,z) = h_2(x_2)$ are locally Lipschitz around $(\bar{w}, \bar{z})$. So, for each $k = 0,1,2$, $\partial^\infty \tilde{h}_k(\bar{w}, \bar{z})$ contains only the zero vector. This implies that condition (A2) is fulfilled as well.

Let $w^* = (w_0^*, w_1^*)$ be an arbitrary element of the regular subdifferential $\widehat{\partial} V(\bar{w})$. Then, Theorem 7 tells us that there exist $x_0^* \in N(\bar{x}_0, C)$, $(\tilde{x}_1^*, \tilde{x}_2^*) \in \mathbb{R}^2$, and $(u_0^*, u_1^*) \in N(\bar{u}, \Omega_1 \times \Omega_2)$ satisfying

$$\begin{cases} \tilde{x}_2^* \in \widehat{\partial} h_2(\bar{x}_2) \\ \tilde{x}_1^* \in \widehat{\partial}_{x_1} h_1(\bar{w}_1, \bar{x}_1, \bar{u}_1) + A_1^* \tilde{x}_2^* \\ x_0^* \in -\widehat{\partial}_{x_0} h_0(\bar{w}_0, \bar{x}_0, \bar{u}_0) - A_0^* \tilde{x}_1^* \\ u_1^* \in -\widehat{\partial}_{u_1} h_1(\bar{w}_1, \bar{x}_1, \bar{u}_1) - B_1^* \tilde{x}_2^* \\ u_0^* \in -\widehat{\partial}_{u_0} h_0(\bar{w}_0, \bar{x}_0, \bar{u}_0) - B_0^* \tilde{x}_1^* \\ w_1^* \in \widehat{\partial}_{w_1} h_1(\bar{w}_1, \bar{x}_1, \bar{u}_1) + T_1^* \tilde{x}_2^* \\ w_0^* \in \widehat{\partial}_{w_0} h_0(\bar{w}_0, \bar{x}_0, \bar{u}_0) + T_0^* \tilde{x}_1^*. \end{cases} \tag{6.2}$$

From the given data, we can obtain $A_0^* = 0$, $B_0^* = 0, T_0^* = 1, A_1^* = 0, B_1^* = -1$, $T_1^* = 0$, $\widehat{\partial}_{x_0} h_0(\bar{w}_0, \bar{x}_0, \bar{u}_0) = \{0\}$, $\widehat{\partial}_{x_1} h_1(\bar{w}_1, \bar{x}_1, \bar{u}_1) = [-1,1]$, $\widehat{\partial} h_2(\bar{x}_2) = \{0\}$, $\widehat{\partial}_{u_0} h_0(\bar{w}_0, \bar{x}_0, \bar{u}_0) = \{0\}$, $\widehat{\partial}_{u_1} h_1(\bar{w}_1, \bar{x}_1, \bar{u}_1) = \{0\}$, $\widehat{\partial}_{w_0} h_0(\bar{w}_0, \bar{x}_0, \bar{u}_0) = \{0\}$, $\widehat{\partial}_{w_1} h_1(\bar{w}_1, \bar{x}_1, \bar{u}_1) = [-1,1]$. Therefore, (6.2) yields $\tilde{x}_2^* = 0$, $\tilde{x}_1^* \in [-1,1]$, $x_0^* = 0$, $u_1^* = 0$, $u_0^* = 0$, $w_1^* \in [-1,1]$ and $w_0^* \in [-1,1]$. Since $N(\bar{x}_0, C) = \{0\}$ and $N(\bar{u}, \Omega_1 \times \Omega_2) = (-\infty, 0] \times \{0\}$, inclusions $x_0^* \in N(\bar{x}_0, C)$ and $(u_0^*, u_1^*) \in N(\bar{u}, \Omega_1 \times \Omega_2)$ are valid also. Because $w^* = (w_0^*, w_1^*)$ is taken arbitrarily in $\widehat{\partial} V(\bar{w})$, we conclude that

$$\widehat{\partial} V(\bar{w}) \subset [-1,1] \times [-1,1].$$

Due to the nondifferentiability of the function $h_1(\cdot)$ at $(\bar{w}_1, \bar{x}_1, \bar{u}_1) = (0, 0, 0)$ let us notice that results in [15, Thm. 1.1], [11, Thm. 1.1], and [37, Thm. 2.1] are not applicable to get an upper estimation for $\widehat{\partial} V(\bar{w})$.

Next, to estimate the limiting subdifferential $\partial V(\bar{w})$ by Theorem 8, we need to prove that the solution map $S(\cdot)$ of (6.1) is V-inner semicontinuous at $(\bar{w}, \bar{z})$. Indeed, by direct computations, we obtain

$$S(w)=\big\{(-u_0, w_0, x_2, u_0, -x_2), x_2 \in \mathbb{R}, u_0 \in [1, 2]\big\} \cup \big\{(-2, w_0, x_2, 1, -x_2), x_2 \in \mathbb{R}\big\},$$

for every $w = (w_0, w_1) \in \mathbb{R}^2$. Let $\{w^n = (w_0^n, w_1^n)\}_{n=1,2,...}$ be an arbitrary sequence converging to $\bar{w}$. Then by choosing $\{z^n := (-1, w_0^n, 0, 1, 0)\}_{n=1}^{\infty}$, we get $z^n \in S(w^n)$ for all $n = 1, 2, ...$ and $z^n \to \bar{z}$. Thus the solution map $S(\cdot)$ of (6.1) is inner semicontinuous at $(\bar{w}, \bar{z})$. Therefore, by Theorem 8, we obtain

$$\partial V(\bar{w}) \subset [-1, 1] \times [-1, 1].$$

We now give an example to illustrate the results of Theorems 9 and 10.

Example 2 Let $X_0 = X_1 = X_2 := \mathbb{R}$, $U_0 = U_1 := \mathbb{R}$, $W_0 = W_1 := \mathbb{R}$, $C := (-\infty, 1]$, $\Omega_0 := [1, +\infty)$, and $\Omega_1 := \mathbb{R}$. Besides, let $A_0 : X_0 \to X_1$, $B_0 : U_0 \to X_1$, $T_0 : W_0 \to X_1$, $A_1 : X_1 \to X_2$, $B_1 : U_1 \to X_2$ and $T_1 : W_1 \to X_2$ be linear operators defined by $A_0 x_0 = 0$, $B_0 u_0 = 0$, $T_0 w_0 = -w_0$, $A_1 x_1 = 0$, $B_1 u_1 = -u_1$, and $T_1 w_1 = 0$. Furthermore, let $h_0 : W_0 \times X_0 \times U_0 \to \mathbb{R}$, $h_1 : W_1 \times X_1 \times U_1 \to \mathbb{R}$, and $h_2 : X_2 \to \mathbb{R}$ be given by

$$\begin{aligned} &h_0(w_0, x_0, u_0) = (x_0 + u_0)^2 + \frac{1}{2} w_0^2, \\ &h_1(w_1, x_1, u_1) = |w_1| + |x_1|, \\ &h_2(x_2) = 0. \end{aligned}$$

Hence, the problem (2.1)–(2.4) depending on $w = (w_0, w_1) \in \mathbb{R}^2$ becomes

$$\begin{cases} (x_0 + u_0)^2 + \frac{1}{2} w_0^2 + |w_1| + |x_1| \to \inf, \\ x_2 = -u_1, \\ x_1 = -w_0, \\ x_0 \in (-\infty, 1], \\ u_0 \in [1, +\infty), \\ u_1 \in \mathbb{R}. \end{cases} \tag{6.3}$$

Since the functions $h_0(\cdot)$, $h_1(\cdot)$, and $h_2(\cdot)$ are all convex, the optimal value function $w \mapsto V(w)$ of (6.3) is convex (see, Theorem 9). Thus, Theorem 10 can be used to estimate the subdifferential in the sense of convex analysis of $V(\cdot)$, provided that condition (A1) is satisfied. To show the latter, we use notations at the beginning of Section 5 and compute from input data of (6.3) to get that

$K = [-\infty, 1] \times \mathbb{R} \times \mathbb{R} \times [1, +\infty) \times \mathbb{R}$ and the linear operators $M : \mathbb{R}^5 \to \mathbb{R}^2$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$ are

$$Mz = \begin{pmatrix} 0\,1\,0\,0\,0 \\ 0\,0\,1\,0\,1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ u_0 \\ u_1 \end{pmatrix} \quad \text{and} \quad Tw = \begin{pmatrix} -1\;0 \\ 0\;0 \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \end{pmatrix}.$$

Take $\tilde{x} = (0, 0) \in \operatorname{rge} T$ and $\hat{z} = (0, 0, 0, 2, 0) \in \operatorname{ri} K$. Then $\tilde{x} = M\hat{z}$ and $\tilde{x} \in M(\operatorname{ri} K) \cap \operatorname{rge} T$. So, condition (A1) holds.

Let $\bar{w} = (\bar{w}_0, \bar{w}_1) = (-1, 1)$ and $\bar{z} = (\bar{x}, \bar{u})$ with $\bar{x} = (\bar{x}_0, \bar{x}_1, \bar{x}_2) = (-1, 1, 0)$ and $\bar{u} = (\bar{u}_0, \bar{u}_1) = (1, 0)$. It is easy to check that $\bar{z} \in S(\bar{w})$ and the functions $h_0(\cdot)$, $h_1(\cdot)$, and $h_2(\cdot)$ are Fréchet differentiable respectively at $(\bar{w}_0, \bar{x}_0, \bar{u}_0)$, $(\bar{w}_1, \bar{x}_1, \bar{u}_1)$, and $\bar{x}_2$. So, if $w^* = (w_0^*, w_1^*) \in \partial V(\bar{w})$, by Theorem 10 there exist $x_0^* \in N(\bar{x}_0, C)$, $(\tilde{x}_1^*, \tilde{x}_2^*) \in \mathbb{R}^2$, and $(u_0^*, u_1^*) \in N(\bar{u}, \Omega_1 \times \Omega_2)$ satisfying

$$\begin{cases} \tilde{x}_2^* = \nabla h_2(\bar{x}_2) \\ \tilde{x}_1^* = \nabla_{x_1} h_1(\bar{w}_1, \bar{x}_1, \bar{u}_1) + A_1^* \tilde{x}_2^* \\ x_0^* = -\nabla_{x_0} h_0(\bar{w}_0, \bar{x}_0, \bar{u}_0) - A_0^* \tilde{x}_1^* \\ u_1^* = -\nabla_{u_1} h_1(\bar{w}_1, \bar{x}_1, \bar{u}_1) - B_1^* \tilde{x}_2^* \\ u_0^* = -\nabla_{u_0} h_0(\bar{w}_0, \bar{x}_0, \bar{u}_0) - B_0^* \tilde{x}_1^* \\ w_1^* = \nabla_{w_1} h_1(\bar{w}_1, \bar{x}_1, \bar{u}_1) + T_1^* \tilde{x}_2^* \\ w_0^* = \nabla_{w_0} h_0(\bar{w}_0, \bar{x}_0, \bar{u}_0) + T_0^* \tilde{x}_1^*. \end{cases} \tag{6.4}$$

From the given data, we get $A_0^* = 0$, $B_0^* = 0$, $T_0^* = -1$, $A_1^* = 0$, $B_1^* = -1$, $T_1^* = 0$, $\nabla_{x_0} h_0(\bar{w}_0, \bar{x}_0, \bar{u}_0) = 0$, $\nabla_{x_1} h_1(\bar{w}_1, \bar{x}_1, \bar{u}_1) = 1$, $\nabla h_2(\bar{x}_2) = 0$, $\nabla_{u_0} h_0(\bar{w}_0, \bar{x}_0, \bar{u}_0) = 0$, $\nabla_{u_1} h_1(\bar{w}_1, \bar{x}_1, \bar{u}_1) = 0$, $\nabla_{w_0} h_0(\bar{w}_0, \bar{x}_0, \bar{u}_0) = -1$, and $\nabla_{w_1} h_1(\bar{w}_1, \bar{x}_1, \bar{u}_1) = 1$. So, (6.4) implies $x_0^* = 0$, $\tilde{x}_1^* = 1$, $\tilde{x}_2^* = 0$, $u_1^* = 0$, $u_0^* = 0$, $w_1^* = 1$ and $w_0^* = -2$. As $N(\bar{x}_0, C) = \{0\}$ and $N(\bar{u}, \Omega_1 \times \Omega_2) = (-\infty, 0] \times \{0\}$, requirements $x_0^* \in N(\bar{x}_0, C)$ and $(u_0^*, u_1^*) \in N(\bar{u}, \Omega_1 \times \Omega_2)$ are satisfied as well. In other words

$$\partial V(\bar{w}) = \{(-2, 1)\}.$$

The latter means that $V(\cdot)$ is Gâteaux differentiable at $\bar{w}$, due to assertion (v) in Theorem 9. We note that Theorem 3 in [2] and Theorem 1 in [15] on estimating the subdifferential of $V(\cdot)$ do not work here, because neither the condition $\ker T^* \subset \ker M^*$ in [2, Thm. 3] nor the requirement that T be surjective in [15, Thm. 1] is valid for problem (6.3). Moreover, due to the nondifferentiability of the function $h_1(\cdot)$ at $(\bar{w}_1, \bar{x}_1, \bar{u}_1) = (0, 0, 0)$, we find that neither [11, Thm. 1.1] nor [37, Thm. 2.1] is applicable to get an upper estimation for $\widehat{\partial} V(\bar{w})$.

7 Concluding Remarks

In this paper, we have obtained various results on differential stability of parametric discrete optimal control problems with possibly nondifferential costs. In the case where the functions describing the cost of the control problem are nonconvex, we have established upper estimates for the regular, the limiting subdifferentials of the optimal value function $V(\cdot)$ under regularity conditions (A1) and (A2), among other assumptions. To our best knowledge, the first condition is the weakest one ever used in the literature for control problems with linear and convex constraints. Meanwhile, the second one is introduced to overcome for the first time the tricky challenge arising from control problems with nonsmooth costs. In the other case where the functions describing the cost of the control problem are convex, besides giving formulas for estimating/computing the subdifferential in the sense of convex analysis of $V(\cdot)$, we have also shown several fundamental properties of $V(\cdot)$ that have not been mentioned in the literature. Last but not least, we have designed some examples showing that our results are applicable while existing results are not.

For further investigation, we are interested in the problem of estimating/computing subdifferentials of the optimal value function $V(\cdot)$ with h_k, $k = 0, ..., N$ being lower regular and the sets C and Ω_i, $i = 0, 1, ..., N-1$ normally regular. As mentioned in this paper, the class of normally regular sets is really bigger than the class of convex sets. To study this problem we can exploit advanced tools and techniques from variational analysis to compute the regular coderivative and limiting coderivative of the constraint map $G(w) \cap K$. Abstract results built up for parametric optimization problems with lower regular objective functions in this paper allow us to hope that new results for this more complicated but interesting problem can be achievable.

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