ON ULTRAGRAPH LEAVITT PATH ALGEBRAS WITH FINITE GELFAND-KIRILLOV DIMENSION

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Abstract. In this article, we prove that every ultragraph Leavitt path algebra is a direct limit of Leavitt path algebras of finite graphs and determine the Gelfand-Kirillov dimension of an ultragraph Leavitt path algebra. We also characterize ultragraph Leavitt path algebras whose simple modules are finitely presented, and show that these algebras have finite Gelfand-Kirillov dimension. Moreover, we construct new classes of simple modules over ultragraph Leavitt path algebras associated with minimal infinite emitters and minimal sinks, which have not yet appeared in the context of Leavitt path algebras of graphs.

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1. Introduction

The study of algebras associated with combinatorial objects is a thriving topic in classical ring theory. One of the key goals of the subject is to establish relationships between combinatorial properties of the initial object and algebraic properties of the associated algebra. Another important direction is the study of the connections with other branches of mathematics, as C^*-algebras and symbolic dynamics. Among interesting examples of algebras associated with combinatorial objects we mention, for example, the following ones: graph C^*-algebras, Leavitt path algebras, higher rank graph algebras, Kumjian-Pask algebras, and ultragraph C^*-algebras (we refer the reader to [1] and [2] for a more comprehensive list).

There is no doubt that, among the non-analytical algebras mentioned above, the Leavitt path algebra associated with a graph figures as the most studied one. For these algebras their structure, and connections with C^*-algebra theory and symbolic dynamics, have been (and still is) studied in detail.

Ultragraphs and ultragraph C^*-algebras were defined by Mark Tomforde in [25] as a unifying approach to C^*-algebras associated with infinite matrices (also

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known as Exel-Laca algebras) and graph C^*-algebras. They have proved to be a key ingredient in the study of Morita equivalence of Exel-Laca and graph C^*-algebras [17]. Recently, Castro, Gonçalves, Royer, Tasca, Wyk, among others, have established nice connections between ultragraph C^*-algebras and the symbolic dynamics of shift spaces over infinite alphabets (see [9], [11], [13] and [24]).

The Leavitt path algebra associated with an ultragraph was defined by Imanfar, Pourabbas and Larki in [16], and by Gonçalves and Royer in [12] in terms of two different definitions. In [8] de Castro, Gonçalves and van Wyk showed that the resulting algebras are isomorphic. As in the C^*-algebraic setting, the ultragraph Leavitt path algebras unify the study of Leavitt path algebras associated with graphs and the algebras associated with infinite matrices. Further to being a convenient way to express both types of algebras mentioned, it was shown in [16, 10] that the class of ultragraph Leavitt path algebras is strictly larger than the class of Leavitt path algebras of graphs.

Since ultragraph Leavitt path algebras form a strictly larger class than Leavitt path algebras of graphs, their study encompasses an extra layer of complexity. Nevertheless, recently several results regarding whether the C^*-algebraic theory of ultragraphs has analogues in the algebraic setting, and whether results about Leavitt path algebras of graphs can be generalized to ultragraph Leavitt path algebras, have been obtained. We mention the following. Gonçalves and Royer [12] realized ultragraph Leavitt path algebras as partial skew group rings. Using this realization they characterized Artinian ultragraph Leavitt path algebras and gave simplicity criteria for these algebras; Gonçalves and Royer [14] extended Chen's construction (see [7]) of simple modules of graph Leavitt path algebras to ultragraph Leavitt path algebras. More namely, they constructed two classes of simple modules of ultragraph Leavitt path algebras associated with sinks v and infinite paths p; de Castro, Gonçalves and van Wyk [8] realized ultragraph Leavitt path algebras as Steinberg algebras. Using this result, Hazrat and the second author [15] constructed additional classes of non-isomorphic simple modules of ultragraph Leavitt path algebras associated with both infinite emitters and pairs (c, f) consisting of closed paths c together with irreducible polynomials $f \in K[x]$; Nam and Nam [22] characterized purely infinite simple ultragraph Leavitt path algebras, and established the Trichotomy Principle for graded simple ultragraph Leavitt path algebras; and Duyen, Gonçalves and the first author [10] proved Exel's Effros-Hahn conjecture on primitive ideals in the ultragraph Leavitt path algebra setting.

The current article is a continuation of this direction. In [3, 4] Alahmedi, Alsulami, Jain and Zelmanov obtained a complete characterization of and a structure theorem for the Leavitt path algebra $L_K(E)$ of a finite graph E having finite Gelfand-Kirillov dimension. Interestingly, it was shown in [5] that this same condition for a finite graph E is equivalent to the Leavitt path algebra $L_K(E)$ whose simple modules are finitely presented. In [23] Rangaswamy gave a complete characterization of Leavitt path algebras of arbitrary graphs with finite Gelfand-Kirillov dimension and those whose simple modules are finitely presented. In [21] Moremo-Fernández and Siles Molina determined the Gelfand-Kirillov dimension of the Leavitt path algebra of an arbitrary graph. Motivated by these interesting results, the main goal of this article is to provide a characterization of ultragraph Leavitt path algebras with finite Gelfand-Kirillov dimension and those whose simple modules are finitely presented, as well as to construct new classes of simple modules of ultragraph Leavitt path algebras which are not isomorphic to these simple modules cited above (even they have not yet appeared in the context of Leavitt path algebras of graphs).

The article is organized as follows. In Section 2, for the reader's convenience, we provide subsequently necessary notions and facts on ultragraphs and ultragraph Leavitt path algebras. We also prove that every ultragraph Leavitt path algebra is a direct limit of Leavitt path algebras of finite graphs (Theorem 2.8). In Section 3, by using Theorem 2.8 and Moremo-Fernández and Siles Molina's result that the Gelfand-Kirillov dimension of algebras commutes with direct limits (see Theorem 3.1 below), we determine the Gelfand-Kirillov dimension of an ultragraph Leavitt path algebra $L_K(\mathcal{G})$ (Theorem 3.4) and show that it is exactly the Gelfand-Kirillov dimension of the Leavitt path algebra of the associated graph of $\mathcal{G}$ (Theorem 3.8). In Section 4, we give new classes of simple modules over ultragraph Leavitt path algebras associated with minimal infinite emitters (Theorem 4.2 and Remark 4.3) and minimal sinks (Theorem 4.5 and Example 4.6), and investigate the finite representation of these simple modules (Proposition 4.4 and Theorem 4.5 (5)) and the simple modules associated with infinite paths (Corollary 4.10). Then, we describe ultragraph Leavitt path algebras whose simple modules are finitely presented, and obtain that these algebras have finite Gelfand-Kirillov dimension (Theorem 4.11).

2. PRELIMINARIES AND SOME USEFUL FACTS

In this section, we recall the definition of an ultragraph Leavitt path algebra and set notation. Also, we prove that every ultragraph Leavitt path algebra is a direct limit of Leavitt path algebras of finite graphs (Theorem 2.8). Consequently, we characterize the von Neumann regularity of an ultragraph Leavitt path algebra which extends [22, Theorem 2.9] to ultragraphs of arbitrary size (Corollary 2.9).

We begin this section by recalling some notions and notes of ultragraph theory introduced by Tomforde in [25] and [26].

Definition 2.1 ([25, Definition 2.1])**.** An *ultragraph* $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ consists of a set of vertices G^0, a set of edges $\mathcal{G}^1$, and functions $s : \mathcal{G}^1 \longrightarrow G^0$ and $r : \mathcal{G}^1 \longrightarrow \mathcal{P}(G^0) \setminus \{\emptyset\}$, where $\mathcal{P}(G^0)$ denotes the set of all subsets of G^0.

In order to define an ultragraph Leavitt path algebra, we need a notion of "generalized vertices".

Definition 2.2 ([25])**.** Let $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ be an ultragraph. Define $\mathcal{G}^0$ to be the smallest subset of $\mathcal{P}(G^0)$ that contains $\{v\}$ for all $v \in G^0$, contains $r(e)$ for all $e \in \mathcal{G}^1$, and is closed under finite unions and finite intersections. Elements of $\mathcal{G}^0$ are called generalized vertices.

A vertex $v \in G^0$ is called a *sink* if $s^{-1}(v) = \emptyset$, and we denote the set of sinks in G^0 by G^0_s. A vertex $v \in G^0$ is called an *infinite emitter* if $s^{-1}(v)$ is infinite. A *singular vertex* is a vertex that is either a sink or an infinite emitter. The set of all singular vertices is denoted by $\mathrm{Sing}(\mathcal{G})$. A vertex $v \in G^0$ is called a *regular vertex* if $s^{-1}(v)$ is finite and non-empty. An ultragraph is called *row-finite* if it has no infinite emitters. An ultragraph $\mathcal{G}$ is called *countable* if G^0 and $\mathcal{G}^1$ are countable sets.

A *finite path* in an ultragraph $\mathcal{G}$ is either an element of $\mathcal{G}^0$ or a sequence $\alpha_1\alpha_2\cdots\alpha_n$ of edges with $s(\alpha_{i+1}) \in r(\alpha_i)$ for all $1 \leq i \leq n-1$ and we say that the path α has *length* $|\alpha| := n$. We consider the elements of $\mathcal{G}^0$ to be paths of length 0. We denote by $\mathcal{G}^*$ the set of all finite paths in $\mathcal{G}$. The maps r and s extend naturally to $\mathcal{G}^*$. Note that when $A \in \mathcal{G}^0$ we define $s(A) = r(A) = A$.

If $\mathcal{G}$ is an ultragraph, then a *closed path* in $\mathcal{G}$ is a path $\alpha = \alpha_1\alpha_2\cdots\alpha_{|\alpha|} \in \mathcal{G}^*$ with $|\alpha| \geq 1$ and $s(\alpha) \in r(\alpha)$. We also say that the closed path α is based at $v = s(\alpha)$. A closed path α is called *simple* if $\alpha \neq \beta^n$ for any closed path β and integer $n \geq 2$. A *cycle* (based at v) is a closed path $\alpha = \alpha_1\alpha_2\cdots\alpha_{|\alpha|}$ (based at v) such that $s(\alpha_i) \neq s(\alpha_j)$ for every $1 \leq i \neq j \leq |\alpha|$. The ultragraph $\mathcal{G}$ is called *acyclic* if $\mathcal{G}$ has no cycles. An *exit* for a cycle α is one of the following:

(1) an edge $e \in \mathcal{G}^1$ such that there exists an i for which $s(e) \in r(\alpha_i)$ but $e \neq \alpha_{i+1}$.
(2) a sink w such that $w \in r(\alpha_i)$ for some i.

In [12] Gonçalves and Royer introduced the Leavitt path algebra of an ultragraph which is an algebraic version of ultragraph C^*-algebras introduced by Mark Tomforde in [25] as an unifying approach to Exel-Laca and graph C^*-algebras.

Definition 2.3 ([12, Definition 2.3])**.** Let $\mathcal{G}$ be an ultragraph and K a field. The *Leavitt path algebra $L_K(\mathcal{G})$ of $\mathcal{G}$ with coefficients in K* is the K-algebra generated by the set $\{s_e, s_e^* \mid e \in \mathcal{G}^1\} \cup \{p_A \mid A \in \mathcal{G}^0\}$, satisfying the following relations for all $A, B \in \mathcal{G}^0$ and $e, f \in \mathcal{G}^1$:

(1) $p_\emptyset = 0, p_A p_B = p_{A\cap B}$ and $p_{A\cup B} = p_A + p_B - p_{A\cap B}$;
(2) $p_{s(e)}s_e = s_e = s_e p_{r(e)}$ and $p_{r(e)}s_e^* = s_e^* = s_e^* p_{s(e)}$;
(3) $s_e^* s_f = \delta_{e,f} p_{r(e)}$;
(4) $p_v = \sum_{s(e)=v} s_e s_e^*$ for any regular vertex v;

where p_v denotes $p_{\{v\}}$ and δ is the Kronecker delta.

It is worth mentioning the following note.

Remark 2.4. (1) There have been different definitions of Leavitt path algebras of ultragraphs, and the difference of these definitions lies in how the set of generalized vertices are defined. Given an ultragraph $\mathcal{G}$, let $\mathcal{B}$ denote the smallest subset of $\mathcal{P}(G^0)$ that contains $\{v\}$ for all $v \in G^0$, contains $r(e)$ for all $e \in \mathcal{G}^1$, and is closed under relative complements, finite unions and finite intersections. We denote by $L_K(\mathcal{G}_r)$ the Leavitt path algebra associated with $\mathcal{G}$ by allowing $A, B \in \mathcal{B}$ in item (1) of Definition 2.3, that means, $L_K(\mathcal{G}_r)$ is the algebra as defined in [16, Definition 2.1]. However, in [8, Proposition 5.2] the authors showed that $L_K(\mathcal{G}_r)$ and $L_K(\mathcal{G})$ are isomorphic to each other.

(2) Every (directed) graph $E = (E^0, E^1, r_E, s_E)$ may be considered as an ultragraph $\mathcal{G}_E = (G^0_E, \mathcal{G}^1_E, r_{\mathcal{G}_E}, s_{\mathcal{G}_E})$, where $G^0_E = E^0$, $\mathcal{G}^1_E = E^1$, and $r_{\mathcal{G}_E}(e) = \{r_E(e)\}$ and $s_{\mathcal{G}_E}(e) = s_E(e)$ for all $e \in E^1$. In this case, we have that $\mathcal{G}^0_E$ is the set of all finite subsets of E^0, and the Leavitt path algebra $L_K(E)$ is naturally isomorphic to $L_K(\mathcal{G}_E)$. We refer the reader to [1] and [2] for more details about Leavitt path algebras of graphs.

We usually denote $s_A := p_A$ for $A \in \mathcal{G}^0$ and $s_\alpha := s_{e_1} \cdots s_{e_n}$ for $\alpha = e_1 \cdots e_n \in \mathcal{G}^*$. It is easy to see that the mappings given by $p_A \longmapsto p_A$ for $A \in \mathcal{G}^0$, and $s_e \longmapsto s^*_e$, $s^*_e \longmapsto s_e$ for $e \in \mathcal{G}^1$, produce an involution on the algebra $L_K(\mathcal{G})$, and for any path $\alpha = \alpha_1 \cdots \alpha_n$ there exists $s^*_\alpha := s^*_{e_n} \cdots s^*_{e_1}$. Also, $L_K(\mathcal{G})$ has the following *universal property*: if $\mathcal{A}$ is a K-algebra generated by a family of elements $\{b_A, c_e, c^*_e \mid A \in \mathcal{G}^0, e \in \mathcal{G}^1\}$ satisfying the relations analogous to (1) - (4) in Definition 2.3, then there always exists a K-algebra homomorphism $\varphi : L_K(\mathcal{G}) \longrightarrow \mathcal{A}$ given by $\varphi(p_A) = b_A$, $\varphi(s_e) = c_e$ and $\varphi(s^*_e) = c^*_e$. Furthermore, we denote another useful properties as follows.

Lemma 2.5 ([12, Theorem 3.10]). *For an ultragraph $\mathcal{G}$ and a field K, then the Leavitt path algebra $L_K(\mathcal{G})$ has the following properties:*

(1) *All elements of the set $\{p_A, s_e, s^*_e \mid A \in \mathcal{G}^0 \setminus \{\emptyset\}, e \in \mathcal{G}^1\}$ are nonzero.*

(2) *$L_K(\mathcal{G})$ is of the form*

$$\mathrm{Span}_K\{s_\alpha p_A s^*_\beta \mid \alpha, \beta \in \mathcal{G}^*, A \in \mathcal{G}^0 \text{ and } r(\alpha) \cap A \cap r(\beta) \neq \emptyset\}.$$

Furthermore, $L_K(\mathcal{G})$ is a $\mathbb{Z}$-graded K-algebra by the grading

$$L_K(\mathcal{G})_n = \mathrm{Span}_K\{s_\alpha p_A s^*_\beta \mid \alpha, \beta \in \mathcal{G}^*, A \in \mathcal{G}^0 \text{ and } |\alpha| - |\beta| = n\} \quad (n \in \mathbb{Z}).$$

Proof. Item (1) follows from [12, Theorem 3.10], and item (2) follows from the last paragraph of the proof of [12, Theorem 3.10]. We should mention that [12, Theorem 3.10] was proved for the case of countable ultragraphs $\mathcal{G}$. However, the assumption on the cardinalilty of the ultragraph was not used in this proof, and so the theorem is valid for ultragraphs of arbitrary size. □

The following lemma is useful to prove the main result of this section.

Lemma 2.6 (cf. [22, Lemma 2.6]). *Let $\mathcal{G}$ be an ultragraph and K a field. Then the algebra $L_K(\mathcal{G})$ is generated by $\left\{s_e, s^*_e \mid e \in \mathcal{G}^1\right\} \cup \{p_v \mid v \in \mathrm{Sing}(\mathcal{G})\}$.*

Proof. The lemma was proved in [22, Lemma 2.6] for the case of countable ultragraphs $\mathcal{G}$. However, the assumption on the cardinalilty of the ultragraph was not used in this proof, and so the statement is valid for ultragraphs of arbitrary size. $\square$

Let $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ be an ultragraph and let F be a finite subset of $\mathcal{G}^1 \cup \mathrm{Sing}(\mathcal{G})$. Write $F^0 := F \cap \mathrm{Sing}(\mathcal{G})$ and $F^1 := F \cap \mathcal{G}^1 = \{e_1, e_2, \ldots, e_n\}$. Following [16], we construct a finite graph G_F as follows. For each $\omega = (\omega_1, \ldots, \omega_n) \in \{0,1\}^n \setminus \{(0,0,\ldots,0)\}$, we define

$$r(\omega) := \bigcap_{\omega_i=1} r(e_i) \setminus \bigcup_{\omega_j=0} r(e_j) \text{ and } R(\omega) := r(\omega) \setminus F^0.$$

Notice that $r(\omega) \cap r(\nu) = \varnothing$ for distinct $\omega, \nu \in \{0,1\}^n \setminus \{(0,0,\ldots,0)\}$. Let

$$\Gamma_0 := \{\omega \in \{0,1\}^n \setminus \{(0,0,\ldots,0)\} \mid \text{there are vertices } v_1, \ldots, v_m \text{ such that } R(\omega) = \{v_1, \ldots, v_m\} \text{ and } \varnothing \neq s^{-1}(v_i) \subseteq F^1 \text{ for } 1 \leq i \leq m\}$$

and

$$\Gamma_F := \{\omega \in \{0,1\}^n \setminus \{(0,0,\ldots,0)\} \mid R(\omega) \neq \varnothing \text{ and } \omega \notin \Gamma_0\}.$$

Now we define the finite graph $G_F = (G_F^0, G_F^1, r_F, s_F)$ as follows:

$$\begin{aligned} G_F^0 &:= F^0 \cup F^1 \cup \Gamma_F, \text{ and} \\ G_F^1 &:= \{(e,f) \in F^1 \times F^1 \mid s(f) \in r(e)\} \\ &\quad \cup \{(e,v) \in F^1 \times F^0 \mid v \in r(e)\} \\ &\quad \cup \{(e,\omega) \in F^1 \times \Gamma_F \mid \omega_i = 1 \text{ when } e = e_i\} \end{aligned}$$

with

$$\begin{array}{lll} s_{G_F}((e,f)) = e & s_{G_F}((e,v)) = e & s_{G_F}((e,\omega)) = e \\ r_{G_F}((e,f)) = f & r_{G_F}((e,v)) = v & r_{G_F}((e,\omega)) = \omega. \end{array}$$

The following lemma gives us criteria for ultragraphs containing cycles and cycles with exits.

Lemma 2.7. *For an ultragraph $\mathcal{G}$, the following statements hold:*

(1) *$\mathcal{G}$ is acyclic if and only if G_F is acyclic for every non-empty finite subset F of $\mathcal{G}^1 \cup \mathrm{Sing}(\mathcal{G})$;*

(2) *$\mathcal{G}$ contains a cycle with an exit if and only if there exists a non-empty finite subset F of $\mathcal{G}^1 \cup \mathrm{Sing}(\mathcal{G})$ such that G_F contains a cycle with an exit.*

Proof. (1) It was proved in [22, Lemma 2.8] under the assumption that $\mathcal{G}$ is countable. However, the assumption on the cardinalilty of the ultragraph was not used in this proof, and so the statement is valid for ultragraphs of arbitrary size.

(2) ($\Longrightarrow$). Assume that $\mathcal{G}$ contains a cycle $\alpha = e_1 e_2 \cdots e_n$ with an exit. We then have the following cases:

Case 1: there exist an edge $e \in \mathcal{G}^1$ and a number $1 \leq i \leq n$ such that $s(e) \in r(e_i)$ but $e \neq e_{i+1}$, where $e_{n+1} := e_1$. Let

$$F := \{e, e_i \mid 1 \leq i \leq n\} \subseteq \mathcal{G}^1.$$

We obtain that the graph G_F contains a cycle $c = (e_1, e_2) \cdots (e_{n-1}, e_n)(e_n, e_1)$ with an exit $f = (e_i, e)$.

Case 2: there exists a sink $w \in G^0$ such that $w \in r(\alpha_i)$ for some $1 \leq i \leq n$. Let $F := \{w, e_i \mid 1 \leq i \leq n\} \subseteq \mathcal{G}^1 \cup \mathrm{Sing}(\mathcal{G})$. We then have that the graph G_F contains a cycle $c = (e_1, e_2) \cdots (e_{n-1}, e_n)(e_n, e_1)$ with an exit $f = (e_i, w)$.

Therefore, in any case, we arrive at the statement.

($\Longleftarrow$). Suppose there exists a non-empty finite subset F of $\mathcal{G}^1 \cup \mathrm{Sing}(\mathcal{G})$ such that G_F contains a cycle c with an exit γ. By renumbering edges in F^1, without loss of generality, we may assume that $F^1 = \{e_1, e_2, \ldots, e_n\}$ and

$$c = (e_1, e_2) \cdots (e_{m-1}, e_m)(e_m, e_1),$$

where $m < n$ and e_i's are in F such that $s_{\mathcal{G}}(e_{i+1}) \in r_{\mathcal{G}}(e_i)$ for all $1 \leq i \leq m-1$, and $s_{\mathcal{G}}(e_1) \in r_{\mathcal{G}}(e_m)$. Consider the closed path $\alpha := e_1 e_2 \cdots e_m$ in $\mathcal{G}$. If $s_{\mathcal{G}}(e_i) \neq s_{\mathcal{G}}(e_j)$ for all $1 \leq i \neq j \leq m$, then we have that α is a cycle in $\mathcal{G}$. Since γ is an exit for c, there exists an number $1 \leq i \leq m$ such that $s_{G_F}(\gamma) = e_i$ and $\gamma \neq (e_i, e_{i+1})$, where $e_{m+1} := e_1$. Consider the following cases:

Case 1: $\gamma = (e, f) \in F^1 \times F^1$ with $s_{\mathcal{G}}(f) \in r_{\mathcal{G}}(e)$. We receive that $e = s_{G_F}(\gamma) = e_i$ and $f \neq e_{i+1}$, and so f is an exit for α.

Case 2: $\gamma = (e, v) \in F^1 \times F^0$ with $v \in r_{\mathcal{G}}(e)$. We have $e = s_{G_F}(\gamma) = e_i$. If v is a sink in $\mathcal{G}$, then v is an exit for α.

Case 3: $\gamma = (e, \omega) \in F^1 \times \Gamma_F$ with $\omega_j = 1$ when $e = e_j$ $(1 \leq j \leq n)$. We obtain that $e = s_{G_F}(\gamma) = e_i$. Since $\omega \in \Gamma_F$, we must have $R(\omega) := r(\omega) \setminus F^0 \neq \varnothing$ and $\omega \notin \Gamma_0$. This implies that $R(\omega) \subseteq r(\omega) = \cap_{\omega_j = 1} r_{\mathcal{G}}(e_j) \setminus \cup_{\omega_k = 0} r_{\mathcal{G}}(e_k) \subseteq r_{\mathcal{G}}(e_i)$, and we also have the following two subcases:

Case 3.1: $R(\omega)$ is infinite. This shows that $r_{\mathcal{G}}(e_i)$ is infinite, and so there exists a vertex $v \in r_{\mathcal{G}}(e_i)$ such that $v \neq s_{\mathcal{G}}(e_{i+1})$. If v is a sink, then v is an exit for α. If v is not sink, then there exists an edge $f \in \mathcal{G}^1$ such that $s_{\mathcal{G}}(f) = v$ and $f \neq e_{i+1}$. Therefore, f is an exit for α.

Case 3.2: $R(\omega) = \{v_1, v_2, \ldots, v_t\} \subseteq G^0$ with either v_l is a sink for some $1 \leq l \leq t$, or there exists an edge $f \in s_{\mathcal{G}}^{-1}(v_k) \setminus F$ for some $1 \leq k \leq t$. If the first case happens, then $v_l \in r_{\mathcal{G}}(e_i)$, and so v_l is an exit for α. If the second case happens, then $f \neq e_{i+1}$ and $s_{\mathcal{G}}(f) = v_k$. Since $R(\omega) \subseteq r_{\mathcal{G}}(e_i)$, $v_k \in r_{\mathcal{G}}(e_i)$. This implies that f is an exit for α.

Consider the case when $s_{\mathcal{G}}(e_i) = s_{\mathcal{G}}(e_j)$ for some $1 \leq i < j \leq m$. Then, there exist two integers k and l such that $i \leq k < l \leq j$, $s_{\mathcal{G}}(e_k) = s_{\mathcal{G}}(e_l)$ and $s_{\mathcal{G}}(e_{k'}) \neq s_{\mathcal{G}}(e_{l'})$ for all $k \leq k' \neq l' \leq l$ with $(k', l') \neq (k, l)$. This implies that $\beta := e_k e_{k+1} \cdots e_l$ is a cycle in $\mathcal{G}$ with an exit e_{l+1}, where $e_{m+1} := e_1$.

Therefore, in any case the ultragraph $\mathcal{G}$ always contains a cycle with exits, thus finishing the proof. □

Let $\mathcal{G}$ be an arbitrary ultragraph. We denote by $\mathcal{F}(\mathcal{G})$ the set of all finite subsets of $\mathcal{G}^1 \cup \mathrm{Sing}(\mathcal{G})$. It is obvious that $\mathcal{F}(\mathcal{G})$ is directed since the union of two elements is their join. Now we are able to present the main result of this section, which plays an important role in the proof of the main result (Theorem 3.4) of the next section.

Theorem 2.8. *Let K be a field, $\mathcal{G}$ an ultragraph, F an element of $\mathcal{F}(\mathcal{G})$, and $B(F)$ the subalgebra of $L_K(\mathcal{G})$ generated by the set $\{p_v, s_e, s_e^* \mid e, v \in F\}$. Then the following statements hold:*
(1) $B(F) \cong L_K(G_F)$;
(2) $L_K(\mathcal{G}) = \bigcup_{F\in\mathcal{F}(\mathcal{G})} B(F) = \varinjlim_{F\in\mathcal{F}(\mathcal{G})} B(F)$;
(3) $L_K(\mathcal{G}) \cong \varinjlim_{F\in\mathcal{F}(\mathcal{G})} L_K(G_F)$.

Proof. (1) We define the elements $\{P_x \mid x \in G_F^0\}$ and $\{S_y, S_y^* \mid y \in G_F^1\}$ of $L_K(\mathcal{G})$ by setting

$$P_x = \begin{cases} s_e s_e^* & \text{if } x = e, \\ p_v(1 - \sum_{f\in F^1} s_f s_f^*) & \text{if } x = v, \\ p_{\bigcap_{w_i=1} r(e_i)}(1 - p_{\bigcup_{w_j=0} r(e_j)})(1 - p_{F^0})(1 - \sum_{f\in F^1} s_f s_f^*) & \text{if } x = \omega, \end{cases}$$

$$S_y = \begin{cases} P_f s_e^* & \text{if } y = (e, f), \\ s_e P_v & \text{if } y = (e, v), \\ s_e P_\omega & \text{if } y = (e, \omega) \end{cases} \qquad \text{and} \qquad S_y^* = \begin{cases} s_e P_f & \text{if } y = (e, f), \\ P_v s_e^* & \text{if } y = (e, v), \\ P_\omega s_e^* & \text{if } y = (e, \omega). \end{cases}$$

By repeating verbatim the argument in the proof of [20, Proposition 4.2], we have $P_x P_{x'} = \delta_{x,x'} P_x$ for all $x \in G_F^0$, $P_{s(y)} S_y = S_y = S_y P_{r(y)}$, $S_y^* P_{s(y)} = S_y^* = P_{r(y)} S_y^*$ and $S_y^* S_{y'} = \delta_{y,y'} P_{r(y)}$ for all $y \in G_F^1$, and $P_x = \sum_{\{y\in G_F^1 \mid x=s(y)\}} S_y S_y^*$ for all regular vertex $x \in G_F^0$, where δ is the Kronecker delta. Then, by the universal property of Leavitt path algebras of graphs, there exists a K-algebra homomorphism $\pi : L_K(G_F) \longrightarrow L_K(\mathcal{G})$ such that $\pi(x) = P_x$, $\pi(y) = S_y$ and $\pi(y^*) = S_y^*$ for all $x \in G_F^0$ and $y \in G_F^1$. Clearly, π is a $\mathbb{Z}$-graded homomorphism. By repeating approach described in the proof of [16, Lemma 2.13 (ii)], we have $P_x \neq 0$ for all $x \in G_F^0$. Then, by [27, Theorem 4.8], π is injective. Moreover, by repeating verbatim the argument in the proof of [20, Proposition 4.2], we obtain that the family $\{Q_x, S_y, S_y^* \mid x \in G_F^0, y \in G_y^1\}$ generates $B(F)$, that means, $\pi(L_K(G_F)) = B(F)$. Therefore, $L_K(G_F)$ is isomorphic to $B(F)$.

(2) It immediately follows from Lemma 2.6.

(3) It follows from items (1) and (2), thus finishing the proof. □

A (not necessarily unital) ring R is called *von Neumann regular* in case for every $r \in R$ there exists $s \in R$ such that $r = rsr$. A *matricial K-algebra* is a finite direct sum of full finite dimensional matrix algebras over the field K. A *locally matricial K-algebra* is a direct limit of matricial K-algebras (with not necessarily-unital transition homomorphisms). In [22, Theorem 2.9] the authors showed that the ultragraph Leavitt path algebras arising from acyclic countable ultragraphs are precisely the von Neumann regular Leavitt path algebras, and in this case they are exactly locally matricial algebras. We close this section with the following corollary, extending this result to Leavitt path algebras of ultragraphs of arbitrary size.

Corollary 2.9. *Let $\mathcal{G}$ be an ultragraph and K a field. Then the following conditions are equivalent:*
(1) *$L_K(\mathcal{G})$ is von Neumann regular;*
(2) *$\mathcal{G}$ is acyclic;*
(3) *$L_K(\mathcal{G})$ is a locally matricial K-algebra.*

Proof. (1)$\Longrightarrow$(2). It was proved in the direction (1)$\Longrightarrow$(2) of [22, Theorem 2.9] under the assumption that $\mathcal{G}$ is countable. However, the assumption on the cardinalilty of the ultragraph was not used in this proof, and so the statement is valid for ultragraphs of arbitrary size.

(2)$\Longrightarrow$(3). By using Lemma 2.7 (1) and Theorem 2.8 (3), and repeating verbatim the argument in the proof of the direction (2)$\Longrightarrow$(3) of [22, Theorem 2.9] we immediately obtain the statement.

(3)$\Longrightarrow$(1). It is well known that every matricial K-algebra is von Neumann regular, and so is a direct limit of such algebras, thus finishing our proof. □

3. The Gelfand-Kirillov dimension of an ultragraph Leavitt path algebra

In this section, based on Theorem 2.8, we determine the Gelfand-Kirillov dimension of an ultragraph Leavitt path algebra $L_K(\mathcal{G})$ (Theorem 3.4) and show that it is exactly the Gelfand-Kirillov dimension of the Leavitt path algebra of the associated graph of $\mathcal{G}$ (Theorem 3.8).

We begin this section by recalling some general notions and facts on the Gelfand-Kirillov dimension of algebras. Given a field K and a finitely generated K-algebra A. *The Gelfand-Kirillov dimension* of A (GKdim(A) for short) is defined to be

$$\mathrm{GKdim}(A) := \limsup_{n\to\infty} \log_n(\dim(V^n)),$$

where V is a finite dimensional subspace of A that generates A as an algebra over K. This definition is independent of the choice of V. If A does not happen to be finitely generated over K, the Gelfand-Kirillov dimension of A is defined to be

$$\mathrm{GKdim}(A) = \sup\{\mathrm{GKdim}(B) \mid B \text{ is a finitely generated subalgebra of } A\}.$$

It is well known (e.g., [18, Lemma 3.1]) that the inequality $\mathrm{GKdim}(B) \le \mathrm{GKdim}(A)$ holds whenever B is a subalgebra of A, or if B is a factor algebra of A. See [18] for a general treatment of the Gelfand-Kirillov dimension. The following result of Moremo-Fernández and Siles Molina will play an important role in our analysis.

Theorem 3.1 ([21, Theorem 3.1]). *The Gelfand-Kirillov dimension of algebras commutes with direct limits.*

A cycle $\alpha = \alpha_1\alpha_2\cdots\alpha_n$ is called an *exclusive cycle* if there does not exist a cycle $\beta = \beta_1\beta_2\cdots\beta_m$ which is different from a cyclic permutation of α and such that $s(\alpha_i), s(\beta_j) \in r(\alpha_{i-1}) \cap r(\beta_{j-1})$ for some $1 \le i \le n$ and $1 \le j \le m$, where $r(\alpha_0) := r(\alpha_n)$ and $r(\beta_0) := r(\beta_m)$. Equivalently, a cycle $\alpha = \alpha_1\alpha_2\cdots\alpha_n$ is called an *exclusive cycle* if there does not exist a cycle $\beta = \beta_1\beta_2\cdots\beta_m$ such that $s(\alpha_i), s(\beta_j) \in r(\alpha_{i-1}) \cap r(\beta_{j-1})$ and $(\alpha_{i-1}, \alpha_i) \neq (\beta_{j-1}, \beta_j)$ for some $1 \le i \le n$ and $1 \le j \le m$. In other case, we say that α is a *non-exclusive cycle*. We say that ultragraph $\mathcal{G}$ satisfies *Condition* (EXC) if every cycle of $\mathcal{G}$ is exclusive. The following lemma provides us criteria for ultragraphs having Condition (EXC).

Lemma 3.2. *An ultragraph $\mathcal{G}$ satisfies Condition* (EXC) if and only if G_F satisfies Condition (EXC) for every non-empty finite subset F of $\mathcal{G}^1 \cup \mathrm{Sing}(\mathcal{G})$.

Proof. ($\Longrightarrow$). Assume that there exists a non-empty finite subset F of $\mathcal{G}^1 \cup \mathrm{Sing}(\mathcal{G})$ such that G_F does not satisfy Condition (EXC). We then obtain that G_F has two cycles c and d having a common vertex such that c is different from a cyclic permutation of d. Write $c = c_1c_2\cdots c_n$ and $d = d_1d_2\cdots d_m$, where $c_i = (e_i, e_{i+1}) \in F^1 \times F^1$ with $s(e_{i+1}) \in r(e_i)$ for all $1 \le i \le n-1$, $c_n = (e_n, e_1)$ with $s(e_1) \in r(e_n)$, and $d_j = (f_j, f_{j+1}) \in F^1 \times F^1$ with $s(f_{j+1}) \in r(f_j)$ for all $1 \le j \le m-1$, $d_m = (f_m, f_1)$ with $s(f_1) \in r(f_m)$. Since c and d have a common vertex, without loss of generality, we may assume that $e_1 = s_{G_F}(c) = s_{G_F}(d) = f_1$. Consider the cycles $\alpha = e_1e_2\cdots e_n$ and $\beta = f_1f_2\cdots f_m$ in $\mathcal{G}$. We then have $s(e_1) = s(f_1) \in r(e_n) \cap r(f_m)$, and so α is a non-exclusive cycle in $\mathcal{G}$. This implies that $\mathcal{G}$ does not satisfy Condition (EXC).

($\Longleftarrow$). Assume that $\mathcal{G}$ does not satisfy Condition (EXC). Then, there exist two cycles $\alpha = \alpha_1\alpha_2\cdots\alpha_n$ and $\beta = \beta_1\beta_2\cdots\beta_m$ such that $s(\alpha_i), s(\beta_j) \in r(\alpha_{i-1}) \cap r(\beta_{j-1})$ and $(\alpha_{i-1}, \alpha_i) \neq (\beta_{j-1}, \beta_j)$ for some $1 \le i \le n$ and $1 \le j \le m$, where $r(\alpha_0) := r(\alpha_n)$ and $r(\beta_0) := r(\beta_m)$. Let $F := \{\alpha_1, \alpha_2, \dots, \alpha_n, \beta_1, \beta_2, \dots, \beta_m\} \subseteq \mathcal{G}^1$. We then have that $\alpha' = (\alpha_i, \alpha_{i+1})\cdots(\alpha_{n-1}, \alpha_n)(\alpha_n, \alpha_1)(\alpha_1, \alpha_2)\cdots(\alpha_{i-1}, \alpha_i)$ and $\beta' = (\alpha_i, \alpha_{i+1})\cdots(\alpha_{n-1}, \alpha_n)(\alpha_n, \alpha_1)(\alpha_1, \alpha_2)\cdots(\alpha_{i-2}, \alpha_{i-1})(\alpha_{i-1}, \beta_j)(\beta_j, \beta_{j+1})$ $\cdots(\beta_{m-1}, \beta_m)(\beta_m, \alpha_1)(\beta_1, \beta_2)\cdots(\beta_{j-2}, \beta_{j-1})(\beta_{j-1}, \alpha_i)$ are two closed paths in G_F with $s_{G_F}(\alpha') = \alpha_i = s_{G_F}(\beta')$, and so G_F does not satisfy Condition (EXC), thus finishing our proof. □

For two exclusive cycles $\alpha = \alpha_1\alpha_2\cdots\alpha_n$ and $\beta = \beta_1\beta_2\cdots\beta_m$, we write $\alpha \Rightarrow \beta$ if there exists a path p such that $s(p) \in r(\alpha_i)$ and $s(\beta_j) \in r(p)$ for some $1 \le i \le n$ and $1 \le j \le m$. A sequence of exclusive cycles $\alpha_1, \alpha_2, \dots, \alpha_k$ is a *chain of cycles of length k* if $\alpha_1 \Rightarrow \alpha_2 \Rightarrow \cdots \Rightarrow \alpha_k$. We say that such a chain has an *exit* if the cycle α_k has an exit. The following lemma provides us criteria for ultragraphs having chains of cycles of finite length and chains of cycles of finite length with exits.

Lemma 3.3. *Let $\mathcal{G}$ be an ultragraph having Condition* (EXC). Then the following statements hold:

(1) $\mathcal{G}$ has a chain of cycles of length t if only if there exists a non-empty finite subset F of $\mathcal{G}^1 \cup \mathrm{Sing}(\mathcal{G})$ such that G_F has a chain of cycles of length t.

(2) $\mathcal{G}$ has a chain of cycles of length t' with an exit if only if there exists a non-empty finite subset F of $\mathcal{G}^1 \cup \mathrm{Sing}(\mathcal{G})$ such that G_F has a chain of cycles of length t' with an exit.

(3) If the maximal length of chains of cycles in $\mathcal{G}$ is equal to t and the maximal length of chains of cycles in $\mathcal{G}$ with an exit is equal to t', then there exists a non-empty finite subset F of $\mathcal{G}^1 \cup \mathrm{Sing}(\mathcal{G})$ such that the maximal length of chains of cycles in G_F is equal to t and the maximal length of chains of cycles in G_F with an exit is equal to t'.

Proof. (1) ($\Longrightarrow$). Assume that $\alpha_1 \Rightarrow \alpha_2 \Rightarrow \cdots \Rightarrow \alpha_t$ is a chain of exclusive cycles of length t in $\mathcal{G}$. Write $\alpha_i = e_1^{(i)} e_2^{(i)} \dots e_{n_i}^{(i)}$, where $e_j^{(i)} \in \mathcal{G}^1$ for all $1 \le i \le t$ and $1 \le j \le n_i$. Then, for each $1 \le i \le t-1$, there exists a path p_i in $\mathcal{G}$ such that $s(p_i) \in r(e_{k_i}^{(i)})$ and $s(e_{k'_i}^{(i+1)}) \in r(p_i)$ for some $1 \le k_i \le n_i$ and $1 \le k'_i \le n_{i+1}$. Write $p_i = f_1^{(i)} f_2^{(i)} \cdots f_{m_i}^{(i)}$ with $f_j^{(i)} \in \mathcal{G}^1$ for all $1 \le i \le t-1$ and $1 \le j \le m_i$. Let

$$F := \{e_j^{(i)}, f_k^{(l)} \mid 1 \le i \le t, 1 \le l \le t-1, 1 \le j \le n_i, 1 \le k \le m_i\} \subseteq \mathcal{G}^1.$$

We then have that $\beta_i = (e_1^{(i)}, e_2^{(i)})(e_2^{(i)}, e_3^{(i)}) \dots (e_{n_i-1}^{(i)}, e_{n_i}^{(i)})(e_{n_i}^{(i)}, e_1^{(i)})$ are exclusive cycles in G_F for all $1 \le i \le t$, by Lemma 3.2. Let

$$q_i := (e_{k_i}^{(i)}, f_1^{(i)})(f_1^{(i)}, f_2^{(i)}) \cdots (f_{m_i-1}^{(i)}, f_{m_i}^{(i)})(f_{m_i}^{(i)}, e_{k'_i}^{(i+1)})$$

for $1 \le i \le t-1$. We then have $s_{G_F}(q_i) = e_{k_i}^{(i)}$ and $r_{G_F}(q_i) = e_{k'_i}^{(i+1)}$ for all $1 \le i \le t-1$. This implies that $\beta_i \Rightarrow \beta_{i+1}$ for all $1 \le i \le t-1$, and so we obtain that $\beta_1 \Rightarrow \beta_2 \Rightarrow \cdots \Rightarrow \beta_t$ is a chain of cycles of lenght t in G_F.

($\Longleftarrow$). Assume that there exists a non-empty finite subset F of $\mathcal{G}^1 \cup \mathrm{Sing}(\mathcal{G})$ such that G_F has a chain of exclusive cycles of length t, say $c_1 \Rightarrow c_2 \Rightarrow \cdots \Rightarrow c_t$. Write

$$c_i = (e_1^{(i)}, e_2^{(i)})(e_2^{(i)}, e_3^{(i)}) \dots (e_{n_i-1}^{(i)}, e_{n_i}^{(i)})(e_{n_i}^{(i)}, e_1^{(i)}),$$

where $e_j^{(i)} \in F$ and $s_{\mathcal{G}}(e_{j+1}^{(i)}) \in r_{\mathcal{G}}(e_j^{(i)})$ for all $1 \le i \le t$ and $1 \le j \le n_i$ (with $e_{n_i+1}^{(i)} := e_1^{(i)}$). For each $1 \le i \le t-1$, we have $c_i \Rightarrow c_{i+1}$, and so there exists a path p_i in G_F such that $s_{G_F}(p_i) = e_{k_i}^{(i)}$ and $r_{G_F}(p_i) = e_{k'_i}^{(i+1)}$ for some $1 \le k_i \le n_i$ and $1 \le k'_i \le n_{i+1}$. Write

$$p_i = (f_1^{(i)}, f_2^{(i)})(f_2^{(i)}, f_3^{(i)}) \cdots (f_{m_i-1}^{(i)}, f_{m_i}^{(i)}),$$

where $f_j^{(i)} \in F$, $f_1^{(i)} = s_{G_F}(p_i) = e_{k_i}^{(i)}$, $f_{m_i}^{(i)} = r_{G_F}(p_i) = e_{k'_i}^{(i+1)}$ and $s_{\mathcal{G}}(f_{j+1}^{(i)}) \in r_{\mathcal{G}}(f_j^{(i)})$ for all $1 \le i \le t-1$ and $1 \le j \le m_i$. Let $\alpha_i = e_1^{(i)} e_2^{(i)} \cdots e_{n_i}^{(i)}$ and $\beta_j := f_1^{(i)} f_2^{(i)} \cdots f_{m_i}^{(i)}$ for all $1 \le i \le t$ and $1 \le j \le t-1$. We then have that α_i's are exclusive cycles in $\mathcal{G}$ such that $s_{\mathcal{G}}(\beta_i) \in r_{\mathcal{G}}(e_{k_i}^{(i)})$ and $s_{\mathcal{G}}(e_{k'_i}^{(i+1)}) \in r_{\mathcal{G}}(\beta_i)$ for all

$1 \leq i \leq t-1$. Therefore, we have a chain of exclusive cycles $\alpha_1 \Rightarrow \alpha_2 \Rightarrow \cdots \Rightarrow \alpha_t$ in $\mathcal{G}$ of length t.

(2) It follows from item (1) and Lemma 2.7.

(3) It follows from items (1) and (2), thus finishing the proof. □

Now we are able to present the first main result of this section which determines the Gelfand-Kirillov dimension of an ultragraph Levitt path algebra and extends Moremo-Fernández and Siles Molina's result (see [21, Theorem 3.21]) to the ultragraph case.

Theorem 3.4. *Let $\mathcal{G}$ be an ultragraph and K a field. Then the following statements hold:*

(1) $\mathrm{GKdim}(L_K(\mathcal{G})) < \infty$ *if and only if* $\mathcal{G}$ *satisfies Condition* (EXC) and the maximal length of chains of cycles in $\mathcal{G}$ is finite.

(2) Assume that $\mathcal{G}$ satisfies Condition (EXC) and the maximal length of chains of cycles in $\mathcal{G}$ is finite, say t, and the maximal length of chains of cycles in $\mathcal{G}$ with an exit is finite, say t'. Then

$$\mathrm{GKdim}(L_K(\mathcal{G})) = \max\{2t-1, 2t'\}.$$

Proof. (1) ($\Longrightarrow$). Assume that $\mathrm{GKdim}(L_K(\mathcal{G})) < \infty$. Then, by Theorem 2.8 (1) and [18, Lemma 3.1], $\mathrm{GKdim}(L_K(G_F)) \leq \mathrm{GKdim}(L_K(\mathcal{G})) < \infty$ for all $F \in \mathcal{F}(\mathcal{G})$, where $\mathcal{F}(\mathcal{G})$ is the set of all finite subsets of $\mathcal{G}^1 \cup \mathrm{Sing}(\mathcal{G})$. By [21, Theorem 3.21 (i)], we obtain that G_F satisfies Condition (EXC) and the maximal length of chains of cycles in G_F is less than or equal to $\mathrm{GKdim}(L_K(\mathcal{G}))$ for all $F \in \mathcal{F}(\mathcal{G})$. This shows that $\mathcal{G}$ satisfies Condition (EXC) and the maximal length of chains of cycles in $\mathcal{G}$ is less than or equal to $\mathrm{GKdim}(L_K(\mathcal{G}))$, by Lemmas 3.2 and 3.3 respectively.

($\Longleftarrow$). Assume that $\mathcal{G}$ satisfies Condition (EXC) and the maximal length of chains of cycles in $\mathcal{G}$ is finite, say t. Then, by Lemmas 3.2 and 3.3 respectively, G_F satisfies Condition (EXC) and the maximal length of chains of cycles in G_F is less than or equal to t for all $F \in \mathcal{F}(\mathcal{G})$. From this observation and [21, Theorem 3.21 (ii)], we receive that $\mathrm{GKdim}(L_K(G_F)) \leq 2t$ for all $F \in \mathcal{F}(\mathcal{G})$. By Theorems 2.8 (3) and 3.1, we have

$$\mathrm{GKdim}(L_K(\mathcal{G})) = \varinjlim_{F \in \mathcal{F}(\mathcal{G})} \mathrm{GKdim}(L_K(G_F)) \leq 2t,$$

as desired.

(2) Since $\mathcal{G}$ satisfies Condition (EXC), G_F satisfies Condition (EXC) for all $F \in \mathcal{F}(\mathcal{G})$. Since the maximal length of chains of cycles in $\mathcal{G}$ is equal to t and by Lemma 3.3 (1), the maximal length of chains of cycles in G_F is less than or equal to t for all $F \in \mathcal{F}(\mathcal{G})$, and there exists an element $F_1 \in \mathcal{F}(\mathcal{G})$ such that the maximal length of chains of cycles in G_{F_1} is equal to t. Moreover, since the maximal length of chains of cycles in $\mathcal{G}$ with an exit is equal to t' and by Lemma 3.3 (2), the maximal length of chains of cycles in G_F with an exit is less than or

equal to t' for all $F \in \mathcal{F}(\mathcal{G})$, and there exists an element $F_2 \in \mathcal{F}(\mathcal{G})$ such that the maximal length of chains of cycles in G_{F_2} with an exit is equal to t'. Then, by [21, Theorem 3.21 (ii)], $\mathrm{GKdim}(L_K(G_F)) \leq \max\{2t-1, 2t'\}$ for all $F \in \mathcal{F}(\mathcal{G})$, and so

$$\mathrm{GKdim}(L_K(\mathcal{G})) = \varinjlim_{F\in\mathcal{F}(\mathcal{G})} \mathrm{GKdim}(L_K(G_F)) \leq \max\{2t-1, 2t'\},$$

by Theorem 2.8 (3). On the other hand, by Lemma 3.3 (3), there exists an element $F' \in \mathcal{F}(\mathcal{G})$ such that the maximal length of chains of cycles in $G_{F'}$ is equal to t and the maximal length of chains of cycles in $G_{F'}$ with an exit is equal to t'. Then, by [21, Theorem 3.21 (ii)], we have $\mathrm{GKdim}(L_K(G_{F'})) = \max\{2t-1, 2t'\}$. By Theorem 2.8 (1) and [18, Lemma 3.1], we obtain that

$$\max\{2t-1, 2t'\} = \mathrm{GKdim}(L_K(G_{F'})) \leq \mathrm{GKdim}(L_K(\mathcal{G})),$$

so $\mathrm{GKdim}(L_K(\mathcal{G})) = \max\{2t-1, 2t'\}$, thus finishing the proof. □

The rest of this section is to show that the Gelfand-Kirillov dimension of the Levitt path algebra of an ultragraph $\mathcal{G}$ is equal to the Gelfand-Kirillov dimension of the Levitt path algebra of the associated graph of $\mathcal{G}$.

Definition 3.5. Let $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ be an ultragraph. The associated graph $E_{\mathcal{G}} = (E^0_{\mathcal{G}}, E^1_{\mathcal{G}}, r_{E_{\mathcal{G}}}, s_{E_{\mathcal{G}}})$ of $\mathcal{G}$ is defined by:

$$E^0_{\mathcal{G}} = G^0, \quad E^1_{\mathcal{G}} = \{(e, v) \mid e \in \mathcal{G}^1, v \in r(e)\}$$
$$s_{E_{\mathcal{G}}}((e,v)) = s(e), \quad r_{E_{\mathcal{G}}}((e,v)) = v.$$

The following lemma provides us criteria for ultragraphs having Condition (EXC) and cycles with exits in terms of their associated graphs.

Lemma 3.6. *For an ultragraph $\mathcal{G}$, the following statements hold:*
(1) $\mathcal{G}$ *satisfies Condition* (EXC) if and only if $E_{\mathcal{G}}$ satisfies Condition (EXC).
(2) $\mathcal{G}$ has a cycle with an exit if and only if $E_{\mathcal{G}}$ has a cycle with an exit.

Proof. (1) ($\Longrightarrow$). Assume that $E_{\mathcal{G}}$ does not satisfy Condition (EXC). We then have that $E_{\mathcal{G}}$ has a non-exclusive cycle $c = c_1c_2\ldots c_n$, that means, there exists a cycle $d = d_1d_2\ldots d_m$ in $E_{\mathcal{G}}$ which is different from a cyclic permutation of c such that $s_{E_{\mathcal{G}}}(c_i) = s_{E_{\mathcal{G}}}(d_j)$ and $c_i \neq d_j$ for some $1 \leq i \leq n$ and $1 \leq j \leq m$. By renumbering edges of c and d, without loss of generality, we may assume that $i = j = 1$. Write $c_i = (e_i, v_i) \in E^1_{\mathcal{G}}$ with $s_{\mathcal{G}}(e_{i+1}) = v_i$, $s_{\mathcal{G}}(e_1) = v_n$, and $d_j = (f_j, w_j) \in E^1_{\mathcal{G}}$ with $s_{\mathcal{G}}(f_{j+1}) = w_j$, $s_{\mathcal{G}}(f_1) = w_m$. We have $v_n = s_{\mathcal{G}}(e_1) = s_{\mathcal{G}}(f_1) = w_m$. Let $\alpha := e_1e_2\cdots e_n$ and $\beta := f_1f_2\cdots f_m$. We then receive that α and β are two cycles in $\mathcal{G}$ with $s_{\mathcal{G}}(\alpha) = v_n = w_m = s_{\mathcal{G}}(\beta)$. Since $c_1 = (e_1, v_1) \neq (f_1, w_1) = d_1$, we have either $e_1 \neq f_1$ or $v_1 \neq w_1$. If $e_1 \neq f_1$, then since $v_n = s_{\mathcal{G}}(e_1) = s_{\mathcal{G}}(f_1) = w_m \in r_{\mathcal{G}}(e_n) \cap r_{\mathcal{G}}(f_m)$, α and β are non-exclusive cycles. If $e_1 = f_1$, then we must have $v_1 \neq w_1$, and so $e_2 \neq f_2$. Since $s_{\mathcal{G}}(e_2) = v_1$ and $s_{\mathcal{G}}(f_2) = w_1 \in r_{\mathcal{G}}(e_1) = r_{\mathcal{G}}(f_1)$, α and β are non-exclusive cycles. Therefore, in any case, $\mathcal{G}$ always contains a non-exclusive cycle, and so $\mathcal{G}$ does not satisfy Condition (EXC).

($\Longleftarrow$). Assume that $\mathcal{G}$ does not satisfy Condition (EXC). We then have that $\mathcal{G}$ contains cycles $\alpha = \alpha_1\alpha_2\cdots\alpha_n$ and $\beta = \beta_1\beta_2\cdots\beta_m$ such that $s(\alpha_i), s(\beta_j) \in r(\alpha_{i-1}) \cap r(\beta_{j-1})$ and $(\alpha_{i-1}, \alpha_i) \neq (\beta_{j-1}, \beta_j)$ for some $1 \le i \le n, 1 \le j \le m$, where $r(\alpha_0) := r(\alpha_n)$ and $r(\beta_0) := r(\beta_m)$. By renumbering edges of α and β, without loss of generality, we may assume that $i = j = 1$. Consider the following two cases:

Case 1: $s(\alpha_1) = s(\beta_1)$. Let $c = (\alpha_1, s(\alpha_2))(\alpha_2, s(\alpha_3))\cdots(\alpha_n, s(\alpha_1))$ and $d = (\beta_1, s(\beta_2))(\beta_2, s(\beta_3))\cdots(\beta_m, s(\beta_1))$. We then have that c and d are two cycles in $E_{\mathcal{G}}$ such that $s_{E_{\mathcal{G}}}(c) = s_{E_{\mathcal{G}}}(d)$ and c is different from a cyclic permutation of d (since $(\alpha_{i-1}, \alpha_i) \neq (\beta_{j-1}, \beta_j)$), and so $E_{\mathcal{G}}$ does not satisfy Condition (EXC).

Case 2: $s(\alpha_1) \neq s(\beta_1)$. Let $c = (\alpha_n, s(\alpha_1))(\alpha_1, s(\alpha_2))\cdots(\alpha_{n-1}, s(\alpha_n))$ and $d = (\alpha_n, s(\beta_1))(\beta_1, s(\beta_2))\cdots(\beta_m, s(\alpha_1))(\alpha_1, s(\alpha_2))(\alpha_2, s(\alpha_3))\cdots(\alpha_{n-1}, s(\alpha_n))$. We obtain that c and d are two cycles in $E_{\mathcal{G}}$ such that $s_{E_{\mathcal{G}}}(c) = s_{E_{\mathcal{G}}}(d)$ and c is different from a cyclic permutation of d. This implies that $E_{\mathcal{G}}$ does not satisfy Condition (EXC).

Thus, in any case we arrive at that $E_{\mathcal{G}}$ does not satisfy Condition (EXC).

(2) ($\Longrightarrow$). Assume that $\mathcal{G}$ has a cycle $\alpha = e_1e_2\cdots e_n$ with an exit. Then, there is either an edge $f \in \mathcal{G}^1$ such that there exists an $1 \le i \le n$ for which $s(f) \in r(e_i)$ but $f \neq e_{i+1}$ (where $e_{n+1} := e_1$), or a sink w such that $w \in r(e_i)$ for some $1 \le i \le n$. If the first case happens, then $E_{\mathcal{G}}$ contains a cycle $c = (\alpha_1, s(\alpha_2))(\alpha_2, s(\alpha_3))\cdots(\alpha_n, s(\alpha_1))$ with an exit $(e_i, s(f))$ when $s(e_{i+1}) \neq s(f)$ and with an exit $(f, s(f))$ when $s(e_{i+1}) = s(f)$. If the second case happens, then $E_{\mathcal{G}}$ contains a cycle $c = (\alpha_1, s(\alpha_2))(\alpha_2, s(\alpha_3))\cdots(\alpha_n, s(\alpha_1))$ with an exit (e_i, w).

($\Longleftarrow$). Assume that $E_{\mathcal{G}}$ has a cycle $c = (\alpha_1, s(\alpha_2))(\alpha_2, s(\alpha_3))\cdots(\alpha_n, s(\alpha_1))$ with an exit (f, w). Then, there exits an $1 \le i \le n$ such that $s_{\mathcal{G}}(f) = s_{\mathcal{G}}(\alpha_{i+1})$ and $(f, w) \neq (\alpha_{i+1}, s_{\mathcal{G}}(\alpha_{i+2}))$. If $f \neq \alpha_{i+1}$, then $\mathcal{G}$ has a cycle $\alpha := e_1e_2\cdots e_n$ with an exit f. If $f = \alpha_{i+1}$, then we must have $w \neq s_{\mathcal{G}}(\alpha_{i+2})$. If w is a sink, then w is an exit for α. If w is not a sink, then every edge $g \in s_{\mathcal{G}}^{-1}(w)$ is an exit for α. Thus, in any case α always has an exit, finishing our proof. $\square$

The following lemma provides us criteria for ultragraphs having chains of cycles of finite length and chains of cycles of finite length with exits in terms of their associated graphs.

Lemma 3.7. *For an ultragraph $\mathcal{G}$ having Condition* (EXC), the following statements hold:

(1) $\mathcal{G}$ has a chain of cycles of length t if and only if $E_{\mathcal{G}}$ has a chain of cycles of length t. Consequently, the maximal length of chains of cycles in $\mathcal{G}$ is equal to t if and only if the maximal length of chains of cycles in $E_{\mathcal{G}}$ is equal to t.

(2) $\mathcal{G}$ has a chain of cycles of length t' with an exit if and only if $E_{\mathcal{G}}$ has a chain of cycles of length t' with an exit. Consequently, the maximal length of chains of

cycles in $\mathcal{G}$ with an exit is equal to t' if and only if the maximal length of chains of cycles in $E_{\mathcal{G}}$ with an exit is equal to t.

Proof. (1) ($\Longrightarrow$). Assume that $\alpha_1 \Rightarrow \alpha_2 \Rightarrow \cdots \Rightarrow \alpha_t$ is a chain of cycles of length t in $\mathcal{G}$. Write $\alpha_i = e_1^{(i)} e_2^{(i)} \cdots e_{n_i}^{(i)}$ with $e_j^{(i)} \in \mathcal{G}^1$ for all $1 \le i \le t, 1 \le j \le n_i$. Then, for each $1 \le i \le t-1$, there exists a path p_i such that $s(p_i) \in r(e_{k_i}^{(i)})$ and $s(e_{k_i'}^{(i+1)}) \in r(p_i)$ for some $1 \le k_i \le n_i$ and $1 \le k_i' \le n_{i+1}$. Write $p_i = f_1^{(i)} f_2^{(i)} \dots f_{m_i}^{(i)}$ with $f_j^{(i)} \in \mathcal{G}^1$ for all $1 \le i \le t-1$ and $1 \le j \le m_i$. Let

$$\beta_i = (e_1^{(i)}, s(e_2^{(i)}))(e_2^{(i)}, s(e_3^{(i)})) \cdots (e_{n_i-1}^{(i)}, s(e_{n_i}^{(i)}))(e_{n_i}^{(i)}, s(e_1^{(i)}))$$

for $1 \le i \le t$, and

$$q_i = (e_{k_i}^{(i)}, s(f_1^{(i)}))(f_1^{(i)}, s(f_2^{(i)})) \cdots (f_{m_i-1}^{(i)}, s(f_{m_i}^{(i)}))(f_{m_i}^{(i)}, s(e_{k_i'}^{(i+1)}))$$

for $1 \le i \le t-1$. We then have β_i's are cycles in $E_{\mathcal{G}}$ with $s_{E_{\mathcal{G}}}(q_i) = s_{\mathcal{G}}(e_j^{(i)})$ and $r_{E_{\mathcal{G}}}(q_i) = s_{\mathcal{G}}(e_{k_i'}^{(i+1)})$ for all $1 \le i \le t-1$, and so $\beta_i \Rightarrow \beta_{i+1}$ for all $1 \le i \le t-1$. This implies that $\beta_1 \Rightarrow \beta_2 \Rightarrow \cdots \Rightarrow \beta_t$ is a chain of cycles of lenght t in $E_{\mathcal{G}}$.

($\Longleftarrow$). Assume that $E_{\mathcal{G}}$ has $c_1 \Rightarrow c_2 \Rightarrow \cdots \Rightarrow c_t$ is a chain of cycles of length t. Then, by renumbering edges of c_i's, without loss of generality, we may assume that for each $1 \le i \le t-1$, there exists a paths p_i in $E_{\mathcal{G}}$ such that $s_{E_{\mathcal{G}}}(p_i) = s_{E_{\mathcal{G}}}(c_i)$ and $r_{E_{\mathcal{G}}}(p_i) = s_{E_{\mathcal{G}}}(c_{i+1})$. Write

$$c_i = (e_1^{(i)}, v_1^{(i)})(e_2^{(i)}, v_2^{(i)}) \cdots (e_{n_i}^{(i)}, v_{n_i}^{(i)})$$

with $v_j^{(i)} \in r_{\mathcal{G}}(e_j^{(i)})$ and $v_j^{(i)} = s_{\mathcal{G}}(e_{j+1}^{(i)})$ for all $1 \le i \le t$ and $1 \le j \le n_i$, where $e_{n_i+1}^{(i)} := e_1^{(i)}$, and write

$$p_i = (f_1^{(i)}, w_1^{(i)})(f_2^{(i)}, w_2^{(i)}) \cdots (f_{m_i}^{(i)}, w_{m_i}^{(i)})$$

with $w_j^{(i)} \in r_{\mathcal{G}}(f_j^{(i)}), v_{n_i}^{(i)} = s_{\mathcal{G}}(f_1^{(i)})$, $w_j^{(i)} = s_{\mathcal{G}}(f_{j+1}^{(i)})$ and $w_{m_i}^{(i)} = s_{\mathcal{G}}(e_1^{(i+1)})$ for all $1 \le i \le t-1$ and $1 \le j \le m_i$, where $f_{m_i+1}^{(i)} := f_1^{(i)}$. Let

$$\alpha_i = e_1^{(i)} e_2^{(i)} \cdots e_{n_i}^{(i)}$$

for all $1 \le i \le t$, and let

$$\beta_i = f_1^{(i)} f_2^{(i)} \cdots f_{m_i}^{(i)}$$

for all $1 \le i \le t-1$. We then have that α_i's are cycles in $\mathcal{G}$ and $s_{\mathcal{G}}(\beta_i) = s_{\mathcal{G}}(f_1^{(i)}) = v_{n_i}^{(i)} \in r_{\mathcal{G}}(e_{n_i}^{(i)}) = r_{\mathcal{G}}(\alpha_i)$, and $s_{\mathcal{G}}(\alpha_{i+1}) = s_{\mathcal{G}}(e_1^{(i+1)}) = w_{m_i}^{(i)} \in r_{\mathcal{G}}(f_{m_i}^{(i)}) = r_{\mathcal{G}}(\beta_i)$ for all $1 \le i \le t-1$. This shows that $\alpha_i \Rightarrow \alpha_{i+1}$ for all $1 \le i \le t-1$. Therefore, we have a chain of cycles $\alpha_1 \Rightarrow \alpha_2 \Rightarrow \cdots \Rightarrow \alpha_t$ in $\mathcal{G}$ of length t.

(2) It follows from item (1) and Lemma 3.6, thus finishing our proof. $\square$

Now we are able to present the second main result of this section, showing that ultragraph Levitt path algebras $L_K(\mathcal{G})$ and Levitt path algebras $L_K(E_{\mathcal{G}})$ of the associated graphs $E_{\mathcal{G}}$ have the same Gelfand-Kirillov dimension.

Theorem 3.8. *Let $\mathcal{G}$ be an ultragraph and K a field. Then* $\mathrm{GKdim}(L_K(\mathcal{G}) = \mathrm{GKdim}(L_K(E_{\mathcal{G}})$.

Proof. It follows from Theorem 3.4, Lemmas 3.6 and 3.7, and [21, Theorem 3.21]. □

4. On ultragraph Leavitt path algebras whose simple modules are finitely presented

In this section we describe ultragraph Leavitt path algebras whose simple modules are finitely presented, and show that these algebras have finite Gelfand-Kirillov dimension (Theorem 4.11). Moreover, we provide new classes of simple modules over ultragraph Leavitt path algebras associated with minimal infinite emitters (Theorem 4.2 and Remark 4.3) and minimal sinks (Theorem 4.5) which have not yet appeared in the context of Leavitt path algebras of graphs, and investigate the finite representation of these modules (Proposition 4.4 and Theorem 4.5 (5)) and the simple modules associated with infinite paths (Corollary 4.10).

We start this section with the notion of minimal infinite emitters, introduced in [24, Definition 3.2]. Let $\mathcal{G}$ be an ultragraph and $V \in \mathcal{G}^0$. We say that V is an *infinite emitter* if the set $\{e \in \mathcal{G}^1 \mid s(e) \in V\}$ is infinite. Otherwise we say that V is a *finite emitter*. The set V is called a *minimal infinite emitter* if V is an infinite emitter, contains no proper subsets in $\mathcal{G}^0$ that are infinite emitters, and contains no proper subsets in $\mathcal{G}^0$ which are finite emitters and have infinite cardinality.

We will construct a new class of simple modules over ultragraph Leavitt path algebras associated with minimal infinite emitters. Let K be a field, $\mathcal{G}$ an ultragraph and V a minimal infinite emitter in $\mathcal{G}^0$. We denote by $S_{V\infty}$ the K-vector space having

$$[V] := \{\alpha \in \mathcal{G}^* \mid |\alpha| \geq 1,\, V \subseteq r(\alpha)\} \cup \{V\}$$

as a basis. For each $A \in \mathcal{G}^0$ we define a linear map $q_A : S_{V\infty} \longrightarrow S_{V\infty}$ such that

$$q_A(\alpha) = \begin{cases} \alpha & \text{if } |\alpha| \geq 1 \text{ and } s(\alpha) \in A, \\ \alpha & \text{if } \alpha = V \text{ and } V \subseteq A, \\ 0 & \text{otherwise,} \end{cases}$$

for all $\alpha \in [V]$. For each $e \in \mathcal{G}^1$ we define linear maps t_e and $t_e^* : S_{V\infty} \longrightarrow S_{V\infty}$ such that, for all $\alpha \in [V]$

$$t_e(\alpha) = \begin{cases} e\alpha & \text{if } |\alpha| \geq 1 \text{ and } s(\alpha) \in r(e), \\ e & \text{if } \alpha = V \text{ and } V \subseteq r(e), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$t_e^*(\alpha) = \begin{cases} \beta & \text{if } \alpha = e\beta \text{ and } |\beta| \geq 1, \\ V & \text{if } \alpha = e \text{ and } V \subseteq r(e), \\ 0 & \text{otherwise.} \end{cases}$$

The above endomorphisms induce a representation of $L_K(\mathcal{G})$ as described below.

Proposition 4.1. *Let K be a field, $\mathcal{G}$ an ultragraph and V a minimal infinite emitter in $\mathcal{G}^0$. Then, there exists a K-algebra homomorphism $\pi_V : L_K(\mathcal{G}) \longrightarrow End_K(S_{V\infty})$ such that $\pi_V(A) = q_A$ for all $A \in \mathcal{G}^0$, $\pi_V(s_e) = t_e$ and $\pi_V(s_e^*) = t_e^*$ for all $e \in \mathcal{G}^1$.*

Proof. We show that the endomorphisms $\{q_A, t_e, t_e^* \mid A \in \mathcal{G}^0,\ e \in \mathcal{G}^1\}$ satisfy the relations analogous to (1) - (4) in Definition 2.3. For (1), it is straightforward to see that $q_A q_B = q_{A\cap B}$ for all A and $B \in \mathcal{G}^0$. Let A, B and C be elements in $\mathcal{G}^0$. We claim that $p_{A\cup B} = p_A + p_B - p_{A\cap B}$. Indeed, for all $\alpha \in [V] \setminus \{V\}$, one considers the following three cases:

Case 1: $s(\alpha) \in (A \setminus B) \cup (B \setminus A)$. We then have $q_{A\cup B}(\alpha) = \alpha = q_A(\alpha) + q_B(\alpha) - q_{A\cap B}(\alpha) = (q_A + q_B - q_{A\cap B})(\alpha)$.

Case 2: $s(\alpha) \in A \cap B$. We then have $q_A(\alpha) = \alpha, q_B(\alpha) = \alpha, q_{A\cap B}(\alpha) = \alpha$ and $q_{A\cup B}(\alpha) = \alpha$. Therefore, $q_{A\cup B}(\alpha) = \alpha = (q_A + q_B - q_{A\cap B})(\alpha)$.

Case 3: $s(\alpha) \notin A \cup B$. We then have $q_{A\cup B}(\alpha) = 0 = (q_A + q_B - q_{A\cap B})(\alpha)$.

Next we prove that $q_{A\cup B}(V) = (q_A + q_B - q_{A\cap B})(V)$. Consider the following two cases:

Case 1: $V \nsubseteq A \cup B$. We then have $q_A(V) = 0, q_B(v) = 0, q_{A\cap B}(v) = 0$ and $q_{A\cup B}(V) = 0$, and so $q_{A\cup B}(V) = 0 = (q_A + q_B - q_{A\cap B})(V)$.

Case 2: $V \subseteq A \cup B$. If V contains an infinite emitter v in G^0, then $V = \{v\}$ (since V is a minimal infinite emitter in $\mathcal{G}^0$). Then it is straightforward to see that $q_{A\cup B}(V) = (q_A + q_B - q_{A\cap B})(V)$. If V does not contain infinite emitters in G^0, then V is an infinite set. Since V is a minimal infinite emitter in $\mathcal{G}^0$, we have either $V \cap D = V$ or $V \cap D$ is finite for all $D \in \mathcal{G}^0$. If $V \cap A$ and $V \cap B$ are finite, then $V = V \cap (A \cup B) = (V \cap A) \cup (V \cap B)$ is finite, a contradiction. Hence, we consider the following three possible subcases.

Case 2.1: $V \subseteq A$ and $V \nsubseteq B$. We have $q_{A\cup B}(V) = V = V + 0 - 0 = (q_A + q_B - q_{A\cap B})(V)$.

Case 2.2: $V \subseteq B$ and $V \nsubseteq A$. We have $q_{A\cup B}(V) = V = 0 + V - 0 = (q_A + q_B - q_{A\cap B})(V)$.

Case 2.3: $V \subseteq A \cap B$. We have $q_{A\cup B}(V) = V = V + V - V = (q_A + q_B - q_{A\cap B})(V)$.

In any case we arrive at that $q_{A\cup B}(\alpha) = (q_A + q_B - q_{A\cap B})(\alpha)$ for all $\alpha \in [V]$, that means, $q_{A\cup B} = (q_A + q_B - q_{A\cap B})$, as desired.

For (2), for each $e \in \mathcal{G}^1$ we have

$$q_{s(e)}t_e(\alpha) = \begin{cases} q_{s(e)}(e\alpha) & \text{if } |\alpha| \geq 1 \text{ and } s(\alpha) \in r(e), \\ q_{s(e)}(e) & \text{if } \alpha = V \text{ and } V \subseteq r(e), \\ 0 & \text{otherwise,} \end{cases}$$
$$= \begin{cases} e\alpha & \text{if } |\alpha| \geq 1 \text{ and } s(\alpha) \in r(e), \\ e & \text{if } \alpha = V \text{ and } V \subseteq r(e), \\ 0 & \text{otherwise,} \end{cases} \quad = t_e(\alpha)$$

for all $\alpha \in [V]$, and

$$t_e q_{r(e)}(\alpha) = \begin{cases} t_e(\alpha) & \text{if } |\alpha| \geq 1 \text{ and } s(\alpha) \in r(e), \\ t_e(\alpha) & \text{if } \alpha = V \text{ and } V \subseteq r(e), \\ 0 & \text{otherwise,} \end{cases}$$
$$= \begin{cases} e\alpha & \text{if } |\alpha| \geq 1 \text{ and } s(\alpha) \in r(e), \\ e & \text{if } \alpha = V \text{ and } V \subseteq r(e), \\ 0 & \text{otherwise,} \end{cases} \quad = t_e(\alpha)$$

for all $\alpha \in [V]$. This shows that $q_{s(e)}t_e = t_e = t_e q_{r(e)}$.

For (3), for each $e \in \mathcal{G}^1$ we have

$$t_e^* t_e(\alpha) = \begin{cases} t_e^*(e\alpha) & \text{if } |\alpha| \geq 1 \text{ and } s(\alpha) \in r(e), \\ t_e^*(e) & \text{if } \alpha = V \text{ and } V \subseteq r(e), \\ 0 & \text{otherwise,} \end{cases}$$
$$= \begin{cases} \alpha & \text{if } |\alpha| \geq 1 \text{ and } s(\alpha) \in r(e), \\ V & \text{if } \alpha = V \text{ and } V \subseteq r(e), \\ 0 & \text{otherwise,} \end{cases} \quad = p_{r(e)}(\alpha)$$

for all $\alpha \in [V]$. Let e and f be two distinct edges in $\mathcal{G}^1$. We then have

$$t_e^* t_f(\alpha) = \begin{cases} t_e^*(f\alpha) & \text{if } |\alpha| \geq 1 \text{ and } s(\alpha) \in r(f), \\ t_e^*(f) & \text{if } \alpha = V \text{ and } V \subseteq r(e), \\ 0 & \text{otherwise,} \end{cases} \quad = 0$$

for all $\alpha \in [V]$.

For (4), let v be a regular vertex in $\mathcal{G}$. We then have $V \nsubseteq \{v\}$ (since V is a minimal infinite emitter in $\mathcal{G}^0$), and so

$$q_v(V) = 0 = \sum_{e \in s^{-1}(v)} t_e t_e^*(V) = (\sum_{e \in s^{-1}(v)} t_e t_e^*)(V).$$

Let $\alpha \in [V] \setminus \{V\}$. If $s(\alpha) \neq v$, then $t_e^*(\alpha) = 0$ for all $e \in s^{-1}(v)$, and so

$$(\sum_{e \in s^{-1}(v)} t_e t_e^*)(\alpha) = \sum_{e \in s^{-1}(v)} t_e t_e^*(\alpha) = \sum_{e \in s^{-1}(v)} (t_e(t_e^*(\alpha)) = 0 = q_v(\alpha).$$

Consider the case when $s(\alpha) = v$. Write $\alpha = e_1 e_2 \ldots e_{|\alpha|}$ with $e_1 \in s^{-1}(v)$. We then have $(\sum\limits_{e \in s^{-1}(v)} t_e t_e^*)(\alpha) = \sum\limits_{e \in s^{-1}(v)} t_e t_e^*(\alpha) = \sum\limits_{e \in s^{-1}(v)} t_e(t_e^*(\alpha)) = t_{e_1}(e_2 \ldots e_{|\alpha|}) = \alpha = q_v(\alpha)$. This implies that $q_v = \sum\limits_{e \in s^{-1}(v)} t_e t_e^*$, proving (4).

By the universal property of ultragraph Leavitt path algebras, there is a unique K-algebra homomorphism $\pi_V : L_K(\mathcal{G}) \longrightarrow End_K(S_{V\infty})$ such that $\pi_V(A) = q_A$ for all $A \in \mathcal{G}^0$, $\pi_V(s_e) = t_e$ and $\pi_V(s_e^*) = t_e^*$ for all $e \in \mathcal{G}^1$, thus finishing the proof. $\square$

We denote the scalar multiplication of $L_K(\mathcal{G})$ on $S_{V\infty}$ by " $\cdot$ ", that means, $r \cdot u = \pi_V(r)(u)$ for all $r \in L_K(\mathcal{G})$ and $u \in S_{V\infty}$. The following result provides simple modules over ultragraph Leavitt path algebras associated with minimal infinite emitters.

Theorem 4.2. *Let K be a field, $\mathcal{G}$ an ultragraph, and V and W minimal infinite emitters in $\mathcal{G}^0$. Then the following statements hold:*
(1) *$S_{V\infty}$ is a simple left $L_K(\mathcal{G})$-module;*
(2) *$S_{V\infty} \cong S_{W\infty}$ as left $L_K(\mathcal{G})$-modules if and only if $V = W$;*
(3) *$End_{L_K(\mathcal{G})}(S_{V\infty}) \cong K$.*

Proof. Let x be a nonzero element in $S_{V\infty}$. Write $x = \sum\limits_{i=1}^{n} k_i \alpha_i$, where $k_i \in K \setminus \{0\}$ and α_i's are distinct elements in $[V]$. Without loss of generality, we may assume that $|\alpha_1| \leq |\alpha_2| \leq \cdots \leq |\alpha_n|$. We then have

$$s_{\alpha_n}^* \cdot x = \pi_V(s_{\alpha_n}^*)(\sum_{i=1}^{n} k_i \alpha_i) = \sum_{i=1}^{n} k_i \pi_V(s_{\alpha_n}^*)(\alpha_i) = k_n V,$$

so $V = k_n^{-1} s_{\alpha_n}^* \cdot x \in L_K(\mathcal{G})x$ and $\alpha = \pi_V(s_\alpha)(V) = s_\alpha \cdot V \in L_K(\mathcal{G})x$ for all $\alpha \in [V]$. This implies that $S_{V\infty} = L_K(\mathcal{G})x$, and so $S_{V\infty}$ is a simple left $L_K(\mathcal{G})$-module, showing (1).

Assume that $\varphi : S_{V\infty} \longrightarrow S_{W\infty}$ is an isomorphism of left $L_K(\mathcal{G})$-modules. Then, we have $0 \neq \varphi(V) = \sum\limits_{i=1}^{n} k_i \alpha_i$, where $k_i \in K \setminus \{0\}$ and α_i's are distinct elements in $[W]$. We claim that $n = 1$ and $\alpha_1 = V$. We assume the converse. Then we may assume that $|\alpha_1|$ is longest among all the $|\alpha_i|$'s. In particular, we have $|\alpha_1| \geq 1$, and so $s_{\alpha_1}^* \cdot V = 0$ and

$$0 = \varphi(s_{\alpha_1}^* \cdot V) = s_{\alpha_1}^* \cdot \varphi(V) = s_{\alpha_1}^* \cdot (\sum_{i=1}^{n} k_i \alpha_i) = k_1 W \neq 0,$$

a contradiction. This shows the claim, and so we obtain that $V = W$ and $End_{L_K(\mathcal{G})}(S_{V\infty}) \cong K$, proving (2) and (3), thus finishing our proof. $\square$

It is worth mentioning the following note.

Remark 4.3. In [15, Lemm 4.4] and [10, Lemma 5.3], the second author and his coauthors defined the simple left $L_K(\mathcal{G})$-module $N_{v\infty}$, where v is an infinite emitter in an ultragraph $\mathcal{G}$. It is obvious that $\{v\}$ is a minimal infinite emitter in $\mathcal{G}^0$ and $S_{\{v\}\infty} = N_{v\infty}$. Using Theorem 4.2 we may construct a new class of simple modules over ultragraph Leavitt path algebras. For example, let $\mathcal{G}$ be the ultragraph such that $G^0 = \{v_n \mid n \in \mathbb{N}\}$ and $\mathcal{G}^1 = \{e_n \mid n \in \mathbb{N}\}$ with $s(e_n) = v_n$ for all $n \in \mathbb{N}$, $r(e_0) = \{v_n \mid n \geq 1\}$ and $r(e_n) = \{v_0, v_n\}$ for all $n \geq 1$. Then $r(e_0)$ is a minimal infinite emitter in $\mathcal{G}$. We refer the reader to [8, Example 3.20] in more details. By Theorem 4.2, $S_{r(e_0)\infty}$ is a simple left $L_K(\mathcal{G})$-modules. This provides a new class of simple modules over ultragraph Leavitt path algebras which has been not appeared in [14, 15, 10].

The following result extends [23, Proposition 2.2] to ultragraph Leavitt path algebras.

Proposition 4.4. *Let K be a field, $\mathcal{G}$ an ultragraph and V a minimal infinite emitter in $\mathcal{G}^0$. Then, the simple left $L_K(\mathcal{G})$-module $S_{V\infty}$ is not finitely presented. Consequently, if every simple left $L_K(\mathcal{G})$-module is finitely presented, then $\mathcal{G}$ is a row-finite ultragraph such that $\mathcal{G}^0$ has no minimal infinite emitters.*

Proof. Assume that $S_{V\infty}$ is finitely presented. Consider the exact sequence

$$0 \to \ker(\varphi) \xrightarrow{\iota} L_K(\mathcal{G})p_V \xrightarrow{\varphi} S_{V\infty} \to 0,$$

where ι is the canonical injection and $\varphi(x) = x{\cdot}V$ for all $x \in L_K(\mathcal{G})p_V$. Since $S_{V\infty}$ is finitely presented and by [19, Schanuel's Lemma], $\ker(\varphi)$ is a finitely generated submodule of the left $L_K(\mathcal{G})$-module $L_K(\mathcal{G})p_V$. Let $x_1, \cdots, x_n$ be the generators of $\ker(\varphi)$. For each $1 \leq t \leq n$, by Lemma 2.5, we can write $x_t = \sum_{i=1}^{m_t} k_i s_{\alpha_i} p_{A_i} s^*_{\beta_i}$ where $m_t \geq 1$, $k_i \in K \setminus \{0\}$, $\alpha_i, \beta_i \in \mathcal{G}^*$, $A_i \in \mathcal{G}^0$ and $r(\alpha_i) \cap A_i \cap r(\beta_i) \neq \emptyset$ for all $1 \leq i \leq m_t$. Consider the following two possible cases.

Case 1: V contains an infinite emitter in G^0. Then, since V is a minimal infinite emitter, $V = \{v\}$. We claim that $|\beta_i| \geq 1$ for all $1 \leq i \leq t$. Because, otherwise, by renumbering the terms of x_t, we may assume that $x_t = \sum_{i=1}^{s} k_i s_{\alpha_i} p_{A_i} p_{B_i} + \sum_{j=s+1}^{m_t} k_j s_{\alpha_j} p_{A_j} s^*_{\beta_j}$, where $v \in r(\alpha_i) \cap A_i \cap B_i \in \mathcal{G}^0$ for all $1 \leq i \leq s$, and $v = s(\beta_j)$ and $|\beta_j| \geq 1$ for all $s{+}1 \leq j \leq m_t$. We then have $x_t = x_t p_v = (\sum_{i=1}^{s} k_i s_{\alpha_i} p_{A_i} p_{B_i} + \sum_{j=s+1}^{m_t} k_j s_{\alpha_j} p_{A_j} s^*_{\beta_j}) p_v = \sum_{i=1}^{s'} k'_i s_{\alpha'_i} p_v + \sum_{j=s+1}^{m_t} k_j s_{\alpha_j} p_{A_j} s^*_{\beta_j}$, where $1 \leq s' \leq s$, $k'_i \in K \setminus \{0\}$ and α'_i's are distinct paths in $\mathcal{G}^*$ with $v \in r(\alpha'_i)$. Since $s^*_{\beta_j} \cdot v = 0$ for all $s + 1 \leq j \leq m_t$, we have $0 = \varphi(x_t) = x_t \cdot v = (\sum_{i=1}^{s'} k_i s_{\alpha'_i} p_v) \cdot v = \sum_{i=1}^{s'} k_i s_{\alpha'_i}$ in $S_{V\infty}$. This implies that $k'_i = 0$ for all $1 \leq i \leq s'$, a contradiction, thus showing the claim. Therefore, each x_t is a K-linear combination of finitely many monomials of the form $s_{\alpha_i} p_{A_i} s^*_{\beta_i}$, and so $\ker(\varphi) = \sum_{i=1}^{m} L_K(\mathcal{G}) s_{\alpha_i} p_{A_i} s^*_{\beta_i}$, where

$m \geq 1$, $\alpha_i, \beta_i \in \mathcal{G}^*$, $A_i \in \mathcal{G}^0$ and $r(\alpha_i) \cap A_i \cap r(\beta_i) \neq \emptyset$, $|\beta_i| \geq 1$ and $v = s(\beta_i)$ for all $1 \leq i \leq m$. Since v is an infinite emitter, there exists an edge $e \in \mathcal{G}^1$ such that $v = s(e)$ and $\beta_i \neq e\gamma$ for all $\gamma \in \mathcal{G}^*$ for all $1 \leq i \leq m$. Since $s_e^* \cdot V = 0$, $s_e^* \in \ker(\varphi)$, and so $s_e^* = \sum_{i=1}^m y_i s_{\alpha_i} p_{A_i} s_{\beta_i}^*$, where $y_i \in L_K(\mathcal{G})$. We then have $p_{r(e)} = s_e^* s_e = (\sum_{i=1}^m y_i s_{\alpha_i} p_{A_i} s_{\beta_i}^*) s_e = \sum_{i=1}^m y_i s_{\alpha_i} p_{A_i} s_{\beta_i}^* s_e = 0$ (since $s_{\beta_i}^* s_e = 0$). On the other hand, by Lemma 2.5 we always have $p_{r(e)} \neq 0$, a contradiction.

Case 2: V does not contain infinite emitters in G^0. We then have either $V \cap A = V$ or $V \cap A$ is finite for all $A \in \mathcal{G}^0$. We note that $x_t p_V = x_t$ for all $1 \leq t \leq n$. Therefore, we have $s(\beta_i) \in V$ for all β_i with $|\beta_i| \geq 1$, and $V \subseteq r(\alpha_i) \cap A_i \cap B_i$ or $r(\alpha_i) \cap A_i \cap B_i$ is a finite subset of V for all $\beta_i = B_i \in \mathcal{G}^0$. Assume that there exists a number $1 \leq t \leq n$ such that

$$x_t = \sum_{i=1}^{s} k_i s_{\alpha_i} p_{A_i} + \sum_{j=s+1}^{r} k_j s_{\alpha_j} p_{B_j} + \sum_{l=r+1}^{m_t} k_l s_{\alpha_l} p_{A_l} s_{\beta_l}^*,$$

where k_j is nonzero in K for all $s+1 \leq j \leq r$, A_i's are finite subsets of $r(\alpha_i) \cap V$ and B_j's are elements in $\mathcal{G}^0$ with $V \subseteq r(\alpha_j) \cap B_j$. We then have $0 = \varphi(x_t) = x_t \cdot V = \sum_{j=s+1}^r k_j s_{\alpha_j}$ in $S_{V\infty}$. This implies that $k_j = 0$ for all $s+1 \leq j \leq r$, a contradiction. Therefore, each x_t is a K-linear combination of finitely many monomials of the forms $s_{\alpha_i'} p_{A_i'}$ and $s_{\alpha_j} p_{A_j} s_{\beta_j}^*$, where A_i' is a finite subset of $V \cap r(\alpha_i')$ and $|\beta_j| \geq 1$, $s(\beta_j) \in V$. Since V is an infinite emitter, there exists an edge $e \in \mathcal{G}^1$ such that $s(e) \in V$, $\beta_j \neq e\gamma$ for all $\gamma \in \mathcal{G}^*$ and for all β_j, and $s(e) \notin A_i'$ for all A_i'. Since $s_e^* \cdot V = 0$, $s_e^* \in \ker(\varphi)$, and so $s_e^* = \sum_{j=1}^m y_j s_{\alpha_j} p_{A_j} s_{\beta_j}^* + \sum_{i=1}^{m'} y_i' s_{\alpha_i'} p_{A_i'}$, where $y_j, y_i' \in L_K(\mathcal{G})$. We then have $p_{r(e)} = s_e^* s_e = (\sum_{j=1}^m y_j s_{\alpha_j} p_{A_j} s_{\beta_j}^* + \sum_{i=1}^{m'} y_i' s_{\alpha_i'} p_{A_i'}) s_e = \sum_{j=1}^m y_j s_{\alpha_j} p_{A_j} s_{\beta_j}^* s_e + \sum_{i=1}^{m'} y_i' s_{\alpha_i'} p_{A_i'} s_e = 0$ (since $s_{\beta_i}^* s_e = 0$ and $s(e) \notin A_i'$). On the other hand, by Lemma 2.5 we always have $p_{r(e)} \neq 0$, a contradiction.

In any case we arrive at a contradiction. This shows that $S_{V\infty}$ is not finitely presented, thus finishing the proof. $\square$

Let $\mathcal{G}$ be an ultragraph. A generalized vertex $A \in \mathcal{G}^0$ is called a *minimal sink* if A is a finite emitter with infinite cardinality and has no subsets (in $\mathcal{G}^0$) with infinite cardinality (see [24, Definition 3.8]). We will construct an additional class of simple modules over ultragraph Leavitt path algebras associated with minimal sinks.

Let K be a field, $\mathcal{G}$ an ultragraph and V a minimal sink in $\mathcal{G}^0$. We denote by $\mathcal{N}_V$ the K-vector space having

$$[V] := \{\alpha \in \mathcal{G}^* \mid |\alpha| \geq 1,\, V \subseteq r(\alpha)\} \cup \{V\}$$

as a basis. By repeating the method established in Proposition 4.1, there exists a K-algebra homomorphism $\pi_V : L_K(\mathcal{G}) \longrightarrow End_K(\mathcal{N}_V)$ such that $\pi_V(A) = q_A$ for all $A \in \mathcal{G}^0$, $\pi_V(s_e) = t_e$ and $\pi_V(s_e^*) = t_e^*$ for all $e \in \mathcal{G}^1$, where endomorphisms q_A $(A \in \mathcal{G}^0)$ and t_e, t_e^* $(e \in \mathcal{G}^1)$ are defined similarly as directly before Proposition

4.1. This forms the scalar multiplication of $L_K(\mathcal{G})$ on $\mathcal{N}_V$ denoted by " $\cdot$ ", that means, $r \cdot u = \pi_V(r)(u)$ for all $r \in L_K(\mathcal{G})$ and $u \in \mathcal{N}_V$. Moreover, we have the following.

Theorem 4.5. *Let K be a field, $\mathcal{G}$ an ultragraph, and V and W minimal sinks in $\mathcal{G}^0$. Then the following statements hold:*
(1) *$\mathcal{N}_V$ is a simple left $L_K(\mathcal{G})$-module;*
(2) *$\mathcal{N}_V \cong \mathcal{N}_W$ as left $L_K(\mathcal{G})$-modules if and only if $V = W$;*
(3) *$End_{L_K(\mathcal{G})}(\mathcal{N}_V) \cong K$;*
(4) *$\mathcal{N}_V \ncong S_{V'\infty}$ for all minimal infinite emitter $V' \in \mathcal{G}^0$;*
(5) *$\mathcal{N}_V$ is not finitely presented.*

Proof. Items (1), (2) and (3) are done similarly as in the proof of Theorem 4.2.

For (4), assume that $\varphi : \mathcal{N}_V \longrightarrow S_{V'\infty}$ is an isomorphism of left $L_K(\mathcal{G})$-modules. We then have $0 \neq \varphi(V) = \sum_{i=1}^{n} k_i\alpha_i$, where $k \in K \setminus \{0\}$ and α_i's are distinct elements in $[V']$. We claim that $n = 1$ and $\alpha_1 = V'$. We assume the converse. Then we may assume that $|\alpha_1|$ is longest among all the $|\alpha_i|$'s. In particular, we have $|\alpha_1| \geq 1$, and so $s^*_{\alpha_1} \cdot V = 0$ and

$$0 = \varphi(s^*_{\alpha_1} \cdot V) = s^*_{\alpha_1} \cdot \varphi(V) = s^*_{\alpha_1} \cdot (\sum_{i=1}^{n} k_i\alpha_i) = k_1V' \neq 0,$$

a contradiction. This shows the claim, and so we obtain that $0 \neq \varphi(V) = k_1V'$. Since $V \neq V'$, we obtain that $V \cap V'$ is finite, and $0 = \varphi(p_{V'} \cdot V) = p_{V'} \cdot \varphi(V) = p_{V'} \cdot (k_1V') = k_1V' \neq 0$, a contradiction. This implies that $\mathcal{N}_V \ncong S_{V'\infty}$ for all minimal infinite emitter $V' \in \mathcal{G}^0$, proving (4).

For (5), assume that $\mathcal{N}_V$ is finitely presented. Consider the exact sequence

$$0 \to \ker(\varphi) \xrightarrow{\iota} L_K(\mathcal{G})p_V \xrightarrow{\varphi} \mathcal{N}_V \to 0,$$

where ι is the canonical injection and $\varphi(x) = x \cdot V$ for all $x \in L_K(\mathcal{G})p_V$. Since $\mathcal{N}_V$ is finitely presented and by [19, Schanuel's Lemma], $\ker(\varphi)$ is a finitely generated submodule of the left $L_K(\mathcal{G})$-module $L_K(\mathcal{G})p_V$. By repeating the method established in Case 2 of the proof of Proposition 4.4, we obtain that $\ker(\varphi)$ is generated by finitely many elements of the forms $s_{\alpha'_i}p_{A'_i}$ and $s_{\alpha_j}p_{A_j}s^*_{\beta_j}$, where A'_i is a finite subset of $V \cap r(\alpha'_i)$ and $|\beta_j| \geq 1$, $s(\beta_j) \in V$. Since V is infinite, there exists a vertex $v \in V$ such that $v \neq s(\beta_j)$ for all β_j, and $v \notin A'_i$ for all A'_i. Since $p_v \cdot V = 0$, $p_v \in \ker(\varphi)$, and so $p_v = \sum_{j=1}^{m} y_j s_{\alpha_j}p_{A_j}s^*_{\beta_j} + \sum_{i=1}^{m'} y'_i s_{\alpha'_i}p_{A'_i}$, where $y_j, y'_i \in L_K(\mathcal{G})$. We then have $p_v = p_vp_v = (\sum_{j=1}^{m} y_j s_{\alpha_j}p_{A_j}s^*_{\beta_j} + \sum_{i=1}^{m'} y'_i s_{\alpha'_i}p_{A'_i})p_v = \sum_{j=1}^{m} y_j s_{\alpha_j}p_{A_j}s^*_{\beta_j}p_v + \sum_{i=1}^{m'} y'_i s_{\alpha'_i}p_{A'_i}p_v = 0$ (since $v \neq s(\beta_i)$ and $v \notin A'_i$). On the other hand, by Lemma 2.5 we always have $p_v \neq 0$, a contradiction, thus finishing the proof. $\square$

For clarification, we illustrate Theorem 4.5 by presenting the following example.

Example 4.6. Let K be a field and $\mathcal{G}$ the ultragraph such that $G^0 = \{v_n \mid n \in \mathbb{N}\}$ and $\mathcal{G}^1 = \{e\}$ with $s(e) = v_0$ and $r(e) = \{v_n \mid n \geq 1\}$. Then $r(e)$ is a minimal sink in $\mathcal{G}^0$. By Theorem 4.2, $\mathcal{N}_{r(e)}$ is a simple left $L_K(\mathcal{G})$-modules, but not finitely presented. This provides a new class of simple modules over ultragraph Leavitt path algebras which has been not appeared in [14, 15, 10].

Let $\mathcal{G}$ be an ultragraph. An *infinite path* in $\mathcal{G}$ is a sequence $e_1e_2\cdots e_n\cdots$ of edges in $\mathcal{G}$ such that $s(e_{i+1}) \in r(e_i)$ for all $i \geq 1$. We denote by $\mathfrak{p}^\infty$ the set of all infinite paths in $\mathcal{G}$. For $p = e_1e_2\cdots e_n\cdots \in \mathfrak{p}^\infty$ and $n \in \mathbb{N}$, we denote by $\tau_{\leq n}(p)$ the finite path $e_1e_2\cdots e_n$, while we denote by $\tau_{>n}(p)$ the infinite path $e_{n+1}e_{n+2}\cdots$. If p and q are infinite paths in $\mathcal{G}$, then we say that p and q are *equivalent* (written $p \sim q$) in case there exist non-negative integers m, n such that $\tau_{>m}(p) = \tau_{>n}(q)$. Clearly $\sim$ is an equivalence on $\mathfrak{p}^\infty$, and we let $[p]$ denote the $\sim$ equivalence class of the infinite path p. Let c be a closed path in $\mathcal{G}$. Then the path $ccc\cdots$ is an infinite path in $\mathcal{G}$, which we denote by c^∞. An infinite path p is called a *rational path* if $p \sim c^\infty$ for some closed path c. An infinite path p is called an *irrational path* if p is not rational.

For $p := e_1\cdots e_n\cdots \in \mathfrak{p}^\infty$, in [14, Proposition 3.9] Gonçalves and Royer defined the simple left $L_K(\mathcal{G})$-module $V_{[p]}$ to be the K-vector space having $[p]$ as a basis and with the scalar multiplication satisfying the following: for all $A \in \mathcal{G}^0$, $e \in \mathcal{G}^1$ and $\alpha \in [p]$,

$$p_A \cdot \alpha = \begin{cases} \alpha & \text{if } s(\alpha) \in A, \\ 0 & \text{otherwise,} \end{cases} \qquad s_e \cdot \alpha = \begin{cases} e\alpha & \text{if } s(\alpha) \in r(e), \\ 0 & \text{otherwise} \end{cases}$$

and

$$s_e^* \cdot \alpha = \begin{cases} \tau_{>1}(\alpha) & \text{if } \alpha = e\tau_{>1}(\alpha), \\ 0 & \text{otherwise.} \end{cases}$$

We should note that in [15, Subsection 4.2] Hazrat and the first author constructed all these modules $V_{[p]}$, by using the realization of ultragraph Leavitt path algebras as Steinberg algebras. The following proposition provides us with another method to construct the modules $V_{[p]}$ where p is an irrational path, which extends [6, Theorem 3.4] to ultragraph Leavitt path algebras.

Proposition 4.7. *Let K be a field and $\mathcal{G}$ an ultragraph, and let $p = e_1\cdots e_n\cdots$ be an irrational path in $\mathcal{G}$ with $v = s(p)$. Let $\epsilon_0 = p_v$ and $\epsilon_i = s_{e_1}\cdots s_{e_i}s_{e_i}^*\cdots s_{e_1}^*$ for all $i \geq 1$. Then, a cyclic left module S_p over $L_K(\mathcal{G})$, generated by x subject to $x = \epsilon_i x$ for all $i \geq 0$, is both simple and isomorphic to $V_{[p]}$. Consequently, one has*

$$\mathrm{Ann}_{L_K(\mathcal{G})}(x) = \bigoplus_{i=0}^{\infty} L_K(\mathcal{G})(\epsilon_i - \epsilon_{i+1}) \oplus L_K(\mathcal{G})(1 - \epsilon_0),$$

where $L_K(\mathcal{G})(1 - \epsilon_0) := \{r - r\epsilon_0 \mid r \in L_K(\mathcal{G})\}$.

Proof. We note that $\epsilon_i \cdot p = p$, as elements in $V_{[p]}$, for all $i \geq 0$. Since $V_{[p]}$ is a simple left $L_K(\mathcal{G})$-module, $V_{[p]}$ is an image of S_p under the map sending $x \in S_p$ to $p \in V_{[p]}$, and so S_p is nonzero. Let $x_0 = p_0 x = x$ and $x_i = s^*_{e_i} \cdots s^*_{e_1} x$ for all $i \geq 1$. We have $x = s_{e_1} \cdots s_{e_i} x_i$ for all $i \geq 1$. Let y be a nonzero element in S_p. Since $S_p = L_K(\mathcal{G})x$, y may be written in the form $y = rx$ and $0 \neq r = \sum_{i=1}^m k_i s_{\alpha_i} p_{A_i} s^*_{\beta_i} \in L_K(\mathcal{G})$, where m is minimal such that $k_i \in K \setminus \{0\}$, $\alpha_i, \beta_i \in \mathcal{G}^*$, $A_i \in \mathcal{G}^0$ and $r(\alpha_i) \cap A_i \cap r(\beta_i) \neq \emptyset$ for all $1 \leq i \leq m$. Let $n \geq \max\{|\beta_i| \mid 1 \leq i \leq m\} + 1$. We then have $y = (\sum_{i=1}^m k_i s_{\alpha_i} p_{A_i} s^*_{\beta_i})x = (\sum_{i=1}^m k_i s_{\alpha_i} p_{A_i} s^*_{\beta_i})\epsilon_n x = (\sum_{i=1}^m k_i s_{\alpha_i} p_{A_i} s^*_{\beta_i} s_{\tau_{\leq n}(p)} s^*_{\tau_{\leq n}(p)})x$. By the minimality of m, $s_{\alpha_i} p_{A_i} s^*_{\beta_i} s_{\tau_{\leq n}(p)} s^*_{\tau_{\leq n}(p)} \neq 0$ for all $1 \leq i \leq m$. In particular, we have $s^*_{\beta_i} s_{\tau_{\leq n}(p)} \neq 0$ for all $1 \leq i \leq m$. Then, for each i, there exists a path $\delta_i \in \mathcal{G}^*$ such that $|\delta_i| \geq 1$, $\tau_{\leq n}(p) = \beta_i \delta_i$ and $s(\delta_i) \in r(\alpha_i) \cap A_i \cap r(\beta_i)$. This implies that

$$y = (\sum_{i=1}^m k_i s_{\alpha_i} p_{A_i} s^*_{\beta_i} s_{\tau_{\leq n}(p)} s^*_{\tau_{\leq n}(p)})x = (\sum_{i=1}^m k_i s_{\alpha_i \delta_i} s^*_{\tau_{\leq n}(p)})x = \sum_{i=1}^m k_i s_{\alpha_i \delta_i} x_n.$$

By the minimality of m, $s_{\alpha_i \delta_i} x_n$'s are nonzero, pairwise different elements in S_p, and so $\alpha_i \delta_i$'s are pairwise different paths of positive length in $\mathcal{G}^*$.

By renumbering paths $\alpha_i \delta_i$, without loss of generality, we may assume that $p_{s(\alpha_1\delta_1)} y = p_{s(\alpha_1\delta_1)}(\sum k_i s_{\alpha_i \delta_i} x_n) = \sum_{i=1}^d k_i s_{\alpha_i \delta_i} x_n$, where $1 \leq d \leq m$, and $s(\alpha_i \delta_i) = s(\alpha_1 \delta_1)$ for all $1 \leq i \leq d$. We note that $s^*_{\alpha_1\delta_1} s_{\alpha_1\delta_1} = p_{r(\alpha_1\delta_1)} = p_{r(\tau_{\leq n}(p))}$ and $s^*_{\alpha_1\delta_1} s_{\alpha_i \delta_i} = 0$ for all $2 \leq i \leq d$, and so

$$k_1^{-1} s_{\tau_{\leq n}(p)} s^*_{\alpha_1\delta_1} p_{s(\alpha_1\delta_1)} y = k_1^{-1} s_{\tau_{\leq n}(p)} s^*_{\alpha_1\delta_1} (\sum_{i=1}^d k_i s_{\alpha_i\delta_i} x_n) = s_{\tau_{\leq n}(p)} x_n = x.$$

This implies that $x \in L_K(\mathcal{G})y$, and hence $S_p = L_K(\mathcal{G})x = L_K(\mathcal{G})y$. Therefore, S_p is both simple and isomorphic to $V_{[p]}$, thus finishing the proof. $\square$

Let $\mathcal{G}$ be an ultragraph and $v, w \in \mathcal{G}^0$. We write $w \geq v$ to mean that there exits a path $\alpha \in \mathcal{G}^*$ with $s(\alpha) = w$ and $v \in r(\alpha)$. We denote by $T_{\mathcal{G}}(v)$ the set $\{w \in G^0 \mid v \geq w\}$. We say there is a *bifucation* at v if $|s^{-1}(v)| \geq 2$. The vertex v is called a *line point* if $|r(e)| = 1$ for all $e \in \mathcal{G}^1$ with $s(e) \in T_{\mathcal{G}}(v)$ and there is no bifucations or a cycle based at any vertex in $T_{\mathcal{G}}(v)$. Let $p = e_1 \cdots e_n \cdots$ be an infinite path in $\mathcal{G}$. We say that p *contains a line point* if there exists a number m such that $s(e_m)$ is a line point.

Corollary 4.8. *Let K be a field, $\mathcal{G}$ an ultragraph and $p = e_1 \cdots e_n \cdots$ an irrational path in $\mathcal{G}$. Then, the simple left $L_K(\mathcal{G})$-module $V_{[p]}$ is finitely presented if and only if p contains a line point.*

Proof. ($\Longrightarrow$). Assume that $V_{[p]}$ is finitely presented. By Proposition 4.7, the direct sum $\bigoplus_{i=0}^{\infty} L_K(\mathcal{G})(\epsilon_i - \epsilon_{i+1})$ is a finite direct sum, where $\epsilon_0 = p_{s(e_1)}$ and $\epsilon_i = s_{e_1} \cdots s_{e_i} s^*_{e_i} \cdots s^*_{e_1}$ for all $i \geq 1$, and so there exists m such that $\epsilon_m = \epsilon_{m+i}$

for all $i \geq 0$. We then have

$$p_{r(e_m)} = s_{e_m}^* \cdots s_{e_1}^* \epsilon_m s_{e_1} \cdots s_{e_m} = s_{e_m}^* \cdots s_{e_1}^* \epsilon_{m+1} s_{e_1} \cdots s_{e_m} = s_{e_{m+1}} s_{e_{m+1}}^*,$$

and

$$p_{s(e_{m+1})} = p_{s(e_{m+1})} p_{r(e_m)} = p_{s(e_{m+1})} s_{e_{m+1}} s_{e_{m+1}}^* = s_{e_{m+1}} s_{e_{m+1}}^* = p_{r(e_m)}.$$

This shows that $r(e_m) = \{s_{e_{m+1}}\}$. Similarly, since $\epsilon_{m+i} = \epsilon_{m+i+1}$ for all $i \geq 0$, we obtain that $r(e_{m+i}) = \{s(e_{m+i+1})\}$ for all i, that means, all these v_i $(i \geq m)$ are line points.

($\Longleftarrow$). Assume that there exists m such that v_m is a line point. We then have $\epsilon_m = \epsilon_{m+i}$ for all $i \geq 0$. By Proposition 4.7, $V_{[p]}$ is finitely presented, thus finishing the proof. $\square$

The following result extends [6, Theorem 3.7] to ultragraph Leavitt path algebras, showing that $V_{[p]}$ is finitely presented for all rational path p.

Proposition 4.9. *Let K be a field and $\mathcal{G}$ an ultragraph, and $p = \pi^\infty$ a rational path with a simple closed path π based at vertex v. Then $V_{[p]} = V_{[\pi^\infty]}$ is isomorphic to a cyclic left $L_K(\mathcal{G})$-module S_π generated by $x \in S_\pi$ subject to $s_\pi x = x$, whence it is both isomorphic to $L_K(\mathcal{G})p_v/L_K(\mathcal{G})(p_v - s_\pi)$ and finitely presented.*

Proof. We have $s_\pi \cdot p = p$ as elements in $V_{[p]}$. Since $V_{[p]}$ is a simple left $L_K(\mathcal{G})$-module, $V_{[p]}$ is an image of S_π under the map sending $x \in S_\pi$ to $p \in V_{[p]}$, and so S_π is nonzero.

We note that $x = s_\pi^n x$ as elements in S_π for all $n \geq 0$, where $\pi^0 := p_v$. Let y be a nonzero element in S_π. Since $S_\pi = L_K(\mathcal{G})x$, y may be written in the form $y = rx$ and $0 \neq r = \sum_{i=1}^m k_i s_{\alpha_i} p_{A_i} s_{\beta_i}^* \in L_K(\mathcal{G})$, where m is minimal such that $k_i \in K \setminus \{0\}$, $\alpha_i, \beta_i \in \mathcal{G}^*$, $A_i \in \mathcal{G}^0$ and $r(\alpha_i) \cap A_i \cap r(\beta_i) \neq \emptyset$ for all $1 \leq i \leq m$.

Let n be a positive integer such that $|\beta_i| < n|\pi|$ for all $1 \leq i \leq m$. We then have

$$y = (\sum_{i=1}^m k_i s_{\alpha_i} p_{A_i} s_{\beta_i}^*)x = (\sum_{i=1}^m k_i s_{\alpha_i} p_{A_i} s_{\beta_i}^*) s_\pi^n x = (\sum_{i=1}^m k_i s_{\alpha_i} p_{A_i} s_{\beta_i}^* s_\pi^n)x.$$

By the minimality of m, $s_{\alpha_i} p_{A_i} s_{\beta_i}^* s_\pi^n \neq 0$ for all $1 \leq i \leq m$. Then, for each i, there exists $\delta_i \in \mathcal{G}^*$ such that $|\delta_i| \geq 1$, $\pi^n = \beta_i \delta_i$ and $s(\delta_i) \in r(\alpha_i) \cap A_i \cap r(\beta_i)$. This implies

$$y = (\sum_{i=1}^m k_i s_{\alpha_i} p_{A_i} s_{\beta_i}^* s_\pi^n)x = (\sum_{i=1}^m k_i s_{\alpha_i \delta_i})x.$$

By the minimality of m, $s_{\alpha_i\delta_i}x$'s are nonzero, pairwise different elements in S_π, and so $\alpha_i\delta_i$'s are pairwise different paths positive length in $\mathcal{G}^*$.

By renumbering paths $\alpha_i\delta_i$, without loss of generality, we may assume that $p_{s(\alpha_1\delta_1)} y = p_{s(\alpha_1\delta_1)}(\sum k_i s_{\alpha_i\delta_i} x) = \sum_{i=1}^d k_i s_{\alpha_i\delta_i} x$, where $1 \leq d \leq m$, and $s(\alpha_i\delta_i) =$

$s(\alpha_1\delta_1)$ for all $1 \le i \le d$. We note that $s^*_{\alpha_1\delta_1}s_{\alpha_1\delta_1} = p_{r(\alpha_1\delta_1)} = p_{r(\pi)}$ and $s^*_{\alpha_1\delta_1}s_{\alpha_i\delta_i} = 0$ for all $2 \le i \le d$, and so

$$k_1^{-1}s^*_{\alpha_1\delta_1}p_{s(\alpha_1\delta_1)}y = k_1^{-1}s^*_{\alpha_1\delta_1}(\sum_{i=1}^{d} k_i s_{\alpha_i\delta_i}x) = p_{r(\pi)}x = p_{r(\pi)}(s_\pi x) = s_\pi x = x.$$

This implies that $x \in L_K(\mathcal{G})y$, and hence $S_\pi = L_K(\mathcal{G})x = L_K(\mathcal{G})y$. Therefore, S_π is both simple and isomorphic to $V_{[p]}$, thus finishing the proof. □

Consequently, we obtain the following corollary.

Corollary 4.10. *Let K be a field, $\mathcal{G}$ an ultragraph and p an infinite path in $\mathcal{G}$. Then, the simple left $L_K(\mathcal{G})$-module $V_{[p]}$ is finitely presented if and only if p either contains a line point or is equivalent to a rational path.*

Proof. It immediately follows from Corollary 4.8 and Proposition 4.9. □

Now we are able to present the main result of this section.

Theorem 4.11. *Let K be a field and $\mathcal{G}$ an ultragraph such that every simple left $L_K(\mathcal{G})$-module is finitely presented. Then the following statements hold:*

(1) *$\mathcal{G}$ is a row-finite ultragraph having Condition* (EXC);

(2) There are neither minimal infinite emitters nor minimal sinks in $\mathcal{G}^0$;

(3) The maximal length of chains of cycles in $\mathcal{G}$ is finite;

(4) Every infinite path in $\mathcal{G}$ either contains a line point or is equivalent to a rational path;

(5) $\mathrm{GKdim}(L_K(\mathcal{G})) < \infty$.

Proof. (1) By Proposition 4.4, $\mathcal{G}$ is a row-finite ultragraph. Assume that $\mathcal{G}$ contains two cycles $\alpha = \alpha_1\alpha_2\cdots\alpha_n$ and $\beta = \beta_1\beta_2\cdots\beta_m$ such that $s(\alpha_i), s(\beta_j) \in r(\alpha_{i-1})\cap r(\beta_{j-1})$ and $(\alpha_{i-1}, \alpha_i) \ne (\beta_{j-1}, \beta_j)$ for some $1 \le i \le n, 1 \le j \le m$, where $r(\alpha_0) := r(\alpha_n)$ and $r(\beta_0) := r(\beta_m)$. By renumbering edges of α and β, without loss of generality, we may assume that $i = j = 1$. We then have an irrational path $p = \alpha\beta\alpha^2\beta^2\cdots\alpha^n\beta^n\cdots$. By Corollary 4.8, the simple left $L_K(\mathcal{G})$-module $V_{[p]}$ is not finitely presented, a contradiction. Therefore, $\mathcal{G}$ satisfies Condition (EXC).

(2) It follows from Proposition 4.4 and Theorem 4.5 (5).

(3) If there exists a chain of cycles of infinite length in $\mathcal{G}$, then this chain may be expanded to an irrational path p in $\mathcal{G}$. This leads to a contradiction since the corresponding simple left $L_K(\mathcal{G})$-module $V_{[p]}$ is not finitely presented, by Corollary 4.8. Thus, the maximal length of chains of cycles in $\mathcal{G}$ is finite.

(4) Let p be an infinite path in $\mathcal{G}$. We then have that the simple left $L_K(\mathcal{G})$-module $V_{[p]}$ is finitely presented. By Corollary 4.10, p either contains a line point or is equivalent to a rational path, as desired.

(5) It follows from items (1) and (3) and Theorem 3.4, thus finishing the proof. □

References

[1] G. Abrams, Leavitt path algebras: the first decade, *Bull. Math. Sci.* **5** (2015), 59-120.

[2] G. Abrams, P. Ara, and M. Siles Molina, *Leavitt path algebras.* Lecture Notes in Mathematics series, Vol. 2191, Springer, London, 2017.

[3] A. Alahmedi, H. Alsulami, S. Jain and Efim I. Zelmanov, Leavitt Path algebras of finite Gelfand-Kirillov dimension, *J. Algebra Appl.* **11** (2012), no. 6, 1250225, 6 pp.

[4] A. Alahmedi, H. Alsulami, S. Jain and Efim I. Zelmanov, Structure of Leavitt path algebras of polynomial growth, *Proc. Natl. Acad. Sci. USA* **110** (2013), 15222-15224.

[5] P. Ara and K.M. Rangaswamy, Finitely presented simple modules over Leavitt path algebras, *J. Algebra*, **417** (2014), 333–352.

[6] P. N. Ánh and T. G. Nam, Special irreducible representations of Leavitt path algebras, *Adv. Math.* **377** (2021), Paper No. 107483, 31 pp.

[7] X.W. Chen, Irreducible representations of Leavitt path algebras, *Forum Math.* **27** (2015), 549–574.

[8] G. G. de Castro, D. Gonçalves and D. W. van Wyk, Ultragraph algebras via labelled graph groupoids, with applications to generated uniqueness theorems, *J. Algebra* **579** (2021), 456-495.

[9] G. G. de Castro, D. Gonçalves and D. W. van Wyk, Topological Full groups of ultragraph groupoids as an isormorphism invariant, *Munster J. Math.* **14**(1), (2021), 165-189.

[10] T. T. H. Duyen, D. Gonçalves and T. G. Nam, On the ideals of ultragraph Leavitt path algebras, arXiv:2109.10440.

[11] D. Gonçalves and D. Royer, Ultragraphs and shift spaces over infinite alphabets, *Bull. Sci. Math.* **141** (2017), 25-45.

[12] D. Gonçalves and D. Royer, Simplicity and chain conditions for ultragraph Leavitt path algebras via partial skew group ring theory, *J. Aust. Math. Soc.* **109** (2020), 299-319.

[13] D. Gonçalves and D. Royer, Infnite alphabet edge shift spaces via ultragraphs and their C^*-algebras, *Int. Math. Res. Not. IMRN* 2019, no. 7, 2177–2203.

[14] D. Gonçalves and D. Royer, Irreducible and permutative representations of ultragraph Leavitt path algebras, *Forum Math.* **32** (2020), 417–431.

[15] R. Hazrat and T. G. Nam, Realizing ultragraph Leavitt path algebras as Steinberg algebras, arXiv:2008.04668 [math.RA], 2020.

[16] M. Imanfar, A. Pourabbas and H. Larki, The Leavitt path algebras of ultragraphs, *Kyungpook Math. J.* **60** (2020), 21–43.

[17] T. Katsura, P. S. Muhly, A. Sims and M. Tomforde, Graph algebras, Exel-Laca algebras, and ultragraph algebras coincide up to Morita equivalence, *J. Reine Angew. Math.* **640** (2010), 135-165.

[18] G. R. Krause and T. H. Lenagan, Growth of algebras and Gelfand-Kirillov dimension, revised edition, Graduate studies in Mathematics Vol. 22 (AMS, 2000).

[19] T.Y. Lam, *Lectures on Modules and Rings.* Graduate Texts in Mathematics series, Vol. 189, Springer-Verlag, New York, 1999.

[20] H. Larki, Primitive ideals and pure infiniteness of ultragraph C^*-algebras, *J. Korean Math. Soc.* **56** (2019), 1–23

[21] J. M. Moremo-Fernández and M. Siles Molina, Graph algebras and the Gelfand-Kirillov dimension, *J. Algebra Appl.* **17** (2018), no. 5, 1850095, 15 pp.

[22] T. G. Nam and N. D. Nam, Purely infinite simple ultragraph Leavitt path algebras, *Mediterr. J. Math.* **19** (2022), no. 1, Paper No. 7, 20 pp.

[23] K. M. Rangaswamy, Leavitt path algebras with finitely presented irreducible representations, *J. Algebra* **447** (2016), 624-648.

[24] F. A. Tasca and D. Gonalves, *KMS states and continuous orbit equivalence for ultragraph shift spaces with sinks*, *Publicacions Matemàtiques* **66** (2022), no 2, 729-787.

[25] M. Tomforde, A unified approach to exel-laca algebras and C^*-algebras associated to graphs, *J. Operator Theory* **50** (2003), 345–368.

[26] M. Tomforde, Simplicity of ultragraph algebras, *Indiana Univ. Math. J.* **52** (2003), 901–925.

[27] M. Tomforde, Uniqueness theorems and ideal structure for Leavitt path algebras, *Journal of Algebra* **318** (2007), 270–299.