Quantitative stability for the complex Monge-Ampère equations

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Abstract

We prove several quantitative stability estimates for solutions of complex Monge-Ampère equations when *both* the cohomology class and the prescribed singularity vary. In a broad sense, our results fit well into the study of degeneration of families of special Kähler metrics. The key mechanism in our method is the pluripotential theory in the space of potentials of finite lower energy.

Keywords: Monge-Ampère equation, convex weights, lower energy, non-pluripolar products.

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1 Introduction

Let (X,ω) be a compact Kähler manifold of dimension n and let α be a big cohomology $(1,1)$-class in X. Let θ be a closed smooth real $(1,1)$-form in α and let μ be a non-pluripolar finite measure on X. Consider the complex Monge-Ampère equation

$$\theta_u^n = \mu, \tag{1.1}$$

where u is a θ-psh function, and $\theta_u := dd^c u + \theta$, and the left-hand side of (1.1) denotes the non-pluripolar self-product of θ_u (see [1, 7, 33, 65]). By monotonicity of non-pluripolar products (see [15, 65, 66]), if (1.1) has a solution, then it is necessary that $\mu(X) \leq \mathrm{vol}(\alpha)$, where $\mathrm{vol}(\alpha)$ denotes the volume of the big class α. When $\mu(X) = \mathrm{vol}(\alpha)$, the equation (1.1) admits a unique solution by [7, 8, 22, 40, 69], and this solution is of minimal singularity in α if μ is sufficiently regular (for example, μ has a L^p ($p > 1$) density with respect to a smooth volume form on X).

One expects that the regularity of solutions agrees well with that of the measure μ. This expectation is true at least for the following two classes of extreme regularities. The first one is the class of measures which are Hölder continuous as a linear functional on the space $\mathrm{PSH}_0(X,\omega)$ of ω-psh functions u with $\int_X u\omega^n = 0$ endowed with L^1-metric (we call such measures *Hölder continuous ones*). The second one is the class of measures of finite lower energy (*i.e,* non-pluripolar measures). These two classes are important because they are two regularities governing the range of measures where (1.1) is solvable (within the framework of the theory of non-pluripolar products of currents). We refer to [18, 24, 26, 31, 45, 43, 49, 51, 52, 57, 62, 63] and references therein for more informations in the setting where $\mu(X) = \mathrm{vol}(\alpha)$.

Consider now the case where the mass of μ is not necessarily equal to $\mathrm{vol}(\alpha)$, *i.e,* where $\mu(X) \leq \mathrm{vol}(\alpha)$. In this case one can still solve (1.1) by putting it in the context of prescribed singularity. We need several notions. Denote by $\mathrm{PSH}(X,\theta)$ the set of θ-psh functions. Let $u_1, u_2 \in \mathrm{PSH}(X,\theta)$. Recall that u_1 is more singular than u_2 if $u_1 \leq u_2 + O(1)$, and u_1 is of the same singularity type as u_2 if $u_1 - u_2$ is bounded.

Let $\phi \in \mathrm{PSH}(X,\theta)$ such that $\phi \leq 0$ and $\int_X \theta_\phi^n > 0$. Denote by $\mathrm{PSH}(X,\theta,\phi)$ the set of θ-psh functions u with $u \leq \phi$. Note that it is slightly different from the usual definition of $\mathrm{PSH}(X,\theta,\phi)$ in which u is only required to be more singular than ϕ. This difference is not essential. We say that ϕ is a *model θ-psh function* (see [15, 53]) if $\phi = P_\theta[\phi]$ and $\int_X \theta_\phi^n > 0$, where

$$P_\theta[\phi] := \big(\sup\{\psi \in \mathrm{PSH}(X,\theta) : \psi \leq 0,\ \psi \leq \phi + O(1)\}\big)^*.$$

The function $P_\theta[\phi]$ is called a roof-top envelope in [15]. By [15], the function $P_\theta[u]$ is a model one for every $u \in \mathrm{PSH}(X,\theta)$ with $\int_X \theta_u^n > 0$, and for every $u \in \mathrm{PSH}(X,\theta,\phi)$ with $\int_X \theta_u^n = \int_X \theta_\phi^n$ we have $P_\theta[u] = P_\theta[\phi]$.

Let ϕ be now a model θ-psh function. Let μ be a non-pluripolar measure with $\mu(X) = \int_X \theta_\phi^n$. We want to solve the equation

$$(dd^c u + \theta)^n = \mu, \tag{1.2}$$

for $u \in \mathrm{PSH}(X,\theta,\phi)$ and $\sup_X(u-\phi) = 0$. We note that since ϕ is a model, if $u \in \mathrm{PSH}(X,\theta)$ such that $u \leq 0$ and $u \leq \phi + O(1)$, then $u \leq \phi$, and $\sup_X(u-\phi) = \sup_X u$. Thus the normalization condition $\sup_X(u-\phi) = 0$ can be rewritten as $\sup_X u = 0$.

The hypothesis that ϕ is model is a minimal requirement so that (1.2) is solvable in a meaningful way; see [15] for an explanation about the nature of this assumption. Let $\mathcal{E}(X,\theta,\phi)$ be the set of $u \in \mathrm{PSH}(X,\theta,\phi)$ such that $\int_X \theta_u^n = \int_X \theta_\phi^n$. By [16] (or [15, 31]), the equation (1.2) admits a unique solution in $\mathcal{E}(X,\theta,\phi)$, and if μ has L^p density then the solution is of the same singularity type as ϕ. Furthermore a characterization of the class of measures μ where (1.2) admits a solution of finite pluricomplex energy was given in [31].

A θ-singularity type (in α) is an equivalence class of θ-psh functions of the same singularity type. The space of θ-singularity types is denoted by $\mathcal{S}(\theta)$(or $\mathcal{S}(\alpha)$ when θ is clear from the context). A natural pseudo-metric $d_{\mathcal{S}(\theta)}$ in $\mathcal{S}(\theta)$ was introduced in [17]. We refer to Section 4 for a recap of this pseudodistance. A model θ-singularity type is by definition the class of a model θ-psh function. By [15, Theorem 1.3], every model θ-singularity type contains a unique model θ-psh function. Hence there is a 1-1 correspondence between model θ-singularity types and model θ-psh functions. For $u \in \mathrm{PSH}(X,\theta)$, we denote by $[u]_\theta$ (or simply $[u]$ when θ is clear) the θ-singularity type of u. To ease the notation we will denote by $d_{\mathcal{S}(\theta)}(u,v)$ the distance $d_{\mathcal{S}(\theta)}\big([u]_\theta,[v]_\theta\big)$.

In Proposition 4.3 (in Section 4), we push further the study of metrics on the space of singularity types by observing that if we embed $\mathcal{S}(\theta)$ into a bigger space $\mathcal{S}(\theta')$ for $\theta' \geq \theta$ (notice that θ' is not necessarily in the cohomology class of θ), then the pseudodistance $d_{\mathcal{S}(\theta)}$ is actually comparable with that induced by $d_{\mathcal{S}(\theta')}$. This allows us to compare singularity types in different cohomology classes without changing the nature of the distance $d_{\mathcal{S}(\theta)}$. By this we will sometimes ignore θ and only write $d_{\mathcal{S}}$. In view of the resolution of (1.2), we are led to the following natural stability question. We fix a $\mathscr{C}^0$-norm on the space of smooth $(1,1)$-forms on X.

Problem 1.1. *Let θ_1, θ_2 be closed smooth real $(1,1)$-forms on X. Let ϕ_j be model θ_j-psh functions and μ_j be a non-pluripolar measure of mass equal to $\int_X \theta_{\phi_j}^n$ for $j = 1,2$. Let u_j be the solution of (1.2) for μ_j, ϕ_j for $j = 1,2$. Compare u_1 with u_2 in terms of $d_{\mathcal{S}}(\phi_1,\phi_2)$, $\|\theta_1 - \theta_2\|_{\mathscr{C}^0}$, and a suitable distance between μ_1, μ_2?*

Here by $d_{\mathcal{S}}(\phi_1,\phi_2)$, we mean $d_{\mathcal{S}(A\omega)}(\phi_1,\phi_2)$, where A is a big constant so that $\theta_j \leq A\omega$ for $j = 1,2$. As discussed above, the condition that $d_{\mathcal{S}(A\omega)}(\phi_1,\phi_2)$ converges to 0 is independent of the choice of A. To get motivated about the above problem, let's consider the following simple situation. Let $(\alpha_j)_j$ be a sequence of cohomology Kähler $(1,1)$-classes converging to a big class α_∞ as $j \to \infty$. We know that there exists a unique closed positive

$(1,1)$-current $T_j \in \alpha_j$ such that $T_j^n = \mathrm{vol}(\alpha_j)\omega^n / \int_X \omega^n$. One thus asks further: what can we say about the convergence of the sequence $(T_j)_j$? Even when α_∞ is also Kähler, it seems that known methods are not sufficient to deal with such a question.

We will develop in this paper a quite satisfactory method to treat the above stability problem. The emphasis of our approach is the quantitative point of view. As it will be clear later, even when one is only interested in obtaining qualitative stability as in the above simplified situation (with varied cohomology classes), it is still essential in proofs to obtain beforehand quantitative stability estimates. To be more precise, *one of the main protagonists in our work is a quantitative stability for solutions to (1.2) of finite lower energy in the setting where the cohomology class and the prescribed singularity are fixed,* i.e, *where θ and ϕ are fixed.* To our best knowledge, no such estimate was established in the literature. Consider in the simplest setting when $\theta = \omega$ and $\phi = 0$, and let $\mu_j = (dd^c u_j + \omega)^n$ be a non-pluripolar measure of mass equal to $\int_X \omega^n$ and $\sup_X u_j = 0$ for $j = 1, 2$. If $\mu_1 = \mu_2$, then it is well-known that $u_1 = u_2$ by [22]. However there has been no available result comparing u_1, u_2 when μ_1, μ_2 are close to each other. This is due to the fact that arguments in [22] (and in other known proofs of this uniqueness property, see ([7, 15, 22, 49]) are non-quantitative.

In another aspect, the stability of solutions when the cohomology class varies is closely related to the question of degenerations of special Kähler metrics on manifolds, or more generally, families of Kähler-Einstein metrics. There is rich literature on this topic; see for example [19, 38, 37, 55, 54, 58]. We would like to stress that although in some typical model of degenerations of Ricci flat Kähler metric an optimal local $\mathscr{C}^\infty$ convergence of potentials (*i.e*, solutions) on some Zariski open subset of the ambient manifold was obtained in [38, 37, 55, 54], it seems that the global convergence of potentials (solutions) has not been well-studied. Our work fits well into this research direction.

The first stability result for varied prescribed singularities, which is not quantitative, was given in [17, Theorem 1.4]. Previously there were several stability results in the fixed prescribed singularity setting in the literature: some are quantitative and some are not. We refer to [2, 6, 23, 42, 44, 34, 49, 63] and references therein for more details. Key technical tools to obtain quantitative stability has been so far (variants of) Kołodziej's capacity method ([42]) and an integration by parts arguments originally in [6]. All of these cited results require the measures in the right-hand side of the Monge-Ampère equations to be sufficiently regular (to be more precise, measures must be at least the Monge-Ampère of θ-psh functions in $\mathcal{E}^1(X, \theta)$).

Finally, we underline that our interest in the stability of solutions also comes from complex dynamics because equilibrium measures associated to holomorphic dynamical systems are, in many important cases, natural Monge-Ampère measures; see [27, 30]. Stability of solutions of (1.1) is hence relevant to the bifurcation theory of these holomorphic dynamical systems (see [5]). We also refer [56] for a recent application of Monge-Ampère equations to dynamical systems and vice versa.

Statement of main results. The first main result of this paper is the following non-quantitative stability theorem:

Theorem 1.2. *Let $(\theta_j)_{j \in \mathbb{N} \cup \{\infty\}}$ be a sequence of closed smooth real $(1,1)$-forms in X such that*

$\theta_j \to \theta_\infty$ in $\mathscr{C}^0$ topology as $j \to \infty$. Let ϕ_j be a model θ_j-psh function for $j \in \mathbb{N} \cup \{\infty\}$ such that

$$d_{\mathcal{S}}(\phi_j, \phi_\infty) \to 0$$

as $j \to \infty$. Let μ_j be a non-pluripolar measure on X such that

$$\mu_j(X) = \int_X (dd^c \phi_j + \theta_j)^n$$

for every j and $\mu_j \to \mu_\infty$ in the mass norm. Let u_j be the θ-psh function satisfying

$$(dd^c u_j + \theta_j)^n = \mu_j, \quad \sup_X (u_j - \phi_j) = 0$$

for $j \in \mathbb{N} \cup \{\infty\}$. Then $u_j \to u_\infty$ in capacity as $j \to \infty$.

Here by $d_{\mathcal{S}}(\phi_j, \phi_\infty)$ we mean the pseudodistance $d_{\mathcal{S}(A\omega)}$ between the $(A\omega)$-singularity types of ϕ_j and ϕ_∞, where $A > 0$ is a big enough constant such that $\theta_j \le A\omega$ for every j. The property $d_{\mathcal{S}}(\phi_j, \phi_\infty) \to 0$ is independent of the choice of A. Moreover, as mentioned above when θ_j is equal to a fixed θ, the pseudometric $d_{\mathcal{S}(A\omega)}$ is comparable with $d_{\mathcal{S}(\theta)}$.

Theorem 1.2 considerably extends [34, Proposition A] (which treats the case where the cohomology class is fixed, $(\phi_j)_j$ is constant and of minimal singularity types, and only the convergence in L^1 was obtained) and [17, Theorem 1.4] which treats the case where again the cohomology class is fixed, and μ_j has L^p density with respect to ω^n; see also [21, Theorem 2.14] for a particular version of Theorem 1.2. The assumption that μ_j has L^p density is crucial in the plurisubharmonic envelope approach in [17, Theorem 1.4]. It is well-known that it is not possible to have $u_j \to u_\infty$ in L^1 if μ_j only converges weakly to μ_∞ in general (see [9, 34] and references therein for examples).

For every Borel set E in X, recall that the capacity of E is given by

$$\text{cap}(E) = \text{cap}_\omega(E) := \sup_{\{w \in \text{PSH}(X,\omega): 0 \le w \le 1\}} \int_E \omega_w^n.$$

We usually remove the subscript ω from cap_ω if ω is clear from the context. There are generalizations of capacity in big cohomology classes, many of them are comparable; see Theorem 3.17 below and [47]. Recall that a sequence of Borel functions $(u_j)_j$ is said to *converge to a Borel function u in capacity* if for every constant $\epsilon > 0$, we have that $\text{cap}(\{|u_j - u| \ge \epsilon\})$ converges to 0 as $j \to \infty$. The convergence in capacity is of great importance in pluripotential theory in part because it implies the convergence of Monge-Ampère operators under reasonable circumstances. To study quantitatively the convergence in capacity, it is convenient to introduce the following distance function on $\text{PSH}(X, \omega)$:

$$d_{\text{cap}}(u, v) := \sup_{w \in \text{PSH}(X,\omega): 0 \le w \le 1} \int_X |u - v|^{1/2} \omega_w^n$$

for every $u, v \in \text{PSH}(X, \omega)$ (note that $d_{\text{cap}}(u, v) < \infty$ thanks to the Chern-Levine-Nirenberg inequality). The number $\frac{1}{2}$ in the definition of d_{cap} can be replaced by any constant in $(0, 1)$.

One can see that for $u_j, u \in \mathrm{PSH}(X,\omega)$ for $j \in \mathbb{N}$, $d_{\text{cap}}(u_j, u) \to 0$ if and only if $|u_j - u| \to 0$ in capacity.

For θ-psh functions u, v, we put

$$d_\theta(u,v) := 2\int_X \theta^n_{\max\{u,v\}} - \int_X \theta^n_u - \int_X \theta^n_v.$$

The function d_θ is comparable to $d_{\mathcal{S}(\theta)}$ (see Proposition 4.3). For quantitative estimates, it is more convenient to use d_θ than $d_{\mathcal{S}(\theta)}$. It is perhaps worth noting that our method to prove the stability results below also implies that d_{cap} is bounded from above by a power of d_θ for model θ-potentials (see Proposition 4.12 for details).

Let $\mathcal{W}^-$ be the set of convex increasing functions $\chi : \mathbb{R}_{\le 0} \to \mathbb{R}_{\le 0}$ so that $\chi(0) = 0$ and $\chi(-\infty) = -\infty$. It follows from [7, Proposition 3.2] that for every non-positive θ-psh function u, there exists $\chi \in \mathcal{W}^-$ and $C > 0$ such that

$$-\int_X \psi\, \theta^n_u \le C,$$

for every $\psi \in \mathrm{PSH}(X,\omega)$ with $\sup_X \psi = 0$. Theorem 1.2 is a consequence of the following much stronger quantitative result:

Theorem 1.3. *Let θ be a closed smooth real $(1,1)$-form such that $\theta \le A\omega$ for a given constant $A \ge 1$. Let $u \in \mathrm{PSH}(X,\theta)$ such that $\sup_X u = 0$ and $\int_X \theta^n_u := \delta > 0$. Let $B \ge A$ and $\tilde\chi \in \mathcal{W}^-$ with $\tilde\chi(-1) = -1$ such that*

$$\int_X -\tilde\chi(\psi)\theta^n_u \le B\delta,$$

for every $\psi \in \mathrm{PSH}(X,(A+1)\omega)$ with $\sup_X \psi = 0$. Let $h(s) := (-\tilde\chi(-s))^{1/2}$ for $s \le 0$. Then, for every constant $0 < \gamma < 1$, there exists a constant $C > 0$ depending only on n, X, ω and γ such that

$$d_{\text{cap}}(u,v)^2 \le C(A\,B)^2 \left(h^{\circ(n)} \left(\frac{\delta}{\|\theta^n_u - \eta^n_v\| + A^n\|\theta - \eta\|_{\mathscr{C}^0} + d_{(A+1)\omega}(u,v)} \right) \right)^{-\gamma}, \qquad (1.3)$$

for every closed smooth real $(1,1)$-form $\eta \le A\omega$ and for each $v \in \mathrm{PSH}(X,\eta)$ with $\sup_X v = 0$.

Here, we denote by $\|\mu - \mu'\|$ the mass norm of $\mu - \mu'$. The condition that $\tilde\chi(-1) = -1$ is merely a normalization one. For an arbitrary $\tilde\chi \in \mathcal{W}^-$, we can consider $\tilde\chi/|\tilde\chi(-1)|$ which satisfies the last requirement. Theorem 1.3 says that under a very weak assumption on the regularity of the Monge-Ampère of u, one can bound from above the distance d_{cap} of u with any other quasi-psh function v.

We now turn our attention to the class of Hölder continuous measures whose definition is recalled below. Let $\mathrm{PSH}_0(X,\omega)$ be the set of ω-psh functions u with $\int_X u\omega^n = 0$. We endow $\mathrm{PSH}_0(X,\omega)$ with the $L^1(\omega^n)$ distance. Let μ be a measure on X such that quasi-psh functions are μ-integrable. We say that μ *is Hölder continuous with Hölder constant A and Hölder exponent γ if it is so as a functional on* $\mathrm{PSH}_0(X,\omega)$, in other words, for every $u_1, u_2 \in \mathrm{PSH}_0(X,\omega)$, we have

$$\int_X |u_1 - u_2| d\mu \le A\|u_1 - u_2\|^\gamma_{L^1(\omega^n)}. \qquad (1.4)$$

This notion was introduced in [26]. By expressing every ω-psh function u as $u = u - \int_X u\omega^n + \int_X u\omega^n$, we deduce from (1.4) that

$$\int_X |u_1 - u_2| d\mu \le \big(A + \mu(X)\big) \max\{\|u_1 - u_2\|^{\gamma}_{L^1(\omega^n)}, \|u_1 - u_2\|_{L^1(\omega^n)}\} \tag{1.5}$$

for every ω-psh function u_1, u_2. Clearly the last inequality also implies that μ is Hölder with Hölder exponent γ and with Hölder constant $\lambda\big(A + \mu(X)\big)$, for some constant λ depending only on (X, ω). Recall that a measure is Hölder continuous if and only if it can be written as $(dd^c u + \omega)^n$ for some Hölder continuous ω-psh function u on X; see [18, 26] and also [43]. We refer to these papers and [39, 46, 52, 62] for examples of Hölder continuous measures. Most basic examples are measures with L^p density or smooth volume forms of (immersed) generic (real) Cauchy-Riemann submanifolds on X.

Recall that the set of Radon measures on X endowed with the weak topology is a metric space with the distance $\mathrm{dist}_{-\delta}$ for $\delta \in [0, \infty)$ defined as follows: for measures μ, μ',

$$\mathrm{dist}_{-\delta}(\mu, \mu') := \sup_{\|v\|_{\mathscr{C}^\delta} \le 1} \big|\langle \mu - \mu', v\rangle\big|, \tag{1.6}$$

where v is a smooth real-valued function on X (see [61, Theorem 6.9]). Note that $\mathrm{dist}_{-\delta}$ induces the same weak topology when $\delta > 0$. When $\delta = 0$, it is the mass norm of $\mu_1 - \mu_2$. We also have the following interpolation inequality: for $0 \le \beta_0 < \beta_1 < \beta_2$,

$$\mathrm{dist}_{-\beta_1} \le \mathrm{dist}_{-\beta_0}^{\frac{\beta_2 - \beta_1}{\beta_2 - \beta_0}} \mathrm{dist}_{-\beta_2}^{\frac{\beta_1 - \beta_0}{\beta_2 - \beta_0}}. \tag{1.7}$$

We refer to [50, 59] for a proof (see also [62]). This kind of estimate is very important in complex dynamics since the appearance of [29] where a more general version of (1.7) for currents was introduced.

Our third main result is as follows:

Theorem 1.4. *Let θ_1, θ_2 be closed smooth real $(1,1)$-forms and A be positive constant at least 1 such that $\theta_j \le A\omega$ for $j = 1, 2$. Let $0 < \delta \le 1$ and $M \ge 1$ be constants and $u_j \in \mathrm{PSH}(X, \theta_j)$ ($j = 1, 2$) such that*

$$\sup_X u_j = 0, \quad \int_X \theta^n_{u_j} \ge \delta,$$

and $\mu_j := (\theta_j + dd^c u_j)^n$ ($j = 1, 2$) are Hölder continuous measures on X with Hölder exponent β and with Hölder constant $M\delta$. Then, there exists a constant $C > 0$ depending only on n, X, ω, A and M such that

$$\big(d_{cap}(u_1, u_2)\big)^2 \le C \left(\frac{\tau^{\beta/8} + \|\theta_1 - \theta_2\|_{\mathscr{C}^0} + d_{(A+1)\omega}(u_1, u_2)}{\delta}\right)^{2^{-n-1}},$$

where $\tau := \mathrm{dist}_{-1}(\mu_1, \mu_2)$.

By interpolation inequality (1.7), an analogous inequality also holds for $\mathrm{dist}_{-\beta}$ in place of dist_{-1} for any constant $\beta > 0$. Our last main result is a generalization of Cegrell-Kołodziej-Xing stability theorem ([9, 68]) which treated the case where $\theta = \omega$ and $\phi = 0$

(and only for the class of potentials of full Monge-Ampère mass). We also underline that the original result in [9, 68] is non-quantitative and Theorem 1.5 already strengthens their results in their setting.

Theorem 1.5. *Let θ_1, θ_2 be closed smooth real $(1,1)$-forms and A be positive constant at least 1 such that $\theta_j \le A\omega$ for $j = 1, 2$. Let $0 < \delta \le 1$ and $u_j \in \mathrm{PSH}(X, \theta_j)$ ($j = 1, 2$) such that $\sup_X u_j = 0$ and $\int_X \theta_{u_j}^n \ge \delta$. Assume that there exists a Radon measure μ on X such that μ vanishes on pluripolar sets and $(\theta_j + dd^c u_j)^n \le \mu$ for $j = 1, 2$. Then, there exists a continuous increasing function $f_\mu : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ depending only on n, X, ω, A, δ and μ such that $f(0) = 0$ and*

$$d_{cap}(u_1, u_2)^2 \le f_\mu \left(\mathrm{dist}_{-1}(\mu_1, \mu_2) + \|\theta_1 - \theta_2\|_{\mathscr{C}^0} + d_{(A+1)\omega}(u_1, u_2) \right),$$

where $\mu_j := (\theta_j + dd^c u_j)^n$ for $j = 1, 2$.

Theorem 1.5 implies particularly that for every model θ-psh function ϕ, the convergence in capacity or in L^1 and the weak convergence of Monge-Ampère measures are equivalent in the class of potentials in $\mathcal{E}(X, \theta, \phi)$ whose Monge-Ampère measures are bounded from above by a fixed non-pluripolar measure. This is more or less the original motivation of Cegrell-Kołodziej in [9].

Finally we note that as an application of Theorem 1.5 or 1.4, one can recover a main result in [17] that the pseudometric space of singularity types of volume bounded from below by a fixed positive constant is complete, we refer to Remark 4.18 in the end of the paper and Subsection 4.3 for details.

Key components in our method. As mentioned above the core of the method developed in this paper is a solution to the quantitative stability for measures of lower energy in the setting where the cohomology class and the prescribed singularity are fixed. We underline that in what follows by convex weights we mean also bounded convex functions, although such functions were not usually considered as weights. This point of view is the key allowing us to treat the general setting when both the cohomology class and the prescribed singularity of solution vary.

Let $\widetilde{\mathcal{W}}^-$ be the set of convex, non-decreasing functions $\chi : \mathbb{R}_{\le 0} \to \mathbb{R}_{\le 0}$ such that $\chi(0) = 0$ and $\chi \ne 0$. Note that χ can be bounded. Obviously $\mathcal{W}^-$ is contained in $\widetilde{\mathcal{W}}^-$. It is crucial in our method that we consider also $\chi \in \widetilde{\mathcal{W}}^-$ which is bounded. Let $M \ge 1$ be a constant and $\mathcal{W}_M^+$ the usual space of increasing concave functions $\chi : \mathbb{R}_{\le 0} \to \mathbb{R}_{\le 0}$ such that $\chi(0) = 0$, $\chi \not\equiv 0$, and $|t\chi'(t)| \le M|\chi(t)|$ for every $t \le 0$.

Let $\varrho := \int_X \theta_\phi^n$. For $\chi \in \widetilde{\mathcal{W}}^- \cup \mathcal{W}_M^+$ and $u \in \mathrm{PSH}(X, \theta, \phi)$, let

$$E^0_{\chi,\theta,\phi}(u) := -\varrho^{-1} \int_X \chi(u - \phi)\theta_u^n$$

which is called *the (normalized) χ-energy* of u (with respect to θ, ϕ). We denote

$$\mathcal{E}_\chi(X, \theta, \phi) := \left\{ u \in \mathcal{E}(X, \theta, \phi) : E_{\chi,\theta,\phi}(u) < \infty \right\},$$

where $\mathcal{E}(X, \theta, \phi)$ is the space of θ-psh functions $u \le \phi$ with $\int_X \theta_u^n = \int_X \theta_\phi^n$. Certainly if χ is bounded, then $\mathcal{E}_\chi(X, \theta, \phi) = \mathcal{E}(X, \theta, \phi)$. We would like to point out however that our method is not about the finiteness of $E^0_{\chi,\theta,\phi}(u)$ but estimating the size of that quantity. Thus whether χ is bounded or not does not make much difference for our later arguments. Put

$$I^0_\chi(u,v) := \varrho^{-1}\int_{\{u<v\}}\chi(u-v)(\theta^n_v-\theta^n_u)+\varrho^{-1}\int_{\{u>v\}}\chi(v-u)(\theta^n_u-\theta^n_v)$$

for $u,v\in\mathcal{E}_\chi(X,\theta,\phi)$. The factor ϱ^{-1} in the defining formulae for $E^0_{\chi,\theta,\phi}(u)$ and $I^0_\chi(u,v)$ plays the role of a normalizing constant. In geometric applications it is important to treat the case where $\varrho\to 0$, *i.e,* to obtain estimates uniformly as $\varrho\to 0$ (here we allow θ or its cohomology class to vary).

Clearly if $\theta^n_u=\theta^n_v$, then $I^0_\chi(u,v)=0$. We will see later that each term in the sum defining $I^0_\chi(u,v)$ is nonnegative. We recall that there is a natural (quasi-)metric on the space $\mathcal{E}_\chi(X,\theta,\phi)$ constructed in [12, 13, 36], and see [14, 20, 60, 67] as well. The functional $I^0_\chi(u,v)$ has an intimate relation with these quasi-metrics. We refer to the end of Section 3 for details on this connection. Here is the first key ingredient in our proof of main results.

Theorem 1.6. *Let θ be a closed smooth real $(1,1)$-form and ϕ be a negative θ-psh function such that $\varrho:=\int_X\theta^n_\phi>0$. Let $\chi,\tilde\chi\in\widetilde{\mathcal{W}}^-\cup\mathcal{W}^+_M$ ($M\geq 1$) such that $\tilde\chi\leq\chi$. Let $B\geq 1$ be a constant and let $u_j,\psi_j\in\mathcal{E}(X,\theta,\phi)$ satisfy $u_1\leq u_2$ and*

$$E^0_{\tilde\chi,\theta,\phi}(u_j)+E^0_{\tilde\chi,\theta,\phi}(\psi_j)\leq B,$$

for $j=1,2$. Then there exists a constant $C_n>0$ depending only on n and M, and a continuous increasing function $f:\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ depending only on $\chi,\tilde\chi$ such that $f(0)=0$ and

$$\int_X -\chi(u_1-u_2)(\theta^n_{\psi_1}-\theta^n_{\psi_2})\leq C_n\varrho B^2 f\big(I^0_\chi(u_1,u_2)\big).$$

The following result is the second key which is a consequence of the first one.

Theorem 1.7. *Let θ be a closed smooth real $(1,1)$-form, and let $A\geq 1$ be a constant such that $\theta\leq A\omega$. Let ϕ be a model θ-psh function. Let $\chi,\tilde\chi\in\widetilde{\mathcal{W}}^-\cup\mathcal{W}^+_M$ ($M\geq 1$) such that $\tilde\chi\leq\chi$. Let $B\geq 1$ be a constant and $u_1,u_2,\psi\in\mathcal{E}(X,\theta,\phi)$ with $\sup_X u_1=\sup_X u_2$ satisfy*

$$E^0_{\tilde\chi,\theta,\phi}(u_1)+E^0_{\tilde\chi,\theta,\phi}(u_2)+E^0_{\tilde\chi,\theta,\phi}(\psi)\leq B.$$

Then, for every constant $m>0$ and $0<\gamma<1$, there exist a constant $C>0$ depending on n,M,X,ω,m and γ, and a function f as in Theorem 1.6 such that

$$\int_X -\chi\big(-|u_1-u_2|\big)\theta^n_\psi\leq -\varrho\chi\left(-\lambda^m\right)+C\varrho B^2_{\gamma,m}\lambda^\gamma,$$

where $\lambda:=f\big(I^0_\chi(u_1,u_2)\big)$ and $B_{\gamma,m}=A^{(1-\gamma)/(2m)}(B-\tilde\chi(-A))(1-\tilde\chi(-1))$.

The condition $\sup_X u_1=\sup_X u_2$ is simply a normalization one. By Theorem 1.7, one sees in particular that if $I^0_\chi(u_1,u_2)\to 0$, then $|u_1-u_2|\to 0$ in L^p for every $p>0$. The function f can be made explicitly; see Theorems 3.1 and 3.2 below for more elaborated versions of these above results.

We note that *the single theorem 1.6 contains the three important results in the pluripotential theory: the uniqueness of solutions of complex Monge-Ampère equations, the domination principle, and the comparison of capacities*. We obtain indeed quantitative (hence stronger) versions of these results for which we refer to Subsection 3.4. Readers also find there a quantitative version of the fact that the convergence in Darvas's metric in $\mathcal{E}_\chi(X, \theta, \phi)$ implies the convergence in capacity. Notice that such an estimate seems to be not reachable by using the usual plurisubharmonic envelope method.

The main novelty of Theorem 1.6 is that it deals with *arbitrary* weights. Similar statements was already known for $\chi(t) = t$ (see [6, 34]). However the proof there only work *exclusively* for this case. One should notice that the weight $\chi(t) = t$ is very special: it is linear and lies in the middle between higher energy weights and lower energy weights. As to the proof of Theorem 1.6, going up to the space of higher energy weights or going down to the space of lower energy weights are equally difficult. We will explain this point in more details in the paragraph after Theorem 1.8 below.

The key in the proof of Theorem 1.6 is Proposition 3.5 in Section 3 a simplified version of which we state here for readers' convenience.

Theorem 1.8. *Let $\chi, \tilde{\chi} \in \widetilde{\mathcal{W}}^- \cup \mathcal{W}^+_M$ such that $\tilde{\chi} \le \chi$ and $\chi \in \mathscr{C}^1(\mathbb{R})$. Let $u_1, u_2, u_3 \in \mathcal{E}(X, \theta, \phi)$ such that $u_1 \le u_2$ and $u_j - \phi$ is bounded ($j = 1, 2, 3$), where ϕ is a negative θ-psh function satisfying $\varrho := \mathrm{vol}(\theta_\phi) > 0$. Then there exist a constant $C_n > 0$ depending only on n and M, and a function f as in Theorem 1.6 such that*

$$\int_X \chi'(u_1 - u_2) d(u_1 - u_2) \wedge d^c(u_1 - u_2) \wedge \theta^{n-1}_{u_3} \le C_n \varrho B^2 f\big(I^0_\chi(u_1, u_2)\big),$$

where $B := \sum_{j=1}^3 \max\{E^0_{\tilde{\chi},\theta,\phi}(u_j), 1\}$.

As far as we know, all of previous works related to Theorem 1.8 only concern with $\chi(t) = t$. In this case, Theorem 1.8 was known with an explicit f and without $\tilde{\chi}$ if ϕ is of minimal singularity in the cohomology class of θ, by [6, 34].

The key ingredients in previous versions of Theorem 1.8 for $\chi(t) = t$ are integration by parts arguments. Direct generalization of such reasoning immediately break down if $\chi \ne \mathrm{id}$: in a more precise but technical level, the integration by parts arguments give terms like $\chi'(u_1 - u_2) d(u_1 - u_3) \wedge d^c(u_1 - u_3)$, such quantity is easy to bound if $\chi = \mathrm{id}$ (hence $\chi' \equiv 1$), but it is no longer the case if $\chi \ne \mathrm{id}$.

In order to prove Theorem 1.8, we still use this strategy but need to use a so-called "monotonicity argument" from [31, 65, 64] to deal with general χ. In a nutshell it is about using intensively the pluri-locality of Monge-Ampère operators together with the monotonicity of pluricomplex energy which allow one to bound from above "Monge-Ampère quantities" of bad potentials by that of nicer potentials. This method is a flexible tool to deal with "low regularity", and was a key in the proof of the convexity of the class of potentials of finite χ-energy in [64], as well as, giving a characterization of the class of Monge-Ampère measures with potentials of finite χ-energy in [31]. Moreover in order to deduce Theorem 1.7 from Theorem 1.6, we use, among other things, an idea from [34] together with a very simple but crucial *lower bound* of the sublevel sets of ω-psh functions;

see Lemma 3.8 below. Such an estimate is of independent interest.

Organization of the paper. In Section 2, we recall the crucial integration by parts formula from [64], auxiliary facts about weights are also collected there. Theorems 1.6, 1.7, and 1.8 are proved in Section 3. We prove Theorems 1.2, 1.3, 1.4 and 1.5 in Subsection 4.4. Proposition 4.13 is proved at the end of the paper.

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2 Preliminaries

2.1 Integration by parts

In this subsection, we recall the integration by parts formula obtained in [64, Theorem 2.6]. This formula will play a key role in our proof of main results later.

Let X be a compact Kähler manifold. Let $T_1, \ldots, T_m$ be closed positive $(1,1)$-currents on X. Let T be a closed positive current of bi-degree (p,p) on X. The *T-relative non-pluripolar product* $\langle \wedge_{j=1}^m T_j \dot{\wedge} T\rangle$ is defined in a way similar to that of the usual non-pluripolar product (see [65]). The product $\langle \wedge_{j=1}^m T_j \dot{\wedge} T\rangle$ is a closed positive current of bi-degree $(m+p, m+p)$; and the wedge product $\langle \wedge_{j=1}^m T_j \dot{\wedge} T\rangle$ as an operator on currents is symmetric with respect to $T_1, \ldots, T_m$ and is homogeneous. In latter applications, we will only use the case where T is the non-pluripolar product of some closed positive $(1,1)$-currents, say, $T = \langle T_{m+1} \wedge \cdots \wedge T_{m+l}\rangle$, where T_j is $(1,1)$-currents for $m+1 \le j \le m+l$. In this case, $\langle T_1 \wedge \cdots \wedge T_m \dot{\wedge} T\rangle$ is simply equal to $\langle \wedge_{j=1}^{m+l} T_j\rangle$. We usually remove the bracket $\langle \quad \rangle$ in the non-pluripolar product to ease the notation.

Recall that a *dsh* function on X is the difference of two quasi-plurisubharmonic (quasi-psh for short) functions on X (see [28]). These functions are well-defined outside pluripolar sets. Let v be a dsh function on X. Let T be a closed positive current on X. We say that v is *T-admissible* if there exist quasi-psh functions φ_1, φ_2 such that $v = \varphi_1 - \varphi_2$ and T has no mass on $\{\varphi_j = -\infty\}$ for $j = 1,2$. In particular, if T has no mass on pluripolar sets, then every dsh function is T-admissible.

Assume now that v is T-admissible. Let φ_1, φ_2 be quasi-psh functions such that $v = \varphi_1 - \varphi_2$ and T has no mass on $\{\varphi_j = -\infty\}$ for $j = 1,2$. Let

$$\varphi_{j,k} := \max\{\varphi_j, -k\}$$

for every $j = 1,2$ and $k \in \mathbb{N}$. Put $v_k := \varphi_{1,k} - \varphi_{2,k}$. Put

$$Q_k := dv_k \wedge d^c v_k \wedge T = dd^c v_k^2 \wedge T - v_k dd^c v_k \wedge T.$$

By the plurifine locality with respect to T ([65, Theorem 2.9]) applied to the right-hand side of the last equality, we have

$$\mathbf{1}_{\cap_{j=1}^2 \{\varphi_j > -k\}} Q_k = \mathbf{1}_{\cap_{j=1}^2 \{\varphi_j > -k\}} Q_s \tag{2.1}$$

for every $s \geq k$. We say that $\langle dv \wedge d^c v \dot{\wedge} T\rangle$ is *well-defined* if the mass of $\mathbf{1}_{\cap_{j=1}^2 \{\varphi_j > -k\}} Q_k$ is uniformly bounded on k. In this case, using (2.1) implies that there exists a positive current Q on X such that for every bounded Borel form Φ with compact support on X such that

$$\langle Q, \Phi\rangle = \lim_{k\to\infty} \langle \mathbf{1}_{\cap_{j=1}^2 \{\varphi_j > -k\}} Q_k, \Phi\rangle,$$

and we define $\langle dv \wedge d^c v \dot{\wedge} T\rangle$ to be the current Q. This agrees with the classical definition if v is the difference of two bounded quasi-psh functions. One can check that this definition is independent of the choice of φ_1, φ_2. By [64, Lemma 2.5], if v is bounded, then $\langle dv \wedge d^c v \dot{\wedge} T\rangle$ is well-defined.

Let w be another T-admissible dsh function. If T is of bi-degree $(n-1, n-1)$, we can also define the current $\langle dv \wedge d^c w \dot{\wedge} T\rangle$ by a similar procedure as above. More precisely, we say $\langle dv \wedge d^c w \dot{\wedge} T\rangle$ is *well-defined* if $\langle dv \wedge d^c v \dot{\wedge} T\rangle$, $\langle dw \wedge d^c w \dot{\wedge} T\rangle$, and $\langle d(v+w) \wedge d^c (v+w) \dot{\wedge} T\rangle$ are well-defined. In this case, as in the classical case of bounded potentials, the defining formula for $\langle dv \wedge d^c w \dot{\wedge} T\rangle$ is obvious:

$$2\langle dv \wedge d^c w \dot{\wedge} T\rangle = \langle d(v+w) \wedge d^c (v+w) \dot{\wedge} T\rangle - \langle dv \wedge d^c v \dot{\wedge} T\rangle - \langle dw \wedge d^c w \dot{\wedge} T\rangle.$$

As above, if v, w are bounded T-admissible, then $\langle dv \wedge d^c w \dot{\wedge} T\rangle$ is well-defined and given by the above formula. The following Cauchy-Schwarz inequality is clear from definition.

Lemma 2.1. *Assume that $\langle dv \wedge d^c w \dot{\wedge} T\rangle$ is well-defined. Then for every positive Borel function χ, we have*

$$\int_X \chi \langle dv \wedge d^c w \dot{\wedge} T\rangle \leq \Big(\int_X \chi \langle dv \wedge d^c v \dot{\wedge} T\rangle\Big)^{1/2} \Big(\int_X \chi \langle dw \wedge d^c w \dot{\wedge} T\rangle\Big)^{1/2}.$$

We put

$$\langle dd^c v \dot{\wedge} T\rangle := \langle dd^c \varphi_1 \dot{\wedge} T\rangle - \langle dd^c \varphi_2 \dot{\wedge} T\rangle$$

which is independent of the choice of φ_1, φ_2. The following integration by parts formula is crucial for us later.

Theorem 2.2. *([64, Theorem 2.6] or [31, Theorem 3.1]) Let T be a closed positive current of bi-degree $(n-1, n-1)$ on X. Let v, w be bounded T-admissible dsh functions on X. If $\chi : \mathbb{R} \to \mathbb{R}$ is a $\mathscr{C}^3$ function then we have*

$$\begin{aligned}\int_X \chi(w) \langle dd^c v \dot{\wedge} T\rangle &= \int_X v \chi''(w) \langle dw \wedge d^c w \dot{\wedge} T\rangle + \int_X v \chi'(w) \langle dd^c w \dot{\wedge} T\rangle \\ &= -\int_X \chi'(w) \langle dw \wedge d^c v \dot{\wedge} T\rangle. \quad (2.2)\end{aligned}$$

Since the case where T is a non-pluripolar product of $(1,1)$-currents plays an important role in the study of the complex Monge-Ampère equation, we present below an equivalent natural way to define the current $\langle d\varphi \wedge d^c \varphi \dot{\wedge} T\rangle$ in this setting. It is just for the purpose of clarification.

Lemma 2.3. *Let $u_1, \dots, u_m$ be negative psh functions on an open subset U in $\mathbb{C}^n$ such that $T := \langle dd^c u_1 \wedge \cdots \wedge dd^c u_m \rangle$ is well-defined. Let v be the difference of two bounded psh functions on U. For $k \in \mathbb{N}$, put $u_{j,k} := \max\{u_j, -k\}$ and*

$$T_k := dd^c u_{1,k} \wedge \cdots \wedge dd^c u_{m,k}.$$

Then we have

$$dv \wedge d^c v \wedge T = dv \wedge d^c v \wedge T_k$$

on $\cap_{j=1}^m \{u_j > -k\}$.

Proof. Put

$$\psi_k := k^{-1} \max\{u_1 + \cdots + u_m, -k\} + 1.$$

Observe $\psi_k T_k = \psi_k T$. Now regularizing v and using the continuity of Monge-Ampère operators of bounded potentials, we obtain

$$\psi_k dv \wedge d^c v \wedge T = \psi_k dv \wedge d^c v \wedge T_k.$$

Hence

$$dv \wedge d^c v \wedge T = dv \wedge d^c v \wedge T_k$$

on $U := \cap_{j=1}^m \{u_j > -k/(2m)\}$ (for $\psi_k \geq 1/2$ on U). Note that $dv \wedge d^c v \wedge T_k = dv \wedge d^c v \wedge T_{k/(2m)}$ on U by the plurifine locality. Thus the desired assertion follows. This finishes the proof. □

Let $T_1, \dots, T_m$ be closed positive $(1,1)$-currents on X. Let $n := \dim X$. *Consider now*

$$T := \langle T_1 \wedge \cdots \wedge T_m \rangle.$$

Note that T has no mass on pluripolar sets. Let φ_1, φ_2 be negative quasi-psh function on X. Let $\varphi_{j,k}$ $(j = 1, 2)$ be as before and $v := \varphi_1 - \varphi_2$. In the moment, we work locally. Let U be an open small enough local chart (biholomorphic to a polydisk in $\mathbb{C}^n$) in X such that $T_j = dd^c u_j$ for $j = 1, \dots, m$, where u_j is negative psh functions on U. Put $u_{j,k} := \max\{u_j, -k\}$ for $k \in \mathbb{N}$, and

$$T_k := dd^c u_{1,k} \wedge \cdots \wedge dd^c u_{m,k}, \quad Q'_k := dv_k \wedge d^c v_k \wedge T_k.$$

Put $A_k := \cap_{j=1}^2 \{\varphi_j > -k\} \cap \cap_{j=1}^m \{u_j > -k\}$. By plurifine properties of Monge-Ampère operators, we have

$$\mathbf{1}_{A_k} Q'_k = \mathbf{1}_{A_k} Q'_s$$

for every $s \geq k$. One can check that the condition that $(\mathbf{1}_{A_k} Q'_k)_k$ is of mass bounded uniformly (on compact subsets in U) in k is independent of the choice of potentials.

Proposition 2.4. *The current $\mathbf{1}_{A_k} Q'_k$ is of mass bounded uniformly in k on compact subsets in U for every U (small enough biholomorphic to a polydisk in $\mathbb{C}^n$) if and only if the current $\langle dv \wedge d^c v \dot{\wedge} T \rangle$ is well-defined. In this case we have*

$$\langle dv \wedge d^c v \dot{\wedge} T \rangle = \lim_{k \to \infty} \mathbf{1}_{A_k} Q'_k. \tag{2.3}$$

Proof. By writing a smooth form of bi-degree $(n-m-1, n-m-1)$ as the difference of two smooth positive forms, we can assume without loss of generality that T is of bi-degree $(n-1, n-1)$ (hence $m = n-1$). Assume that $\langle dv \wedge d^c v \dot{\wedge} T\rangle$ is well-defined. We will check that $\mathbf{1}_{A_k} Q'_k$ is of mass bounded uniformly in k on compact subsets in U. Let χ be a nonnegative smooth function compactly supported on U. Put

$$\psi := \varphi_1 + \varphi_2 + u_1 + \cdots + u_m, \quad \psi_k := k^{-1} \max\{\psi, -k\} + 1.$$

and $\varphi_{jk} := \max\{\varphi_j, -k\}$ for $1 \le j \le 2$. Observe that $0 \le \psi_k \le 1$ and if $\psi_k > 0$, then $\varphi_j > -k$ for $1 \le j \le 2$; and

$$\psi_k(x) \ge 1 - s/k \tag{2.4}$$

for every $x \in A_{s/(m+2)}$ and $1 \le s \le k$. Recall $v_k := \varphi_{1k} - \varphi_{2k}$ which is bounded (but not necessarily uniformly in k). Observe that $\langle dv \wedge d^c v \dot{\wedge} T\rangle$ has no mass on pluripolar sets because T is so (see for example [65, Lemma 2.1]). Put $Q''_k := \psi_k Q_k = \psi_k \mathbf{1}_{A_k} Q'_k$ By (2.4) and Lemma 2.3, we have

$$\begin{aligned} \langle dv \wedge d^c v \dot{\wedge} T\rangle &= \lim_{k\to\infty} \psi_k dv_k \wedge d^c v_k \wedge T \\ &= \lim_{k\to\infty} \psi_k dv_k \wedge d^c v_k \wedge T_k = \lim_{k\to\infty} Q''_k \end{aligned} \tag{2.5}$$

on U. On the other hand, by (2.4) again, we see that the claim that Q''_k is of mass uniformly bounded on compact subsets in U is equivalent to that $\mathbf{1}_{A_k} Q'_k$ is so. This together with (2.5) yields the desired assertion.

Conversely, suppose now that $\mathbf{1}_{A_k} Q'_k$ is of mass bounded uniformly in k on compact subsets in U for every U. Thus there exists a positive current R on U such that $\mathbf{1}_{A_k} R = \mathbf{1}_{A_k} Q'_k$ for every k and U. Set

$$\tilde\psi := \varphi_1 + \varphi_2, \quad \tilde\psi_k := k^{-1} \max\{\tilde\psi, -k\} + 1.$$

Let $s \in \mathbb{N}$ with $s \ge k$. Observe

$$\psi_s R = \tilde\psi_k \psi_s R + (1 - \tilde\psi_k)\psi_s R.$$

The second term in the right-hand side of the last inequality tends to 0 (uniformly in s) because $\tilde\psi_k$ converges pointwise to 1 outside a pluripolar set and R has no mass on pluripolar sets. Using Lemma 2.3, we have

$$\begin{aligned} \tilde\psi_k \psi_s R &= \tilde\psi_k \psi_s dv_s \wedge d^c v_s \wedge T_s \\ &= \tilde\psi_k \psi_s dv_s \wedge d^c v_s \wedge T = \tilde\psi_k \psi_s dv_k \wedge d^c v_k \wedge T, \end{aligned}$$

here we used the plurifine topology properties with respect to T (see [65, Theorem 2.9]), thanks to the fact that $\varphi_{j,k} = \varphi_{j,s}$ on $\{\tilde\psi_k \ne 0\}$ for $j = 1, 2$ (recall $s \ge k$), and they are bounded psh functions. Letting $s \to \infty$ gives

$$\tilde\psi_k R = \tilde\psi_k \mathbf{1}_{\cup_{j=1}^m \{u_j > -\infty\}} dv_k \wedge d^c v_k \wedge T = \tilde\psi_k dv_k \wedge d^c v_k \wedge T$$

because the current $dv_k \wedge d^c v_k \wedge T$ has no mass on pluripolar sets. Now letting $k \to \infty$ gives the desired assertion. This finishes the proof. $\square$

Thanks to Proposition 2.4, we can use the right-hand side of (2.3) to define $\langle dv\wedge d^c v\dot{\wedge} T\rangle$ in the case where T is the non-pluripolar product of some closed positive $(1,1)$-currents. By the same reason, in this case, we will use the expression $dv\wedge d^c w\wedge T_1\wedge\ldots\wedge T_{n-1}$ to denote $\big\langle dv\wedge d^c w\dot{\wedge}\langle T_1\wedge\cdots\wedge T_{n-1}\rangle\big\rangle$ whenever it is well-defined.

2.2 Auxiliary facts on weights

In this subsection, we present some facts about weights needed for the proofs of main results.

Recall that $\widetilde{\mathcal{W}}^-$ is the set of all convex, non-decreasing functions $\chi:\mathbb{R}_{\le 0}\to\mathbb{R}_{\le 0}$ such that $\chi(0)=0$ and $\chi\not\equiv 0$. Let $M\ge 1$ be a constant and $\mathcal{W}^+_M$ the usual space of increasing concave functions $\chi:\mathbb{R}_{\le 0}\to\mathbb{R}_{\le 0}$ such that $\chi(0)=0$, $\chi\not\equiv 0$, and $|t\chi'(t)|\le M|\chi(t)|$ for every $t\le 0$. We have the following lemmas.

Lemma 2.5. *Let $c>0$, $0<\delta<1$ and $\chi:\mathbb{R}\to\mathbb{R}$ such that $\chi(t)=ct$ for every $t\ge-\delta$ and $\chi|_{(-\infty,0]}\in\widetilde{\mathcal{W}}^-\cup\mathcal{W}^+_M$ ($M\ge 1$). Let g be a smooth radial cut-off function supported in $[-1,1]$ on $\mathbb{R}$,* i.e, *$g(t)=g(-t)$ for $t\in\mathbb{R}$, $0\le g\le 1$ and $\int_{\mathbb{R}}g(t)dt=1$. Put $g_\epsilon(t):=\epsilon^{-1}g(\epsilon t)$ for every constant $\epsilon>0$ and $\chi_\epsilon:=\chi*g_\epsilon$ (the convolution of χ with g_ϵ). Then the following assertions are true:*

(i) if $\chi\in\widetilde{\mathcal{W}}^-$, then $\chi_\epsilon|_{(-\infty,0]}\in\widetilde{\mathcal{W}}^-$ for every $0<\epsilon<\delta$, $\chi_\epsilon\searrow\chi$ as $\epsilon\searrow 0$ and $\sup(\chi_\epsilon-\chi)\le c\epsilon$;

(ii) if $\chi\in\mathcal{W}^+_M$ and $0<\epsilon<\delta^2/2$ then $\chi_\epsilon|_{(-\infty,0]}\in\mathcal{W}^+_{M/(1-\delta)}$. Moreover, if $0<\epsilon<\delta^2/8$ then

$$\overline{\chi}_\epsilon:=\chi_\epsilon(\cdot+\epsilon)-c\epsilon\in\mathcal{W}^+_{M/(1-\delta)^2},\quad \overline{\chi}_\epsilon\ge\chi-c\epsilon,$$

and $\overline{\chi}_\epsilon$ converges uniformly to χ as $\epsilon\to 0$ on compact subsets in $\mathbb{R}$.

Proof. The part (i) follows from [31, Lemma 2.1]. The part (ii) can be obtained more or less by similar arguments as in the last reference. We provide details for readers' convenience. It is clear that χ_ϵ is a concave, increasing function with $\chi_\epsilon(0)=0$. We will show that

$$\chi'_\epsilon(t)\le\frac{M}{1-\delta}\frac{\chi_\epsilon(t)}{t},\tag{2.6}$$

for every $t<0$ and $0<\epsilon<\delta^2/2$.

If $t<-\dfrac{\delta}{2}$ then we have

$$\begin{aligned}\chi'_\epsilon(t)=\int_{-\epsilon}^{\epsilon}\chi'(t-s)g_\epsilon(s)ds\le\int_{-\epsilon}^{\epsilon}\frac{M\chi(t-s)}{t-s}g_\epsilon(s)ds&\le\int_{-\epsilon}^{\epsilon}\frac{M\chi(t-s)}{t+\epsilon}g_\epsilon(s)ds\\&=\frac{M\chi_\epsilon(t)}{t+\epsilon}\\&=\frac{Mt}{t+\epsilon}\frac{\chi_\epsilon(t)}{t}\\&\le\frac{M}{1-\delta}\frac{\chi_\epsilon(t)}{t},\end{aligned}$$

for every $0 < \epsilon < \delta^2/2$.

On the other hand, if $t \geq -\dfrac{\delta}{2}$, then $\chi_\epsilon(t) = \chi(t) = ct$ for every $0 < \epsilon < \delta^2/2$. As a consequence, we have

$$\chi_\epsilon'(t) = \chi'(t) \leq \frac{M\chi(t)}{t} = M\frac{\chi_\epsilon(t)}{t}.$$

Thus, (2.6) follows. Hence, $\chi_\epsilon|_{(-\infty,0]} \in \mathcal{W}^+_{M/(1-\delta)}$.

Now, we consider $\overline{\chi}_\epsilon$. Since χ is increasing, one sees that $\overline{\chi}_\epsilon \geq \chi - c\epsilon$ and $\overline{\chi}_\epsilon$ converges uniformly to χ as $\epsilon \to 0$ on compact subsets in $\mathbb{R}$. It remains to show that $\overline{\chi}_\epsilon \in \mathcal{W}^+_{M(1+\delta)/(1-\delta)}$ for every $0 < \epsilon < \delta^2/8$. Note that

$$\overline{\chi}_\epsilon = h_\epsilon * g_\epsilon,$$

where $h_\epsilon(t) = \chi(t+\epsilon) - c\epsilon$. The function $\overline{\chi}_\epsilon(t)$ is concave, increasing and $\overline{\chi} + \epsilon(0) = 0$.

If $-\delta/2 \leq t < 0$ then $h_\epsilon(t) = \chi(t) = ct$ for every $0 < \epsilon < \delta^2/2$. Therefore

$$h_\epsilon'(t) = \chi'(t) \leq \frac{M\chi(t)}{t} = M\frac{h_\epsilon(t)}{t}.$$

If $t < -\delta/2$ then

$$\begin{aligned} h_\epsilon'(t) = \chi'(t+\epsilon) \leq M\frac{\chi(t+\epsilon)}{t+\epsilon} \leq M\frac{\chi(t+\epsilon) - c\epsilon}{t+\epsilon} = M\frac{h_\epsilon(t)}{t+\epsilon} &= \frac{Mt}{t+\epsilon}\frac{h_\epsilon(t)}{t} \\ &\leq \frac{M}{1-\delta}\frac{h_\epsilon(t)}{t}, \end{aligned}$$

for every $0 < \epsilon < \delta^2/2$.

Then, for every $0 < \epsilon < \delta^2/2$, we have $h_\epsilon \in \mathcal{W}^+_{M/(1-\delta)}$ and $h_\epsilon = ct$ for every $t \geq -\delta/2$. Hence, for every $0 < \epsilon < \delta^2/8$, we have

$$\overline{\chi}_\epsilon = h_\epsilon * g_\epsilon \in \mathcal{W}^+_{\frac{M}{(1-\delta)(1-\delta/2)}} \subset \mathcal{W}^+_{\frac{M}{(1-\delta)^2}}.$$

The proof is completed. □

Lemma 2.6. *Let $\chi, \tilde{\chi} \in \widetilde{\mathcal{W}}^- \cup \mathcal{W}^+_M$ ($M \geq 1$) such that $\tilde{\chi} \leq \chi$. Then, there exist sequences of functions $\chi_j, \tilde{\chi}_j \in \widetilde{\mathcal{W}}^- \cup \mathcal{W}^+_{M_j}$ (with $M_j \searrow M$ as $j \to \infty$) satisfying the following conditions:*

- *$\chi_j \in \mathscr{C}^\infty(\mathbb{R})$ for every j;*
- *$\chi_j \geq \tilde{\chi}_j$ and $\chi_j \geq \chi - 2^{-j}$ for every j big enough;*
- *$\tilde{\chi} - 2^{-j} \leq \tilde{\chi}_j \leq \tilde{\chi}$ on $(-\infty, -1]$ for every j big enough;*
- *χ_j converges uniformly to χ on compact subsets in $\mathbb{R}_{\leq 0}$.*

Proof. We split the proof into two cases.

Case 1: $\chi \in \widetilde{\mathcal{W}}^-$.
For every $j \geq 1$, we denote

$$\overline{\chi}_j(t) = \begin{cases} \max\{\chi(t), c_j t\} & \text{if} \quad t < 0, \\ c_j t & \text{if} \quad t \geq 0, \end{cases}$$

where

$$c_j := \frac{-\chi(-2^{-j})}{2^{-j}}.$$

Then $\overline{\chi}_j$ satisfies the hypothesis of Lemma 2.5 for $\delta := 2^{-j}$. Let g be a smooth radial cut-off function supported in $[-1, 1]$ on $\mathbb{R}$, *i.e,* $g(t) = g(-t)$ for $t \in \mathbb{R}$, $0 \leq g \leq 1$ and $\int_{\mathbb{R}} g(t)dt = 1$. For every $j \geq 1$, we define

$$\chi_j = \overline{\chi}_j * g_{4^{-j-1}} \quad \text{and} \quad \tilde{\chi}_j = \tilde{\chi}.$$

By Lemma 2.5, we have χ_j and $\tilde{\chi}_j$ satisfy the desired conditions.

Case 2: $\chi \in \mathcal{W}_M^+$.
Since $\chi \geq \tilde{\chi}$, we also have $\tilde{\chi} \in \mathcal{W}_M^+$. Assume that g and c_j are as in Case 1. For every $j \geq 1$, we define

$$\overline{\chi}_j(t) = \begin{cases} \min\{\chi(t), c_j t\} & \text{if} \quad t < 0, \\ c_j t & \text{if} \quad t \geq 0, \end{cases}$$

and

$$\chi_j(t) = (\overline{\chi}_j(\cdot + 4^{-j-1}) * g_{4^{-j-1}})(t) - c_j 4^{-j-1}.$$

We also denote $\tilde{\chi}_j(t) = \min\{\tilde{\chi}(t), \chi_j(t)\}$. By Lemma 2.5, we have χ_j and $\tilde{\chi}_j$ satisfy the desired conditions. The proof is completed. □

Let ϕ be a negative θ-psh function. We denote by $\mathrm{PSH}(X, \theta, \phi)$ the set of θ-psh functions $u \leq \phi$. Recall that by monotonicity, we always have $\int_X \theta_u^n \leq \int_X \theta_\phi^n$, where for every θ-psh function v, we put $\theta_v := dd^c v + \theta$. We also define by $\mathcal{E}(X, \theta, \phi)$ the set of $u \in \mathrm{PSH}(X, \theta, \phi)$ of full Monge-Ampère mass with respect to ϕ, *i.e,* $\int_X \theta_u^n = \int_X \theta_\phi^n$.

Let $\chi \in \widetilde{\mathcal{W}}^- \cup \mathcal{W}_M^+$, and $u \in \mathrm{PSH}(X, \theta, \phi)$. We put

$$E_{\chi,\theta,\phi}(u) := \int_X -\chi(u - \phi)\theta_u^n.$$

We also define by $\mathcal{E}_\chi(X, \theta, \phi)$ the set of $u \in \mathcal{E}(X, \theta, \phi)$ with $E_{\chi,\theta,\phi}(u) < \infty$.

Lemma 2.7. *Let $\chi \in \widetilde{\mathcal{W}}^- \cup \mathcal{W}_M^+$ and $u_1, u_2 \in \mathcal{E}_\chi(X, \theta, \phi)$. Then there exists a constant $C_1 > 0$ depending only on n and M such that*

$$-\int_X \chi(u_1 - \phi)\theta_{u_2}^n \leq C_1 \sum_{j=1}^{2} E_{\chi,\phi,\theta}(u_j),$$

and

$$E_{\chi,\theta,\phi}\big(au_1+(1-a)\big)u_2 \le C_1 \sum_{j=1}^{2} E_{\chi,\phi,\theta}(u_j),$$

for every $0<a<1$. Furthermore if $u_1 \ge u_2$, then

$$E_{\chi,\phi,\theta}(u_1) \le C_2 E_{\chi,\phi,\theta}(u_2),$$

for some constant C_2 depending only on n and M.

Proof. The first and third inequalities are from [31, Lemma 3.2] (see also [33, Propositions 2.3, 2.5] for the case where $\phi=0$ and θ is a Kähler form). The second desired inequality was implicitly proved in the proof of convexity of finite energy classes in [64, Proposition 3.3] (in a much broader context). Alternatively one can use properties of envelopes in [15] to get the same conclusion. We prove here the second desired inequality using ideas from [64] for readers' convenience.

Considering $u_j-\epsilon$ for $\epsilon>0$ instead of u_j, and taking $\epsilon\to 0$ later, without loss of generality, we can assume that $u_j<\phi\le 0$ for $j=1.2$. By replacing u_j,θ by $u_j-\phi$, θ_ϕ respectively, we can assume that $\phi=0$, but θ is no longer a smooth form but a closed positive $(1,1)$-current. This change causes no trouble for us. Let $v:=au_1+(1-a)u_2$. Observe that $X\subset\{u_1<u_2\}\cup\{u_1>2u_2\}$. Hence

$$\begin{aligned}
E_{\chi,\theta}(v) &\le \int_{\{u_1<u_2\}} -\chi(v)\theta_v^n + \int_{\{u_1>2u_2\}} -\chi(v)\theta_v^n \\
&\le \int_{\{u_1<u_2\}} -\chi(v)\theta_v^n + \int_{\{u_1>2u_2\}} -\chi(v)\theta_v^n \\
&\le \sum_{k=0}^{n}\left(\int_{\{u_1<u_2\}} -\chi(u_1)\theta_{u_1}^k\wedge\theta_{u_2}^{n-k} + \int_{\{u_1>2u_2\}} -\chi((2-a/2)u_2)\theta_{u_1}^k\wedge\theta_{u_2}^{n-k}\right) \\
&\le \sum_{k=0}^{n}\int_{\{u_1<u_2\}} -\chi(u_1)\theta_{u_1}^k\wedge\theta_{\max\{u_1,u_2\}}^{n-k} + \\
&\quad+\sum_{k=0}^{n}\int_{\{u_1>2u_2\}} -2^{k+1}\chi(u_2)\theta_{\max\{u_1/2,u_2\}}^k\wedge\theta_{u_2}^{n-k} \\
&\le \sum_{k=0}^{n}\left(\int_X -\chi(u_1)\theta_{u_1}^k\wedge\theta_{\max\{u_1,u_2\}}^{n-k} + 2^{k+1}\int_X -\chi(u_2)\theta_{\max\{u_1/2,u_2\}}^k\wedge\theta_{u_2}^{n-k}\right) \\
&\lesssim E_{\chi,\theta}(u_1)+E_{\chi,\theta}(\max\{u_1,(u_1+u_2)/2\})+E_{\chi,\theta}(u_2)+E_{\chi,\theta}(\max\{u_1/4+u_2/2,u_2\}) \\
&\lesssim E_{\chi,\theta}(u_1)+E_{\chi,\theta}(u_2),
\end{aligned}$$

where the two last estimates hold due to the first and third inequalities of the lemma. This finishes the proof. □

Lemma 2.8. *Let $\chi,\tilde\chi\in\widetilde{\mathcal{W}}^-\cup\mathcal{W}_M^+$ such that $\tilde\chi\le\chi$ and let $u_1,u_2,...,u_{n+1}\in\mathcal{E}(X,\theta,\phi)$. Denote $\varrho:=\mathrm{vol}(\theta_\phi)$. Then there exists a constant $C>0$ depending only on n and M such that*

$$-\int_X \chi(\epsilon(u_1-\phi))\theta_{u_2}\wedge...\wedge\theta_{u_{n+1}} \le C\,B\varrho(1-\tilde\chi(-1))Q_0(\epsilon),$$

for every $0 < \epsilon \leq 1$*, where*

$$B = 1 + \max_{1\leq j\leq n+1} E_{\tilde{\chi},\theta,\phi}(u_j)/\varrho \quad \textit{and} \quad Q_0(\epsilon) := \sup_{\{t\leq -1\}} \frac{\chi(\epsilon t)}{\tilde{\chi}(t)}.$$

Proof. Let L be the left-hand side of the desired inequality. We have

$$\begin{aligned}
L &\leq -\int_{\{u_1\geq\phi-1\}} \chi(\epsilon(u_1-\phi))\theta_{u_2}\wedge\ldots\wedge\theta_{u_{n+1}} - \int_{\{u_1<\phi-1\}} \chi(\epsilon(u_1-\phi))\theta_{u_2}\wedge\ldots\wedge\theta_{u_{n+1}} \\
&\leq -\chi(-\epsilon)\varrho - Q_0(\epsilon)\int_{\{u_1<\phi-1\}} \tilde{\chi}(u_1-\phi)\theta_{u_2}\wedge\ldots\wedge\theta_{u_{n+1}} \\
&\leq -\varrho Q_0(\epsilon)\tilde{\chi}(-1) - Q_0(\epsilon)\int_X \tilde{\chi}(u_1-\phi)\theta_{u_2}\wedge\ldots\wedge\theta_{u_{n+1}} \\
&\leq -\varrho Q_0(\epsilon)\tilde{\chi}(-1) + CQ_0(\epsilon)\max_{1\leq j\leq n+1} E_{\tilde{\chi},\theta,\phi}(u_j),
\end{aligned}$$

where $C > 0$ depends only on n and M. The last estimate holds due to Lemma 2.7. Thus the desired inequality follows. □

By the convexity/concavity and by the assumption $\tilde{\chi} \leq \chi$, we have

$$\begin{cases} Q_0(\epsilon) \geq \epsilon Q_0(1) & \text{if} \quad \chi \in \widetilde{\mathcal{W}}^-, \\ Q_0(\epsilon) \leq \epsilon Q_0(1) & \text{if} \quad \chi \in \mathcal{W}_M^+, \end{cases} \tag{2.7}$$

for every $0 < \epsilon \leq 1$. Moreover, if $\chi \in \widetilde{\mathcal{W}}^-$ and $\chi(t)/\tilde{\chi}(t) \to 0$ as $t \to -\infty$, then by the definition of Q_0, we also have

$$Q_0(\epsilon) \leq \frac{\chi(-\epsilon^{1/2})}{\tilde{\chi}(-1)} + \sup_{\{t\leq -\epsilon^{-1/2}\}} \frac{\chi(t)}{\tilde{\chi}(t)} \overset{\epsilon\to 0^+}{\longrightarrow} 0. \tag{2.8}$$

Let $u_1, u_2 \in \mathcal{E}_\chi(X,\theta,\phi)$, and $v := \max\{u_1, u_2\}$. Put

$$\nu(u_1,u_2) := \chi(-|u_1-u_2|)(\theta_{u_2}^n - \theta_{u_1}^n),$$

and

$$I_\chi(u_1,u_2) := \int_{\{u_1<u_2\}} \nu(u_1,u_2) + \int_{\{u_1>u_2\}} \nu(u_2,u_1) = \int_X \nu(u_1,v) + \int_X \nu(u_2,v). \tag{2.9}$$

Proposition 2.9. *Let* $\chi \in \widetilde{\mathcal{W}}^- \cup \mathcal{W}_M^+$*. Let* ϕ *is a negative* θ*-psh function and* $u_1, u_2 \in \mathcal{E}_\chi(X,\theta,\phi)$*. Then*

$$I_\chi(u_1,u_2) \geq 0.$$

Proof. Denote $\mu = \theta^n_{u_2} - \theta^n_{u_1}$. Since χ is absolutely continuous, we have χ is differentiable almost everywhere and $-\chi(t) = \int_t^0 \chi'(s)ds$ for every $t < 0$. Hence

$$\begin{aligned}\int_{\{u_1<u_2\}} \nu(u_1,u_2) &= -\int_{\{u_1<u_2\}} \left(\int_{u_1-u_2}^0 \chi'(t)dt\right) d\mu \\ &= -\int_{\{u_1<u_2\}} \left(\int_{-\infty}^0 \chi'(t)\mathbf{1}_{\{u_1<u_2+t\}}dt\right) d\mu \\ &= -\int_{-\infty}^0 \chi'(t)\mu\{u_1 < u_2+t\}dt.\end{aligned}$$

Moreover, it follows from [16, Lemma 2.3] that $\mu\{u_1 < u_2 + t\} \leq 0$ for every $t \leq 0$. Hence

$$\int_{\{u_1<u_2\}} \nu(u_1,u_2) = -\int_{-\infty}^0 \chi'(t)\mu\{u_1 < u_2+t\}dt \geq 0.$$

Similarly, we have

$$\int_{\{u_2<u_1\}} \nu(u_2,u_1) \geq 0.$$

Thus

$$I_\chi(u_1,u_2) = \int_{\{u_1<u_2\}} \nu(u_1,u_2) + \int_{\{u_2<u_1\}} \nu(u_2,u_1) \geq 0.$$

□

3 Stability estimates for fixed singularity type

3.1 Main results

Let $\chi, \tilde{\chi} \in \widetilde{\mathcal{W}}^- \cup \mathcal{W}^+_M$ $(M \geq 1)$ such that $\tilde{\chi} \leq \chi$. For each constant $t \geq 0$, we denote

$$Q(t) = Q_{\chi,\tilde{\chi}}(t) := \begin{cases} 1 \quad \text{if} \quad t \geq 1, \\ (Q_0(t)/Q_0(1))^{1/2} \quad \text{if} \quad 0 < t < 1 \quad \text{and} \quad \chi \in \widetilde{\mathcal{W}}^-, \\ t^{1/2} \quad \text{if} \quad 0 < t < 1 \quad \text{and} \quad \chi \in \mathcal{W}^+_M, \\ \lim_{s\to 0^+} Q(s) \quad \text{if} \quad t = 0. \end{cases} \tag{3.1}$$

where Q_0 is defined as in Lemma 2.8. We remove the subscript $\chi, \tilde{\chi}$ from $Q_{\chi,\tilde{\chi}}$ if $\chi, \tilde{\chi}$ are clear from the context. Note that Q is increasing continuous function in t and

$$Q(0) = 0 \quad \text{if either} \quad \chi, \tilde{\chi} \in \mathcal{W}^+_M \quad \text{or} \quad \lim_{t\to-\infty} \frac{\chi(t)}{\tilde{\chi}(t)} = 0. \tag{3.2}$$

Now, we state the main results of this section. For the convenience, we normalize energies with respect to $\varrho := \int_X \theta^n_\phi$ as follows

$$E^0_{\tilde{\chi},\theta,\phi} := \varrho^{-1} E_{\tilde{\chi},\theta,\phi}, \quad I^0_\chi(u_1,u_2) = \varrho^{-1} I_\chi(u_1,u_2).$$

Theorem 3.1. *Let θ be a closed smooth real $(1,1)$-form and ϕ be a negative θ-psh function such that $\varrho := \mathrm{vol}(\theta_\phi) > 0$. Let $\chi, \tilde{\chi} \in \widetilde{\mathcal{W}}^- \cup \mathcal{W}_M^+$ ($M \geq 1$) such that $\tilde{\chi} \leq \chi$. Let $B \geq 1$ be a constant and let $u_j, \psi_j \in \mathcal{E}(X, \theta, \phi)$ satisfy $u_1 \leq u_2$ and*

$$E^0_{\tilde{\chi},\theta,\phi}(u_j) + E^0_{\tilde{\chi},\theta,\phi}(\psi_j) \leq B,$$

for $j = 1, 2$. Then there exists a constant $C_n > 0$ depending only on n and M such that

$$\int_X -\chi(u_1 - u_2)(\theta^n_{\psi_1} - \theta^n_{\psi_2}) \leq C_n \varrho B^2 (1 - \tilde{\chi}(-1))^2 Q^{\circ n}\big(I^0_\chi(u_1, u_2)\big), \tag{3.3}$$

where Q is defined by (3.1)*, and $Q^{\circ n} := Q \circ Q \circ \cdots \circ Q$ (n-iterate of Q).*

Since the measure $\theta^n_{\psi_1} - \theta^n_{\psi_2}$ is not positive, we need the following consequence of the above theorem for later applications on stability estimates.

Theorem 3.2. *Let θ be a closed smooth real $(1,1)$-form and ϕ be a negative θ-psh function such that $\phi = P_\theta[\phi]$, $\varrho := \mathrm{vol}(\theta_\phi) > 0$ and $\theta \leq A\omega$ for some constant $A \geq 1$. Let $\chi, \tilde{\chi} \in \widetilde{\mathcal{W}}^- \cup \mathcal{W}_M^+$ ($M \geq 1$) such that $\tilde{\chi} \leq \chi$. Let $B \geq 1$ be a constant and $u_1, u_2, \psi \in \mathcal{E}(X, \theta, \phi)$ satisfying*

$$E^0_{\tilde{\chi},\theta,\phi}(u_1) + E^0_{\tilde{\chi},\theta,\phi}(u_2) + E^0_{\tilde{\chi},\theta,\phi}(\psi) \leq B,$$

for $j = 1, 2$. Then, for every constant $m > 0$ and $0 < \gamma < 1$, there exists a constant $C > 0$ depending on n, M, X, ω, m and γ such that

$$\int_X -\chi\big(-|u_1 - u_2|\big)\theta^n_\psi \leq -\varrho\chi\left(-|a_1 - a_2| - \lambda^m\right) + C\varrho A^{(1-\gamma)/m}(B - \tilde{\chi}(-A))^2 (1 - \tilde{\chi}(-1))^2 \lambda^\gamma,$$

where $a_j := \sup_X u_j$ and $\lambda = Q^{\circ n}\big(I^0_\chi(u_1, u_2)\big)$.

3.2 Proof of Theorem 3.1

Here is the first step in the proof of Theorem 3.1.

Lemma 3.3. *If Theorem 3.1 holds for u_j, ψ_j of the same singularity type as ϕ, then it holds for the general case.*

Proof. Let u_j, ψ_j ($j = 1, 2$) be as in the statement of Theorem 3.1. For every $k > 0$, we denote $u_{j,k} := \max\{u_j, \phi - k\}$ and $\psi_{j,k} = \max\{\psi_j, \phi - k\}$. By Lemma 2.7, there exists a constant $C_1 > 0$ depending only on n and M such that

$$E^0_{\tilde{\chi},\theta,\phi}(u_{j,k}) + E^0_{\tilde{\chi},\theta,\phi}(\psi_{j,k}) \leq C_1 B,$$

for $j = 1, 2$ and for every $k > 0$. Therefore, by the assumption, there exists a constant $C_2 > 0$ depending only on n and M such that

$$\int_X -\chi(u_{1,k} - u_{2,k})(\theta^n_{\psi_{1,l}} - \theta^n_{\psi_{2,l}}) \leq C_2 \varrho B^2 (1 - \tilde{\chi}(-1))^2 Q^{\circ(n)}(I^0_\chi(u_{1,k}, u_{2,k})),$$

for every $k, l > 0$. Letting $l \to \infty$ and using [17, Theorem 2.2], we get

$$\int_X -\chi(u_{1,k} - u_{2,k})(\theta^n_{\psi_1} - \theta^n_{\psi_2}) \leq C_2 \varrho B^2 (1 - \tilde{\chi}(-1))^2 Q^{\circ(n)}(I^0_\chi(u_{1,k}, u_{2,k})) \tag{3.4}$$

for every $k > 0$. We will show that

$$I_\chi(u_1, u_2) = \lim_{k\to\infty} I_\chi(u_{1,k}, u_{2,k}). \tag{3.5}$$

Denote

$$f := \chi(u_1 - u_2)(\theta^n_{u_2} - \theta^n_{u_1}), \quad f_k := \chi(u_{1,k} - u_{2,k})(\theta^n_{u_{2,k}} - \theta^n_{u_{1,k}}).$$

We have

$$\begin{aligned} I_\chi(u_{1,k}, u_{2,k}) = \int_X f_k &= \int_{\{u_1 > \phi - k\}} f_k + \int_{\{u_1 \leq \phi - k\}} f_k \\ &= \int_{\{u_1 > \phi - k\}} f + \int_{\{u_1 \leq \phi - k\}} f_k \\ &= I_\chi(u_1, u_2) - \int_{\{u_1 \leq \phi - k\}} f + \int_{\{u_1 \leq \phi - k\}} f_k. \end{aligned}$$

Then

$$\begin{aligned} |I_\chi(u_{1,k}, u_{2,k}) - I_\chi(u_1, u_2)| &= \left| \int_{\{u_1 \leq \phi - k\}} f + \int_{\{u_1 \leq \phi - k\}} f_k \right| \\ &\leq \int_{\{u_1 \leq \phi - k\}} \mu + \int_{\{u_1 \leq \phi - k\}} -\chi(u_{1,k} - u_{2,k})(\theta^n_{u_{2,k}} + \theta^n_{u_{1,k}}) \\ &\leq \int_{\{u_1 \leq \phi - k\}} \mu + \int_{\{u_1 \leq \phi - k\}} -\chi(-k)(\theta^n_{u_{2,k}} + \theta^n_{u_{1,k}}), \end{aligned}$$

where $\mu = -\chi(u_1 - \phi)(\theta^n_{u_1} + \theta^n_{u_2})$. By Lemma 2.7, we have $\int_X \mu < \infty$. Then it follows from Lebesgue's dominated convergence theorem that $\lim_{k\to\infty} \int_{\{u_1 \leq \phi - k\}} \mu = 0$. Therefore,

$$\limsup_{k\to\infty} |I_\chi(u_{1,k}, u_{2,k}) - I_\chi(u_1, u_2)| \leq \limsup_{k\to\infty} \int_{\{u_1 \leq \phi - k\}} -\chi(-k)(\theta^n_{u_{1,k}} + \theta^n_{u_{2,k}}). \tag{3.6}$$

By the fact that

$$\int_X \theta^n_{u_{1,k}} = \int_X \theta^n_{u_{1,k}} = \int_X \theta^n_\phi, \quad \mathbf{1}_{\{u_1 \leq \phi - k\}} \theta^n_{u_{j,k}} = \mathbf{1}_{\{u_1 \leq \phi - k\}} \theta^n_{u_j} \quad (j = 1, 2),$$

we have

$$-\chi(-k) \int_{\{u_1 \leq \phi - k\}} (\theta^n_{u_{1,k}} + \theta^n_{u_{2,k}}) = -\chi(-k) \int_{\{u_1 \leq \phi - k\}} (\theta^n_{u_1} + \theta^n_{u_2}) \leq \int_{\{u_1 \leq \phi - k\}} \mu. \tag{3.7}$$

By using (3.6), (3.7) and the fact $\lim_{k\to\infty} \int_{\{u_1 \leq \phi - k\}} \mu = 0$, we get (3.5). Now, combining (3.4) and (3.5), we obtain

$$\int_X -\chi(u_1 - u_2)(\theta^n_{\psi_1} - \theta^n_{\psi_2}) \leq C_2 \varrho B^2 (1 - \tilde{\chi}(-1))^2 Q^{\circ(n)}(I^0_\chi(u_1, u_2)).$$

The proof is completed. □

Lemma 3.4. *Let $M \geq 1$ and $\chi, \tilde{\chi} \in \widetilde{\mathcal{W}}^- \cup \mathcal{W}_M^+$ such that $\tilde{\chi} \leq \chi$ and $\chi \in \mathscr{C}^1(\mathbb{R})$. Let $u_1, u_2, ..., u_{n+2} \in \mathcal{E}(X, \theta, \phi)$ such that $u_1 \leq u_2$ and $u_j - \phi$ is bounded ($j = 1, 2, ..., n+2$), where ϕ is a negative θ-psh function satisfying $\varrho := \mathrm{vol}(\theta_\phi) > 0$. Denote*

$$T = \theta_{u_4} \wedge ... \wedge \theta_{u_{n+2}}, \quad I = \left| \int_X \chi'(u_1 - u_2) d(u_1 - u_2) \wedge d^c(u_1 - u_3) \wedge T \right|,$$

and

$$J = \int_X \chi'(u_1 - u_2) d(u_1 - u_2) \wedge d^c(u_1 - u_2) \wedge T.$$

Then there exists $C > 0$ depending only on n and M such that

$$I \leq C\varrho B(1 - \tilde{\chi}(-1)) Q(J/\varrho),$$

where $B := \sum_{j=1}^{n+2} \max\{E^0_{\tilde{\chi},\theta,\phi}(u_j), 1\}$ and Q is defined by (3.1).

Clearly if $\chi \in \widetilde{\mathcal{W}}^-$, then the above constant C does not depend on M.

Proof. In this proof, we use the symbols $\lesssim$ and $\gtrsim$ for inequalities modulo a constant depending only on n and M. By Theorem 2.2 and Lemma 2.7, we have

$$I = \left| \int_X -\chi(u_1 - u_2) dd^c(u_1 - u_3) \wedge T \right| \lesssim \varrho B = \varrho B Q(1).$$

Therefore, without loss of generality, we can assume that $J/\varrho < 1$. Approximating u_3 by $u_3 - \delta$ with $\delta \searrow 0$, we can assume that $u_3 < \phi$ on X.

For each $0 < \epsilon < 1/2$ we denote

$$U(\epsilon) = \{u_1 - u_2 < \epsilon(u_1 + u_3 - 2\phi)\}, V(\epsilon) = \{u_1 - u_2 > \epsilon(u_1 + u_3 - 2\phi)\},$$

and $\Gamma(\epsilon) = \{u_1 - u_2 = \epsilon(u_1 + u_3 - 2\phi)\}$. Since $\Gamma(\epsilon_1) \cap \Gamma(\epsilon_2) = \emptyset$ for every $\epsilon_1 \neq \epsilon_2$ (note $u_3 < \phi$), we have

$$\int_{\Gamma(\epsilon)} d(u_1 - u_3) \wedge d^c(u_1 - u_3) \wedge T = 0, \tag{3.8}$$

for almost everywhere $\epsilon \in (0, 1/2)$.

Let $0 < \epsilon < 1/2$ be a constant satisfying (3.8). To simplify the notation, from now on, we write U, V, Γ for $U(\epsilon), V(\epsilon), \Gamma(\epsilon)$ respectively. Denote

$$\tilde{u}_1 = \frac{u_1 + \epsilon u_3}{1 + \epsilon}, \qquad \tilde{u}_2 = \max\left\{\frac{u_2 + \epsilon u_3}{1 + \epsilon}, \quad \frac{(1 - \epsilon)u_1 + 2\epsilon\phi}{1 + \epsilon}\right\} \quad \text{and} \quad \tilde{\varphi} = \tilde{u}_1 - \tilde{u}_2.$$

Then $\varphi := (u_1 - u_2) = (1+\epsilon)\tilde{\varphi}$ on U. Hence

$$
\begin{aligned}
I &= \left| \int_X -\chi(\varphi) dd^c(u_1 - u_3) \wedge T \right| \\
&\leq \left| \int_U -\chi(\varphi) dd^c(u_1 - u_3) \wedge T \right| + \left| \int_{X \backslash U} -\chi(\varphi) dd^c(u_1 - u_3) \wedge T \right| \\
&\leq \left| \int_U -\chi((1+\epsilon)\tilde{\varphi}) dd^c(u_1 - u_3) \wedge T \right| + \left| \int_{X \backslash U} -\chi(\varphi)(\theta_{u_1} + \theta_{u_3}) \wedge T \right| \\
&\leq \left| \int_U -\chi((1+\epsilon)\tilde{\varphi}) dd^c(u_1 - u_3) \wedge T \right| + \left| \int_{X \backslash U} -\chi(\epsilon(u_1 + u_3 - 2\phi))(\theta_{u_1} + \theta_{u_3}) \wedge T \right| \\
&:= I_1 + I_2,
\end{aligned}
$$

where in the last inequality we used the fact that χ is increasing and $\varphi \geq \epsilon(u_1 + u_2 - 2\phi)$ on $X \backslash U$. By Lemma 2.7, we have $E^0_{\tilde{\chi},\theta,\phi}\left(\frac{u_1 + u_3}{2}\right) \lesssim B$. Therefore, it follows from Lemma 2.8 that

$$
I_2 \leq 2 \int_X -\chi\left(2\epsilon\left(\frac{u_1 + u_3}{2} - \phi\right)\right) \theta_{(u_1+u_3)/2} \wedge T \lesssim B\varrho(1 - \tilde{\chi}(-1))Q_0(2\epsilon). \tag{3.9}
$$

In order to estimate I_1, we divide it into two terms

$$
\begin{aligned}
I_1 &\leq \left| \int_X -\chi((1+\epsilon)\tilde{\varphi}) dd^c(u_1 - u_3) \wedge T \right| + \left| \int_{X \backslash U} -\chi((1+\epsilon)\tilde{\varphi}) dd^c(u_1 - u_3) \wedge T \right| \\
&:= I_3 + I_4.
\end{aligned}
$$

Note that $\tilde{u}_1 - \tilde{u}_2 = \epsilon(u_1 + u_3 - 2\phi)/(1+\epsilon)$ on $X \setminus U$. Hence

$$
I_4 \leq \int_{X \backslash U} -\chi((1+\epsilon)\tilde{\varphi})(\theta_{u_1} + \theta_{u_3}) \wedge T \leq \int_{X \backslash U} -\chi(\epsilon(u_1 + u_2 - 2\phi))(\theta_{u_1} + \theta_{u_3}) \wedge T.
$$

Using Lemma 2.8 again, we get

$$
I_4 \lesssim B\varrho(1 - \tilde{\chi}(-1))Q_0(2\epsilon). \tag{3.10}
$$

Using integration by parts, we have

$$
I_3 = (1+\epsilon) \left| \int_X \chi'((1+\epsilon)\tilde{\varphi}) d\tilde{\varphi} \wedge d^c(u_1 - u_3) \wedge T \right|.
$$

Moreover, by Cauchy-Schwarz inequality and by the choice of ϵ (see (3.8)), we get

$$
\int_\Gamma \chi'((1+\epsilon)\tilde{\varphi}) d\tilde{\varphi} \wedge d^c(u_1 - u_3) \wedge T = 0.
$$

Hence

$$
I_3 = (1+\epsilon) \left| \int_{U \cup V} \chi'((1+\epsilon)\tilde{\varphi}) d\tilde{\varphi} \wedge d^c(u_1 - u_3) \wedge T \right| \leq (1+\epsilon)(I_5 I_6)^{1/2} \tag{3.11}
$$

where

$$I_5 = \int_{U\cup V} \chi'((1+\epsilon)\tilde{\varphi}) d(u_1 - u_3) \wedge d^c(u_1 - u_3) \wedge T,$$

and

$$I_6 = \int_{U\cup V} \chi'((1+\epsilon)\tilde{\varphi}) d\tilde{\varphi} \wedge d^c\tilde{\varphi} \wedge T.$$

Since $(1+\epsilon)\tilde{\varphi} \leq \epsilon(u_1 + u_3 - 2\phi)$, if $\chi \in \widetilde{\mathcal{W}}^-$ (hence χ' is nonnegative and increasing on $\mathbb{R}_{\leq 0}$) then

$$\begin{aligned}
I_5 &\leq \int_X \chi'(\epsilon(u_1 + u_3 - 2\phi)) d(u_1 - u_3) \wedge d^c(u_1 - u_3) \wedge T \\
&\lesssim \int_X \chi'(\epsilon(u_1 + u_3 - 2\phi)) d(u_1 - \phi) \wedge d^c(u_1 - \phi) \wedge T \\
&+ \int_X \chi'(\epsilon(u_1 + u_3 - 2\phi)) d(u_3 - \phi) \wedge d^c(u_3 - \phi) \wedge T \\
&\leq \int_X \chi'(\epsilon(u_1 - \phi)) d(u_1 - \phi) \wedge d^c(u_1 - \phi) \wedge T + \int_X \chi'(\epsilon(u_3 - \phi)) d(u_3 - \phi) \wedge d^c(u_3 - \phi) \wedge T \\
&= \epsilon^{-1} \int_X \chi(\epsilon(u_1 - \phi)) dd^c(u_1 - \phi) \wedge T + \epsilon^{-1} \int_X \chi(\epsilon(u_3 - \phi)) dd^c(u_3 - \phi) \wedge T \\
&\lesssim B\varrho(1 - \tilde{\chi}(-1))\epsilon^{-1} Q_0(\epsilon),
\end{aligned}$$

where the last estimate holds due to Lemma 2.8.

Denote $v_1 := (u_1 + 2u_3)/3$ and $v_2 := (2u_1 + u_3)/3$. Since

$$(1+\epsilon)(\tilde{u}_1 - \tilde{u}_2) \geq u_1 + u_3 - 2\phi, \quad u_1 - u_3 = -(v_1 - v_2)/3,$$

one sees that if $\chi \in \mathcal{W}_M^+$ (hence χ' is nonnegative and decreasing in $\mathbb{R}_{\leq 0}$) then

$$\begin{aligned}
I_5 &\leq \int_X \chi'((u_1 + u_3 - 2\phi)) d(u_1 - u_3) \wedge d^c(u_1 - u_3) \wedge T \\
&\lesssim \int_X \chi'((u_1 + u_3 - 2\phi)) \big(d(v_1 - \phi) \wedge d^c(v_1 - \phi) + d(v_2 - \phi) \wedge d^c(v_2 - \phi)\big) \wedge T \\
&\leq \int_X \chi'(3(v_1 - \phi)) d(v_1 - \phi) \wedge d^c(v_1 - \phi) \wedge T + \int_X \chi'(3(v_2 - \phi)) d(v_2 - \phi) \wedge d^c(v_2 - \phi) \wedge T \\
&= \frac{1}{3} \int_X -\chi(3(v_1 - \phi)) dd^c(v_1 - \phi) \wedge T + \frac{1}{3} \int_X -\chi(3(v_2 - \phi)) dd^c(v_2 - \phi) \wedge T \\
&\leq \int_X -\chi(3(v_1 - \phi))(\theta_{v_1} + \theta_\phi) \wedge T + \int_X -\chi(3(v_2 - \phi))(\theta_{v_1} + \theta_\phi) \wedge T \\
&\leq 3^M \int_X -\chi(v_1 - \phi)(\theta_{v_1} + \theta_\phi) \wedge T + 3^M \int_X -\chi(v_2 - \phi)(\theta_{v_1} + \theta_\phi) \wedge T \\
&\lesssim B\varrho,
\end{aligned}$$

where the two last estimates hold due to Lemma 2.7 and the fact

$$\log(-\chi(3t)) - \log(-\chi(t)) = \int_t^{3t} \frac{\chi'(s)}{\chi(s)} ds \leq \int_t^{3t} \frac{M}{s} ds = M \log 3,$$

for every $\chi \in \mathcal{W}_M^+$ and $t \leq 0$. Combining the estimates in both cases, we obtain

$$I_5 \lesssim B\varrho(1-\tilde{\chi}(-1))\frac{Q(\epsilon)^2}{\epsilon}, \tag{3.12}$$

where we used the inequality $Q(\epsilon) \geq \epsilon^{1/2}$ if $\chi \in \widetilde{W}^-$. Now, we estimate I_6. Since U, V are open in the plurifine topology and

$$(1+\epsilon)\tilde{\varphi} = \begin{cases} \varphi \quad \text{on} \quad U \\ \epsilon(u_1+u_3-2\varphi) \quad \text{on} \quad V \end{cases},$$

we have

$$\begin{aligned} I_6 &= \int_U \chi'((1+\epsilon)\tilde{\varphi})d\tilde{\varphi} \wedge d^c\tilde{\varphi} \wedge T + \int_V \chi'((1+\epsilon)\tilde{\varphi})d\tilde{\varphi} \wedge d^c\tilde{\varphi} \wedge T \\ &= (1+\epsilon)^{-2}\int_U \chi'(\varphi)d\varphi \wedge d^c\varphi \wedge T \\ &+ \frac{\epsilon^2}{(1+\epsilon)^2}\int_V \chi'(\epsilon(u_1+u_3-2\varphi))d(u_1+u_3-2\varphi) \wedge d^c(u_1+u_3-2\varphi) \wedge T \\ &\leq J + \epsilon^2\int_X \chi'(\epsilon(u_1+u_3-2\varphi))d(u_1+u_3-2\varphi) \wedge d^c(u_1+u_3-2\varphi) \wedge T \\ &= J + \epsilon\int_X -\chi(\epsilon(u_1+u_3-2\varphi))dd^c(u_1+u_3-2\varphi) \wedge T. \end{aligned}$$

Therefore, it follows from Lemma 2.8 that

$$I_6 \lesssim J + B\varrho(1-\tilde{\chi}(-1))\epsilon Q_0(2\epsilon). \tag{3.13}$$

Combining (3.9), (3.11), (3.10), (3.12) and (3.13), we get

$$\begin{aligned} I \leq I_1 + I_2 &\leq I_3 + I_4 + I_2 \\ &\lesssim (I_5 I_6)^{1/2} + I_4 + I_2 \\ &\lesssim \left(B\varrho(1-\tilde{\chi}(-1))\epsilon^{-1}J\right)^{1/2} Q(\epsilon) + B\varrho(1-\tilde{\chi}(-1))\epsilon Q(2\epsilon)^2. \end{aligned}$$

Letting $\epsilon \searrow J/(2\varrho)$ (and ϵ satisfies (3.8)), we obtain

$$I \lesssim B\varrho(1-\tilde{\chi}(-1))Q(J/\varrho).$$

The proof is completed. □

Proposition 3.5. *Let $\chi, \tilde{\chi} \in \widetilde{\mathcal{W}}^- \cup \mathcal{W}_M^+$ such that $\tilde{\chi} \leq \chi$ and $\chi \in \mathscr{C}^1(\mathbb{R})$. Let $u_1, u_2, u_3 \in \mathcal{E}(X, \theta, \phi)$ such that $u_1 \leq u_2$ and $u_j - \phi$ is bounded ($j = 1, 2, 3$), where ϕ is a negative θ-psh function satisfying $\varrho := \mathrm{vol}(\theta_\phi) > 0$. Then there exists a constant $C_n > 0$ depending only on n and M such that*

$$\int_X \chi'(u_1-u_2)d(u_1-u_2) \wedge d^c(u_1-u_2) \wedge \theta_{u_3}^{n-1} \leq C_n\varrho B^2(1-\tilde{\chi}(-1))^2 Q^{\circ(n-1)}\big(I_\chi^0(u_1,u_2)\big), \tag{3.14}$$

where $B := \sum_{j=1}^3 \max\{E^0_{\tilde{\chi},\theta,\phi}(u_j), 1\}$ and Q is defined by (3.1).

Proof. Let

$$\varphi := u_1 - u_2, \quad T := \sum_{j=1}^{n-1} \theta_{u_1}^j \wedge \theta_{u_2}^{n-1-j},$$

and

$$T_{k,l} := \theta_{u_1}^k \wedge \theta_{u_2}^l \wedge \theta_{u_3}^{n-k-l-1}, \quad L_{k,l} := \int_X \chi'(\varphi) d\varphi \wedge d^c\varphi \wedge T_{k,l}.$$

Observe

$$\theta_{u_2}^n - \theta_{u_1}^n = -dd^c\varphi \wedge T$$

and

$$L_{k,n-1-k} \leq \int_X \chi'(\varphi) d\varphi \wedge d^c\varphi \wedge T = \varrho I_\chi^0(u_1, u_2) \tag{3.15}$$

by integration by parts. We now prove by inverse induction on $m := k + l$ that

$$L_{k,l} \leq C_{m,n} \varrho B^2 (1 - \tilde{\chi}(-1))^2 Q^{\circ(n-1-k-l)}\big(I_\chi^0(u_1, u_2)\big), \tag{3.16}$$

for some constant $C_{m,n} > 1$ depending only on m, n and M. The desired assertion (3.14) is the case where $k = l = 0$. In what follows we use the symbols $\lesssim$ and $\gtrsim$ for inequalities modulo a constant depending only on n and M. We have checked (3.16) for $k + l = n - 1$. Suppose that (3.16) holds for $k + l = m$ with $0 < m \leq n - 1$. We will verify it for $L_{k-1,l}$, where $k + l = m$ and $k > 1$. The case $L_{k,l-1}$ is done similarly.

Denote $S_{k-1,l} = \theta_{u_1}^{k-1} \wedge \theta_{u_2}^l \wedge \theta_{u_3}^{n-k-l-1}$. Then

$$L_{k-1,l} - L_{k,l} = \int_X \chi'(\varphi) d\varphi \wedge d^c\varphi \wedge dd^c(u_3 - u_1) \wedge S_{k-1,l}.$$

Using integration by parts, we have

$$\begin{aligned}
L_{k-1,l} - L_{k,l} &= \int_X -\chi(\varphi) dd^c(\varphi) \wedge dd^c(u_3 - u_1) \wedge S_{k-1,l} \\
&= \int_X -\chi(\varphi) dd^c(u_3 - u_1) \wedge T_{k,l} - \int_X -\chi(\varphi) dd^c(u_3 - u_1) \wedge T_{k-1,l+1} \\
&= \int_X \chi'(\varphi) d\varphi \wedge d^c(u_3 - u_1) \wedge T_{k,l} - \int_X \chi'(\varphi) d\varphi \wedge d^c(u_3 - u_1) \wedge T_{k-1,l+1}
\end{aligned}$$

Therefore, it follows from Lemma 3.4 that

$$L_{k-1,l} - L_{k,l} \lesssim \varrho B(1 - \tilde{\chi}(-1)) \left(Q(L_{k,l}/\varrho) + Q(L_{k-1,l+1}/\varrho) \right).$$

Hence, by using the inductive hypothesis, we get

$$\begin{aligned}
L_{k-1,l} &\lesssim \varrho B^2 (1 - \tilde{\chi}(-1))^2 Q^{\circ(n-1-m)}\big(I_\chi^0(u_1, u_2)\big) \\
&\quad + \varrho B(1 - \tilde{\chi}(-1)) Q\left(C_{m,n} B^2 (1 - \tilde{\chi}(-1))^2 Q^{\circ(n-1-m)}\big(I_\chi^0(u_1, u_2)\big) \right) \\
&\lesssim \varrho B^2 (1 - \tilde{\chi}(-1))^2 Q^{\circ(n-m)}\big(I_\chi^0(u_1, u_2)\big).
\end{aligned}$$

Here we use the fact $Q(t_1) \leq (t_1/t_2)^{1/2} Q(t_2)$ for every $t_1 > t_2 > 0$ (see Lemma 3.6).

Thus, (3.16) holds for $L_{k-1,l}$. This finishes the proof. □

Lemma 3.6. *The function* $h(t) = \dfrac{(Q(t))^2}{t}$ *is non-increasing in* $\mathbb{R}_{>0}$.

Proof. If $\chi \in \mathcal{W}_M^+$ then we have

$$h(t) = \begin{cases} \dfrac{1}{t} & \text{if} \quad t \geq 1, \\ 1 & \text{if} \quad 0 < t < 1, \end{cases}$$

is a non-increasing function.

We consider the case $\chi \in \widetilde{\mathcal{W}}^-$. We have

$$h(t) = \begin{cases} \dfrac{1}{t} & \text{if} \quad t \geq 1, \\ \dfrac{Q_0(t)}{tQ_0(1)} & \text{if} \quad 0 < t < 1. \end{cases}$$

It is clear that h is decreasing in $[1, \infty)$. We need to show that h is non-increasing in $(0, 1)$. Since χ is convex, we have

$$\frac{\chi(t_1 s)}{t_1 s} \leq \frac{\chi(t_2 s)}{t_2 s},$$

for every $0 < t_2 < t_1 < 1$ and $s < 0$. Dividing both sides of the last estimate by $\tilde{\chi}(s)/s$, we get

$$\frac{\chi(t_1 s)}{t_1 \tilde{\chi}(s)} \leq \frac{\chi(t_2 s)}{t_2 \tilde{\chi}(s)}.$$

Taking the supremum of both sides, we obtain

$$\frac{Q_0(t_1)}{t_1} = \sup_{s \leq -1} \frac{\chi(t_1 s)}{t_1 \tilde{\chi}(s)} \leq \sup_{s \leq -1} \frac{\chi(t_2 s)}{t_2 \tilde{\chi}(s)} = \frac{Q_0(t_2)}{t_2}.$$

Then $h(t_1) \leq h(t_2)$. Hence, h is non-increasing in $(0, 1)$. The proof is completed. □

End of the proof of Theorem 3.1. By Lemma 3.3 and Lemma 2.6, the problem is reduced to the case where $\chi \in \mathscr{C}^1(\mathbb{R})$ and u_j, ψ_j are of the same singularity type as ϕ.

Let L be the left-hand side of the desired inequality. We have

$$\begin{aligned} L &= \int_X -\chi(u_1 - u_2)(\theta_{\psi_1}^n - \theta_{u_1}^n) - \int_X -\chi(u_1 - u_2)(\theta_{\psi_2}^n - \theta_{u_1}^n) \\ &= \int_X -\chi(u_1 - u_2) dd^c(\psi_1 - u_1) \wedge T_1 - \int_X -\chi(u_1 - u_2) dd^c(\psi_2 - u_1) \wedge T_2 \\ &= L_1 - L_2, \end{aligned}$$

where $T_j = \sum_{l=0}^{n-1} \theta_{\psi_j}^l \wedge \theta_{u_1}^{n-l-1}$. Using integration by parts and Lemma 3.4, we get

$$\begin{aligned} L_1 &= \int_X \chi'(u_1 - u_2) d(u_1 - u_2) \wedge d^c(\psi_1 - u_1) \wedge T_1 \\ &\leq C_1 \varrho B(1 - \tilde{\chi}(-1)) Q \left(\varrho^{-1} \int_X \chi'(u_1 - u_2) d(u_1 - u_2) \wedge d^c(u_1 - u_2) \wedge T_1 \right), \end{aligned}$$

where $C_1 > 0$ depends only on n and M. Moreover, it follows from Proposition 3.5 that

$$\varrho^{-1}\int_X \chi'(u_1-u_2)d(u_1-u_2)\wedge d^c(u_1-u_2)\wedge T_1 \leq C_2B^2(1-\tilde{\chi}(-1))^2Q^{\circ(n-1)}\left(I^0_\chi(u_1,u_2)\right),$$

where $C_2 > 1$ depends only on n and M. Then

$$L_1 \leq C_3\varrho B^2(1-\tilde{\chi}(-1))^2Q^{\circ n}\left(I^0_\chi(u_1,u_2)\right),$$

where $C_3 > 0$ depends only on n and M. Here we use the fact $Q(t_1) \leq (t_1/t_2)^{1/2}Q(t_2)$ for every $t_1 > t_2 > 0$.

By the same arguments, we also have

$$-L_2 \leq C_4\varrho B^2(1-\tilde{\chi}(-1))^2Q^{\circ n}\left(I^0_\chi(u_1,u_2)\right),$$

where $C_4 > 0$ depends only on n and M.

Hence

$$L = L_1 - L_2 \leq (C_3+C_4)\varrho B^2(1-\tilde{\chi}(-1))^2Q^{\circ n}\left(I^0_\chi(u_1,u_2)\right).$$

The proof is completed. □

3.3 Proof of Theorem 3.2

Recall that for every Borel set E in X, we define

$$\mathrm{cap}_{\theta,\phi}(E) := \sup\left\{\int_E \theta_h^n : \quad h\in \mathrm{PSH}(X,\theta), \quad \phi-1\leq h\leq \phi\right\}.$$

The following is an improvement of results from [16, 15] (see also [7, 41]).

Theorem 3.7. *Let $A \geq 1$ be a constant and let θ be a closed smooth real $(1,1)$-form such that $\theta \leq A\omega$. Let $\phi \in \mathrm{PSH}(X,\theta)$ and $0 \leq f \in L^p(X)$ for some constant $p > 1$ such that $\phi = P[\phi]$ and $0 < \int_X f\omega^n = \int_X \theta_\phi^n := \varrho$. Assume $u \in \mathcal{E}(X,\theta,\phi)$ satisfies $\sup_X(u-\phi) = 0$ and $\theta_u^n = fdV$. Then, there exists a constant $C \geq 1$ depending only on X,ω,n and p such that*

$$u \geq \phi - C\,A\left(\log\|f\,\mathrm{vol}_\omega(X)^q/\varrho\|_{L^p} + \log A + 1\right), \tag{3.17}$$

where $\mathrm{vol}_\omega(X) := \int_X \omega^n$ *and* $q = \dfrac{p}{p-1}$.

By Hölder inequalities, one sees that

$$1 = \int_X \frac{f}{\varrho}\omega^n \leq \|f/\varrho\|_{L^p}\left(\mathrm{vol}_\omega(X)\right)^q,$$

and then $\log\|f\,\mathrm{vol}_\omega(X)^q/\varrho\|_{L^p} \geq 0$.

Proof. Without loss of generality, we can assume that $\mathrm{vol}_\omega(X) = 1$. Recall that there exists a constant $\nu > 0$ depending only on X, ω such that

$$\int_X \exp\left(-\psi/\nu\right)\omega^n \leq C_0^2,$$

for every $\psi \in \mathrm{PSH}(X,\omega)$ with $\sup_X \psi = 0$, where $C_0 > 0$ is a constant depending only on X and ω. Consequently, one gets

$$\int_X \exp\left(-\psi/(A\nu)\right)\omega^n \leq C_0^2,$$

for every $\psi \in \mathrm{PSH}(X,\theta) \subset \mathrm{PSH}(X, A\omega)$ with $\sup_X \psi = 0$. By the same arguments as in the proof of [15, Proposition 4.30] (use [16, Lemma 3.9] instead of [15, Lemma 4.9]), we have

$$\int_E \omega^n \leq C_0 \exp\left(-\frac{1}{2A\nu}\left(\frac{\mathrm{cap}_{\theta,\phi}(E)}{\varrho}\right)^{-1/n}\right),$$

for every Borel set $E \subset X$. Therefore, by the Hölder inequality and the fact $e^{-1/t} \leq m! t^m$ for every $m \in \mathbb{N}$ and every $t > 0$, there exists $A_0 > 0$ depending only on X, ω, n and p such that

$$\varrho^{-1}\int_E \theta_u^n = \int_E (f/\varrho)\omega^n \leq \|f/\varrho\|_{L^p}\left(\int_E \omega^n\right)^{1/q} \leq A_0 A^{2n}\|f/\varrho\|_{L^p}\frac{\mathrm{cap}_{\theta,\phi}(E)^2}{\varrho^2}, \tag{3.18}$$

for every Borel set $E \subset X$, where $1/p + 1/q = 1$. On the other hand, denoting $b = (A\nu q)^{-1}$ and $B_0 = (C_0)^{1/q}$, we have

$$\varrho^{-1}\int_X e^{-bw}\theta_u^n \leq \|f/\varrho\|_{L^p}\left(\int_X e^{-bqw}dV\right)^{1/q} \leq B_0\|f/\varrho\|_{L^p}, \tag{3.19}$$

for every $w \in \mathrm{PSH}(X,\theta)$ with $\sup_X w = 0$.

For every $h \in \mathrm{PSH}(X,\theta)$ with $\phi - 1 \leq h \leq \phi$, for each $0 \leq t \leq 1$ and $s > 0$, we have

$$\begin{aligned} t^n \int\limits_{\{u<\phi-t-s\}} \theta_h^n \leq \int\limits_{\{u<(1-t)\phi+th-s\}} \theta_{(1-t)\phi+th}^n &\leq \int\limits_{\{u<(1-t)\phi+th-s\}} \theta_u^n \\ &\leq \int\limits_{\{u<\phi-s\}} \theta_u^n, \end{aligned}$$

where the third estimate holds due to the comparison principle [16, Lemma 2.3]. Then

$$t^n\,\mathrm{cap}_\phi(u < \phi - t - s) \leq \int\limits_{\{u<\phi-s\}} \theta_u^n, \tag{3.20}$$

for every $0 \leq t \leq 1$, $s > 0$. Therefore, it follows from (3.18) that

$$t^n\,\varrho^{-1}\mathrm{cap}_\phi(u < \phi - t - s) \leq A_1\,\varrho^{-2}\mathrm{cap}_\phi(u < \phi - s)^2,$$

where $A_1 = A_0 A^{2n}\|f/\varrho\|_{L^p}$. Putting $g(s) = \varrho^{-1/n}\mathrm{cap}_\phi(u < \phi - s)^{1/n}$, the above inequality becomes

$$tg(t+s) \leq A_1^{1/n} g(s)^2.$$

Hence, it follows from [32, Lemma 2.4 and Remark 2.5] that if $g(s_0) < 1/(2A_1^{1/n})$ then $g(s) = 0$ for all $s \geq s_0 + 2$. Moreover, by (3.20) and the condition (3.19), we have

$$g(s+1)^n \leq \varrho^{-1} \int\limits_{\{u<\phi-s\}} \theta_u^n \leq \varrho^{-1} \int\limits_X e^{b(\phi-u-s)}\theta_u^n \leq B_1\, e^{-bs},$$

for every $s > 0$, where $B_1 = B_0\|f/\varrho\|_{L^p}$. Then $g(s+1) < 1/(2A_1^{1/n})$ provided that

$$s > \frac{n\log 2 + \log A_1}{b} + \frac{\log B_1}{b}.$$

Hence $g(s) = 0$ for every

$$s \geq \frac{n\log 2 + \log A_1}{b} + \frac{\log B_1}{b} + 4.$$

Thus

$$u \geq \phi - \left(\frac{n\log 2 + \log A_1}{b} + \frac{\log B_1}{b} + 4\right) = \phi - C_1 \log\|f/\varrho\|_{L^p} - C_2,$$

where $C_1 = \frac{2}{b} = 2\nu q A$ and

$$\begin{aligned} C_2 &= 4 + \frac{n\log 2 + \log A_0 + \log B_0 + 2n\log A}{b} \\ &= 4 + 2\nu q(n\log 2 + \log A_0 + \log B_0 + 2n\log A)A. \end{aligned}$$

The proof is finished. □

Lemma 3.8. *There exists a constant $C > 0$ depending only on n, X and ω such that for every $u \in \mathrm{PSH}(X,\omega)$ satisfying $\sup_X u = 0$ and for every constant $0 < t \leq 1$, one has*

$$\int_{\{u>-t\}} \omega^n \geq Ct^{2n}. \tag{3.21}$$

Proof. Let $(U_j, \varphi_j)_{j=1}^m$ such that $U_j \subset X$ are open, $\varphi_j : 4\mathbb{B} \longrightarrow U_j$ are biholomorphic and $\cup_{j=1}^m \varphi_j(\mathbb{B}) = X$ (where $\mathbb{B}$ is the open unit ball in $\mathbb{C}^n$), and there is a smooth psh function ρ_j in U_j such that $dd^c\rho_j = \omega$ for $1 \leq j \leq m$. Denote

$$C_\rho = \sup_{1\leq j\leq m} \sup_{2\mathbb{B}} \|\nabla(\rho_j \circ \varphi_j)\|.$$

Assume $u(z_0) = 0$. Then there exists $1 \le j_0 \le m$ such that $z_0 \in \varphi_{j_0}(\mathbb{B})$. Denote $w_0 = \varphi_{j_0}^{-1}(z_0)$, $\widehat{u}(w) = u \circ \varphi_{j_0}(w)$ and $\widehat{\rho}(w) = \rho_{j_0} \circ \varphi_{j_0}(w) - \rho_{j_0} \circ \varphi_{j_0}(w_0)$. By the plurisubharmonicity of $\widehat{u} + \widehat{\rho}$, for every $t > 0$ and $0 < r < 1$, we have

$$\begin{aligned}
0 = (\widehat{u} + \widehat{\rho})(w_0) &\le \frac{1}{\mathrm{vol}_{\mathbb{C}^n}(r\mathbb{B})} \int_{r\mathbb{B}} (\widehat{u} + \widehat{\rho}) dV_{2n} \\
&\le C_\rho r + \frac{1}{c_{2n} r^{2n}} \int_{r\mathbb{B}} \widehat{u} dV_{2n} \\
&\le C_\rho r - \frac{t}{c_{2n} r^{2n}} \int_{r\mathbb{B} \cap \{\widehat{u} \le -t\}} dV_{2n} \\
&\le C_\rho r - t + \frac{t}{c_{2n} r^{2n}} \int_{r\mathbb{B} \cap \{\widehat{u} > -t\}} dV_{2n} \\
&\le C_\rho r - t + \frac{C_\omega t}{r^{2n}} \mathrm{vol}_\omega(\{u > -t\}),
\end{aligned}$$

where $c_{2n} = \mathrm{vol}_{\mathbb{C}^n}(\mathbb{B})$ and $C_\omega > 0$ is a constant depending only on n, X, ω. It follows that

$$\mathrm{vol}_\omega(\{u > -t\}) \ge \frac{r^{2n}}{C_\omega} \left(1 - \frac{C_\rho r}{t}\right).$$

Hence, for every $0 < t < 1$, by choosing $r = \frac{t}{1 + C_\rho}$, we have

$$\mathrm{vol}_\omega(\{u > -t\}) \ge C t^{2n},$$

where $C = \dfrac{1}{C_\omega (1 + C_\rho)^{2n+1}}$ depends only on n, X and ω. □

End of the proof of Theorem 3.2. Without loss of generality, we can assume that $u_1 \le u_2$. Denote $W_t = \{u_1 > a_1 - t\}$ for $0 < t \le 1$. We have

$$\int_{W_t} -\chi(u_1 - u_2)\omega^n \le \int_{W_t} -\chi(u_1 - a_2)\omega^n \le -b_t \chi(a_1 - a_2 - t), \tag{3.22}$$

where $b_t := \mathrm{vol}(W_t)$.

It follows from Lemma 3.8 that $W_t \ne \emptyset$. Moreover,

$$b_t := \int_{W_t} \omega^n \ge C_1 \left(\frac{t}{A}\right)^{2n}, \tag{3.23}$$

where $C_1 > 0$ is a constant depending only on n, X and ω. By [16, Theorem A] (see also [31, Theorem 3]), there exists a unique $\varphi \in \mathcal{E}(X, \theta, \phi)$ with $\sup_X(\varphi - \phi) = 0$ such that

$$\theta_\varphi^n = \frac{\varrho}{b_t} \mathbf{1}_{W_t} \omega^n.$$

It follows from Theorem 3.7 that

$$\phi - C_2 A \left(-\log t + \log A + 1\right) \le \varphi \le \phi, \tag{3.24}$$

for some constant $C_2 \geq 1$ depending only on n, X and ω. Thus, we have

$$E^0_{\tilde{\chi},\theta,\phi}(\varphi) \leq -\tilde{\chi}\big(C_2\, A\, (-\log t + \log A + 1)\,\big) \leq -C_3 \left(\log \frac{Ae}{t}\right)^M \tilde{\chi}(-A),$$

where $C_3 > 0$ depends only on n, X, ω and M.

Hence, it follows from Theorem 3.1 that

$$\int_X -\chi(u_1 - u_2)(\theta^n_\psi - \theta^n_\varphi) \leq C_4 \varrho \left(\log \frac{Ae}{t}\right)^{2M} (B - \tilde{\chi}(-A))^2 (1 - \tilde{\chi}(-1))^2 \lambda, \tag{3.25}$$

where $\lambda = Q^{\circ(n)}(I^0_\chi(u_1, u_2))$ and $C_4 > 0$ depends only on n, X, ω and M.

Combining (3.22) and (3.25), we get

$$\int_X -\chi(u_1 - u_2)\theta^n_\psi \leq -\varrho\chi(a_1 - a_2 - t) + C_4 \varrho \left(\log \frac{Ae}{t}\right)^{2M} (B - \tilde{\chi}(-A))^2 (1 - \tilde{\chi}(-1))^2 \lambda.$$

Letting $t \to \lambda^m$, we get

$$\begin{aligned}
\int_X -\chi(u_1 - u_2)\theta^n_\psi &\leq -\varrho\chi(a_1 - a_2 - \lambda^m) + C_4 \varrho \left(\log \frac{Ae}{\lambda^m}\right)^{2M} (B - \tilde{\chi}(-A))^2 (1 - \tilde{\chi}(-1))^2 \lambda \\
&\leq -\varrho\chi(a_1 - a_2 - \lambda^m) + C_5 \varrho \frac{A^{(1-\gamma)/m}}{\lambda^{1-\gamma}} (B - \tilde{\chi}(-A))^2 (1 - \tilde{\chi}(-1))^2 \lambda \\
&\leq -\varrho\chi(a_1 - a_2 - \lambda^m) + C_5 \varrho A^{(1-\gamma)/m} \lambda^{1-\gamma} (B - \tilde{\chi}(-A))^2 (1 - \tilde{\chi}(-1))^2 \lambda^\gamma,
\end{aligned}$$

where $C_5 > 0$ depends only on n, X, ω, M, m and γ.

The proof is completed. □

3.4 Applications

3.4.1 A quantitative version for the domination principle

Theorem 3.9. *Let $A \geq 1$ be a constant and let $\theta \leq A\omega$ be a closed smooth real $(1,1)$-form and ϕ be a model θ-psh function, and $\varrho := \mathrm{vol}(\theta_\phi) > 0$. Let $B \geq 1$ be a constant, $\tilde{\chi} \in \mathcal{W}^-$ and $u_1, u_2 \in \mathcal{E}(X, \theta, \phi)$ such that $\tilde{\chi}(-1) = -1$ and*

$$E^0_{\tilde{\chi},\theta,\phi}(u_1) + E^0_{\tilde{\chi},\theta,\phi}(u_2) \leq B.$$

Assume that there exists a constant $0 \leq c < 1$ and a Radon measure μ on X satisfying $\theta^n_{u_1} \leq c\theta^n_{u_2} + \varrho\mu$ on $\{u_1 < u_2\}$ and $c_\mu := \int_{\{u_1 < u_2\}} d\mu \leq 1$. Then there exists a constant $C > 0$ depending only on n, X and ω such that

$$cap_\omega\{u_1 < u_2 - \epsilon\} \leq \frac{C\, \mathrm{vol}(X)(A + B)^2}{\epsilon(1 - c)h^{\circ n}(1/c_\mu)},$$

for every $0 < \epsilon < 1$, where $h(s) = (-\tilde{\chi}(-s))^{1/2}$ for every $0 \leq s \leq \infty$.

In particular, if $c_\mu = 0$ then $cap_\omega\{u_1 < u_2 - \epsilon\} = 0$ for every $\epsilon > 0$, and then $u_1 \geq u_2$ on whole X.

The standard domination principle corresponds to the case where $c = 0$ and $\mu := 0$. A non-quantitative version of this domination principle in the non-Kähler setting was obtained in [48]. In order to prove Theorem 3.9, we need the following result which is an immediate consequence of the Chern-Levine-Nirenberg inequality:

Proposition 3.10. *Let $\theta \leq A\omega$ be a closed smooth real $(1,1)$-form (where $A \geq 1$ is a constant) and ϕ be a model θ-psh function with $\varrho := \int_X \theta_\phi^n > 0$. Let $0 \leq w \leq 1$ is an ω-psh function and ψ is the unique solution to the problem*

$$\begin{cases} u \in \mathcal{E}(X, \theta, \phi), \\ \theta_u^n = \dfrac{\varrho}{\mathrm{vol}(X)} (dd^c w + \omega)^n, \\ \sup_X u = 0. \end{cases} \tag{3.26}$$

Then there exists a constant $C > 0$ depending only on X and ω such that

$$\int_X |\psi| \theta_\psi^n \leq C\, A\varrho.$$

Proof of Theorem 3.9. Let w be an arbitrary ω-psh function satisfying $0 \leq w \leq 1$ and ψ is the unique solution to (3.26). Denote $v = \max\{u_1, u_2\}$ and $\chi(t) = \max\{t, -1\} \geq \tilde{\chi}(t)$. By Theorem 3.1 and Proposition 3.10, there exists a constant $C_1 > 0$ depending only on n, X an ω such that

$$I_1 := \int_X -\chi(u_1 - v)(\theta_\psi^n - \theta_{u_1}^n) \leq C_1 \varrho (A+B)^2 Q^{\circ(n)}(I_\chi^0(u_1, v)), \tag{3.27}$$

and

$$I_2 := \int_X -\chi(u_1 - v)(\theta_{u_2}^n - \theta_{u_1}^n) \leq C_1 \varrho (A+B)^2 Q^{\circ(n)}(I_\chi^0(u_1, v)). \tag{3.28}$$

Moreover, by the fact $\theta_v^n = \theta_{u_2}^n$ on $\{u_1 < u_2\}$ and by the assumption $\theta_{u_1}^n \leq c\theta_{u_2}^n + \varrho\mu$ on $\{u_1 < u_2\}$, we have

$$I_\chi^0(u_1, v) = \varrho^{-1} \int_{\{u_1 < u_2\}} -\chi(u_1 - v)(\theta_{u_1}^n - \theta_{u_2}^n) \leq \varrho^{-1} \int_{\{u_1 < u_2\}} -\chi(u_1 - v)(\theta_{u_1}^n - c\theta_{u_2}^n) \leq c_\mu. \tag{3.29}$$

Combining (3.27), (3.28) and (3.29), we get

$$\begin{aligned}
(1-c) \int_X -\chi(u_1 - v)\theta_\psi^n &= \int_X -\chi(u_1 - v)(\theta_{u_1}^n - c\theta_{u_2}^n) + (1-c)I_1 + cI_2 \\
&\leq \int_X -\chi(u_1 - v)(\theta_{u_1}^n - c\theta_{u_2}^n) + C_1 \varrho (A+B)^2 (1 - \tilde{\chi}(-1))^2 Q^{\circ n}(c_\mu) \\
&\leq \varrho c_\mu + C_1 \varrho (A+B)^2 Q^{\circ n}(c_\mu) \\
&\leq C\varrho (A+B)^2 Q^{\circ n}(c_\mu),
\end{aligned}$$

where $C = C_1 + 1$. Hence

$$\int_{\{u_1 < u_2 - \epsilon\}} \theta_w^n = \frac{\mathrm{vol}(X)}{\varrho} \int_{\{u_1 < u_2 - \epsilon\}} \theta_\psi^n \leq \frac{C\, \mathrm{vol}(X)(A+B)^2 Q^{\circ n}(c_\mu)}{(1-c)\epsilon},$$

for every $0 < \epsilon < 1$. Since w is arbitrary, it follows that

$$\mathrm{cap}_\omega\{u_1 < u_2 - \epsilon\} \leq \frac{C\,\mathrm{vol}(X)(A+B)^2 Q^{\circ n}(c_\mu)}{(1-c)\epsilon}. \tag{3.30}$$

Moreover, by the definition of χ and the formula of Q, we have

$$Q(s) = \frac{1}{(-\tilde{\chi}(-1/s))^{1/2}} = \frac{1}{h(1/s)},$$

for every $0 < s \leq 1$, and $Q(0) = 0$. Then

$$Q^{\circ n}(s) = \frac{1}{h^{\circ n}(1/s)}, \tag{3.31}$$

for every $0 \leq s \leq 1$. The proof is completed. □

3.4.2 A quantitative version of Dinew's uniqueness theorem

Theorem 3.11. *Let $A \geq 1$ be a constant. Let $\theta \leq A\omega$ be a closed smooth real $(1,1)$-form and let ϕ be a model θ-psh function such that $\varrho := \mathrm{vol}(\theta_\phi) > 0$. Let $B \geq 1$, $\tilde{\chi} \in \mathcal{W}^-$ and $u_1, u_2 \in \mathcal{E}(X, \theta, \phi)$ such that $\tilde{\chi}(-1) = -1$ and*

$$E^0_{\tilde{\chi},\theta,\phi}(u_1) + E^0_{\tilde{\chi},\theta,\phi}(u_2) \leq B.$$

Then, for every $0 < \gamma < 1$, there exists $C > 0$ depending only on n, X, ω and γ such that

$$d_{cap}(u_1, u_2)^2 \leq C\left(A + |a_1 - a_2|\right)\left(|a_1 - a_2| + A(A+B)^2\tau^\gamma\right),$$

where $a_j := \sup_X u_j$, $\tau = \dfrac{1}{h^{\circ n}(\varrho/\|\theta^n_{u_1} - \theta^n_{u_2}\|)}$ and $h(s) = (-\tilde{\chi}(-s))^{1/2}$.

Note that if $\chi(t) = \max\{t, -1\}$ then $I^0_\chi(u_1, u_2) \leq \varrho^{-1}\|\theta^n_{u_1} - \theta^n_{u_2}\|$. Therefore, Theorem 3.11 is a consequence of the following:

Theorem 3.12. *Let $\theta \leq A\omega$ be a closed smooth real $(1,1)$-form ($A \geq 1$) and let ϕ be a model θ-psh function such that $\mathrm{vol}(\theta_\phi) > 0$. Let $B \geq 1$, $\tilde{\chi} \in \widetilde{\mathcal{W}}^-$ and $u_1, u_2 \in \mathcal{E}(X, \theta, \phi)$ such that $\tilde{\chi}(-1) = -1$ and*

$$E^0_{\tilde{\chi},\theta,\phi}(u_1) + E^0_{\tilde{\chi},\theta,\phi}(u_2) \leq B.$$

Denote $\chi(t) = \max\{t, -1\}$. Then, for every $0 < \gamma < 1$, there exists $C > 0$ depending only on n, X, ω and γ such that

$$d_{cap}(u_1, u_2)^2 \leq C\left(A + |a_1 - a_2|\right)\left(|a_1 - a_2| + A(A+B)^2\lambda^\gamma\right), \tag{3.32}$$

where $a_j := \sup_X u_j$, $\lambda = \dfrac{1}{h^{\circ n}(1/I^0_\chi(u_1, u_2))}$ and $h(s) = (-\tilde{\chi}(-s))^{1/2}$.

Proof. Suppose that w is an arbitrary ω-psh function satisfying $0 \leq w \leq 1$ and ψ is the unique solution to the problem

$$\begin{cases} u \in \mathcal{E}(X, \theta, \phi), \\ \theta_u^n = \dfrac{\varrho}{\mathrm{vol}(X)}(dd^c w + \omega)^n, \\ \sup_X u = 0. \end{cases} \tag{3.33}$$

Recall that $\lambda = Q^{\circ n}_{\chi,\tilde{\chi}}(I^0_\chi(u_1, u_2))$, and one has $-\tilde{\chi}(-A) \leq A$ because $\tilde{\chi}(-1) = -1$. It follows from Theorem 3.2 and Proposition 3.10 that, for every $0 < \gamma < 1$, there exists $C_1 > 0$ depending only on n, X, ω and γ such that

$$I := \int_X -\chi\big(-|u_1 - u_2|\big)\theta_\psi^n \leq -\varrho\chi\left(-|a_1 - a_2| - \lambda\right) + C_1\varrho A(A+B)^2\lambda^\gamma. \tag{3.34}$$

Moreover

$$\begin{aligned} \frac{\varrho}{\mathrm{vol}(X)}\int_X |u_1 - u_2|^{1/2}(\omega + dd^c w)^n &= \int_X |u_1 - u_2|^{1/2}\theta_\psi^n \\ &= \int_{\{|u_1-u_2|\leq 1\}} |u_1 - u_2|^{1/2}\theta_\psi^n + \int_{\{|u_1-u_2|>1\}} |u_1 - u_2|^{1/2}\theta_\psi^n \end{aligned}$$

which is less than or equal to

$$\leq I^{1/2}\left(\left(\int_{\{|u_1-u_2|\leq 1\}}\theta_\psi^n\right)^{1/2} + \left(\int_{\{|u_1-u_2|>1\}}|u_1 - u_2|\theta_\psi^n\right)^{1/2}\right),$$

where the last estimate holds due to the Cauchy-Schwarz inequality. Moreover, it follows from Chern-Levine-Nirenberg inequality ([42]) that

$$\begin{aligned} \int_X |u_1 - a_1 - u_2 + a_2|\theta_\psi^n &= \frac{\varrho}{\mathrm{vol}(X)}\int_X |u_1 - a_1 - u_2 + a_2|(dd^c w + \omega)^n \\ &\leq C_2\varrho(\|u_1 - a_1\|_{L^1(X)} + \|u_2 - a_2\|_{L^1(X)}) \\ &\leq \varrho C_3 A, \end{aligned}$$

where $C_2, C_3 > 0$ depend only on X and ω. Here, the last estimate holds due to the compactness of $\{u \in \mathrm{PSH}(X, \omega) : \sup_X u = 0\}$ in $L^1(X)$.

Hence, we have

$$\frac{\varrho}{\mathrm{vol}(X)}\int_X |u_1 - u_2|^{1/2}(\omega + dd^c w)^n \leq C_4 I^{1/2}\varrho^{1/2}(A + |a_1 - a_2|)^{1/2}, \tag{3.35}$$

where $C_4 > 0$ depends only on X and ω.

Combining (3.34) and (3.35), we get

$$\begin{aligned} \left(\int_X |u_1 - u_2|^{1/2}(\omega + dd^c w)^n\right)^2 &\leq C_5(A + |a_1 - a_2|)\left(-\chi\left(-|a_1 - a_2| - \lambda\right) + A(A+B)^2\lambda^\gamma\right) \\ &\leq C_5(A + |a_1 - a_2|)\left(|a_1 - a_2| + \lambda + A(A+B)^2\lambda^\gamma\right) \\ &\leq C_6(A + |a_1 - a_2|)\left(|a_1 - a_2| + A(A+B)^2\lambda^\gamma\right), \end{aligned}$$

where $C_5, C_6 > 0$ depend only on n, X, ω and γ. Since w is arbitrary, we obtain desired inequality. The proof is completed. □

Remark 3.13. *If $B \geq A$ then the inequality* (3.32) *is equivalent to*

$$d_{cap}(u_1,u_2)^2 \leq \widetilde{C}\,(A+|a_1-a_2|)\left(|a_1-a_2| + A\,B^2\lambda^\gamma\right),$$

where $\tilde{C} > 0$ depends only on n, X, ω and γ.

3.4.3 Relation to Darvas's metrics on the space of potentials of finite energy

Let $\chi \in \mathcal{W}^- \cup \mathcal{W}^+_M$. Let θ be a closed smooth real $(1,1)$-form in a big cohomology class. When θ is Kähler, it was proved in [10, 11, 13] that there is a natural metric d_χ on $\mathcal{E}_\chi(X,\theta)$ which makes the last space to be a complete metric space. When $\chi(t) = t$, such metrics have a long history and play an important role in the study of complex Monge-Ampère equations. We refer to these last references and [3, 4] for more details. We now draw the connection between $I_\chi(u,v)$ and the metric on $\mathcal{E}_\chi(X,\theta)$. Let

$$\tilde{I}_\chi(u,v) = \int_{\{u<v\}} -\chi(u-v)(\theta_v^n + \theta_u^n) + \int_{\{u>v\}} -\chi(v-u)(\theta_u^n + \theta_v^n) \geq I_\chi(u,v).$$

By [10, 11, 13], there exists a constant $C > 0$ such that

$$C^{-1}\tilde{I}_\chi(u,v) \leq d_\chi(u,v) \leq C\tilde{I}_\chi(u,v)$$

for every $u, v \in \mathcal{E}_\chi(X,\theta)$ and θ is Kähler. It was proved in [36] (and also [10, 14, 20, 60, 67]) that $\tilde{I}_\chi(u,v)$ satisfies a quasi-triangle inequality, and the convergence in $\tilde{I}_\chi(u,v)$ implies the convergence in capacity by using the plurisubharmonic envelope. Such a method is not quantitative. We present below quantitative version of this fact by using our approach.

Theorem 3.14. *Let $\theta \leq A\omega$ be a closed smooth real $(1,1)$-form ($A \geq 1$ is a constant) and ϕ be a model θ-psh function with $\varrho := \mathrm{vol}(\theta_\phi) > 0$. Let $B \geq 1$, $\tilde{\chi} \in \mathcal{W}^-$ and $u_1, u_2 \in \mathcal{E}(X,\theta,\phi)$ such that $|\sup_X u_1 - \sup_X u_2| \leq A$, $\tilde{\chi}(-1) = -1$ and*

$$E^0_{\tilde{\chi},\theta,\phi}(u_1) + E^0_{\tilde{\chi},\theta,\phi}(u_2) \leq B.$$

Then there exist $C > 0$ depending only on n, X and ω such that

$$d_{cap}(u_1,u_2)^2 \leq \frac{C\,A\,(A+B)^2}{h^{\circ n}(\varrho/\tilde{I}_{\tilde{\chi}}(u_1,u_2))},$$

where $h(s) = (-\tilde{\chi}(-s))^{1/2}$ for every $0 \leq s \leq \infty$.

One sees from the above estimate that if $\tilde{I}_{\tilde{\chi}}(u_1,u_2)$ is small, then so is $d_{\text{cap}}(u_1,u_2)$ (uniformly in $u_1, u_2 \in \mathcal{E}(X,\theta,\phi)$ of $\tilde{\chi}$-energy bounded by a fixed constant).

Proof. Let $\chi(t) = \max\{t, -1\}$. Suppose that w is an arbitrary ω-psh function satisfying $0 \leq w \leq 1$. By the proof of Theorem 3.12 (see (3.35)), there exists $C_1 > 0$ depending only on X and ω such that

$$\left(\int_X |u_1 - u_2|^{1/2}(\omega + dd^c w)^n\right)^2 \leq C_1 A \varrho^{-1} \int_X -\chi\big(-|u_1 - u_2|\big)\theta_\psi^n, \tag{3.36}$$

where ψ is defined by (3.33). Moreover, it follows from Theorem 3.1 and Proposition 3.10 that

$$\int_X -\chi\big(-|u_1 - u_2|\big)\theta_\psi^n \leq \tilde{I}_\chi(u_1, u_2) + C_2\varrho(A + B)^2 Q^{\circ(n)}_{\chi,\tilde{\chi}}(I^0_\chi(u_1, u_2)),$$

where $C_2 > 0$ depends only on n. Therefore, by the facts $Q^{\circ(n)}(s) = \dfrac{1}{h^{\circ(n)}(1/s)}$ and $I_\chi(u_1, u_2) \leq \tilde{I}_\chi(u_1, u_2) \leq \tilde{I}_{\tilde{\chi}}(u_1, u_2))$, we obtain

$$\int_X -\chi\big(-|u_1 - u_2|\big)\theta_\psi^n \leq \frac{C_3\varrho(A + B)^2}{h^{\circ(n)}(\varrho/\tilde{I}_{\tilde{\chi}}(u_1, u_2))}, \tag{3.37}$$

where $C_3 > 0$ depends only on n, X and ω. Combining (3.36) and (3.37), we get

$$\left(\int_X |u_1 - u_2|^{1/2}(\omega + dd^c w)^n\right)^2 \leq \frac{C\, A\, (A + B)^2}{h^{\circ(n)}(\varrho/\tilde{I}_{\tilde{\chi}}(u_1, u_2))},$$

where $C > 0$ depends only on n, X and ω. Since w is arbitrary, we get the desired inequality. The proof is completed. □

When $\tilde{\chi} \in \mathcal{W}^+_M$, our estimate is more explicit.

Theorem 3.15. *Let $\theta \leq A\omega$ be a closed smooth real $(1,1)$-form ($A \geq 1$) and ϕ be a model θ-psh function such that $\varrho := \mathrm{vol}(\theta_\phi) > 0$. Let $B \geq 1$, $\tilde{\chi} \in \mathcal{W}^+_M$ ($M \geq 1$) and $u_1, u_2 \in \mathcal{E}(X, \theta, \phi)$ such that $\tilde{\chi}(-1) = -1$ and*

$$E^0_{\tilde{\chi},\theta,\phi}(u_1) + E^0_{\tilde{\chi},\theta,\phi}(u_2) \leq B.$$

Then there exists $C > 0$ depending only on n and M such that

$$\int_X -\tilde{\chi}(-|u_1 - u_2|)\theta_\psi^n \leq C\varrho B^2 \left(\tilde{I}_{\tilde{\chi}}(u_1, u_2)/\varrho\right)^{2^{-n}},$$

for every $\psi \in \mathrm{PSH}(X, \theta)$ with $\phi - 1 \leq \psi \leq \phi$. Moreover, if $\sup_X u_1 = \sup_X u_2$ then there exists $C' > 0$ depending on n, X, ω, A and M such that

$$\tilde{I}_{\tilde{\chi}}(u_1, u_2) \leq C'\varrho A^{1/2} B^2 \left(I^0_{\tilde{\chi}}(u_1, u_2)\right)^{2^{-n-1}}.$$

Proof. The case $I^0_{\tilde{\chi}}(u_1, u_2) \geq 1$ is trivial. It remains to consider the case $I^0_{\tilde{\chi}}(u_1, u_2) < 1$.

Denote $v = \max\{u_1, u_2\}$. By Lemma 2.7, we have $v \in \mathcal{E}(X, \theta, \phi)$ and $E^0_{\tilde{\chi},\theta,\phi}(v) \leq C_1 B$, where $C_1 > 0$ depends only on n and M. Taking $\chi = \tilde{\chi}$ and using Theorem 3.1, we get

$$\int_X -\tilde{\chi}(u_j - v)\theta_\psi^n \leq \int_X -\tilde{\chi}(u_j - v)\theta_{u_j}^n + C_2\varrho B^2 \big(I^0_{\tilde{\chi}}(u_j, v)\big)^{2^{-n}}, \tag{3.38}$$

for $j=1,2$, where $C_2>0$ depends on n and M. Note that

$$\int_X -\tilde{\chi}(u_1-v)\theta_{u_1}^n + \int_X -\tilde{\chi}(u_2-v)\theta_{u_2}^n \le \int_X -\tilde{\chi}(-|u_1-u_2|)(\theta_{u_1}^n+\theta_{u_2}^n) = \tilde{I}_{\tilde{\chi}}(u_1,u_2),$$

and

$$I_{\tilde{\chi}}^0(u_1,v)+I_{\tilde{\chi}}^0(u_2,v)=I_{\tilde{\chi}}^0(u_1,u_2)\le \varrho^{-1}\tilde{I}_{\tilde{\chi}}(u_1,u_2).$$

Hence, by (3.38), we get

$$\begin{aligned}\int_X -\tilde{\chi}(-|u_1-u_2|)\theta_\psi^n &= \int_X -\tilde{\chi}(u_1-v)\theta_\psi^n + \int_X -\tilde{\chi}(u_2-v)\theta_\psi^n \\ &\le \int_X -\tilde{\chi}(u_1-v)\theta_{u_1}^n + \int_X -\tilde{\chi}(u_2-v)\theta_{u_2}^n \\ &+ C_2\varrho B^2\left(\big(I_{\tilde{\chi}}^0(u_1,v)\big)^{2^{-n}} + \big(I_{\tilde{\chi}}^0(u_2,v)\big)^{2^{-n}}\right) \\ &\le \tilde{I}_{\tilde{\chi}}(u_1,u_2) + 2C_2\varrho B^2\big(\tilde{I}_{\tilde{\chi}}(u_1,u_2)/\varrho\big)^{2^{-n}} \\ &\le C_3\varrho B^2\big(\tilde{I}_{\tilde{\chi}}(u_1,u_2)/\varrho\big)^{2^{-n}},\end{aligned}$$

where $C_3>0$ depends on n and M. Here, the last estimate holds due to the fact $\tilde{I}_{\tilde{\chi}}(u_1,u_2)\le \varrho B$.

Now, we consider the case $\sup_X u_1=\sup_X u_2$. By Theorem 3.2 (choose $m=1$ and $\gamma=1/2$), there exists $C_4>0$ depending only on n,X,ω and M such that

$$\begin{aligned}\tilde{I}_{\tilde{\chi}}(u_1,u_2) &\le \int_X -\tilde{\chi}(-|u_1-u_2|)(\theta_{u_1}^n+\theta_{u_2}^n) \\ &\le -2\varrho\,\tilde{\chi}\left(-\big(I_{\tilde{\chi}}^0(u_1,u_2)\big)^{2^{-n}}\right) + C_4\varrho A^{1/2}B^2\big(I_{\tilde{\chi}}^0(u_1,u_2)\big)^{2^{-n-1}}.\end{aligned} \tag{3.39}$$

Moreover, since $\tilde{\chi}$ is concave, we have

$$\frac{\tilde{\chi}(t)}{t}\le\frac{\tilde{\chi}(-1)}{-1}=1,$$

for every $-1<t<0$. Hence, by (3.39), we have

$$\begin{aligned}\tilde{I}_{\tilde{\chi}}(u_1,u_2) &\le 2\varrho\,\big(I_{\tilde{\chi}}^0(u_1,u_2)\big)^{2^{-n}} + C_4\varrho A^{1/2}B^2\big(I_{\tilde{\chi}}^0(u_1,u_2)\big)^{2^{-n-1}} \\ &\le (2+C_4)\varrho A^{1/2}B^2\big(I_{\tilde{\chi}}^0(u_1,u_2)\big)^{2^{-n-1}}.\end{aligned}$$

The proof is completed. □

3.4.4 Comparison of capacities

For every Borel subset E in X and for every $\varphi\in\mathrm{PSH}(X,\theta)$, one denotes

$$\mathrm{cap}_{\theta,\varphi}(E) := \sup\Big\{\int_E \theta_\psi^n : \psi\in\mathrm{PSH}(X,\theta),\quad \varphi-1\le\psi\le\varphi\Big\}.$$

In [47], Lu showed that if φ_j ($j = 1, 2$) is a θ_j-psh function with $\int_X (\theta_j + dd^c\varphi_j)^n > 0$ then there exists a continuous function $f : \mathbb{R}_{\leq 0} \to \mathbb{R}_{\geq 0}$ with $f(0) = 0$ such that $\text{cap}_{\theta_1,\varphi_1}(E) \leq f(\text{cap}_{\theta_2,\varphi_2}(E))$ for every Borel set $E \subset X$. By using Theorem 3.2, we will reprove Lu's result for the case where φ_j is a model θ_j-psh function. Moreover, we also provide a specific form of f.

First, we need the following lemma:

Lemma 3.16. *Let $A, B > 0$ be constants. Let θ be a closed smooth real $(1, 1)$-form representing a big cohomology class such that $\theta \leq A\omega$. Assume that u, v are θ-psh functions satisfying $v \leq u \leq v + B$. Then,*

$$\int_X (-\psi)\theta_u^n \leq \int_X (-\psi)\theta_v^n + nA^n B \int_X \omega^n,$$

for every negative $A\omega$-psh function ψ.

Proof. Using approximations, we can assume that ψ is smooth. Denote

$$T = \sum_{l=0}^{n-1} \theta_u^l \wedge \theta_v^{n-l-1}.$$

We have $\theta_u^n - \theta_v^n = dd^c(u - v) \wedge T$. Moreover, using integration by parts (Theorem 2.2), we get

$$\int_X (-\psi)dd^c(u - v) \wedge T = \int_X (u - v)dd^c(-\psi) \wedge T \leq A \int_X (u - v)\omega \wedge T \leq nA^n B \int_X \omega^n.$$

Hence

$$\int_X (-\psi)\theta_u^n \leq \int_X (-\psi)\theta_v^n + nA^n B \int_X \omega^n.$$

□

Theorem 3.17. *(Comparison of capacities) Assume that $\theta_1, \theta_2 \leq A\omega$ are closed smooth real $(1, 1)$-forms representing big cohomology classes and, for $j = 1, 2$, ϕ_j is a model θ_j-psh function satisfying $\int_X (\theta_j + dd^c\phi_j)^n = \varrho_j > 0$. Then, for every $0 < \gamma < 1$, there exists $C > 0$ depending only on n, X, ω, A and γ such that*

$$\frac{cap_{\theta_1,\phi_1}(E)}{\varrho_1} \leq C\left(\frac{cap_{\theta_2,\phi_2}(E)}{\varrho_2}\right)^{2^{-n}\gamma},$$

for every Borel set $E \subset X$.

Proof. By the inner regularity of capacities (see [15, Lemma 4.2]), we only need consider the case where E is compact. Since the case $\text{cap}_{\theta_2,\phi_2}(E) = \varrho_2$ is trivial, we can also assume that $\text{cap}_{\theta_2,\phi_2}(E) < \varrho_2$. In particular, by [16, Proposition 3.7] and [17, Lemma 2.7], we have

$$\sup_X h^*_{E,\theta_2,\phi_2} = \sup_X (h^*_{E,\theta_2,\phi_2} - \phi_2) = 0,$$

where

$$h_{E,\theta_2,\phi_2} = \sup\{w \in \mathrm{PSH}(X,\theta_2) : w|_E \le \phi_2 - 1, w \le \phi_2\}.$$

Set $\chi(t) = \tilde{\chi}(t) = t$. We will use Theorem 3.2 for $u_1 = (h_{E,\theta_2,\phi_2})^*$ and $u_2 = \phi_2$. It is clear that $E^0_{\tilde{\chi},\theta_2,\phi_2}(u_2) = 0$ and $u_1 = u_2 - 1$ on $E \setminus N$, where N is a pluripolar set. Moreover, it follows from [16, Proposition 3.7] that

$$I^0_{\chi}(u_1,u_2) \le E^0_{\tilde{\chi},\theta_2,\phi_2}(u_1) = \varrho_2^{-1}\mathrm{cap}_{\theta_2,\phi_2}(E) \le 1.$$

By Theorem 3.2, for every $0 < \gamma < 1$ and $B \ge 1$, there exists $C > 0$ depending only on X, ω, n, A and γ such that

$$\int_E \theta_\psi^n \le \int_X \chi(-|u_1 - u_2|)\theta_\psi^n \le C\varrho_2 A(A+B)^2 \left(\mathrm{cap}_{\theta_2,\phi_2}(E)/\varrho_2\right)^{2^{-n}\gamma}, \tag{3.40}$$

for every compact set E and for each $\psi \in \mathcal{E}(X,\theta_2,\phi_2)$ with $E^0_{\tilde{\chi},\theta_2,\phi_2}(\psi) \le B$. Let $\varphi \in \mathcal{E}(X,\theta_1,\phi_1)$ such that $\phi_1 - 1 \le \varphi \le \phi_1$ and $\int_E(\theta_1 + dd^c\varphi)^n \ge \dfrac{1}{2}\mathrm{cap}_{\theta_1,\phi_1}(E)$. By [16], there exists a unique function $\psi_0 \in \mathcal{E}(X,\theta_2,\phi_2)$ such that $\sup_X \psi_0 = 0$ and $(\theta_2 + dd^c\psi_0)^n = \dfrac{\varrho_2}{\varrho_1}(\theta_1 + dd^c\varphi)^n$. When $\psi = \psi_0$, we have

$$\int_E \theta_\psi^n \ge \frac{\varrho_2}{2\varrho_1}\mathrm{cap}_{\theta_1,\phi_1}(E). \tag{3.41}$$

Moreover, by using Lemma 3.16 for φ, ϕ_1 and using the fact that $(\theta_2 + dd^c\phi_2)^n \le \mathbf{1}_{\{\phi_2=0\}}\theta_2^n$ (see [15, Theorem 3.8]), we have

$$\varrho_1 E^0_{\tilde{\chi},\theta_2,\phi_2}(\psi_0) = \int_X (\phi_2 - \psi_0)(\theta_1 + dd^c\varphi)^n \le \int_X (-\psi_0)(\theta_1 + dd^c\phi_1)^n + nA^n \int_X \omega^n \le B, \tag{3.42}$$

where $B \ge 1$ depends only on A, X, ω, n. Combining (3.40), (3.41) and (3.42), we get

$$\begin{aligned}\mathrm{cap}_{\theta_1,\phi_1}(E) \le \frac{2\varrho_1}{\varrho_2}\int_E \theta_{\psi_0}^n &\le \frac{2\varrho_1}{\varrho_2}\int_X \chi(-|u_1 - u_2|)\theta_{\psi_0}^n \\ &\le 2C\varrho_1 A(A+B)^2 \left(\mathrm{cap}_{\theta_2,\phi_2}(E)/\varrho_2\right)^{2^{-n}\gamma}.\end{aligned}$$

The proof is completed. □

4 Stability estimates for varied singularity type and cohomology class

4.1 Pseudo-metric on the space of singularity types

We first recall some facts about the pseudo-metric on the space of singularity types. Let α be a big cohomology class and θ a smooth closed $(1,1)$-form in α. Let $\mathcal{S}(\theta)$ be the space of singularity types of θ-psh functions and

$$\mathcal{S}_\delta(\theta) := \{[u] \in \mathcal{S}(\theta) : \int_X \theta_u^n \ge \delta\}.$$

The pseudo-distance $d_{\mathcal{S}}$ on $\mathcal{S}$ was introduced in [17], and it satisfies

$$d_{\mathcal{S}(\theta)}([u],[v]) \le \sum_{j=0}^{n}\left(2\int_X \theta^j_{V_\theta}\wedge\theta^{n-j}_{\max\{u,v\}} - \int_X \theta^j_{V_\theta}\wedge\theta^{n-j}_{u} - \int_X \theta^j_{V_\theta}\wedge\theta^{n-j}_{v}\right) \le C d_{\mathcal{S}(\theta)}([u],[v]), \tag{4.1}$$

where $C > 1$ depends only on n. Here V_θ is the upper envelope of all non-positive θ-psh functions:

$$V_\theta := \sup\{\varphi \in \mathrm{PSH}(X,\theta) : \varphi \le 0 \quad \text{on} \quad X\}.$$

For all θ-psh functions u, v, we put

$$d_\theta(u,v) := 2\int_X \theta^n_{\max\{u,v\}} - \int_X \theta^n_u - \int_X \theta^n_v.$$

In particular, if $u \le v$ then $d_\theta(u,v) = \int_X \theta^n_v - \int_X \theta^n_u$. By (4.1), we have

$$d_\theta(u,v) \le C d_{\mathcal{S}(\theta)}([u],[v]),$$

where $C = C(n) > 0$. Moreover, if $\theta = A\omega$ for some $A > 0$ then, we have

$$d_{\mathcal{S}(A\omega)}([u],[v]) \le A^n d_{(A+1)\omega}(u,v),$$

for every $u, v \in \mathrm{PSH}(X, A\omega)$. In the sequel, we provide more properties of d_θ.

Lemma 4.1. *Let u_1, u_2 be θ-psh functions. Let θ' be a smooth real closed $(1,1)$-form such that $\theta' \ge \theta$. Then*

$$d_\theta(u_1,u_2) \le d_{\theta'}(u_1,u_2).$$

Proof. By the fact $d_\eta(u_1,u_2) = d_\eta(u_1,\max\{u_1,u_2\}) + d_\eta(u_2,\max\{u_1,u_2\})$ for $\eta = \theta, \theta'$, the problem is reduced to the case $u_1 \le u_2$. Then we have

$$d_\eta(u_1,u_2) = \int_X (\eta + dd^c u_2)^n - \int_X (\eta + dd^c u_1)^n,$$

for $\eta = \theta, \theta'$. Moreover,

$$(\theta' + dd^c u_j)^n - (\theta + dd^c u_j)^n = (\theta' - \theta) \wedge \sum_{l=0}^{n-1} (\theta' + dd^c u_j)^l \wedge (\theta + dd^c u_j)^{n-l-1},$$

for $j = 1, 2$. Hence

$$d_{\theta'}(u_1,u_2) - d_\theta(u_1,u_2) = \int_X (\theta' - \theta)\wedge T_2 - \int_X (\theta'-\theta)\wedge T_1,$$

where

$$T_j = \sum_{l=0}^{n-1} (\theta' + dd^c u_j)^l \wedge (\theta + dd^c u_j)^{n-l-1}.$$

Thus, by the monotonicity of non-pluripolar products [15, Theorem 1.1], we obtain

$$d_{\theta'}(u_1,u_2) - d_\theta(u_1,u_2) \ge 0.$$

The proof is completed. □

Lemma 4.2. *Let $\delta > 0, A > 0$ be constants. Let u, v be θ-psh functions such that $u \leq v$ and $\int_X \theta_u^n \geq \delta$. Let ψ be an η-psh function, where η is a closed smooth $(1,1)$-form. Assume that $\theta \leq A\omega, \eta \leq A\omega$. Then there exists a constant C depending only on n, ω such that*

$$\left| \int_X \theta_u^m \wedge \eta_\psi^{n-m} - \int_X \theta_v^m \wedge \eta_\psi^{n-m} \right| \leq CA^n \left(\frac{d_\theta(u,v)}{\delta} \right)^{1/n}.$$

Proof. This is essentially the proof of [17, Proposition 4.8]. Note that by monotonicity we have

$$d_\theta(u,v) = \int_X \theta_v^n - \int_X \theta_u^n, \quad \int_X \theta_v^m \wedge \eta_\psi^{n-m} \geq \int_X \theta_u^m \wedge \eta_\psi^{n-m}.$$

Without loss of generality, we can assume $d_\theta(u,v) \leq \delta/2^{n+2}$. If $d_\theta(u,v) = 0$, then using $u \leq v$ and [15], we get $P_\theta[u] = P_\theta[v]$. In this case the left-hand side of the desired inequality is also zero. Hence from now on we assume $d_\theta(u,v) > 0$.

Let $b > 2$ be a constant such that $\delta/d_\theta(u,v) < 2b^n < 2\delta/d_\theta(u,v)$. We have

$$b^n \int_X \theta_u^n \geq (b^n - 1) \int_X \theta_v^n.$$

By this and [17, Lemma 4.3], we obtain $w_b := P_\theta(bu + (1-b)v) \in \mathrm{PSH}(X, \theta)$. Observe

$$b^{-1}w_b + (1-b^{-1})v \leq b^{-1}(bu + (1-b)v) + (1-b^{-1})v = u.$$

Combining this with monotonicity of non-pluripolar products gives

$$\int_X \theta_u^m \wedge \eta_\psi^{n-m} \geq \int_X \theta_{b^{-1}w_b + (1-b^{-1})v}^m \wedge \eta_\psi^{n-m} \geq (1-b^{-1})^m \int_X \theta_v^m \wedge \eta_\psi^{n-m}.$$

It follows that

$$\int_X \theta_u^m \wedge \eta_\psi^{n-m} - \int_X \theta_v^m \wedge \eta_\psi^{n-m} \geq -mb^{-1} \int_X \theta_v^m \wedge \eta_\psi^{n-m} \geq -nb^{-1}A^n \int_X \omega^n$$

by monotonicity. Hence

$$\left| \int_X \theta_u^m \wedge \eta_\psi^{n-m} - \int_X \theta_v^m \wedge \eta_\psi^{n-m} \right| \leq Cb^{-1} \leq 2^{1/n} C \left(\frac{d_\theta(u,v)}{\delta} \right)^{1/n},$$

where $C := nA^n \int_X \omega^n$. This finishes the proof. ☐

By Lemma 4.2, we have

Proposition 4.3. *Let α, θ be as above. Then there exists a constant $C > 0$ such that*

$$C^{-1}\delta\, d_{\mathcal{S}(\theta)}([u],[v])^n \leq d_\theta(u,v) \leq C d_{\mathcal{S}(\theta)}([u],[v])$$

for every $[u],[v] \in \mathcal{S}_\delta(\alpha)$. Moreover if θ' is a smooth real closed $(1,1)$-form and A is a positive constant such that

$$\theta' \leq A\omega, \quad \theta \leq A\omega,$$

for some constant $A > 0$, then there exists a constant $C_1 > 0$ depending only on A, ω such that

$$\delta\big(d_{\theta'}(u,v)\big)^n \leq C_1 d_\theta(u,v),$$

for every $u, v \in \mathcal{S}_\delta(\alpha)$.

Proof. The first desired assertion is clear from Lemma 4.2. Also by the same lemma, one gets

$$\delta\big(d_{A\omega}(u,v)\big)^n \leq C_1 d_\theta(u,v),$$

for every $u,v \in \mathcal{S}_\delta(\alpha)$, and some constant C_1 independent of u,v,δ. This coupled with Lemma 4.1 gives the last desired inequality. The proof is complete. □

If θ' is another closed smooth form in α, then $\mathcal{S}_\delta(\theta)$ and $\mathcal{S}_\delta(\theta')$ are isometric under the map $u \mapsto u+\varphi$, where φ is a smooth function such that $dd^c\varphi = \theta' - \theta$. Hence in general in order to study singularity types in α, it is enough to fix a smooth form in α.

4.2 The case of fixed cohomology

In this subsection, we will study the stability question when solutions are in the same cohomology class. Let θ, η be closed smooth real $(1,1)$-forms representing big cohomology classes. For every $\chi \in \widetilde{\mathcal{W}}^-$ and $u \in \mathrm{PSH}(X,\theta)$, we denote

$$\tilde{E}_{\chi,\eta,\theta}(u) = \sup\left\{\int_X -\chi(\psi)\theta_u^n : \psi \in \mathrm{PSH}^-(X,\eta),\, \sup_X \psi = 0\right\}, \tag{4.2}$$

where we recall that $\mathrm{PSH}^-(X,\eta)$ is the space of negative η-psh functions on X. If χ is bounded then it is clear that $\tilde{E}_{\chi,\eta,\theta}(u) < \infty$ for every $u \in \mathrm{PSH}(X,\theta)$. Moreover, it follows from [7, Proposition 3.2] that for every $u \in \mathrm{PSH}(X,\theta)$, there exists $\chi \in \mathcal{W}^-$ such that $\tilde{E}_{\chi,\eta,\theta}(u) < \infty$.

For every constant $B > 0$ and for every $\chi \in \widetilde{\mathcal{W}}^-$, we define

$$\tilde{\mathcal{E}}_{\chi,\eta,B}(X,\theta) = \{u \in \mathrm{PSH}^-(X,\theta) : \tilde{E}_{\chi,\eta,\theta}(u) \leq B\}. \tag{4.3}$$

For the convenience, in the case $\eta = \theta$, we denote $\tilde{E}_{\chi,\theta}(u) := \tilde{E}_{\chi,\theta,\theta}(u)$ and $\tilde{\mathcal{E}}_{\chi,B}(X,\theta) = \tilde{\mathcal{E}}_{\chi,\theta,B}(X,\theta)$.

If $u,v \in \tilde{\mathcal{E}}_{\chi,B}(X,\theta)$ then we also denote

$$I_\chi(u,v) = \int_{\{u<v\}} -\chi(u-v)(\theta_u^n - \theta_v^n) + \int_{\{v<u\}} -\chi(v-u)(\theta_v^n - \theta_u^n). \tag{4.4}$$

In general, $I_\chi(u,v)$ may be negative. However, by Lemma 4.7 below (observer that there always exists $\tilde{\chi} \in \mathcal{W}^-$ such that both $\tilde{E}_{\tilde{\chi},\theta}(u), \tilde{E}_{\tilde{\chi},\theta}(v)$ are finite), if $\inf \chi = -1$ then $I_\chi(u,v)$ is bounded from below by $-d_\theta(u,v)$.

Lemma 4.4. *Let $\chi \in \widetilde{\mathcal{W}}^-$. Assume that u, ϕ are negative θ-psh functions satisfying $u \leq \phi$. Denote $u_k = \max\{u, \phi - k\}$ for every $k > 0$. Then*

$$\int_X -\chi(u_k - \phi)\theta_{u_k}^n = \int_X -\chi(u-\phi)\theta_u^n - \chi(-k) d_\theta(u,\phi),$$

for every $k > 0$.

Proof. Since $\theta^n_{u_k} = \theta^n_u$ in $\{u > \phi - k\}$ and $u_k = \phi - k$ in $\{u \le \phi - k\}$, we have

$$\int_X -\chi(u_k - \phi)\theta^n_{u_k} = \int_{\{u\le\phi-k\}} -\chi(-k)\theta^n_{u_k} + \int_{\{u>\phi-k\}} -\chi(u-\phi)\theta^n_u. \tag{4.5}$$

Since $\int_X \theta^n_\phi = \int_X \theta^n_{u_k}$, we have

$$\begin{aligned}\int_{\{u\le\phi-k\}} -\chi(-k)\theta^n_{u_k} &= \int_X -\chi(-k)\theta^n_{u_k} + \int_{\{u>\phi-k\}} \chi(-k)\theta^n_{u_k} \\ &= \int_X -\chi(-k)\theta^n_\phi + \int_{\{u>\phi-k\}} \chi(-k)\theta^n_u.\end{aligned} \tag{4.6}$$

Combining (4.5) and (4.6), we get

$$\begin{aligned}\int_X -\chi(u_k-\phi)\theta^n_{u_k} &= \int_X -\chi(-k)\theta^n_\phi + \int_{\{u>\phi-k\}} \chi(-k)\theta^n_u + \int_{\{u>\phi-k\}} -\chi(u-\phi)\theta^n_u \\ &= -\chi(-k)d_\theta(\phi,u) + \int_{\{u\le\phi-k\}} -\chi(-k)\theta^n_u + \int_{\{u>\phi-k\}} -\chi(u-\phi)\theta^n_u \\ &= -\chi(-k)d_\theta(\phi,u) + \int_X -\chi(u-\phi)\theta^n_u.\end{aligned}$$

The proof is completed. □

Lemma 4.5. *Let $\chi, \tilde\chi \in \widetilde{\mathcal{W}}^-$ such that $\inf_{\mathbb{R}_{<0}} \chi = -1$. Assume that u_1, u_2, u_3, ϕ are negative θ-psh functions satisfying $u_1 \le u_2 \le \phi$ and $u_1 \le u_3 \le \phi$. Denote $u_{j,k} = \max\{u_j, \phi - k\}$ for every $k > 1$ and $j = 1, 2, 3$. Then*

$$\int_X -\chi(u_{1,k} - u_{2,k})\theta^n_{u_{3,k}} \le \int_X -\chi(u_1 - u_2)\theta^n_{u_3} + d_\theta(u_3,\phi) + \frac{1}{\tilde\chi(-k)}\int_X \tilde\chi(u_1-\phi)\theta^n_{u_3},$$

for every $k > 1$. In particular, if $\sup_X u_1 = 0$ and $u_3 \in \tilde{\mathcal{E}}_{\tilde\chi,B}(X,\theta)$ for some $B > 0$ then

$$\int_X -\chi(u_{1,k} - u_{2,k})\theta^n_{u_{3,k}} \le \int_X -\chi(u_1 - u_2)\theta^n_{u_3} + d_\theta(u_3,\phi) - \frac{B}{\tilde\chi(-k)},$$

for every $k > 1$.

Proof. Denote

$$I_k := \int_X -\chi(u_{1,k} - u_{2,k})\theta^n_{u_{3,k}}.$$

Since $\theta^n_{u_{3,k}} = \theta^n_{u_3}$ in $\{u_1 > \phi - k\} \subset \{u_3 > \phi - k\}$, we have

$$\begin{aligned}I_k &= \int_{\{u_1>\phi-k\}} -\chi(u_1-u_2)\theta^n_{u_3} + \int_{\{u_1\le\phi-k\}} -\chi(u_{1,k}-u_{2,k})\theta^n_{u_{3,k}} \\ &\le \int_{\{u_1>\phi-k\}} -\chi(u_1-u_2)\theta^n_{u_3} + \int_{\{u_1\le\phi-k\}} -\chi(-k)\theta^n_{u_{3,k}} \\ &= \int_{\{u_1>\phi-k\}} -\chi(u_1-u_2)\theta^n_{u_3} + \int_{\{u_1\le\phi-k\}} \theta^n_{u_{3,k}}.\end{aligned}$$

Then, by the fact $\int_X \theta_\phi^n = \int_X \theta_{u_{3,k}}^n$, we get

$$\begin{aligned} I_k &\leq \int_{\{u_1>\phi-k\}} -\chi(u_1-u_2)\theta_{u_3}^n + \int_X \theta_\phi^n - \int_{\{u_1>\phi-k\}} \theta_{u_3}^n \\ &= \int_{\{u_1>\phi-k\}} -\chi(u_1-u_2)\theta_{u_3}^n + d_\theta(u_3,\phi) + \int_{\{u_1\leq\phi-k\}} \theta_{u_3}^n \\ &\leq \int_X -\chi(u_1-u_2)\theta_{u_3}^n + d_\theta(u_3,\phi) + \frac{1}{\tilde{\chi}(-k)} \int_X \tilde{\chi}(u_1-\phi)\theta_{u_3}^n. \end{aligned}$$

The proof is completed. □

Lemma 4.6. *Let* $\chi, \tilde{\chi} \in \widetilde{\mathcal{W}}^-$ *such that* $\inf_{\mathbb{R}_{<0}} \chi = -1$. *Assume that* u_1, u_2, u_3, ϕ *are negative* θ*-psh functions satisfying* $u_1 \leq u_2 \leq \phi$ *and* $u_1 \leq u_3 \leq \phi$. *Denote* $u_{j,k} = \max\{u_j, \phi - k\}$ *for every* $k > 1$ *and* $j = 1, 2, 3$. *Then*

$$\int_X -\chi(u_{1,k}-u_{2,k})\theta_{u_{3,k}}^n \geq \int_X -\chi(u_1-u_2)\theta_{u_3}^n - \frac{1}{\tilde{\chi}(-k)} \int_X \tilde{\chi}(u_1-\phi)\theta_{u_3}^n,$$

for every $k > 1$. *In particular, if* $\sup_X u_1 = 0$ *and* $u_3 \in \tilde{\mathcal{E}}_{\tilde{\chi},B}(X,\theta)$ *for some* $B > 0$ *then*

$$\int_X -\chi(u_{1,k}-u_{2,k})\theta_{u_{3,k}}^n \geq \int_X -\chi(u_1-u_2)\theta_{u_3}^n + \frac{B}{\tilde{\chi}(-k)},$$

for every $k > 1$.

Proof. Since $\theta_{u_{3,k}}^n = \theta_{u_3}^n$ in $\{u_1 > \phi - k\}$, we have

$$\begin{aligned} \int_X -\chi(u_{1,k}-u_{2,k})\theta_{u_{3,k}}^n &\geq \int_{\{u_1>\phi-k\}} -\chi(u_{1,k}-u_{2,k})\theta_{u_{3,k}}^n \\ &= \int_{\{u_1>\phi-k\}} -\chi(u_1-u_2)\theta_{u_3}^n \\ &= \int_X -\chi(u_1-u_2)\theta_{u_3}^n - \int_{\{u_1\leq\phi-k\}} -\chi(u_1-u_2)\theta_{u_3}^n \\ &\geq \int_X -\chi(u_1-u_2)\theta_{u_3}^n - \int_{\{u_1\leq\phi-k\}} \theta_{u_3}^n \\ &\geq \int_X -\chi(u_1-u_2)\theta_{u_3}^n - \frac{1}{\tilde{\chi}(-k)} \int_X \tilde{\chi}(u_1-\phi)\theta_{u_3}^n. \end{aligned}$$

This finishes the proof. □

Lemma 4.7. *Let* θ *be a closed smooth real* $(1,1)$*-form representing a big cohomology class. Let* $\chi, \tilde{\chi} \in \widetilde{\mathcal{W}}^-$ *such that* $\inf_{\mathbb{R}_{<0}} \chi = -1$. *Assume that* $B > 0$ *and* $u_1, u_2 \in \tilde{\mathcal{E}}_{\tilde{\chi},B}(X,\theta)$ *with* $\sup_X u_1 = \sup_X u_2 = 0$. *Denote* $\phi = P_\theta[\max\{u_1, u_2\}]$ *and* $u_{j,k} = \max\{u_j, \phi - k\}$ *for every* $k > 1$ *and* $j = 1, 2$. *Then*

$$E_{\tilde{\chi},\theta,\phi}(u_{j,k}) \leq B - \tilde{\chi}(-k) d_\theta(u_j, \phi), \tag{4.7}$$

and
$$\frac{4B}{\tilde{\chi}(-k)} \leq I_\chi(u_{1,k}, u_{2,k}) - I_\chi(u_1, u_2) \leq -\frac{4B}{\tilde{\chi}(-k)} + d_\theta(u_1, u_2), \tag{4.8}$$
for every $k > 1$.

Proof. The first inequality is obtained directly from Lemma 4.4. It remains to prove (4.8). For $j = 1, 2$ and $k > 1$, we denote
$$I_{1,j} = \int_{\{u_1 < u_2\}} -\chi(u_{1,k} - u_{2,k})\theta^n_{u_{j,k}} + \int_{\{u_1 < u_2\}} \chi(u_1 - u_2)\theta^n_{u_j},$$
and
$$I_{2,j} = \int_{\{u_2 < u_1\}} -\chi(u_{2,k} - u_{1,k})\theta^n_{u_{j,k}} + \int_{\{u_1 < u_2\}} \chi(u_2 - u_1)\theta^n_{u_j}.$$
We have
$$I_\chi(u_{1,k}, u_{2,k}) - I_\chi(u_1, u_2) = (I_{1,1} - I_{1,2}) + (I_{2,2} - I_{2,1}) := I_1 + I_2. \tag{4.9}$$
We will estimate I_1 and I_2. By Lemmas 4.5 and 4.6 (replace u_2, u_3 and ϕ, respectively, by $\max\{u_1, u_2\}$, u_1 and $\max\{u_1, u_2\}$), we have
$$\frac{B}{\tilde{\chi}(-k)} \leq I_{1,1} \leq d_\theta(u_1, \max\{u_1, u_2\}) - \frac{B}{\tilde{\chi}(-k)}. \tag{4.10}$$
Using Lemmas 4.5 and 4.6 again (replace u_2, u_3 and ϕ by $\max\{u_1, u_2\}$), we get
$$\frac{B}{\tilde{\chi}(-k)} \leq \int_{\{u_1 < u_2\}} -\chi(u_{1,k} - u_{2,k})\theta^n_{u_{2,k}} + \int_{\{u_1 < u_2\}} \chi(u_1 - u_2)\theta^n_{u_2} \leq -\frac{B}{\tilde{\chi}(-k)}. \tag{4.11}$$
Combining (4.10) and (4.11), we obtain
$$\frac{2B}{\tilde{\chi}(-k)} \leq I_1 \leq -\frac{2B}{\tilde{\chi}(-k)} + d_\theta(u_1, \max\{u_1, u_2\}). \tag{4.12}$$
Similar, we have
$$\frac{2B}{\tilde{\chi}(-k)} \leq I_2 \leq -\frac{2B}{\tilde{\chi}(-k)} + d_\theta(u_2, \max\{u_1, u_2\}). \tag{4.13}$$
Combining (4.9), (4.12) and (4.13), we have
$$\frac{4B}{\tilde{\chi}(-k)} \leq I_\chi(u_{1,k}, u_{2,k}) - I_\chi(u_1, u_2) \leq -\frac{4B}{\tilde{\chi}(-k)} + d_\theta(u_1, u_2).$$
The proof is completed. □

The following theorem is the key step to prove the main results in the case of fixed cohomology:

Theorem 4.8. *Let $\theta \leq A\omega$ be a closed smooth real $(1,1)$-form representing a big cohomology class ($A \geq 1$). Let $0 < \delta < 1$, $B \geq A$, $\tilde{\chi} \in \widetilde{\mathcal{W}}^-$ and $u_1, u_2 \in \tilde{\mathcal{E}}_{\tilde{\chi}, B\delta}(X, \theta)$ such that $\inf \tilde{\chi} < \tilde{\chi}(-1) = -1$, $\sup_X u_1 = \sup_X u_2 = 0$ and $\int_X \theta_{u_1}^n + \int_X \theta_{u_2}^n \geq 2\delta$. Let $\epsilon > 0$ be a constant such that*

$$\inf \tilde{\chi} < \frac{-4B\delta}{\epsilon + d_\theta(u_1, u_2)}.$$

Then, for every $0 < \gamma < 1$, there exists $C > 0$ depending only on n, X, ω and γ such that

$$d_{cap}(u_1, u_2)^2 \leq C(AB)^2 \left(h^{\circ n} \left(\frac{\delta}{|I_\chi(u_1, u_2)| + \epsilon + d_\theta(u_1, u_2)} \right) \right)^{-\gamma},$$

where $\chi(t) = \max\{t, -1\}$ and $h(s) = (-\tilde{\chi}(-s))^{1/2}$.

Proof. Without loss of generality, we can assume that

$$\frac{4B\delta}{\epsilon + d_\theta(u_1, u_2)} \geq 1.$$

Denote $\phi = P_\theta[\max\{u_1, u_2\}]$ and $u_{j,k} = \max\{u_j, \phi - k\}$ for every $k > 1$ and $j = 1, 2$. By Theorem 3.12, Remark 3.13 and Lemma 4.7, we get

$$d_{\text{cap}}(u_{1,k}, u_{2,k})^2 \leq C_1 A^2 \left(B - \frac{\tilde{\chi}(-k) d_\theta(u_1, u_2)}{\delta} \right)^2 \left(h^{\circ n} \left(\frac{\delta}{I_\chi(u_1, u_2) - \dfrac{4B\delta}{\tilde{\chi}(-k)} + d_\theta(u_1, u_2)} \right) \right)^{-\gamma},$$

for every $k > 1$, where $C_1 > 0$ depends only on n, X, ω and γ.

Let $k_0 > 1$ such that $\tilde{\chi}(-k_0) = \dfrac{-4B\delta}{\epsilon + d_\theta(u_1, u_2)}$. We have

$$d_{\text{cap}}(u_{1,k_0}, u_{2,k_0})^2 \leq 25 C_1 (AB)^2 \left(h^{\circ(n)} \left(\frac{\delta}{I_\chi(u_1, u_2) + \epsilon + 2d_\theta(u_1, u_2)} \right) \right)^{-\gamma}. \tag{4.14}$$

On the other hand, for every $\varphi \in \text{PSH}(X, \omega)$ with $0 \leq \varphi \leq 1$, we have,

$$\begin{aligned} \left(\int_X |u_j - u_{j,k_0}|^{1/2} \omega_\varphi^n \right)^2 &= \left(\int_{\{u_j < \phi - k_0\}} |u_j - u_{j,k_0}|^{1/2} \omega_\varphi^n \right)^2 \\ &\leq \frac{1}{k_0} \left(\int_{\{u_j < \phi - k_0\}} |u_j| \omega_\varphi^n \right)^2 \\ &\leq \frac{C_2 A^2}{k_0}, \end{aligned}$$

for $j = 1, 2$, where $C_2 > 0$ depends only on X and ω. The last inequality holds due to the Chern-Levine-Nirenberg inequality.

Hence, by the facts $t \geq -\tilde{\chi}(-t)$ for every $t \geq 1$ and $s \leq h(s)$ for every $0 < s < 1$, we get

$$d_{\text{cap}}(u_j, u_{j,k_0})^2 \leq \frac{C_2 A^2}{k_0} \leq \frac{C_2 A^2 (\epsilon + d_\theta(u_1, u_2))}{4B\delta} \leq C_2 A^2 \left(h^{\circ n} \left(\frac{\delta}{\epsilon + d_\theta(u_1, u_2)} \right) \right)^{-1}. \tag{4.15}$$

Combining (4.14) and (4.15), we obtain

$$d_{\mathrm{cap}}(u_1,u_2)^2 \le C_3(AB)^2\left(h^{\circ n}\left(\frac{\delta}{|I_\chi(u_1,u_2)|+\epsilon+d_\theta(u_1,u_2)}\right)\right)^{-\gamma},$$

where $C_3>0$ depends only on n, X, ω and γ.

The proof is completed. □

Now, we prove Theorem 1.3 for the case of fixed cohomology:

Theorem 4.9. *Let $\theta \le A\omega$ be a closed smooth real $(1,1)$-form representing a big cohomology class ($A \ge 1$). Let $u \in \mathrm{PSH}(X,\theta)$ such that $\sup_X u = 0$ and $\int_X \theta_u^n := \delta > 0$. Assume $u \in \tilde{\mathcal{E}}_{\tilde\chi,B\delta}(X,\theta)$, where $B \ge A$ is a given constant and $\tilde\chi \in \mathcal{W}^-$ with $\tilde\chi(-1)=-1$. Then, for every $0<\gamma<1$, there exists $C>0$ depending only on n, X, ω and γ such that*

$$d_{cap}(u,v)^2 \le C(A\,B)^2\left(h^{\circ n}\left(\frac{\delta}{\|\theta_u^n-\theta_v^n\|+d_\theta(u,v)}\right)\right)^{-\gamma},$$

for every $v \in \mathrm{PSH}(X,\theta)$ with $\sup_X v = 0$, where $h(s) = (-\tilde\chi(-s))^{1/2}$.

Proof. Put

$$t_0 = \|\theta_u^n-\theta_v^n\| + d_\theta(u,v).$$

Denote

$$M=\frac{5B\delta}{t_0},\quad \tilde\chi_M(s)=\max\{\tilde\chi(s),-M\}\quad\text{and}\quad h_M(s)=(-\tilde\chi_M(-s))^{1/2}.$$

We have $v \in \tilde{E}_{\tilde\chi_M,B\delta+Mt_0}(X,\theta)$ and $Mt_0 = 5B\delta$. Since $\inf\tilde\chi_M = -M < \dfrac{-4B\delta}{\|\theta_u^n-\theta_v^n\|+d_\theta(u,v)}$, it follows from Theorem 4.8 that

$$d_{\mathrm{cap}}(u,v)^2 \le C_1(AB)^2\left(h_M^{\circ n}\left(\frac{\delta}{|I_\chi(u,v)|+\|\theta_u^n-\theta_v^n\|+d_\theta(u,v)}\right)\right)^{-\gamma},$$

where $\chi(s)=\max\{s,-1\}$ and $C_1>0$ depends only on n, X, ω and γ. Since $|I_\chi(u,v)| \le \|\theta_u^n-\theta_v^n\|$ and $B \ge 1$, it follows that

$$d_{\mathrm{cap}}(u,v)^2 \le C_2(A\,B)^2\left(h_M^{\circ n}\left(\frac{\delta}{\|\theta_u^n-\theta_v^n\|+d_\theta(u,v)}\right)\right)^{-\gamma},$$

where $C_2 = 4C_1$. By the fact $h_M(t) = h(t) \le M$ for every $0<t\le M$, we obtain

$$d_{\mathrm{cap}}(u,v)^2 \le C_2(A\,B)^2\left(h^{\circ n}\left(\frac{\delta}{\|\theta_u^n-\theta_v^n\|+d_\theta(u,v)}\right)\right)^{-\gamma}.$$

The proof is completed. □

In order to prove the next main result, we need the following lemma:

Lemma 4.10. *Let $u : B_{R+r} := \{z \in \mathbb{C}^n : |z| < R + r\} \to [-\infty, 0]$ be a measurable function such that $A := \int_{B_{R+r}} e^{-u} dV < \infty$, where $R > r > 0$ and dV denotes the Lebesgue measure on $\mathbb{C}^n$. Assume that h is a non-negative smooth function on $\mathbb{C}^n$ satisfying $\int_{\mathbb{C}^n} h dV = 1$ and $\mathrm{Supp}(h) \subset B_\epsilon$ for some $\epsilon \in (0, r)$. Then, for every $0 < a < 1$, there exists $C > 0$ depending only on R, n, a and A such that*

$$\left| \int_{B_R} (u * h - u) dV \right| \leq C\epsilon^a.$$

Proof. We have

$$\begin{aligned}\left| \int_{B_R} (u * h - u) dV \right| &= \left| \int_{B_R} \int_{B_\epsilon} h(w)(u(z-w) - u(z)) dV_w dV_z \right| \\ &= \left| \int_{B_\epsilon} h(w) \int_{B_R} (u(z-w) - u(z)) dV_w dV_z \right| \\ &\leq \int_{B_\epsilon} h(w) |\hat{u}(-w) - \hat{u}(0)| dV_w,\end{aligned}$$

where $\hat{u}(w) = \int_{\{|\xi - w| < R\}} (-u)(\xi) dV_\xi$. Moreover, for every $w \in B_\epsilon$ and $k > 0$, we have

$$\begin{aligned}&\left| \int_{B_R} \max\{u(z-w), k\} dV_z - \int_{B_R} \max\{u(z), k\} dV_z \right| \\ &\qquad = \left| \int_{B_R(-w)} \max\{u(z), k\} dV_z - \int_{B_R} \max\{u(z), k\} dV_z \right| \leq C_n R^{2n-1} k |w|,\end{aligned}$$

where $C_n > 0$ is a constant depending only on n, and

$$\int_{B_R} |u(z-w) - \max\{u(z-w), -k\}| dV_z \leq \int_{\{u < -k\}} (-u-k) dV \leq \int_{B_{R+r}} e^{-u-k} dV = Ae^{-k}.$$

Then

$$\left| \int_{B_R} (u * h - u) dV \right| \leq \int_{B_\epsilon} h(w) |\hat{u}(-w) - \hat{u}(0)| dV_w \leq C_n R^{2n-1} k\epsilon + Ae^{-k}.$$

Choosing $k = \epsilon^{a-1}$, we get the desired inequality. □

The following result can be considered as a generalization of Theorems 1.4 and 1.5 for the case of fixed cohomology:

Theorem 4.11. *Let θ be a closed smooth real $(1,1)$-form such that $\theta \leq A\omega$ for a given constant $A \geq 1$. Let $0 < \delta < 1$, $B \geq A$, $\tilde{\chi} \in \mathcal{W}^-$ and $u_1, u_2 \in \tilde{\mathcal{E}}_{\tilde{\chi}, B\delta}(X, \theta)$ such that $\tilde{\chi}(-1) = -1$, $\sup_X u_1 = \sup_X u_2 = 0$ and $\int_X \theta_{u_1}^n + \int_X \theta_{u_2}^n \geq 2\delta$. Assume that there exists a concave increasing function $H : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that, for $j = 1, 2$,*

$$\int_X \min\{|\psi_1 - \psi_2|, 1\} \theta_{u_j}^n \leq H(\|\psi_1 - \psi_2\|_{L^1(X)}), \tag{4.16}$$

for every $\psi_1, \psi_2 \in \mathrm{PSH}(X,\omega)$ with $\sup_X \psi_1 = \sup_X \psi_2 = 0$. Then, for every $0 < \gamma < 1$, there exists $C > 0$ depending only on n, X, ω and γ such that

$$d_{cap}(u_1,u_2)^2 \le C(A\,B)^2 \left(h^{\circ n}\left(\frac{\delta}{A(x^{1/2}+H(x^{1/8}))+d_\theta(u_1,u_2)}\right)\right)^{-\gamma},$$

where $x := \mathrm{dist}_{-1}(\theta^n_{u_1}, \theta^n_{u_2})$ and $h(s) = (-\tilde{\chi}(-s))^{1/2}$.

Proof. Denote $\chi(t) = \max\{t, -1\}$. Since $\inf \tilde{\chi} = -\infty$, it follows from Theorem 4.8 that

$$d_{\mathrm{cap}}(u_1,u_2)^2 \le C_0(A\,B)^2 \left(h^{\circ n}\left(\frac{\delta}{|I_\chi(u_1,u_2)|+d_\theta(u_1,u_2)}\right)\right)^{-\gamma}, \tag{4.17}$$

where $C_0 > 0$ depends only on n, X, ω and γ. We will estimate $|I_\chi(u_1,u_2)|$.

For every $k > 0$ and $j = 1, 2$, we have

$$\|u_j - \max\{u_j,-k\}\|_{L^1(X)} \le \int_{\{u_j<-k\}} (-u_j)dV \le \left(\int_{\{u_j<-k\}} u_j^2 dV\right)^{1/2} \left(\int_{\{u_j<-k\}} dV\right)^{1/2}.$$

Then, it follows from the Skoda integrability theorem that

$$\|u_j - \max\{u_j,-k\}\|_{L^1(X)} \le C_1 A^{3/2} k^{-1/2}, \tag{4.18}$$

where $C_1 > 0$ depends only on X, ω. For every $k > 0$ and for each $0 < \epsilon < 1$, by using the standard convolution and a partition of unit, we can find a smooth function $u_{j,k,\epsilon} \in \mathrm{PSH}(X,(A+1)\omega)$ such that

$$\|u_{j,k,\epsilon}\|_{C^1(X)} \le \frac{C_2 k}{\epsilon}, \tag{4.19}$$

and

$$\|u_{j,k,\epsilon} - \max\{u_j,-k\}\|_{L^1(X)} \le C_2 A \epsilon^{1/2}, \tag{4.20}$$

where $C_2 > C_1$ depends only on X, ω. Here, the last inequality holds due to Lemma 4.10. Combining (4.18) and (4.20), we have

$$\|u_j - u_{j,k,\epsilon}\|_{L^1(X)} \le C_2 A(\epsilon^{1/2} + (A/k)^{1/2}). \tag{4.21}$$

Recall that

$$\begin{aligned} I_\chi(u_1,u_2) &= \int_{\{u_1<u_2\}} \min\{|u_1-u_2|,1\}(\theta^n_{u_1}-\theta^n_{u_2}) + \int_{\{u_2<u_1\}} \min\{|u_1-u_2|,1\}(\theta^n_{u_2}-\theta^n_{u_1}) \\ &= \int_X \max\{\min\{u_2-u_1,1\},-1\}(\theta^n_{u_1}-\theta^n_{u_2}). \end{aligned}$$

By the fact that

$$\max\{t_1,t_3\} - \max\{t_2,t_3\} = \min\{-t_2,-t_3\} - \min\{-t_1,-t_3\} \le \max\{t_1-t_2, 0\},$$

we have

$$|\max\{\min\{u_2-u_1,1\},-1\} - \max\{\min\{u_{2,k,\epsilon}-u_{1,k,\epsilon},1\},-1\}| \le |u_2-u_1-u_{2,k,\epsilon}+u_{1,k,\epsilon}|,$$

for every $k > 0$ and $0 < \epsilon < 1$.Since the LHS of the last inequality is bounded by 2, it follows that

$$|\max\{\min\{u_2 - u_1, 1\}, -1\} - \max\{\min\{u_{2,k,\epsilon} - u_{1,k,\epsilon}, 1\}, -1\}| \leq \Psi_{k,\epsilon},$$

where

$$\Psi_{k,\epsilon} = \min\{|u_2 - u_1 - u_{2,k,\epsilon} + u_{1,k,\epsilon}|, 2\}.$$

Therefore

$$|I_X(u_1, u_2)| \leq \left| \int_X \Phi_{k,\epsilon}(\theta^n_{u_1} - \theta^n_{u_2}) \right| + \int_X \Psi_{k,\epsilon}(\theta^n_{u_1} + \theta^n_{u_2}), \tag{4.22}$$

where $\Phi_{k,\epsilon} = \max\{\min\{u_{2,k,\epsilon} - u_{1,k,\epsilon}, 1\}, -1\}$. By (4.19), we have

$$\left| \int_X \Phi_{k,\epsilon}(\theta^n_{u_1} - \theta^n_{u_2}) \right| \leq \frac{C_2 k}{\epsilon} \operatorname{dist}_{-1}(\theta^n_{u_1}, \theta^n_{u_2}). \tag{4.23}$$

Moreover, by (4.21) and (4.16), we have

$$\int_X \Psi_{k,\epsilon}(\theta^n_{u_1} + \theta^n_{u_2}) \leq 4(A+1)H\left(\frac{C_2 A}{A+1}(\epsilon^{1/2} + (A/k)^{1/2})\right). \tag{4.24}$$

Combining (4.22), (4.23) and (4.24), we have

$$|I_X(u_1, u_2)| \leq \frac{C_2 k}{\epsilon} \operatorname{dist}_{-1}(\theta^n_{u_1}, \theta^n_{u_2}) + 4(A+1)H\left(\frac{C_2 A}{A+1}(\epsilon^{1/2} + (A/k)^{1/2})\right).$$

Denoting $x = \operatorname{dist}_{-1}(\theta^n_{u_1}, \theta^n_{u_2})$ and choosing $\epsilon = A/k = (2C_2)^{-2} x^{1/4}$, we obtain

$$|I_X(u_1, u_2)| \leq C_3 A(x^{1/2} + H(x^{1/8})), \tag{4.25}$$

where $C_3 > 1$ depends only on X and ω. Combining (4.17) and (4.25), we obtain the desired inequality. The proof is completed. □

4.3 Application to the space of singularity types

In this part we apply quantitative stability theorems in the previous subsection to deduce some properties of the pseudometric space of singularity types in a big cohomology class.

Proposition 4.12. *Let $\theta \leq A\omega$ be a closed smooth real $(1,1)$-form representing a big cohomology class ($A \geq 1$). Assume that u_1 and u_2 are model θ-psh functions such that $\int_X \theta^n_{u_1} + \int_X \theta^n_{u_2} \geq 2\delta > 0$, where $\delta > 0$ is a constant. Then, for every $0 < \gamma < 1$, there exists $C > 0$ depending only on n, X, ω and γ such that*

$$d_{cap}(u_1, u_2)^2 \leq C \frac{A^{2n+4}}{\delta^2} \left(\frac{d_\theta(u_1, u_2)}{\delta}\right)^{2^{-n}\gamma}.$$

The above result implies in particular that for model potentials, the convergence in $d_{\mathcal{S}}$ is stronger than that in capacity. The last non-quantitative fact follows also from [17, Theorem 5.6].

Proof. By [15, Theorem 3.8], we have

$$\theta_{u_j}^n \leq \mathbf{1}_{\{u_j=0\}}\theta^n \leq A^n\omega^n,$$

for $j = 1, 2$. Therefore, there exists $C_1 > 0$ depending only on X and ω such that

$$\int_X (-\psi)\theta_{u_j}^n \leq C_1 A^{n+1},$$

for every $\psi \in \mathrm{PSH}(X, \theta) \subset \mathrm{PSH}(X, A\omega)$ with $\sup_X \psi = 0$. Using Theorem 4.8 for $\tilde{\chi}(t) = t$, we get

$$d_{\mathrm{cap}}(u_1, u_2)^2 \leq C_2 \left(A\frac{C_1 A^{n+1}}{\delta}\right)^2 \left(\frac{|I_\chi(u_1, u_2)| + d_\theta(u_1, u_2)}{\delta}\right)^{2^{-n}\gamma}, \tag{4.26}$$

where $\chi(t) = \max\{t, -1\}$ and $C_2 > 0$ is a constant depending only on n, X, ω and γ. Since $\theta_{u_j}^n \leq \mathbf{1}_{\{u_j=0\}}\theta^n$, we have

$$\int_{\{u_1<u_2\}} -\chi(u_1 - u_2)\theta_{u_1}^n = \int_{\{u_2<u_1\}} -\chi(u_2 - u_1)\theta_{u_2}^n = 0.$$

Therefore

$$I_\chi(u_1, u_2) \leq 0. \tag{4.27}$$

Moreover, it follows from Lemma 4.7 that

$$I_\chi(u_1, u_2) \geq -d_\theta(u_1, u_2). \tag{4.28}$$

Combining (4.26), (4.27) and (4.28), we obtain

$$d_{\mathrm{cap}}(u_1, u_2)^2 \leq C_3 \frac{A^{2n+4}}{\delta^2} \left(\frac{d_\theta(u_1, u_2)}{\delta}\right)^{2^{-n}\gamma},$$

where $C_3 > 0$ depends only on n, X, ω and γ. The proof is completed. □

By using Proposition 4.12, we recover the following result which is obtained in [17] (with a different proof).

Proposition 4.13. *Let $\delta > 0$ be a constant. Let $\mathcal{S}_\delta(\theta)$ be the subset of $\mathcal{S}(\theta)$ consisting of $[u] \in \mathcal{S}_\theta$ such that $\int_X \theta_u^n \geq \delta$. Then $(\mathcal{S}_\delta(\theta), d_{\mathcal{S}})$ is a complete (pseudo)-metric space.*

Proof. Let $([u_j])_j$ be a Cauchy sequence in $\mathcal{S}_\delta(\theta)$ (recall $[u_j]$ denotes the singularity type of a θ-psh function u_j with $\sup_X u_j = 0$), *i.e,* for every constant $\epsilon > 0$, there exists $k_\epsilon \in \mathbb{N}$ such that $d_\theta(u_j, u_k) \leq \epsilon$ for every $j \geq k_\epsilon$, and $k \geq k_\epsilon$. We need to prove that there exists a class $[u_\infty] \in \mathcal{S}_\delta(\theta)$ so that $d_\theta(u_j, u_\infty) \to 0$ as $j \to \infty$. By using contradiction, it suffices to prove it for some subsequence of $(u_j)_j$. Hence we can assume safely that

$$d_\theta(u_j, u_{j+1}) \leq 4^{-n2^{n+1}},$$

because one can always extract a subsequence of $(u_j)_j$ with that property.

Since $d_\theta\big(u, P_\theta[u]\big) = 0$ for every $u \in \mathrm{PSH}(X,\theta)$, without loss of generality, we can assume that $u_j = P_\theta[u_j]$ for every $j \in \mathbb{N}$, in other words, u_j's are model θ-psh functions. Consequently, by Proposition 4.12 (with $\gamma = 1/2$), we get

$$d_{\mathrm{cap}}(u_j, u_{j+1}) \le 2^{-n} C_1,$$

for every j, where $C_1 > 0$ is a constant depending only on n, X, ω, θ and δ. Therefore, there exists a θ-psh function u_∞ such that u_j converges to u_∞ in capacity as $j \to \infty$.

Moreover, it follows from [15, Theorem 3.8] that

$$\theta^n_{u_j} \le \mathbf{1}_{\{u_j=0\}} \theta^n \le C_2 \omega^n, \tag{4.29}$$

for some constant $C_2 > 0$ independent of j. This coupled with Lemma 4.14 below yields that

$$\theta^n_{u_j} \to \theta^n_{u_\infty}, \tag{4.30}$$

as $j \to \infty$. It is clear that $\int_X \theta^n_{u_\infty} \ge \delta$. It remains to show that $d_\theta(u_j, u_\infty) \to 0$ as $j \to \infty$.

Since

$$d_\theta(u_j, u_k) = 2\int_X \theta^n_{\max\{u_j,u_k\}} - \int_X \theta^n_{u_j} - \int_X \theta^n_{u_k},$$

using (4.30) and the fact that $\max\{u_j, u_k\} \to \max\{u_j, u_\infty\}$ in capacity as $k \to \infty$ (for $u_k \to u_\infty$ in capacity), one gets

$$\liminf_{k\to\infty} d_\theta(u_j, u_k) \ge 2\int_X \theta^n_{\max\{u_j,u_\infty\}} - \int_X \theta^n_{u_j} - \int_X \theta^n_{u_\infty} = d_\theta(u_j, u_\infty).$$

It follows that $d_\theta(u_j, u_\infty) \to 0$ as $j \to \infty$. In other words, $[u_j] \to [u_\infty]$ in the topology induced by the pseudo-metric $d_{\mathcal{S}}$ (we note that $[u_\infty]$ might not be unique, but the singularity type of its envelope $P_\theta[u_\infty]$ is unique). □

The following lemma is probably known. We present a proof for readers' convenience.

Lemma 4.14. *Let Ω be an open subset in $\mathbb{C}^n$. Let $(u_j)_j$ be a sequence of psh functions converging to a psh function u_∞ in capacity in Ω. Assume that the non-pluripolar product $(dd^c u_j)^n$ is well-defined for $1 \le j \le \infty$, and there exists a non-pluripolar Radon measure μ on Ω such that $(dd^c u_j)^n \le \mu$ for every j. Then $(dd^c u_j)^n$ converges weakly to $(dd^c u_\infty)^n$ as $j \to \infty$.*

Proof. Let ν be a limit measure of the sequence $\big((dd^c u_j)^n\big)_j$ as $j \to \infty$. We need to check that $\nu = (dd^c u_\infty)^n$. Observe that $\nu \ge (dd^c u_\infty)^n$ because $u_j \to u_\infty$ in capacity. It remains to verify the converse inequality.

Let $g \ge 0$ be a smooth function with compact support in Ω. Put $u_{jk} := \max\{u_j, -k\}$ for $1 \le j \le \infty$. We get $u_{jk} \to u_{\infty k}$ in the capacity as $j \to \infty$. It follows that $(dd^c u_{jk})^n \to (dd^c u_{\infty k})^n$ as $j \to \infty$, and

$$\limsup_{j\to\infty} \int_\Omega g \mathbf{1}_{\{u_j \ge -k+1\}} (dd^c u_{jk})^n \le \int_\Omega g \mathbf{1}_{\{u_\infty \ge -k+1\}} (dd^c u_{\infty k})^n.$$

By this and the equality $\mathbf{1}_{\{u_j>-k\}}(dd^c u_{jk})^n = (dd^c u_j)^n$, we get

$$\limsup_{j\to\infty}\int_\Omega g\mathbf{1}_{\{u_j>-k+1\}}(dd^c u_j)^n \le \int_\Omega g\mathbf{1}_{\{u_\infty>-k\}}(dd^c u_{\infty k})^n = \int_\Omega g\mathbf{1}_{\{u_\infty>-k\}}(dd^c u_\infty)^n \tag{4.31}$$

which converges to 0 as $k\to\infty$. On the other hand, by hypothesis, one gets

$$\mathbf{1}_{\{u_j<-k\}}(dd^c u_j)^n \le \mathbf{1}_{\{u_j<-k\}}\mu \le \mathbf{1}_{\{u<-k\}}\mu + \mathbf{1}_{\{u_j-u\le 1\}}\mu.$$

Combining this with [33, Lemma 4.5] (we use here the fact that μ is non-pluripolar) gives

$$\limsup_{j\to\infty}\int_\Omega g\mathbf{1}_{\{u_j<-k\}}(dd^c u_j)^n \le \int_\Omega g\mathbf{1}_{\{u<-k\}}\mu$$

Letting $k\to\infty$, we obtain

$$\limsup_{j\to\infty}\int_\Omega g\mathbf{1}_{\{u_j<-k\}}(dd^c u_j)^n \to 0$$

as $k\to\infty$. Combining the last inequality with (4.31) yields

$$\limsup_{j\to\infty}\int_\Omega g(dd^c u_j)^n \le \int_\Omega g(dd^c u_\infty)^n.$$

Hence $(dd^c u_j)^n \to dd^c u_\infty$ as $j\to\infty$. This finishes the proof. □

4.4 The case of varied cohomology

We first explain how to deduce Theorem 1.2 from Theorem 1.3.

End of the proof of Theorem 1.2. Since $\theta_j\to\theta_\infty$ in $\mathscr{C}^0$-norm, there exists a constant $A\ge 1$ so that $\theta_j\le A\omega$ for every $j\in\mathbb{N}\cup\{\infty\}$. By [7, Proposition 3.2], there exists $\tilde\chi\in\mathcal{W}^-$ such that

$$\sup_{\psi\in \mathrm{PSH}(X,(A+1)\omega):\sup_X\psi=0}\int_X -\tilde\chi(\psi)d\mu_\infty < \infty.$$

By considering $\tilde\chi/|\tilde\chi(-1)|$ instead of $\tilde\chi$, we can assume that $\tilde\chi(-1)=-1$. This allows us to apply Theorem 1.3 to $u:=u_\infty$, $v:=u_j$, $\theta:=\theta_\infty$, and $\eta:=\theta_j$, and we note that

$$d_{(A+1)\omega}(u,v) = d_{(A+1)\omega}(u_j,u_\infty) = d_{(A+1)\omega}(\phi_j,\phi_\infty)\to 0$$

as $j\to\infty$ by the hypothesis. We thus obtain $d_{\mathrm{cap}}(u_j,u_\infty)\to 0$ as $j\to\infty$. The desired convergence hence follows. The proof is finished. □

We now continue with the proof of Theorem 1.3.

End of the proof of Theorem 1.3. Put $\epsilon = \|\theta - \eta\|_{\mathscr{C}^0}$. Then, there exists $C_1 \geq 1$ depending only on X and ω such that

$$\theta \leq \eta + C_1\epsilon\omega \leq \theta + 2C_1\epsilon\omega. \tag{4.32}$$

Note that, by Chern-Levine-Nirenberg inequality and by the compactness of $\{w \in \mathrm{PSH}(X,\omega) : \sup_X w = 0\}$ in $L^1(X)$, there exists $C_\omega > 0$ depending only on X and ω such that

$$d_{\mathsf{cap}}(u,v)^2 \leq C_\omega A.$$

Therefore, if $0 < \dfrac{\delta}{2C_1} \leq \epsilon$ then the desired inequality (1.3) holds. Hence, without loss of generality, we can assume that

$$\epsilon < \frac{\delta}{2C_1},$$

and, as a consequence, we have $\tilde{\theta} := \theta + C_1\epsilon\omega \leq (A+1)\omega$.

It follows from [16, Theorem 4.7] that there exists a unique $\tilde{u} \in \mathcal{E}(X, \tilde{\theta}, P_{\tilde{\theta}}[u])$ such that

$$\begin{cases} \tilde{\theta}^n_{\tilde{u}} = c\theta^n_u, \\ \sup_X \tilde{u} = 0, \end{cases} \tag{4.33}$$

where $c = \dfrac{\int_X \tilde{\theta}^n_u}{\int_X \theta^n_u} \geq 1$. Observe that $\int_X \tilde{\theta}^n_{\tilde{u}} \geq c\delta$ and $\tilde{u} \in \tilde{E}_{\tilde{\chi},Bc\delta}(X,\tilde{\theta})$. It follows from Theorem 4.9 that

$$d_{\mathsf{cap}}(\tilde{u},u)^2 \leq C_2(A+1)^2B^2\left(h^{\circ n}\left(\frac{\delta}{\|\tilde{\theta}^n_{\tilde{u}} - \tilde{\theta}^n_u\| + d_{\tilde{\theta}}(\tilde{u},u)}\right)\right)^{-\gamma}, \tag{4.34}$$

and

$$d_{\mathsf{cap}}(\tilde{u},v)^2 \leq C_2(A+1)^2B^2\left(h^{\circ n}\left(\frac{\delta}{\|\tilde{\theta}^n_{\tilde{u}} - \tilde{\theta}^n_v\| + d_{\tilde{\theta}}(\tilde{u},v)}\right)\right)^{-\gamma}, \tag{4.35}$$

where $C_2 > 0$ depends only on n, X, ω and γ. Since $P_{\tilde{\theta}}[u] = P_{\tilde{\theta}}[\tilde{u}]$, we have

$$d_{\tilde{\theta}}(\tilde{u},u) = 0 \quad \text{and} \quad d_{\tilde{\theta}}(\tilde{u},v) = d_{\tilde{\theta}}(u,v). \tag{4.36}$$

Combining (4.34), (4.35) and (4.36), we get

$$d_{\mathsf{cap}}(u,v)^2 \leq C_3(A\,B)^2\left(h^{\circ n}\left(\frac{\delta}{\|\tilde{\theta}^n_{\tilde{u}} - \tilde{\theta}^n_u\| + \|\tilde{\theta}^n_{\tilde{u}} - \tilde{\theta}^n_v\| + d_{\tilde{\theta}}(u,v)}\right)\right)^{-\gamma}, \tag{4.37}$$

where $C_3 > 0$ depends only on n, X, ω and γ. By (4.32), we have

$$\theta^n_u \leq \tilde{\theta}^n_u \leq \theta^n_u + C_4\epsilon(\theta_u + \omega)^n,$$

and

$$\eta^n_v \leq \tilde{\theta}^n_v \leq (\eta_v + 2C_1\epsilon\omega)^n \leq \eta^n_v + C_4\epsilon(\eta_v + \omega)^n,$$

where $C_4 > 0$ depends only on X and ω. Therefore

$$\|\theta_u^n - \tilde{\theta}_u^n\| + \|\eta_v^n - \tilde{\theta}_v^n\| \leq C_5(A+1)^n \operatorname{vol}(X)\epsilon, \tag{4.38}$$

where $C_5 > 0$ depends only on X and ω. Moreover,

$$\|\theta_u^n - \tilde{\theta}_{\tilde{u}}^n\| = (c-1)\int_X \theta_u^n = \int_X (\tilde{\theta}_u^n - \theta_u^n) \leq \|\theta_u^n - \tilde{\theta}_u^n\|. \tag{4.39}$$

Combining (4.38) and (4.39), we get

$$\begin{aligned}\|\tilde{\theta}_{\tilde{u}}^n - \tilde{\theta}_u^n\| + \|\tilde{\theta}_{\tilde{u}}^n - \tilde{\theta}_v^n\| &\leq \|\theta_u^n - \tilde{\theta}_u^n\| + 2\|\theta_u^n - \tilde{\theta}_{\tilde{u}}^n\| + \|\theta_u^n - \eta_v^n\| + \|\eta_v^n - \tilde{\theta}_v^n\| \\ &\leq 3\|\theta_u^n - \tilde{\theta}_u^n\| + \|\eta_v^n - \tilde{\theta}_v^n\| + \|\theta_u^n - \eta_v^n\| \\ &\leq 3C_5(A+1)^n \operatorname{vol}(X)\epsilon + \|\theta_u^n - \eta_v^n\|.\end{aligned}$$

Hence, by (4.37), we obtain

$$\begin{aligned}d_{\text{cap}}(u,v)^2 &\leq C_3(A\,B)^2\left(h^{\circ n}\left(\frac{\delta}{3C_5(A+1)^n \operatorname{vol}(X)\epsilon + \|\theta_u^n - \eta_v^n\| + d_{\tilde{\theta}}(u,v)}\right)\right)^{-\gamma} \\ &\leq C_6(A\,B)^2\left(h^{\circ n}\left(\frac{\delta}{A^n\epsilon + \|\theta_u^n - \eta_v^n\| + d_{(A+1)\omega}(u,v)}\right)\right)^{-\gamma},\end{aligned}$$

where $C_6 > 0$ depends only on n, X, ω and γ. Here we use the facts $d_{\tilde{\theta}}(u,v) \leq d_{(A+1)\omega}(u,v)$ (see Lemma 4.1) and $h(t) \leq h(Mt) \leq M\,h(t)$ for every $M \geq 1$ and $t > 0$. The proof is completed. □

In the sequel, we will proceed to prove Theorems 1.4 and 1.5.

Theorem 4.15. *Let $\theta_1, \theta_2 \leq A\omega$ be closed smooth real $(1,1)$-forms ($A \geq 1$). Let $0 < \delta < 1$, $B \geq 1$, $\tilde{\chi} \in \mathcal{W}^-$ and $u_j \in \tilde{\mathcal{E}}_{\tilde{\chi},(A+1)\omega,B\delta}(X,\theta_j)$ ($j = 1,2$) such that $\tilde{\chi}(-1) = -1$, $\sup_X u_j = 0$ and $\int_X \theta_{u_j}^n \geq \delta$. Assume that there exists a concave increasing function $H : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that, for $j = 1,2$,*

$$\int_X \min\{|\psi_1 - \psi_2|, 1\}(\theta_j + dd^c u_j)^n \leq H(\|\psi_1 - \psi_2\|_{L^1(X)}), \tag{4.40}$$

for every $\psi_1, \psi_2 \in \operatorname{PSH}(X,\omega)$ with $\sup_X \psi_1 = \sup_X \psi_2 = 0$. Then, for every $0 < \gamma < 1$, there exists $C > 0$ depending only on n, X, ω and γ such that

$$d_{cap}(u_1,u_2)^2 \leq C(A\,B)^2\left(h^{\circ n}\left(\frac{\delta}{A(\tau^{1/2} + H(\tau^{1/8})) + A^n\|\theta_1 - \theta_2\|_{\mathscr{C}^0} + d_{(A+1)\omega}(u_1,u_2)}\right)\right)^{-\gamma},$$

where $\tau = \operatorname{dist}_{-1}((\theta_1 + dd^c u_1)^n, (\theta_2 + dd^c u_2)^n)$ and $h(s) = (-\tilde{\chi}(-s))^{1/2}$.

Proof. Without loss of generality, we can assume that $\int_X (\theta_2 + dd^c u_2)^n \geq \int_X (\theta_1 + dd^c u_1)^n$. Denote $\mu_1 = (\theta_1 + dd^c u_1)^n$, $\mu_2 = (\theta_2 + dd^c u_2)^n$ and $c = \dfrac{\mu_1(X)}{\mu_2(X)} \leq 1$. It follows from [16, Theorem 4.7] that there exists a unique $u_3 \in \mathcal{E}(X, \theta_1, P_{\theta_1}[u_1])$ such that

$$\begin{cases}(\theta_1 + dd^c u_3)^n = c\mu_2, \\ \sup_X u_3 = 0.\end{cases} \tag{4.41}$$

By Theorem 4.11, we have

$$d_{\mathrm{cap}}(u_1,u_3)^2 \le C_1(A\,B)^2\left(h^{\circ n}\left(\frac{\delta}{A(x^{1/2}+H(x^{1/8}))+d_{\theta_1}(u_1,u_3)}\right)\right)^{-\gamma}, \tag{4.42}$$

where $x := \mathrm{dist}_{-1}(\mu_1, c\mu_2)$ and $C_1 > 0$ depends only on n, X, ω and γ.

By Theorem 4.15, we have

$$d_{\mathrm{cap}}(u_2,u_3)^2 \le C_2(A\,B)^2\left(h^{\circ n}\left(\frac{\delta}{(1-c)\|\mu_2\|+A^n\|\theta_1-\theta_2\|_{\mathscr{C}^0}+d_{(A+1)\omega}(u_2,u_3)}\right)\right)^{-\gamma} \tag{4.43}$$

where $C_2 > 0$ depends only on n, X, ω and γ.

Combining (4.42), (4.43) and using the fact $d_{\theta_1}(u_1,u_3) = d_{(A+1)\omega}(u_1,u_3) = 0$, we get

$$d_{\mathrm{cap}}(u_1,u_2)^2 \le C_3(A\,B)^2\left(h^{\circ n}\left(\frac{\delta}{A(x^{1/2}+H(x^{1/8}))+(1-c)\|\mu_2\|+R}\right)\right)^{-\gamma}, \tag{4.44}$$

where $R = A^n\|\theta_1-\theta_2\|_{\mathscr{C}^0} + d_{(A+1)\omega}(u_1,u_2)$ and $C_2 > 0$ is a constant depending only on n, X, ω and γ.

Note that

$$(1-c)\|\mu_2\| = \int_X d\mu_2 - \int_X d\mu_1 \le \mathrm{dist}_{-1}(\mu_1,\mu_2) = \tau. \tag{4.45}$$

Then

$$x = \mathrm{dist}_{-1}(\mu_1,c\mu_2) \le \mathrm{dist}_{-1}(\mu_1,\mu_2) + (1-c)\|\mu_2\| \le 2\,\mathrm{dist}_{-1}(\mu_1,\mu_2) = 2\tau. \tag{4.46}$$

Combining (4.44), (4.45) and (4.46), we get

$$d_{\mathrm{cap}}(u_1,u_2)^2 \le C_3(A\,B)^2\left(h^{\circ n}\left(\frac{\delta}{A(\tau^{1/2}+H(\tau^{1/8}))+A^n\|\theta_1-\theta_2\|_{\mathscr{C}^0}+d_{(A+1)\omega}(u_1,u_2)}\right)\right)^{-\gamma},$$

where $C_3 > 0$ depends only on n, X, ω and γ.

This finishes the proof. □

End of the proof of Theorem 1.4. By the assumption, we have $\mu_j := (\theta_j + dd^c u_j)^n$ satisfies (4.40) for $H(t) = M\delta t^\beta$ and $j = 1, 2$. Moreover, it follows from [26, Proposition 4.4] that, for every $\psi \in \mathrm{PSH}(X,\omega)$ with $\sup_X \psi = 0$,

$$\int_X -\psi\mu_j \le B\delta,$$

where $B > 0$ depends on X, ω, M and β. Hence, by using Theorem 4.15 (choose $\gamma = 1/2$), we have

$$d_{\mathrm{cap}}(u_1,u_2)^2 \le C\left(\frac{\tau^{\beta/8}+\|\theta_1-\theta_2\|_{\mathscr{C}^0}+d_{(A+1)\omega}(u_1,u_2)}{\delta}\right)^{2^{-n-1}},$$

where $\tau = \mathrm{dist}_{-1}(\mu_1,\mu_2)$ and $C > 0$ is a constant depending only on n, X, ω, A, M and β.

The proof is completed. □

In order to prove Theorem 1.5, we need the following lemma:

Lemma 4.16. *Let μ be a Radon measure on X vanishing on every pluripolar set. Then, there exists a concave, non-decreasing function $H : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $H(0) = 0$ such that*

$$\int_X \min\{|u - v|, 1\} d\mu \leq H\left(\|u - v\|_{L^1(X)}\right),$$

for every $u, v \in \mathrm{PSH}(X, \omega)$ with $\sup_X u = \sup_X v = 0$.

Proof. For every $t > 0$, we denote

$$h(t) = \sup\left\{\int_X \min\{|u - v|, 1\} d\mu : u, v \in \mathrm{PSH}_{\sup}(X, \omega), \|u - v\|_{L^1(X)} \leq t\right\},$$

where $\mathrm{PSH}_{\sup}(X, \omega) = \{w \in \mathrm{PSH}(X, \omega) : \sup_X w = 0\}$. Then h is non-decreasing. We will show that

$$\lim_{t \to 0^+} h(t) = 0. \tag{4.47}$$

Indeed, if $\lim\limits_{t \to 0^+} h(t) = 2\epsilon > 0$ then there exist sequences $u_j, v_j \in \mathrm{PSH}_{\sup}(X, \omega)$ such that $\|u_j - v_j\|_{L^1(X)} \to 0$ as $j \to \infty$ and

$$\int_X \min\{|u_j - v_j|, 1\} d\mu \geq \epsilon, \tag{4.48}$$

for every j. By the compactness of $\mathrm{PSH}_{\sup}(X, \omega)$, we can assume that $u_j, v_j \to w \in \mathrm{PSH}_{\sup}(X, \omega)$ as $j \to \infty$. Then, it follows from Lemma 4.17 below that

$$\lim_{j \to \infty} \int_X \min\{|u_j - w|, 1\} d\mu = \lim_{j \to \infty} \int_X \min\{|v_j - w|, 1\} d\mu = 0,$$

and it follows that

$$\lim_{j \to \infty} \int_X \min\{|u_j - v_j|, 1\} d\mu = 0.$$

This contradicts with (4.48). Hence, (4.47) is true.

Now, we put

$$M = \sup\{\int_X -w\omega^n : w \in \mathrm{PSH}_{\sup}(X, \omega)\}.$$

For every $m > \frac{1}{2M}$, we also define

$$k_m = \sup\left\{\frac{h(s)}{s} : \frac{1}{m} \leq t \leq 2M\right\} \qquad \text{and} \qquad H_m(t) = k_m t + h(1/m).$$

Then $H_m(t) \geq h(t)$ for every $t \geq 0$ and $\lim_{t \to 0^+} H_m = h(1/m)$. Set $H(t) = \inf_m H_m(t)$. We have H is a concave, non-decreasing function satisfying $H(0) = 0$ and $H \geq h$. In particular,

$$\int_X \min\{|u - v|, 1\} d\mu \leq H\left(\|u - v\|_{L^1(X)}\right),$$

for every $u, v \in \mathrm{PSH}_{\sup}(X, \omega)$.

The proof is completed. □

Lemma 4.17. *Let μ be a Radon measure on X vanishing on every pluripolar set. Assume that u_j, $j \in \mathbb{N} \cup \{\infty\}$, are negative θ-psh functions satisfying $u_j \to u_\infty$ in $L^1(X)$ as $j \to \infty$. Then*

$$\int_X \min\{|u_j - u_\infty|, 1\} d\mu \to 0,$$

as $j \to \infty$.

Proof. Denote $B = \sup_j \|u_j\|_{L^1}$. By Chern-Levine-Nirenberg inequality, there exists $C > 0$ such that

$$\text{cap}\{u_j < -k\} \leq \frac{BC}{k},$$

for every $j \in \mathbb{N} \cup \{\infty\}$ and $k > 0$. Since μ vanishes on pluripolar sets, by [33, Lemma 4.5], there exists $w \in \text{PSH}(X, \omega) \cap L^\infty(X)$ such that $\mu = f\omega_w^n$ for some nonnegative function $f \in L^1(\omega_w^n)$. Let $M > 0$ be a big enough constant such that

$$\int_{\{f>M\}} d\mu < \epsilon/6.$$

We have

$$\begin{aligned}
\mu(\{u_j < -k\}) &= \int_{\{f>M\} \cap \{u_j < -k\}} d\mu + \int_{\{f>M\} \cap \{u_j < -k\}} d\mu \\
&\leq \int_{\{f \leq M\} \cap \{u_j < -k\}} d\mu + \int_{\{f>M\}} d\mu \\
&\leq M(\sup_X w - \inf_X w)\text{cap}\{u_j < -k\} + \epsilon/6.
\end{aligned}$$

It follows that for each $\epsilon > 0$, there exists $k_0 \geq 1$ such that

$$\mu(\{u_j < -k\}) \leq \epsilon/3 \tag{4.49}$$

for every $j \in \mathbb{N} \cup \{\infty\}$ and $k \geq k_0$. Denote $u_{j,k} = \max\{u_j, -k\}$ and $v_{j,k} = \max\{u_{j,k}, u_{\infty,k}\}$. Then for every k, we have $u_{j,k} \to u_{\infty,k}$ in $L^1(X)$ and $v_{j,k} \to u_{\infty,k}$ in capacity as $j \to \infty$. It follows from [35, Lemma 11.5] that

$$\int_X \max\{u_{j,k} - u_{\infty,k}, 0\} d\mu = \int_X (v_{j,k} - u_{\infty,k}) d\mu \to 0,$$

and

$$\int_X (u_{j,k} - u_{\infty,k}) d\mu \to 0,$$

as $j \to \infty$. Combining the last two convergences gives

$$\int_X |u_{j,k} - u_{\infty,k}| d\mu \to 0,$$

as $j \to \infty$. Choose j_0 such that

$$\int_X |u_{j,k_0} - u_{\infty,k_0}| d\mu < \frac{\epsilon}{3},$$

for every $j > j_0$. Using the last inequality and (4.49), we have

$$\int_X \min\{|u_j - u_\infty|, 1\} d\mu \leq \int_{\{u_j, u_\infty \geq -k_0\}} |u_j - u_\infty| d\mu + \mu(\{u_j < -k\}) + \mu(\{u_\infty < -k\})$$
$$\leq \int_{\{u_j, u_\infty \geq -k_0\}} |u_{j,k_0} - u_{\infty,k_0}| d\mu + \frac{2\epsilon}{3} \leq \epsilon,$$

for every $j > j_0$. Thus $\int_X \min\{|u_{a_j} - u_\infty|, 1\} d\mu \to 0$ as $j \to \infty$. □

End of the proof of Theorem 1.5. By Lemma 4.16, there exists a concave, non-decreasing function $H : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ depending only on μ, X and ω such that $H(0) = 0$ and

$$\int_X \min\{|\psi_1 - \psi_2|, 1\} d\mu \leq H\left(\|\psi_1 - \psi_2\|_{L^1(X)}\right),$$

for every $\psi_1, \psi_2 \in \mathrm{PSH}(X, \omega)$ with $\sup_X \psi_1 = \sup_X \psi_2 = 0$.

Moreover, it follows from [7, Proposition 3.2] that there exist a constant $B > 0$ and a function $\tilde{\chi} \in \mathcal{W}^-$ depending on X, ω and μ such that

$$\int -\tilde{\chi}(\psi) d\mu \leq B,$$

for every $\psi \in \mathrm{PSH}(X, \omega)$ with $\sup_X \psi = 0$. In particular, $u_j \in \tilde{\mathcal{E}}_{\tilde{\chi},(A+1)\omega,(A+1)B}(X, \theta_j)$ for $j = 1, 2$. Hence, by Theorem 4.15, there exists $C > 0$ depending only on n, X and ω such that

$$d_{\mathrm{cap}}(u_1, u_2)^2 \leq \frac{C(A+1)^2 B^2}{\delta^2} \left(h^{\circ n} \left(\frac{\delta}{A(\tau^{1/2} + H(\tau^{1/8})) + A^n \|\theta_1 - \theta_2\|_{\mathscr{C}^0} + d_{(A+1)\omega}(u_1, u_2)} \right) \right)^{-1/2},$$

where $\tau = \mathrm{dist}_{-1}(\mu_1, \mu_2)$ and $h(s) = (-\tilde{\chi}(-s))^{1/2}$. Denote

$$f_\mu(t) = \frac{C(A+1)^2 B^2}{\delta^2} \left(h^{\circ n} \left(\frac{\delta}{A(t^{1/2} + H(t^{1/8})) + A^n t} \right) \right)^{-1/2}.$$

We obtain

$$d_{\mathrm{cap}}(u_1, u_2)^2 \leq f_\mu\left(\mathrm{dist}_{-1}(\mu_1, \mu_2) + \|\theta_1 - \theta_2\|_{\mathscr{C}^0} + d_{(A+1)\omega}(u_1, u_2)\right).$$

The proof is completed. □

Remark 4.18. *We explain how to prove Proposition 4.13 using either Theorem 1.5 or 1.4 (in place of Proposition 4.12). This is almost identical to the proof of Proposition 4.13 presented above: the new arguments are only required to show that there is a subsequence of $(u_j)_j$ which is convergent in capacity. To this end, we can assume u_j's are model as we did in the above proof of Proposition 4.13. Next we extract a subsequence $(u_{j_s})_s$ of $(u_j)_j$ such that u_{j_s} converges to some u in L^1, and $\theta^n_{u_{j_s}}$ is convergent. Now applying either Theorem 1.5 or 1.4 (thanks to (4.29)), one sees that the sequence $(u_{j_s})_s$ is convergent in capacity. Consequently $u_{j_s} \to u$ in capacity (see, e.g., [25, Lemma 2.2]).*

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