ZAGIER-HOFFMAN'S CONJECTURES IN POSITIVE CHARACTERISTIC

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Abstract. Multiples zeta values and alternating multiple zeta values in positive characteristic were introduced by Thakur and Harada as analogues of classical multiple zeta values of Euler and Euler sums. In this paper we determine all linear relations among alternating multiple zeta values and settle the main goals of these theories. As a consequence we completely establish Zagier-Hoffman's conjectures in positive characteristic formulated by Todd and Thakur which predict the dimension and an explicit basis of the span of multiple zeta values of Thakur of fixed weight.

Contents

Introduction

0.1. Classical setting.

0.1.1. *Multiple zeta values.* Multiple zeta values of Euler (MZV's for short) are real positive numbers given by

$$\zeta(n_1,\dots,n_r)=\sum_{0<k_1<\dots<k_r}\frac{1}{k_1^{n_1}\dots k_r^{n_r}},\quad \text{where } n_i\geq 1, n_r\geq 2.$$

Here r is called the depth and $w=n_1+\dots+n_r$ is called the weight of the presentation $\zeta(n_1,\dots,n_r)$. These values covers the special values $\zeta(n)$ for $n\geq 2$ of the Riemann zeta function and have been studied intensively especially in the last three decades with important and deep connections to different branches of

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mathematics and physics, for example arithmetic geometry, knot theory and higher energy physics. We refer the reader to [7, 39] for more details.

The main goal of this theory is to understand all $\mathbb{Q}$-linear relations among MZV's. Goncharov conjectures that all $\mathbb{Q}$-linear relations among MZV's can be derived from those among MZV's of the same weight. As the next step, precise conjectures formulated by Zagier [39] and Hoffman [21] predict the dimension and an explicit basis for the $\mathbb{Q}$-vector space $\mathcal{Z}_k$ spanned by MZV's of weight k for $k \in \mathbb{N}$.

Conjecture 0.1 (Zagier's conjecture)**.** *We define a Fibonacci-like sequence of integers d_k as follows. Letting $d_0 = 1, d_1 = 0$ and $d_2 = 1$ we define $d_k = d_{k-2} + d_{k-3}$ for $k \geq 3$. Then for $k \in \mathbb{N}$ we have*

$$\dim_{\mathbb{Q}} \mathcal{Z}_k = d_k.$$

Conjecture 0.2 (Hoffman's conjecture)**.** *The $\mathbb{Q}$-vector space $\mathcal{Z}_k$ is generated by the basis consisting of MZV's of weight k of the form $\zeta(n_1, \dots, n_r)$ with $n_i \in \{2, 3\}$.*

The algebraic part of these conjectures which concerns upper bounds for $\dim_{\mathbb{Q}} \mathcal{Z}_k$ was solved by Terasoma [30], Deligne-Goncharov [16] and Brown [5] using the theory of mixed Tate motives.

Theorem 0.3 (Deligne-Goncharov, Terasoma)**.** *For $k \in \mathbb{N}$ we have $\dim_{\mathbb{Q}} \mathcal{Z}_k \leq d_k$.*

Theorem 0.4 (Brown)**.** *The $\mathbb{Q}$-vector space $\mathcal{Z}_k$ is generated by MZV's of weight k of the form $\zeta(n_1, \dots, n_r)$ with $n_i \in \{2, 3\}$.*

Unfortunately, the transcendental part which concerns lower bounds for $\dim_{\mathbb{Q}} \mathcal{Z}_k$ is completely open. We refer the reader to [7, 15, 39] for more details and more exhaustive references.

0.1.2. *Alternating multiple zeta values.* There exists a variant of MZV's called the alternating multiple zeta values (AMZV's for short), also known as Euler sums. They are real numbers given by

$$\zeta\begin{pmatrix} \epsilon_1 & \dots & \epsilon_r \\ n_1 & \dots & n_r \end{pmatrix} = \sum_{0<k_1<\dots<k_r} \frac{\epsilon_1^{k_1} \dots \epsilon_r^{k_r}}{k_1^{n_1} \dots k_r^{n_r}}$$

where $\epsilon_i \in \{\pm 1\}$, $n_i \in \mathbb{N}$ and $(n_r, \epsilon_r) \neq (1, 1)$. Similar to MZV's, these values have been studied by Broadhurst, Deligne–Goncharov, Hoffman, Kaneko–Tsumura and many others due to many connections in different contexts. We refer the reader to [19, 22, 40] for more references.

As before, it is expected that all $\mathbb{Q}$-linear relations among AMZV's can be derived from those among AMZV's of the same weight. In particular, it is natural to ask whether one could formulate similar conjectures to those of Zagier and Hoffman for AMZV's of fixed weight. By the work of Deligne-Goncharov [16], the sharp upper bounds are achieved:

Theorem 0.5 (Deligne-Goncharov)**.** *For $k \in \mathbb{N}$ if we denote by $\mathcal{A}_k$ the $\mathbb{Q}$-vector space spanned by AMZV's of weight k, then $\dim_{\mathbb{Q}} \mathcal{A}_k \leq F_{k+1}$. Here F_n is the n-th Fibonacci number defined by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 1$.*

The fact that the previous upper bounds are sharp was also explained by Deligne in [14] using a variant of a conjecture of Grothendieck. In the direction of extending Brown's theorem for AMZV's, there exist several sets of generators for $\mathcal{A}_k$ (see for example [11, 14]). However, we mention that these generators are only linear combinations of AMZV's.

Finally, we know nothing about non trivial lower bounds for $\dim_{\mathbb{Q}} \mathcal{A}_k$.

0.2. **Function field setting.**

0.2.1. *MZV's of Thakur and analogues of Zagier-Hoffman's conjectures.* By analogy between number fields and function fields, based on the pioneering work of Carlitz [8], Thakur [31] defined analogues of multiple zeta values in positive characteristic. We now need to introduce some notations. Let $A = \mathbb{F}_q[\theta]$ be the polynomial ring in the variable θ over a finite field $\mathbb{F}_q$ of q elements of characteristic $p > 0$. We denote by A_+ the set of monic polynomials in A. Let $K = \mathbb{F}_q(\theta)$ be the fraction field of A equipped with the rational point ∞. Let K_∞ be the completion of K at ∞ and $\mathbb{C}_\infty$ be the completion of a fixed algebraic closure $\overline{K}$ of K at ∞. We denote by v_∞ the discrete valuation on K corresponding to the place ∞ normalized such that $v_\infty(\theta) = -1$, and by $|\cdot|_\infty = q^{-v_\infty}$ the associated absolute value on K. The unique valuation of $\mathbb{C}_\infty$ which extends v_∞ will still be denoted by v_∞.

Let $\mathbb{N} = \{1, 2, \dots\}$ be the set of positive integers and $\mathbb{Z}^{\geq 0} = \{0, 1, 2, \dots\}$ be the set of non-negative integers. In [8] Carlitz introduced the Carlitz zeta values $\zeta_A(n)$ for $n \in \mathbb{N}$ given by

$$\zeta_A(n) := \sum_{a \in A_+} \frac{1}{a^n} \in K_\infty$$

which are analogues of classical special zeta values in the function field setting. For any tuple of positive integers $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$, Thakur [31] defined the characteristic p multiple zeta value (MZV for short) $\zeta_A(\mathfrak{s})$ or $\zeta_A(s_1, \dots, s_r)$ by

$$\zeta_A(\mathfrak{s}) := \sum \frac{1}{a_1^{s_1} \dots a_r^{s_r}} \in K_\infty$$

where the sum runs through the set of tuples $(a_1, \dots, a_r) \in A_+^r$ with $\deg a_1 > \dots > \deg a_r$. We call r the depth of $\zeta_A(\mathfrak{s})$ and $w(\mathfrak{s}) = s_1 + \dots + s_r$ the weight of $\zeta_A(\mathfrak{s})$. We note that Carlitz zeta values are exactly depth one MZV's. Thakur [32] showed that all the MZV's do not vanish. We refer the reader to [3, 4, 17, 24, 25, 29, 31, 33, 34, 35, 36, 38] for more details about these objects.

As in the classical setting, the main goal of the theory is to understand all linear relations over K among MZV's. We now state analogues of Zagier-Hoffman's conjectures in positive characteristic formulated by Thakur in [35, §8] and by Todd in [37].

For $w \in \mathbb{N}$ we denote by $\mathcal{Z}_w$ the K-vector space spanned by the MZV's of weight w. We denote by $\mathcal{T}_w$ the set of $\zeta_A(\mathfrak{s})$ where $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ of weight w with $1 \leq s_i \leq q$ for $1 \leq i \leq r-1$ and $s_r < q$.

Conjecture 0.6 (Zagier's conjecture in positive characteristic)**.** *Letting*

$$d(w) = \begin{cases} 1 & \text{if } w = 0, \\ 2^{w-1} & \text{if } 1 \leq w \leq q-1, \\ 2^{w-1} - 1 & \text{if } w = q, \end{cases}$$

we put $d(w) = \sum_{i=1}^{q} d(w-i)$ *for* $w > q$. *Then for any* $w \in \mathbb{N}$, *we have*

$$\dim_K \mathcal{Z}_w = d(w).$$

Conjecture 0.7 (Hoffman's conjecture in positive characteristic)**.** *A K-basis for $\mathcal{Z}_w$ is given by $\mathcal{T}_w$ consisting of $\zeta_A(s_1, \dots, s_r)$ of weight w with $s_i \le q$ for $1 \le i < r$, and $s_r < q$.*

In [27] one of the authors succeeded in proving the algebraic part of these conjectures (see [27, Theorem A]): for all $w \in \mathbb{N}$, we have

$$\dim_K \mathcal{Z}_w \le d(w).$$

This part is based on shuffle relations for MZV's due to Chen and Thakur and some operations introduced by Todd. For the transcendental part, he used the Anderson-Brownawell-Papanikolas criterion in [2] and proved sharp lower bounds for small weights $w \le 2q-2$ (see [27, Theorem D]). It was already noted that it is very difficult to extend his method for general weights (see [27] for more details).

0.2.2. *AMZV's in positive characteristic.* Recently, Harada [19] introduced the alternating multiple zeta values in positive characteristic (AMZV's) as follows. Letting $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^n$ and $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_r) \in (\mathbb{F}_q^\times)^n$, we define

$$\zeta_A \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} = \sum \frac{\varepsilon_1^{\deg a_1} \dots \varepsilon_r^{\deg a_r}}{a_1^{s_1} \dots a_r^{s_r}} \in K_\infty$$

where the sum runs through the set of tuples $(a_1, \dots, a_r) \in A_+^r$ with $\deg a_1 > \dots > \deg a_r$. The numbers r and $w(\mathfrak{s}) = s_1 + \dots + s_r$ are called the depth and the weight of $\zeta_A \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}$, respectively. Harada [19] extended basic properties of MZV's to AMZV's, i.e., non-vanishing, shuffle relations, period interpretation and linear independence. Again the main goal of this theory is to determine all linear relations over K among AMZV's. It is natural to ask whether one could extend the previous work on analogues of Zagier-Hoffman's conjectures to this setting. More precisely, if for $w \in \mathbb{N}$ we denote by $\mathcal{AZ}_w$ the K-vector space spanned by the AMZV's of weight w, then we would like to determine the dimensions of $\mathcal{AZ}_w$ and exhibit some nice bases of these vector spaces.

0.3. **Main results.**

0.3.1. *Statements of the main results.* In this manuscript we present complete answers to all the previous conjectures and problems raised in §0.2.

First, for all w we calculate the dimension of $\mathcal{AZ}_w$ and give an explicit basis in the spirit of Hoffman.

Theorem A. *We define a Fibonacci-like sequence $s(w)$ as follows. We put*

$$s(w) = \begin{cases} (q-1)q^{w-1} & \text{if } 1 \le w < q, \\ (q-1)(q^{w-1}-1) & \text{if } w = q, \end{cases}$$

and for $w > q$, $s(w) = (q-1)\sum_{i=1}^{q-1} s(w-i) + s(w-q)$. Then for all $w \in \mathbb{N}$,

$$\dim_K \mathcal{AZ}_w = s(w).$$

Further, we can exhibit a Hoffman-like basis of $\mathcal{AZ}_w$.

Second, we give a proof of both Conjectures 0.6 and 0.7 which generalizes the previous work of the fourth author [27].

Theorem B. *For all $w \in \mathbb{N}$, $\mathcal{T}_w$ is a K-basis for $\mathcal{Z}_w$. In particular,*

$$\dim_K \mathcal{Z}_w = d(w).$$

We recall that analogues of Goncharov's conjectures in positive characteristic were proved in [9]. As a consequence, we completely determine all linear relations over K among MZV's and AMZV's and settle the main goals of these theories.

0.3.2. *Ingredients of the proofs.*

Let us emphasize here that Theorem A is much harder than Theorem B and that it is not enough to work inside the setting of AMZV's. On the one hand, although it is straightforward to extend the algeraic part for AMZV's following the same line in [27, §2 and §3], we only obtain a weak version of Brown's theorem in this setting. More precisely, we get a set of generators for $\mathcal{AZ}_w$ but it is too large to be a basis of this vector space. For small weights, we find ad hoc arguments to produce a smaller set of generators but it does not work for arbitrary weights (see §4.1.3). Roughly speaking, in [27, §2 and §3] we have an algorithm which moves forward so that we can express any AMZV's as a linear combination of generators. But we lack some precise controls on coefficients in these expressions so that we cannot go backward and change bases. On the other hand, the transcendental part for AMZV's shares the same difficulties with the case of MZV's as noted before.

In this paper we use *a completely new approach* which is based on the study of alternating Carlitz multiple polylogarithms (ACMPL's for short) defined as follows. We put $\ell_0 := 1$ and $\ell_d := \prod_{i=1}^d (\theta - \theta^{q^i})$ for all $d \in \mathbb{N}$. For any tuple $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ and $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_r) \in (\mathbb{F}_q^\times)^r$, we introduce the corresponding alternating Carlitz multiple polygarithm by

$$\operatorname{Li} \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} := \sum_{d_1 > \dots > d_r \geq 0} \frac{\varepsilon_1^{d_1} \dots \varepsilon_r^{d_r}}{\ell_{d_1}^{s_1} \dots \ell_{d_r}^{s_r}} \in K_\infty.$$

The key result is establishing a non trivial connection among AMZV's and ACMPL's which allows us to go back and forth among these objects (see Theorem 4.3). To do so, following [27, §2 and §3] we use stuffle relations to develop an algebraic theory for ACMPL's and obtain a weak version of Brown's theorem, i.e., a set of generators for the K-vector space $\mathcal{AL}_w$ spanned by ACMPL's of weight w. We observe that this set of generators is exactly the same as that for AMZV's. Thus $\mathcal{AL}_w = \mathcal{AZ}_w$, which provides a dictionary among AMZV's and ACMPL's.

We then determine all K-linear relations among ACMPL's (see Theorem 3.6). The proof we give here, although using similar tools as in [27], differs at some crucial points and requires *three new ingredients*.

The first new ingredient is constructing a suitable Hoffman-like basis $\mathcal{AS}_w$ of $\mathcal{AL}_w$. In fact, our transcendental method dictates that we have to find a full system of bases $\mathcal{AS}_w$ of $\mathcal{AL}_w$ indexed by weights w with strong constraints as given in Theorem 2.4. The failure of finding such a system of bases is the main obstruction for generalizing [27, Theorem D] (see §4.1 and [27, Remark 6.3] for more details).

The second new ingredient is formulating and proving (a strong version of) Brown's theorem for AMCPL's (see Theorem 1.10). As mentioned before, the method in [27] only yields a weak version of Brown's theorem for ACMPL's as the set of generators is not a basis. Roughly speaking, given any ACMPL's we can express it as a linear combination of generators. The fact that stuffle relations for ACMPL's are "simpler" than shuffle relations for AMZV's give more precise information on coefficients of these expressions. Consequently, we show that a certain transition matrix is invertible and obtain Brown's theorem for ACMPL's. This settles the algebraic part for ACMPL's.

The last new ingredient is proving the transcendental part for ACMPL's in full generality, i.e., the ACMPL's in $\mathcal{AS}_w$ are linearly independent over K (see Theorem 3.4). We emphasize that we do need the full strength of the algebraic part to prove the transcendental part. The proof follows the same line in [27, §4 and §5] which is formulated in a more general setting in §2. First, we have to consider not only linear relations among ACMPL's in $\mathcal{AS}_w$ but also those among ACMPL's in $\mathcal{AS}_w$ and the suitable power $\widetilde{\pi}^w$ of the Carlitz period $\widetilde{\pi}$. Second, starting from such a non trivial relation we apply the Anderson-Brownawell-Papanikolas criterion in [2] and reduce to solve a system of σ-linear equations. While in [27, §4 and §5] this system does not have a non trivial solution which allows us to conclude, our system has a unique solution for even w (i.e., $q-1$ divides w). This means that for such w up to a scalar there is a unique linear relation among ACMPL's in $\mathcal{AS}_w$ and $\widetilde{\pi}^w$. The final step consists of showing that in this unique relation, the coefficient of $\widetilde{\pi}^w$ is non zero. Unexpectedly, this is a consequence of Brown's theorem for AMCPL's mentioned above.

0.3.3. *Plan of the paper.* We briefly explain the organization of the manuscript.

- In §1 we recall the definition and basic properties of ACMPL's. We then develop an algebraic theory for these objects and obtain weak and strong Brown's theorems (see Proposition 1.9 and Theorem 1.10).
- In §2 we generalize some transcendental results in [27] and give statements in a more general setting (see Theorem 2.4).
- In §3 we prove transcendental results for ACMPL's and completely determine all linear relations among ACMPL's (see Theorems 3.4 and 3.6).
- Finally, in §4 we present two applications and prove the main results, i.e., Theorems A and B. The first application is to prove the aforementioned connection among ACMPL's and AMZV's and then to determine all linear relations among AMZV's in positive characteristic (see §4.1). The second application is a proof of Zagier-Hoffman's conjectures in positive characteristic which generalizes the main results of [27] (see §4.2).

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1. Weak and strong Brown's theorems for ACMPL's

In this section we first extend the work of [27] and develop an algebraic theory for ACMPL's. Next we prove a weak version of Brown's theorem for ACMPL's (see Theorem 1.9) which gives a set of generators for the K-vector space spanned by ACMPL's of weight w. The techniques of Sections 1.1-1.3 are similar to [27] and the expert reader could skip the details.

Contrary to what happens in [27], it turns out that the previous set of generators is too large to be a basis. Consequently, in §1.4 we introduce another set of generators and prove a strong version of Brown's theorem for ACMPL's (see Theorem 1.10).

1.1. **Analogues of power sums.**

1.1.1. We recall and introduce some notation in [27]. Letting $\mathfrak{s} = (s_1, \dots, s_n) \in \mathbb{N}^n$ and $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n) \in (\mathbb{F}_q^\times)^n$, we set $\mathfrak{s}_- := (s_2, \dots, s_n)$ and $\boldsymbol{\varepsilon}_- := (\varepsilon_2, \dots, \varepsilon_n)$. A positive array $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}$ is an array of the form

$$\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} = \begin{pmatrix} \varepsilon_1 & \dots & \varepsilon_n \\ s_1 & \dots & s_n \end{pmatrix}.$$

For $i \in \mathbb{N}$ we define $T_i(\mathfrak{s})$ to be the tuple $(s_1 + \dots + s_i, s_{i+1}, \dots, s_n)$. Further, for tuples of positive integers $\mathfrak{s}, \mathfrak{t}$ and for $i \in \mathbb{N}$, if $T_i(\mathfrak{s}) \le T_i(\mathfrak{t})$, then $T_k(\mathfrak{s}) \le T_k(\mathfrak{t})$ for all $k \ge i$. This notion extends straightforward to positive arrays.

Let $\mathfrak{s} = (s_1, \dots, s_r)$ be a tuple of positive integers. We denote by $0 \le i \le r$ the biggest integer such that $s_j \le q$ for all $1 \le j \le i$ and define *the initial tuple* $\operatorname{Init}(\mathfrak{s})$ of $\mathfrak{s}$ to be the tuple

$$\operatorname{Init}(\mathfrak{s}) := (s_1, \dots, s_i).$$

In particular, if $s_1 > q$, then $i = 0$ and $\operatorname{Init}(\mathfrak{s})$ is the empty tuple.

For two different tuples $\mathfrak{s}$ and $\mathfrak{t}$, we consider the lexicographical order for initial tuples and write $\operatorname{Init}(\mathfrak{t}) \preceq \operatorname{Init}(\mathfrak{s})$ (resp. $\operatorname{Init}(\mathfrak{t}) \prec \operatorname{Init}(\mathfrak{s})$, $\operatorname{Init}(\mathfrak{t}) \succeq \operatorname{Init}(\mathfrak{s})$ and $\operatorname{Init}(\mathfrak{t}) \succ \operatorname{Init}(\mathfrak{s})$).

1.1.2. We recall the power sums studied by Thakur [34]. For $d \in \mathbb{Z}$ and for $\mathfrak{s} = (s_1, \dots, s_n) \in \mathbb{N}^n$ we introduce

$$S_d(\mathfrak{s}) = \sum_{\substack{a_1, \dots, a_n \in A_+ \\ d = \deg a_1 > \dots > \deg a_n \ge 0}} \frac{1}{a_1^{s_1} \dots a_n^{s_n}} \in K$$

and

$$S_{<d}(\mathfrak{s}) = \sum_{\substack{a_1, \dots, a_n \in A_+ \\ d > \deg a_1 > \dots > \deg a_n \ge 0}} \frac{1}{a_1^{s_1} \dots a_n^{s_n}} \in K.$$

We also recall $\ell_0 := 1$ and $\ell_d := \prod_{i=1}^d (\theta - \theta^{q^i})$ for all $d \in \mathbb{N}$. Letting $\mathfrak{s} = (s_1, \dots, s_n) \in \mathbb{N}^n$, for $d \in \mathbb{Z}$, we define analogues of power sums by

$$\operatorname{Si}_d(\mathfrak{s}) = \sum_{d = d_1 > \dots > d_n \ge 0} \frac{1}{\ell_{d_1}^{s_1} \dots \ell_{d_n}^{s_n}} \in K,$$

and

$$\mathrm{Si}_{<d}(\mathfrak{s}) = \sum_{d>d_1>\dots>d_n\geq 0} \frac{1}{\ell_{d_1}^{s_1}\dots\ell_{d_n}^{s_n}} \in K.$$

Let $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} = \begin{pmatrix} \varepsilon_1 & \dots & \varepsilon_n \\ s_1 & \dots & s_n \end{pmatrix}$ be a positive array. For $d \in \mathbb{Z}$, we define

$$S_d \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} = \sum_{\substack{a_1,\dots,a_n\in A_+ \\ d=\deg a_1>\dots>\deg a_n\geq 0}} \frac{\varepsilon_1^{\deg a_1}\dots\varepsilon_n^{\deg a_n}}{a_1^{s_1}\dots a_n^{s_n}} \in K$$

and

$$S_{<d} \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} = \sum_{\substack{a_1,\dots,a_n\in A_+ \\ d>\deg a_1>\dots>\deg a_n\geq 0}} \frac{\varepsilon_1^{\deg a_1}\dots\varepsilon_n^{\deg a_n}}{a_1^{s_1}\dots a_n^{s_n}} \in K.$$

We also introduce

$$\mathrm{Si}_d \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} = \sum_{d=d_1>\dots>d_n\geq 0} \frac{\varepsilon_1^{d_1}\dots\varepsilon_n^{d_n}}{\ell_{d_1}^{s_1}\dots\ell_{d_n}^{s_n}} \in K,$$

and

$$\mathrm{Si}_{<d} \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} = \sum_{d>d_1>\dots>d_n\geq 0} \frac{\varepsilon_1^{d_1}\dots\varepsilon_n^{d_n}}{\ell_{d_1}^{s_1}\dots\ell_{d_n}^{s_n}} \in K.$$

One verifies easily the following formulas:

$$\mathrm{Si}_d \begin{pmatrix} \varepsilon \\ s \end{pmatrix} = \varepsilon^d \,\mathrm{Si}_d(s),$$

$$\mathrm{Si}_d \begin{pmatrix} 1 & \dots & 1 \\ s_1 & \dots & s_n \end{pmatrix} = \mathrm{Si}_d(s_1,\dots,s_n),$$

$$\mathrm{Si}_{<d} \begin{pmatrix} 1 & \dots & 1 \\ s_1 & \dots & s_n \end{pmatrix} = \mathrm{Si}_{<d}(s_1,\dots,s_n),$$

$$\mathrm{Si}_d \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} = \mathrm{Si}_d \begin{pmatrix} \varepsilon_1 \\ s_1 \end{pmatrix} \mathrm{Si}_{<d} \begin{pmatrix} \boldsymbol{\varepsilon}_- \\ \mathfrak{s}_- \end{pmatrix}.$$

Then we define the alternating Carlitz multiple polygarithm (ACMPL for short) as follows

$$\mathrm{Li} \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} = \sum_{d\geq 0} \mathrm{Si}_d \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} = \sum_{d_1>\dots>d_n\geq 0} \frac{\varepsilon_1^{d_1}\dots\varepsilon_n^{d_n}}{\ell_{d_1}^{s_1}\dots\ell_{d_n}^{s_n}} \in K_\infty.$$

We agree also that $\mathrm{Li} \begin{pmatrix} \emptyset \\ \emptyset \end{pmatrix} = 1$. We call $\mathrm{depth}(\mathfrak{s}) = n$ the depth, $w(\mathfrak{s}) = s_1+\dots+s_n$ the weight and $\chi(\boldsymbol{\varepsilon}) = \varepsilon_1\dots\varepsilon_n$ the character of $\mathrm{Li} \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}$.

Lemma 1.1. *For all* $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}$ *as above such that* $s_i \leq q$ *for all* i*, we have*

$$S_d \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} = \mathrm{Si}_d \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} \quad \text{for all } d \in \mathbb{Z}.$$

Therefore,

$$\zeta_A \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} = \mathrm{Li} \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}.$$

Proof. We denote by $\mathfrak{J}$ the set of all positive arrays $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} = \begin{pmatrix} \varepsilon_1 & \dots & \varepsilon_n \\ s_1 & \dots & s_n \end{pmatrix}$ for some n such that $s_1, \dots, s_n \leq q$.

The second statement follows at once from the first statement. We prove the first statement by induction on $\operatorname{depth}(\mathfrak{s})$. For $\operatorname{depth}(\mathfrak{s}) = 1$, we let $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} = \begin{pmatrix} \varepsilon \\ s \end{pmatrix}$ with $s \leq q$. It follows from special cases of power sums in [33, §3.3] that for all $d \in \mathbb{Z}$,

$$S_d \begin{pmatrix} \varepsilon \\ s \end{pmatrix} = \frac{\varepsilon^d}{\ell_d^s} = \operatorname{Si}_d \begin{pmatrix} \varepsilon \\ s \end{pmatrix}.$$

Suppose that the first statement holds for all arrays $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} \in \mathfrak{J}$ with $\operatorname{depth}(\mathfrak{s}) = n-1$ and for all $d \in \mathbb{Z}$. Let $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} = \begin{pmatrix} \varepsilon_1 & \dots & \varepsilon_n \\ s_1 & \dots & s_n \end{pmatrix}$ be an element of $\mathfrak{J}$. Note that if $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} \in \mathfrak{J}$, then $\begin{pmatrix} \boldsymbol{\varepsilon}_- \\ \mathfrak{s}_- \end{pmatrix} \in \mathfrak{J}$. It follows from induction hypothesis and the fact $s_1 \leq q$ that for all $d \in \mathbb{Z}$

$$S_d \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} = S_d \begin{pmatrix} \varepsilon_1 \\ s_1 \end{pmatrix} S_{<d} \begin{pmatrix} \boldsymbol{\varepsilon}_- \\ \mathfrak{s}_- \end{pmatrix} = \operatorname{Si}_d \begin{pmatrix} \varepsilon_1 \\ s_1 \end{pmatrix} \operatorname{Si}_{<d} \begin{pmatrix} \boldsymbol{\varepsilon}_- \\ \mathfrak{s}_- \end{pmatrix} = \operatorname{Si}_d \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}.$$

This proves the lemma. □

Let $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}, \begin{pmatrix} \boldsymbol{\epsilon} \\ \mathfrak{t} \end{pmatrix}$ be two positive arrays. We set $s_i = 0$ and $\varepsilon_i = 1$ for all $i > \operatorname{depth}(\mathfrak{s})$; $t_i = 0$ and $\epsilon_i = 1$ for all $i > \operatorname{depth}(\mathfrak{t})$. We define the following operation

$$\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} + \begin{pmatrix} \boldsymbol{\epsilon} \\ \mathfrak{t} \end{pmatrix} := \begin{pmatrix} \boldsymbol{\varepsilon}\boldsymbol{\epsilon} \\ \mathfrak{s} + \mathfrak{t} \end{pmatrix},$$

where $\boldsymbol{\varepsilon}\boldsymbol{\epsilon}$ and $\mathfrak{s}+\mathfrak{t}$ are defined by component multiplication and component addition, respectively. We say that $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} \leq \begin{pmatrix} \boldsymbol{\epsilon} \\ \mathfrak{t} \end{pmatrix}$ if the following conditions are satisfied:

(1) $\chi(\boldsymbol{\varepsilon}) = \chi(\boldsymbol{\epsilon})$,
(2) $w(\mathfrak{s}) = w(\mathfrak{t})$,
(3) $s_1 + \cdots + s_i \leq t_1 + \cdots + t_i$ for all $i \in \mathbb{N}$.

1.1.3. We now consider some formulas related to analogues of power sums. It is easily seen that

$$\operatorname{Si}_d \begin{pmatrix} \varepsilon \\ s \end{pmatrix} \operatorname{Si}_d \begin{pmatrix} \epsilon \\ t \end{pmatrix} = \operatorname{Si}_d \begin{pmatrix} \varepsilon\epsilon \\ s+t \end{pmatrix},$$

hence, for $\mathfrak{t} = (t_1, \dots, t_n)$,

$$\operatorname{Si}_d \begin{pmatrix} \varepsilon \\ s \end{pmatrix} \operatorname{Si}_d \begin{pmatrix} \boldsymbol{\epsilon} \\ \mathfrak{t} \end{pmatrix} = \operatorname{Si}_d \begin{pmatrix} \varepsilon\epsilon_1 & \boldsymbol{\epsilon}_- \\ s+t_1 & \mathfrak{t}_- \end{pmatrix}. \tag{1.1}$$

More generally, we deduce the following proposition which will be used frequently later.

Proposition 1.2. *Let* $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}, \begin{pmatrix} \boldsymbol{\epsilon} \\ \mathfrak{t} \end{pmatrix}$ *be two positive arrays. Then*

(1) *There exist* $f_i \in \mathbb{F}_p$ *and arrays* $\begin{pmatrix} \boldsymbol{\mu}_i \\ \mathfrak{u}_i \end{pmatrix}$ *with* $\begin{pmatrix} \boldsymbol{\mu}_i \\ \mathfrak{u}_i \end{pmatrix} \le \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} + \begin{pmatrix} \boldsymbol{\epsilon} \\ \mathfrak{t} \end{pmatrix}$ *and* $\operatorname{depth}(\mathfrak{u}_i) \le \operatorname{depth}(\mathfrak{s}) + \operatorname{depth}(\mathfrak{t})$ *for all* i *such that*

$$\operatorname{Si}_d \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} \operatorname{Si}_d \begin{pmatrix} \boldsymbol{\epsilon} \\ \mathfrak{t} \end{pmatrix} = \sum_i f_i \operatorname{Si}_d \begin{pmatrix} \boldsymbol{\mu}_i \\ \mathfrak{u}_i \end{pmatrix} \quad \textit{for all } d \in \mathbb{Z}.$$

(2) *There exist* $f_i' \in \mathbb{F}_p$ *and arrays* $\begin{pmatrix} \boldsymbol{\mu}_i' \\ \mathfrak{u}_i' \end{pmatrix}$ *with* $\begin{pmatrix} \boldsymbol{\mu}_i' \\ \mathfrak{u}_i' \end{pmatrix} \le \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} + \begin{pmatrix} \boldsymbol{\epsilon} \\ \mathfrak{t} \end{pmatrix}$ *and* $\operatorname{depth}(\mathfrak{u}_i') \le \operatorname{depth}(\mathfrak{s}) + \operatorname{depth}(\mathfrak{t})$ *for all* i *such that*

$$\operatorname{Si}_{<d} \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} \operatorname{Si}_{<d} \begin{pmatrix} \boldsymbol{\epsilon} \\ \mathfrak{t} \end{pmatrix} = \sum_i f_i' \operatorname{Si}_{<d} \begin{pmatrix} \boldsymbol{\mu}_i' \\ \mathfrak{u}_i' \end{pmatrix} \quad \textit{for all } d \in \mathbb{Z}.$$

(3) *There exist* $f_i'' \in \mathbb{F}_p$ *and arrays* $\begin{pmatrix} \boldsymbol{\mu}_i'' \\ \mathfrak{u}_i'' \end{pmatrix}$ *with* $\begin{pmatrix} \boldsymbol{\mu}_i'' \\ \mathfrak{u}_i'' \end{pmatrix} \le \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} + \begin{pmatrix} \boldsymbol{\epsilon} \\ \mathfrak{t} \end{pmatrix}$ *and* $\operatorname{depth}(\mathfrak{u}_i'') \le \operatorname{depth}(\mathfrak{s}) + \operatorname{depth}(\mathfrak{t})$ *for all* i *such that*

$$\operatorname{Si}_d \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} \operatorname{Si}_{<d} \begin{pmatrix} \boldsymbol{\epsilon} \\ \mathfrak{t} \end{pmatrix} = \sum_i f_i'' \operatorname{Si}_d \begin{pmatrix} \boldsymbol{\mu}_i'' \\ \mathfrak{u}_i'' \end{pmatrix} \quad \textit{for all } d \in \mathbb{Z}.$$

Proof. The proof follows the same line as in [27, Proposition 2.1]. We omit the details. □

We denote by $\mathcal{AL}$ the K-vector space generated by the ACMPL's and by $\mathcal{AL}_w$ the K-vector space generated by the ACMPL's of weight w. It follows from Proposition 1.2 that $\mathcal{AL}$ is a K-algebra under the multiplication of K_∞.

1.2. **Operators $\mathcal{B}^*$, $\mathcal{C}$ and $\mathcal{BC}$.** In this section we define operators $\mathcal{B}^*$ and $\mathcal{C}$ of Todd [37] and the operator $\mathcal{BC}$ of Ngo Dac [27] in the case of ACMPL's.

Definition 1.3. A binary relation is a K-linear combination of the form

$$\sum_i a_i \operatorname{Si}_d \begin{pmatrix} \boldsymbol{\varepsilon}_i \\ \mathfrak{s}_i \end{pmatrix} + \sum_i b_i \operatorname{Si}_{d+1} \begin{pmatrix} \boldsymbol{\epsilon}_i \\ \mathfrak{t}_i \end{pmatrix} = 0 \quad \text{for all } d \in \mathbb{Z},$$

where $a_i, b_i \in K$ and $\begin{pmatrix} \boldsymbol{\varepsilon}_i \\ \mathfrak{s}_i \end{pmatrix}, \begin{pmatrix} \boldsymbol{\epsilon}_i \\ \mathfrak{t}_i \end{pmatrix}$ are positive arrays of the same weight.

A binary relation is called a fixed relation if $b_i = 0$ for all i.

We denote by $\mathfrak{BR}_w$ the set of all binary relations of weight w. One verifies at once that $\mathfrak{BR}_w$ is a K-vector space under the addition and multiplication of K. It follows from the fundamental relation in [33, §3.4.6] and Lemma 1.1, an important example of binary relations

$$R_\varepsilon\colon \quad \operatorname{Si}_d \begin{pmatrix} \varepsilon \\ q \end{pmatrix} + \varepsilon^{-1} D_1 \operatorname{Si}_{d+1} \begin{pmatrix} \varepsilon & 1 \\ 1 & q-1 \end{pmatrix} = 0,$$

where $D_1 = \theta^q - \theta$.

For later definitions, let $R \in \mathfrak{BR}_w$ be a binary relation of the form

$$R(d)\colon \quad \sum_i a_i \operatorname{Si}_d \begin{pmatrix} \boldsymbol{\varepsilon}_i \\ \mathfrak{s}_i \end{pmatrix} + \sum_i b_i \operatorname{Si}_{d+1} \begin{pmatrix} \boldsymbol{\epsilon}_i \\ \mathfrak{t}_i \end{pmatrix} = 0,$$

where $a_i, b_i \in K$ and $\begin{pmatrix} \boldsymbol{\varepsilon}_i \\ \mathfrak{s}_i \end{pmatrix}, \begin{pmatrix} \boldsymbol{\epsilon}_i \\ \mathfrak{t}_i \end{pmatrix}$ are positive arrays of the same weight. We now define some operators on K-vector spaces of binary relations.

1.2.1. *Operators* $\mathcal{B}^*$. Let $\begin{pmatrix} \sigma \\ v \end{pmatrix}$ be a positive array. We define an operator

$$\mathcal{B}^*_{\sigma,v} \colon \mathfrak{BR}_w \longrightarrow \mathfrak{BR}_{w+v}$$

as follows: for each $R \in \mathfrak{BR}_w$, the image $\mathcal{B}^*_{\sigma,v}(R) = \operatorname{Si}_d \begin{pmatrix} \sigma \\ v \end{pmatrix} \sum_{j<d} R(j)$ is a fixed relation of the form

$$\begin{aligned}
0 &= \operatorname{Si}_d \begin{pmatrix} \sigma \\ v \end{pmatrix} \left(\sum_i a_i \operatorname{Si}_{<d} \begin{pmatrix} \boldsymbol{\varepsilon}_i \\ \mathfrak{s}_i \end{pmatrix} + \sum_i b_i \operatorname{Si}_{<d+1} \begin{pmatrix} \boldsymbol{\epsilon}_i \\ \mathfrak{t}_i \end{pmatrix} \right) \\
&= \sum_i a_i \operatorname{Si}_d \begin{pmatrix} \sigma \\ v \end{pmatrix} \operatorname{Si}_{<d} \begin{pmatrix} \boldsymbol{\varepsilon}_i \\ \mathfrak{s}_i \end{pmatrix} + \sum_i b_i \operatorname{Si}_d \begin{pmatrix} \sigma \\ v \end{pmatrix} \operatorname{Si}_{<d} \begin{pmatrix} \boldsymbol{\epsilon}_i \\ \mathfrak{t}_i \end{pmatrix} + \sum_i b_i \operatorname{Si}_d \begin{pmatrix} \sigma \\ v \end{pmatrix} \operatorname{Si}_d \begin{pmatrix} \boldsymbol{\epsilon}_i \\ \mathfrak{t}_i \end{pmatrix} \\
&= \sum_i a_i \operatorname{Si}_d \begin{pmatrix} \sigma & \boldsymbol{\varepsilon}_i \\ v & \mathfrak{s}_i \end{pmatrix} + \sum_i b_i \operatorname{Si}_d \begin{pmatrix} \sigma & \boldsymbol{\epsilon}_i \\ v & \mathfrak{t}_i \end{pmatrix} + \sum_i b_i \operatorname{Si}_d \begin{pmatrix} \sigma\epsilon_{i1} & \boldsymbol{\epsilon}_{i-} \\ v + t_{i1} & \mathfrak{t}_{i-} \end{pmatrix}.
\end{aligned}$$

The last equality follows from (1.1).

Let $\begin{pmatrix} \Sigma \\ V \end{pmatrix} = \begin{pmatrix} \sigma_1 & \dots & \sigma_n \\ v_1 & \dots & v_n \end{pmatrix}$ be a positive array. We define an operator $\mathcal{B}^*_{\Sigma,V}(R)$ by

$$\mathcal{B}^*_{\Sigma,V}(R) = \mathcal{B}^*_{\sigma_1,v_1} \circ \cdots \circ \mathcal{B}^*_{\sigma_n,v_n}(R).$$

Lemma 1.4. *Let* $\begin{pmatrix} \Sigma \\ V \end{pmatrix} = \begin{pmatrix} \sigma_1 & \dots & \sigma_n \\ v_1 & \dots & v_n \end{pmatrix}$ *be a positive array. Then* $\mathcal{B}^*_{\Sigma,V}(R)$ *is of the form*

$$\begin{aligned}
&\sum_i a_i \operatorname{Si}_d \begin{pmatrix} \Sigma & \boldsymbol{\varepsilon}_i \\ V & \mathfrak{s}_i \end{pmatrix} + \sum_i b_i \operatorname{Si}_d \begin{pmatrix} \Sigma & \boldsymbol{\epsilon}_i \\ V & \mathfrak{t}_i \end{pmatrix} \\
&\qquad + \sum_i b_i \operatorname{Si}_d \begin{pmatrix} \sigma_1 & \dots & \sigma_{n-1} & \sigma_n\epsilon_{i1} & \boldsymbol{\epsilon}_{i-} \\ v_1 & \dots & v_{n-1} & v_n + t_{i1} & \mathfrak{t}_{i-} \end{pmatrix} = 0.
\end{aligned}$$

Proof. The proof is straightforward. We omit this proof. □

1.2.2. *Operators* $\mathcal{C}$. Let $\begin{pmatrix} \Sigma \\ V \end{pmatrix}$ be a positive array of weight v. We define an operator

$$\mathcal{C}_{\Sigma,V}(R) \colon \mathfrak{BR}_w \longrightarrow \mathfrak{BR}_{w+v}$$

as follows: for each $R \in \mathfrak{BR}_w$, the image $\mathcal{C}_{\Sigma,V}(R) = R(d) \operatorname{Si}_{<d+1} \begin{pmatrix} \Sigma \\ V \end{pmatrix}$ is a binary relation of the form

$$\begin{aligned}
0 &= \left(\sum_i a_i \operatorname{Si}_d \begin{pmatrix} \boldsymbol{\varepsilon}_i \\ \mathfrak{s}_i \end{pmatrix} + \sum_i b_i \operatorname{Si}_{d+1} \begin{pmatrix} \boldsymbol{\epsilon}_i \\ \mathfrak{t}_i \end{pmatrix} \right) \operatorname{Si}_{<d+1} \begin{pmatrix} \Sigma \\ V \end{pmatrix} \\
&= \sum_i a_i \operatorname{Si}_d \begin{pmatrix} \boldsymbol{\varepsilon}_i \\ \mathfrak{s}_i \end{pmatrix} \operatorname{Si}_d \begin{pmatrix} \Sigma \\ V \end{pmatrix} + \sum_i a_i \operatorname{Si}_d \begin{pmatrix} \boldsymbol{\varepsilon}_i \\ \mathfrak{s}_i \end{pmatrix} \operatorname{Si}_{<d} \begin{pmatrix} \Sigma \\ V \end{pmatrix} + \sum_i b_i \operatorname{Si}_{d+1} \begin{pmatrix} \boldsymbol{\epsilon}_i \\ \mathfrak{t}_i \end{pmatrix} \operatorname{Si}_{<d+1} \begin{pmatrix} \Sigma \\ V \end{pmatrix}
\end{aligned}$$

$$= \sum_i f_i \operatorname{Si}_d \begin{pmatrix} \boldsymbol{\mu}_i \\ \mathfrak{u}_i \end{pmatrix} + \sum_i f'_i \operatorname{Si}_{d+1} \begin{pmatrix} \boldsymbol{\mu}'_i \\ \mathfrak{u}'_i \end{pmatrix}.$$

The last equality follows from Proposition 1.2.

In particular, the following proposition gives the form of $\mathcal{C}_{\Sigma,V}(R_\varepsilon)$.

Proposition 1.5. *Let $\begin{pmatrix} \Sigma \\ V \end{pmatrix}$ be a positive array with $V = (v_1, V_-)$ and $\Sigma = (\sigma_1, \Sigma_-)$. Then $\mathcal{C}_{\Sigma,V}(R_\varepsilon)$ is of the form*

$$\operatorname{Si}_d \begin{pmatrix} \varepsilon\sigma_1 & \Sigma_- \\ q+v_1 & V_- \end{pmatrix} + \operatorname{Si}_d \begin{pmatrix} \varepsilon & \Sigma \\ q & V \end{pmatrix} + \sum_i b_i \operatorname{Si}_{d+1} \begin{pmatrix} \varepsilon & \boldsymbol{\epsilon}_i \\ 1 & \mathfrak{t}_i \end{pmatrix} = 0,$$

where $b_i \in K$ and $\begin{pmatrix} \boldsymbol{\epsilon}_i \\ \mathfrak{t}_i \end{pmatrix}$ are positive arrays satisfying $\begin{pmatrix} \boldsymbol{\epsilon}_i \\ \mathfrak{t}_i \end{pmatrix} \leq \begin{pmatrix} 1 \\ q-1 \end{pmatrix} + \begin{pmatrix} \Sigma \\ V \end{pmatrix}$ for all i.

Proof. We see that $\mathcal{C}_{\Sigma,V}(R_\varepsilon)$ is of the form

$$\operatorname{Si}_d \begin{pmatrix} \varepsilon \\ q \end{pmatrix} \operatorname{Si}_d \begin{pmatrix} \Sigma \\ V \end{pmatrix} + \operatorname{Si}_d \begin{pmatrix} \varepsilon \\ q \end{pmatrix} \operatorname{Si}_{<d} \begin{pmatrix} \Sigma \\ V \end{pmatrix} + \varepsilon^{-1} D_1 \operatorname{Si}_{d+1} \begin{pmatrix} \varepsilon & 1 \\ 1 & q-1 \end{pmatrix} \operatorname{Si}_{<d+1} \begin{pmatrix} \Sigma \\ V \end{pmatrix} = 0.$$

It follows from (1.1) and Proposition 1.2 that

$$\operatorname{Si}_d \begin{pmatrix} \varepsilon \\ q \end{pmatrix} \operatorname{Si}_d \begin{pmatrix} \Sigma \\ V \end{pmatrix} + \operatorname{Si}_d \begin{pmatrix} \varepsilon \\ q \end{pmatrix} \operatorname{Si}_{<d} \begin{pmatrix} \Sigma \\ V \end{pmatrix} = \operatorname{Si}_d \begin{pmatrix} \varepsilon\sigma_1 & \Sigma_- \\ q+v_1 & V_- \end{pmatrix} + \operatorname{Si}_d \begin{pmatrix} \varepsilon & \Sigma \\ q & V \end{pmatrix},$$

$$\varepsilon^{-1} D_1 \operatorname{Si}_{d+1} \begin{pmatrix} \varepsilon & 1 \\ 1 & q-1 \end{pmatrix} \operatorname{Si}_{<d+1} \begin{pmatrix} \Sigma \\ V \end{pmatrix} = \sum_i b_i \operatorname{Si}_{d+1} \begin{pmatrix} \varepsilon & \boldsymbol{\epsilon}_i \\ 1 & \mathfrak{t}_i \end{pmatrix},$$

where $b_i \in K$ and $\begin{pmatrix} \boldsymbol{\epsilon}_i \\ \mathfrak{t}_i \end{pmatrix}$ are positive arrays satisfying $\begin{pmatrix} \boldsymbol{\epsilon}_i \\ \mathfrak{t}_i \end{pmatrix} \leq \begin{pmatrix} 1 \\ q-1 \end{pmatrix} + \begin{pmatrix} \Sigma \\ V \end{pmatrix}$ for all i. This proves the proposition. ☐

1.2.3. *Operators $\mathcal{BC}$.* Let $\varepsilon \in \mathbb{F}_q^\times$. We define an operator

$$\mathcal{BC}_{\varepsilon,q} \colon \mathfrak{BR}_w \longrightarrow \mathfrak{BR}_{w+q}$$

as follows: for each $R \in \mathfrak{BR}_w$, the image $\mathcal{BC}_{\varepsilon,q}(R)$ is a binary relation given by

$$\mathcal{BC}_{\varepsilon,q}(R) = \mathcal{B}^*_{\varepsilon,q}(R) - \sum_i b_i \mathcal{C}_{\boldsymbol{\epsilon}_i,\mathfrak{t}_i}(R_\varepsilon).$$

Let us clarify the definition of $\mathcal{BC}_{\varepsilon,q}$. We know that $\mathcal{B}^*_{\varepsilon,q}(R)$ is of the form

$$\sum_i a_i \operatorname{Si}_d \begin{pmatrix} \varepsilon & \boldsymbol{\varepsilon}_i \\ q & \mathfrak{s}_i \end{pmatrix} + \sum_i b_i \operatorname{Si}_d \begin{pmatrix} \varepsilon & \boldsymbol{\epsilon}_i \\ q & \mathfrak{t}_i \end{pmatrix} + \sum_i b_i \operatorname{Si}_d \begin{pmatrix} \varepsilon\epsilon_{i1} & \boldsymbol{\epsilon}_{i-} \\ q+t_{i1} & \mathfrak{t}_{i-} \end{pmatrix} = 0.$$

Moreover, $\mathcal{C}_{\boldsymbol{\epsilon}_i,\mathfrak{t}_i}(R_\varepsilon)$ is of the form

$$Si_d \begin{pmatrix} \varepsilon & \boldsymbol{\epsilon}_i \\ q & \mathfrak{t}_i \end{pmatrix} + \operatorname{Si}_d \begin{pmatrix} \varepsilon\epsilon_{i1} & \boldsymbol{\epsilon}_{i-} \\ q+t_{i1} & \mathfrak{t}_{i-} \end{pmatrix} + \varepsilon^{-1} D_1 \operatorname{Si}_{d+1} \begin{pmatrix} \varepsilon \\ 1 \end{pmatrix} \operatorname{Si}_{<d+1} \begin{pmatrix} 1 \\ q-1 \end{pmatrix} \operatorname{Si}_{<d+1} \begin{pmatrix} \boldsymbol{\epsilon}_i \\ \mathfrak{t}_i \end{pmatrix} = 0.$$

Combining with Proposition 1.2 (2), we have that $\mathcal{BC}_{\varepsilon,q}(R)$ is of the form

$$\sum_i a_i \operatorname{Si}_d \begin{pmatrix} \varepsilon & \boldsymbol{\varepsilon}_i \\ q & \mathfrak{s}_i \end{pmatrix} + \sum_{i,j} b_{ij} \operatorname{Si}_{d+1} \begin{pmatrix} \varepsilon & \boldsymbol{\epsilon}_{ij} \\ 1 & \mathfrak{t}_{ij} \end{pmatrix} = 0,$$

where $b_{ij} \in K$ and $\begin{pmatrix} \boldsymbol{\epsilon}_{ij} \\ \mathfrak{t}_{ij} \end{pmatrix}$ are positive arrays satisfying $\begin{pmatrix} \boldsymbol{\epsilon}_{ij} \\ \mathfrak{t}_{ij} \end{pmatrix} \leq \begin{pmatrix} 1 \\ q-1 \end{pmatrix} + \begin{pmatrix} \boldsymbol{\epsilon}_{i} \\ \mathfrak{t}_{i} \end{pmatrix}$ for all j.

1.3. **A weak version of Brown's theorem for ACMP's.**

1.3.1. *Preparatory results.*

Proposition 1.6. *1) Let* $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} = \begin{pmatrix} \varepsilon_1 & \dots & \varepsilon_n \\ s_1 & \dots & s_n \end{pmatrix}$ *be a positive array such that* $\mathrm{Init}(\mathfrak{s}) = (s_1, \dots, s_{k-1})$ *for some* $1 \leq k \leq n$, *and let* ε *be an element in* $\mathbb{F}_q^\times$. *Then* $\mathrm{Li}\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}$ *can be decomposed as follows*

$$\mathrm{Li}\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} = \underbrace{-\mathrm{Li}\begin{pmatrix} \boldsymbol{\varepsilon}' \\ \mathfrak{s}' \end{pmatrix}}_{\textit{type 1}} + \underbrace{\sum_i b_i \,\mathrm{Li}\begin{pmatrix} \boldsymbol{\epsilon}_i' \\ \mathfrak{t}_i' \end{pmatrix}}_{\textit{type 2}} + \underbrace{\sum_i c_i \,\mathrm{Li}\begin{pmatrix} \boldsymbol{\mu}_i \\ \mathfrak{u}_i \end{pmatrix}}_{\textit{type 3}},$$

where $b_i, c_i \in A$ *divisible by* D_1 *such that for all* i, *the following properties are satisfied:*

- *For all arrays* $\begin{pmatrix} \boldsymbol{\epsilon} \\ \mathfrak{t} \end{pmatrix}$ *appearing on the right hand side,*

$$\mathrm{depth}(\mathfrak{t}) \geq \mathrm{depth}(\mathfrak{s}) \quad \textit{and} \quad T_k(\mathfrak{t}) \leq T_k(\mathfrak{s}).$$

- *For the array* $\begin{pmatrix} \boldsymbol{\varepsilon}' \\ \mathfrak{s}' \end{pmatrix}$ *of type* 1 *with respect to* $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}$, *we have*

$$\begin{pmatrix} \boldsymbol{\varepsilon}' \\ \mathfrak{s}' \end{pmatrix} = \begin{pmatrix} \varepsilon_1 & \dots & \varepsilon_{k-1} & \varepsilon & \varepsilon^{-1}\varepsilon_k & \varepsilon_{k+1} & \dots & \varepsilon_n \\ s_1 & \dots & s_{k-1} & q & s_k - q & s_{k+1} & \dots & s_n \end{pmatrix}.$$

  Moreover, for all $k \leq \ell \leq n$,

$$s_1' + \dots + s_\ell' < s_1 + \dots + s_\ell.$$

- *For the array* $\begin{pmatrix} \boldsymbol{\epsilon}' \\ \mathfrak{t}' \end{pmatrix}$ *of type* 2 *with respect to* $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}$, *for all* $k \leq \ell \leq n$,

$$t_1' + \dots + t_\ell' < s_1 + \dots + s_\ell.$$

- *For the array* $\begin{pmatrix} \boldsymbol{\mu} \\ \mathfrak{u} \end{pmatrix}$ *of type* 3 *with respect to* $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}$, *we have* $\mathrm{Init}(\mathfrak{s}) \prec \mathrm{Init}(\mathfrak{u})$.

2) Let $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} = \begin{pmatrix} \varepsilon_1 & \dots & \varepsilon_k \\ s_1 & \dots & s_k \end{pmatrix}$ *be a positive array such that* $\mathrm{Init}(\mathfrak{s}) = \mathfrak{s}$ *and* $s_k = q$. *Then* $\mathrm{Li}\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}$ *can be decomposed as follows:*

$$\mathrm{Li}\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} = \underbrace{\sum_i b_i \,\mathrm{Li}\begin{pmatrix} \boldsymbol{\epsilon}_i' \\ \mathfrak{t}_i' \end{pmatrix}}_{\textit{type 2}} + \underbrace{\sum_i c_i \,\mathrm{Li}\begin{pmatrix} \boldsymbol{\mu}_i \\ \mathfrak{u}_i \end{pmatrix}}_{\textit{type 3}},$$

where $b_i, c_i \in A$ *divisible by* D_1 *such that for all* i, *the following properties are satisfied:*

- *For all arrays* $\begin{pmatrix} \boldsymbol{\epsilon} \\ \mathfrak{t} \end{pmatrix}$ *appearing on the right hand side,*

$$\operatorname{depth}(\mathfrak{t}) \geq \operatorname{depth}(\mathfrak{s}) \quad \textit{and} \quad T_k(\mathfrak{t}) \leq T_k(\mathfrak{s}).$$

- *For the array* $\begin{pmatrix} \boldsymbol{\epsilon}' \\ \mathfrak{t}' \end{pmatrix}$ *of type* 2 *with respect to* $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}$,

$$t'_1 + \dots + t'_k < s_1 + \dots + s_k.$$

- *For the array* $\begin{pmatrix} \boldsymbol{\mu} \\ \mathfrak{u} \end{pmatrix}$ *of type* 3 *with respect to* $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}$, *we have* $\operatorname{Init}(\mathfrak{s}) \prec \operatorname{Init}(\mathfrak{u})$.

Proof. For Part 1, since $\operatorname{Init}(\mathfrak{s}) = (s_1, \dots, s_{k-1})$, we get $s_k > q$. Set $\begin{pmatrix} \Sigma \\ V \end{pmatrix} = \begin{pmatrix} \varepsilon^{-1}\varepsilon_k & \varepsilon_{k+1} & \dots & \varepsilon_n \\ s_k - q & s_{k+1} & \dots & s_n \end{pmatrix}$. By Proposition 1.5, $\mathcal{C}_{\Sigma,V}(R_\varepsilon)$ is of the form

$$\operatorname{Si}_d \begin{pmatrix} \varepsilon_k & \dots & \varepsilon_n \\ s_k & \dots & s_n \end{pmatrix} + \operatorname{Si}_d \begin{pmatrix} \varepsilon & \varepsilon^{-1}\varepsilon_k & \varepsilon_{k+1} & \dots & \varepsilon_n \\ q & s_k - q & s_{k+1} & \dots & s_n \end{pmatrix} + \sum_i b_i \operatorname{Si}_{d+1} \begin{pmatrix} \varepsilon & \boldsymbol{\epsilon}_i \\ 1 & \mathfrak{t}_i \end{pmatrix} = 0, \tag{1.2}$$

where $b_i \in A$ divisible by D_1 and $\begin{pmatrix} \boldsymbol{\epsilon}_i \\ \mathfrak{t}_i \end{pmatrix}$ are positive arrays satisfying for all i,

$$\begin{pmatrix} \boldsymbol{\epsilon}_i \\ \mathfrak{t}_i \end{pmatrix} \leq \begin{pmatrix} 1 \\ q-1 \end{pmatrix} + \begin{pmatrix} \Sigma \\ V \end{pmatrix} = \begin{pmatrix} \varepsilon^{-1}\varepsilon_k & \varepsilon_{k+1} & \dots & \varepsilon_n \\ s_k - 1 & s_{k+1} & \dots & s_n \end{pmatrix}.$$

For $m \in \mathbb{N}$, we denote by $q^{\{m\}}$ the sequence of length m with all terms equal to q. We agree by convention that $q^{\{0\}}$ is the empty sequence. Setting $s_0 = 0$, we may assume that there exists a maximal index j with $0 \leq j \leq k-1$ such that $s_j < q$, hence $\operatorname{Init}(\mathfrak{s}) = (s_1, \dots, s_j, q^{\{k-j-1\}})$.

Then the operator $\mathcal{BC}_{\varepsilon_{j+1},q} \circ \dots \circ \mathcal{BC}_{\varepsilon_{k-1},q}$ applied to the relation (1.2) gives

$$\begin{aligned} &\operatorname{Si}_d \begin{pmatrix} \varepsilon_{j+1} & \dots & \varepsilon_{k-1} & \varepsilon_k & \dots & \varepsilon_n \\ q & \dots & q & s_k & \dots & s_n \end{pmatrix} \\ &\quad + \operatorname{Si}_d \begin{pmatrix} \varepsilon_{j+1} & \dots & \varepsilon_{k-1} & \varepsilon & \varepsilon^{-1}\varepsilon_k & \epsilon_{k+1} & \dots & \epsilon_n \\ q & \dots & q & q & s_k - q & s_{k+1} & \dots & s_n \end{pmatrix} \\ &\quad + \sum_i b_{i_1 \dots i_{k-j}} \operatorname{Si}_{d+1} \begin{pmatrix} \varepsilon_{j+1} & \boldsymbol{\epsilon}_{i_1 \dots i_{k-j}} \\ 1 & \mathfrak{t}_{i_1 \dots i_{k-j}} \end{pmatrix} = 0, \end{aligned}$$

where $b_{i_1 \dots i_{k-j}} \in A$ are divisible by D_1 and for $2 \leq l \leq k-j$, $\begin{pmatrix} \boldsymbol{\epsilon}_{i_1 \dots i_l} \\ \mathfrak{t}_{i_1 \dots i_l} \end{pmatrix}$ are positive arrays satisfying

$$\begin{pmatrix} \boldsymbol{\epsilon}_{i_1 \dots i_l} \\ \mathfrak{t}_{i_1 \dots i_l} \end{pmatrix} \leq \begin{pmatrix} 1 \\ q-1 \end{pmatrix} + \begin{pmatrix} \varepsilon_{k-l+2} & \boldsymbol{\epsilon}_{i_1 \dots i_{l-1}} \\ 1 & \mathfrak{t}_{i_1 \dots i_{l-1}} \end{pmatrix} = \begin{pmatrix} \varepsilon_{k-l+2} & \boldsymbol{\epsilon}_{i_1 \dots i_{l-1}} \\ q & \mathfrak{t}_{i_1 \dots i_{l-1}} \end{pmatrix}.$$

Thus

$$\begin{pmatrix} \boldsymbol{\epsilon}_{i_1 \dots i_{k-j}} \\ \mathfrak{t}_{i_1 \dots i_{k-j}} \end{pmatrix} \leq \begin{pmatrix} \varepsilon_{j+2} & \dots & \varepsilon_{k-1} & \varepsilon & \varepsilon^{-1}\varepsilon_k & \varepsilon_{k+1} & \dots & \varepsilon_n \\ q & \dots & q & q & s_k - 1 & s_{k+1} & \dots & s_n \end{pmatrix}.$$

Letting $\begin{pmatrix} \Sigma' \\ V' \end{pmatrix} = \begin{pmatrix} \varepsilon_1 & \dots & \varepsilon_j \\ s_1 & \dots & s_j \end{pmatrix}$, by Proposition 1.4, we continue to apply $\mathcal{B}^*_{\Sigma',V'}$ to the above relation and get

$$\begin{aligned} \operatorname{Si}_d \begin{pmatrix} \boldsymbol{\epsilon} \\ \mathfrak{s} \end{pmatrix} &+ \operatorname{Si}_d \begin{pmatrix} \varepsilon_1 & \dots & \varepsilon_{k-1} & \varepsilon & \varepsilon^{-1}\varepsilon_k & \varepsilon_{k+1} & \dots & \varepsilon_n \\ s_1 & \dots & s_{k-1} & q & s_k - q & s_{k+1} & \dots & s_n \end{pmatrix} \\ &+ \sum_i b_{i_1\dots i_{k-j}} \operatorname{Si}_d \begin{pmatrix} \varepsilon_1 & \dots & \varepsilon_j & \varepsilon_{j+1} & \boldsymbol{\epsilon}_{i_1\dots i_{k-j}} \\ s_1 & \dots & s_j & 1 & \mathfrak{t}_{i_1\dots i_{k-j}} \end{pmatrix} \\ &+ \sum_i b_{i_1\dots i_{k-j}} \operatorname{Si}_{d+1} \begin{pmatrix} \varepsilon_1 & \dots & \varepsilon_{j-1} & \varepsilon_j\varepsilon_{j+1} & \boldsymbol{\epsilon}_{i_1\dots i_{k-j}} \\ s_1 & \dots & s_{j-1} & s_j + 1 & \mathfrak{t}_{i_1\dots i_{k-j}} \end{pmatrix} = 0. \end{aligned}$$

Hence

$$\begin{aligned} \operatorname{Li} \begin{pmatrix} \boldsymbol{\epsilon} \\ \mathfrak{s} \end{pmatrix} &+ \operatorname{Li} \begin{pmatrix} \varepsilon_1 & \dots & \varepsilon_{k-1} & \varepsilon & \varepsilon^{-1}\varepsilon_k & \varepsilon_{k+1} & \dots & \varepsilon_n \\ s_1 & \dots & s_{k-1} & q & s_k - q & s_{k+1} & \dots & s_n \end{pmatrix} \\ &+ \sum_i b_{i_1\dots i_{k-j}} \operatorname{Li} \begin{pmatrix} \varepsilon_1 & \dots & \varepsilon_j & \varepsilon_{j+1} & \boldsymbol{\epsilon}_{i_1\dots i_{k-j}} \\ s_1 & \dots & s_j & 1 & \mathfrak{t}_{i_1\dots i_{k-j}} \end{pmatrix} \\ &+ \sum_i b_{i_1\dots i_{k-j}} \operatorname{Li} \begin{pmatrix} \varepsilon_1 & \dots & \varepsilon_{j-1} & \varepsilon_j\varepsilon_{j+1} & \boldsymbol{\epsilon}_{i_1\dots i_{k-j}} \\ s_1 & \dots & s_{j-1} & s_j + 1 & \mathfrak{t}_{i_1\dots i_{k-j}} \end{pmatrix} = 0. \end{aligned}$$

The verification of positive arrays of type 1, type 2, type 3 with respect to $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}$ follows the same line as in [27]. We omit the details.

For Part 2, the proof follows the same line as in Part 1. We omit this proof. ☐

We recall the following definition of [27] (see [27, Definition 3.1]):

Definition 1.7. Let $k \in \mathbb{N}$ and $\mathfrak{s}$ be a tuple of positive integers. We say that $\mathfrak{s}$ is k-admissible if it satisfies the following two conditions:

1) $s_1, \dots, s_k \leq q$.
2) $\mathfrak{s}$ is not of the form $(s_1, \dots, s_r)$ with $r \leq k$, $s_1, \dots, s_{r-1} \leq q$, and $s_r = q$.

Here we recall $s_i = 0$ for $i > \operatorname{depth}(\mathfrak{s})$.

A positive array is k-admissible if the corresponding tuple is k-admissible.

Proposition 1.8. *For all $k \in \mathbb{N}$ and for all arrays $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}$, $\operatorname{Li} \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}$ can be expressed as a K-linear combination of $\operatorname{Li} \begin{pmatrix} \boldsymbol{\epsilon} \\ \mathfrak{t} \end{pmatrix}$'s of the same weight such that $\mathfrak{t}$ is k-admissible.*

Proof. The proof follow the same line as that of [27, Theorem A]. ☐

1.3.2. *A set of generators $\mathcal{AT}_w$ for ACMPL's.* We recall that $\mathcal{AL}_w$ is the K-vector space generated by ACMPL's of weight w. We denote by $\mathcal{AT}_w$ the set of all ACMPL's $\operatorname{Li} \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} = \operatorname{Li} \begin{pmatrix} \varepsilon_1 & \dots & \varepsilon_n \\ s_1 & \dots & s_n \end{pmatrix}$ of weight w such that $s_1, \dots, s_{n-1} \leq q$ and $s_n < q$.

We put $t(w) = |\mathcal{AT}_w|$. Then one verifies that

$$t(w) = \begin{cases} (q-1)q^{w-1} & \text{if } 1 \leq w < q, \\ (q-1)(q^{w-1} - 1) & \text{if } w = q, \end{cases}$$

and for $w > q$,

$$t(w) = (q-1)\sum_{i=1}^{q} t(w-i).$$

We are ready to state a weak version of Brown's theorem for ACMPL's.

Proposition 1.9. *The set of all elements* $\mathrm{Li}\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix}$ *such that* $\mathrm{Li}\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix} \in \mathcal{AT}_w$ *forms a set of generators for* $\mathcal{AL}_w$*.*

Proof. The result follows immediately from Proposition 1.8 in the case of $k = w$. □

1.4. A strong version of Brown's theorem for ACMPL's.

1.4.1. *Another set of generators* $\mathcal{AS}_w$ *for ACMPL's.* We consider the set $\mathcal{J}_w$ consisting of positive tuples $\mathfrak{s} = (s_1, \dots, s_n)$ of weight w such that $s_1, \dots, s_{n-1} \le q$ and $s_n < q$, together with the set $\mathcal{J}'_w$ consisting of positive tuples $\mathfrak{s} = (s_1, \dots, s_n)$ of weight w such that $q \nmid s_i$ for all i. Then there is a bijection

$$\iota \colon \mathcal{J}'_w \longrightarrow \mathcal{J}_w$$

given as follows: for each tuple $\mathfrak{s} = (s_1, \dots, s_n) \in \mathcal{J}'_w$, since $q \nmid s_i$, we can write $s_i = h_i q + r_i$ where $0 < r_i < q$ and $h_i \in \mathbb{Z}^{\ge 0}$. The image $\iota(\mathfrak{s})$ is the tuple

$$\iota(\mathfrak{s}) = (\underbrace{q, \dots, q}_{h_1 \text{ times}}, r_1, \dots, \underbrace{q, \dots, q}_{h_n \text{ times}}, r_n).$$

Let $\mathcal{AS}_w$ denote the set of ACMPL's $\mathrm{Li}\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix}$ such that $\mathfrak{s} \in \mathcal{J}'_w$. We note that in general, $\mathcal{AS}_w$ is much smaller than $\mathcal{AT}_w$. The only exceptions are when $q = 2$ or $w < q$.

1.4.2. *Cardinality of* $\mathcal{AS}_w$*.* We now compute $s(w) = |\mathcal{AS}_w|$. To do so we denote by $\mathcal{AJ}_w$ the set consisting of positive arrays $\begin{pmatrix}\varepsilon_1 & \dots & \varepsilon_n\\ s_1 & \dots & s_n\end{pmatrix}$ of weight w such that $q \nmid s_i$ for all i and by $\mathcal{AJ}^1_w$ the set consisting of positive arrays $\begin{pmatrix}\varepsilon_1 & \dots & \varepsilon_n\\ s_1 & \dots & s_n\end{pmatrix}$ of weight w such that $s_1, \dots, s_{n-1} \le q, s_n < q$ and $\varepsilon_i = 1$ whenever $s_i = q$ for $1 \le i \le n$. We construct a map

$$\varphi \colon \mathcal{AJ}_w \longrightarrow \mathcal{AJ}^1_w$$

as follows: for each array $\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix} = \begin{pmatrix}\varepsilon_1 & \dots & \varepsilon_n\\ s_1 & \dots & s_n\end{pmatrix} \in \mathcal{AJ}_w$, since $q \nmid s_i$, we can write $s_i = h_i q + r_i$ where $0 < r_i < q$ and $h_i \in \mathbb{Z}^{\ge 0}$. The image $\varphi\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix}$ is the array

$$\varphi\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix} = \left(\underbrace{\begin{pmatrix}1 & \dots & 1\\ q & \dots & q\end{pmatrix}}_{h_1 \text{ times}}\begin{pmatrix}\varepsilon_1\\ r_1\end{pmatrix} \dots \underbrace{\begin{pmatrix}1 & \dots & 1\\ q & \dots & q\end{pmatrix}}_{h_n \text{ times}}\begin{pmatrix}\varepsilon_n\\ r_n\end{pmatrix}\right).$$

It is easily seen that φ is a bijection, hence $|\mathcal{AS}_w| = |\mathcal{AJ}_w| = |\mathcal{AJ}^1_w|$. Thus $s(w) = |\mathcal{AJ}^1_w|$. One verifies that

$$s(w) = \begin{cases}(q-1)q^{w-1} & \text{if } 1 \le w < q,\\ (q-1)(q^{w-1}-1) & \text{if } w = q,\end{cases}$$

and for $w > q$,

$$s(w) = (q-1)\sum_{i=1}^{q-1} s(w-i) + s(w-q).$$

1.4.3. We state a strong version of Brown's theorem for ACMPL's.

Theorem 1.10. *The set $\mathcal{AS}_w$ forms a set of generators for $\mathcal{AL}_w$. In particular,*

$$\dim_K \mathcal{AL}_w \leq s(w).$$

Proof. For each $\mathrm{Li}\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} \in \mathcal{AT}_w$, since $\mathfrak{s} \in \mathcal{J}_w$ and $\iota \colon \mathcal{J}'_w \to \mathcal{J}_w$ is a bijection, there exists a tuple $\mathfrak{s}' \in \mathcal{J}'_w$ such that $\mathfrak{s} = \iota(\mathfrak{s}')$. By Proposition 1.6 and Proposition 1.9, it follows that there exists a tuple $\boldsymbol{\varepsilon}' \in (\mathbb{F}_q^\times)^{\mathrm{depth}(\mathfrak{s}')}$ such that $\mathrm{Li}\begin{pmatrix} \boldsymbol{\varepsilon}' \\ \mathfrak{s}' \end{pmatrix}$ can be expressed as follows

$$\mathrm{Li}\begin{pmatrix} \boldsymbol{\varepsilon}' \\ \mathfrak{s}' \end{pmatrix} = \sum a_{\boldsymbol{\epsilon},\mathfrak{t}}^{\boldsymbol{\varepsilon}',\mathfrak{s}'} \mathrm{Li}\begin{pmatrix} \boldsymbol{\epsilon} \\ \mathfrak{t} \end{pmatrix},$$

where $\begin{pmatrix} \boldsymbol{\epsilon} \\ \mathfrak{t} \end{pmatrix}$ ranges over all elements of $\mathcal{AJ}_w$ and $a_{\boldsymbol{\epsilon},\mathfrak{t}}^{\boldsymbol{\varepsilon}',\mathfrak{s}'} \in A$ satisfying

$$a_{\boldsymbol{\epsilon},\mathfrak{t}}^{\boldsymbol{\varepsilon}',\mathfrak{s}'} \equiv \begin{cases} \pm 1 \pmod{D_1} & \text{if } \begin{pmatrix} \boldsymbol{\epsilon} \\ \mathfrak{t} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}, \\ 0 \pmod{D_1} & \text{otherwise.} \end{cases}$$

Note that $\mathrm{Li}\begin{pmatrix} \boldsymbol{\varepsilon}' \\ \mathfrak{s}' \end{pmatrix} \in \mathcal{AS}_w$. Thus the transition matrix from the set consisting of such $\mathrm{Li}\begin{pmatrix} \boldsymbol{\varepsilon}' \\ \mathfrak{s}' \end{pmatrix}$ as above (we allow repeated elements) to the set consisting of $\mathrm{Li}\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}$ with $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} \in \mathcal{AJ}_w$ is invertible. It then follows again from Proposition 1.9 that $\mathcal{AS}_w$ is a set of generators for $\mathcal{AL}_w$, as desired. ☐

2. Dual t-motives and linear independence

We continue with the notation given in the Introduction. Further, letting t be another independent variable, we denote by $\mathbb{T}$ the Tate algebra in the variable t with coefficients in $\mathbb{C}_\infty$ equipped with the Gauss norm $\|.\|_\infty$, and by $\mathbb{L}$ the fraction field of $\mathbb{T}$.

2.1. Dual t-motives.

We recall the notion of dual t-motives due to Anderson (see [6, §4] and [20, §5] for more details). We refer the reader to [1] for the related notion of t-motives.

For $i \in \mathbb{Z}$ we consider the i-fold twisting of $\mathbb{C}_\infty((t))$ defined by

$$\begin{aligned} \mathbb{C}_\infty((t)) &\to \mathbb{C}_\infty((t)) \\ f = \sum_j a_j t^j &\mapsto f^{(i)} := \sum_j a_j^{q^i} t^j. \end{aligned}$$

We extend i-fold twisting to matrices with entries in $\mathbb{C}_\infty((t))$ by twisting entry-wise.

Let $\overline{K}[t,\sigma]$ be the non-commutative $\overline{K}[t]$-algebra generated by the new variable σ subject to the relation $\sigma f = f^{(-1)}\sigma$ for all $f \in \overline{K}[t]$.

Definition 2.1. An effective dual t-motive is a $\overline{K}[t,\sigma]$-module $\mathcal{M}'$ which is free and finitely generated over $\overline{K}[t]$ such that for $\ell \gg 0$ we have

$$(t-\theta)^{\ell}(\mathcal{M}'/\sigma\mathcal{M}') = \{0\}.$$

We mention that effective dual t-motives are called Frobenius modules in [10, 13, 19, 23]. Note that Hartl and Juschka [20, §4] introduced a more general notion of dual t-motives. In particular, effective dual t-motives are always dual t-motives.

Throughout this paper we will always work with effective dual t-motives. Therefore, we will sometimes drop the word "effective" where there is no confusion.

Let $\mathcal{M}$ and $\mathcal{M}'$ be two effective dual t-motives. Then a morphism of effective dual t-motives $\mathcal{M} \to \mathcal{M}'$ is just a homomorphism of left $\overline{K}[t,\sigma]$-modules. We denote by $\mathcal{F}$ the category of effective dual t-motives equipped with the trivial object $\mathbf{1}$.

We say that an object $\mathcal{M}$ of $\mathcal{F}$ is given by a matrix $\Phi \in \mathrm{Mat}_r(\overline{K}[t])$ if $\mathcal{M}$ is a $\overline{K}[t]$-module free of rank r and the action of σ is represented by the matrix Φ on a given $\overline{K}[t]$-basis for $\mathcal{M}$. We say that an object $\mathcal{M}$ of $\mathcal{F}$ is uniformizable or rigid analytically trivial if there exists a matrix $\Psi \in \mathrm{GL}_r(\mathbb{T})$ satisfying $\Psi^{(-1)} = \Phi\Psi$. The matrix Ψ is called a rigid analytic trivialization of $\mathcal{M}$.

We now recall the Anderson-Brownawell-Papanikolas criterion which is crucial in the sequel (see [2, Theorem 3.1.1]).

Theorem 2.2 (Anderson-Brownawell-Papanikolas)**.** *Let $\Phi \in \mathrm{Mat}_\ell(\overline{K}[t])$ be a matrix such that $\det\Phi = c(t-\theta)^s$ for some $c \in \overline{K}$ and $s \in \mathbb{Z}^{\geq 0}$. Let $\psi \in \mathrm{Mat}_{\ell\times 1}(\mathcal{E})$ be a vector satisfying $\psi^{(-1)} = \Phi\psi$ and $\rho \in \mathrm{Mat}_{1\times\ell}(\overline{K})$ such that $\rho\psi(\theta) = 0$. Then there exists a vector $P \in \mathrm{Mat}_{1\times\ell}(\overline{K}[t])$ such that*

$$P\psi = 0 \quad \textit{and} \quad P(\theta) = \rho.$$

2.2. Some constructions of dual t-motives.

2.2.1. *General case.* We briefly review some constructions of dual t-motives introduced in [10] (see also [9, 13, 19]). Let $\mathfrak{s} = (s_1,\dots,s_r) \in \mathbb{N}^r$ be a tuple of positive integers and $\mathfrak{Q} = (Q_1,\dots,Q_r) \in \overline{K}[t]^r$ satisfying the condition

$$(\|Q_1\|_\infty/|\theta|_\infty^{\frac{qs_1}{q-1}})^{q^{i_1}} \dots (\|Q_r\|_\infty/|\theta|_\infty^{\frac{qs_r}{q-1}})^{q^{i_r}} \to 0 \tag{2.1}$$

as $0 \leq i_r < \dots < i_1$ and $i_1 \to \infty$.

We consider the dual t-motives $\mathcal{M}_{\mathfrak{s},\mathfrak{Q}}$ and $\mathcal{M}'_{\mathfrak{s},\mathfrak{Q}}$ attached to $(\mathfrak{s},\mathfrak{Q})$ given by the matrices

$$\Phi_{\mathfrak{s},\mathfrak{Q}} = \begin{pmatrix} (t-\theta)^{s_1+\dots+s_r} & 0 & 0 & \dots & 0 \\ Q_1^{(-1)}(t-\theta)^{s_1+\dots+s_r} & (t-\theta)^{s_2+\dots+s_r} & 0 & \dots & 0 \\ 0 & Q_2^{(-1)}(t-\theta)^{s_2+\dots+s_r} & \ddots & & \vdots \\ \vdots & & \ddots & (t-\theta)^{s_r} & 0 \\ 0 & \dots & 0 & Q_r^{(-1)}(t-\theta)^{s_r} & 1 \end{pmatrix}$$
$$\in \mathrm{Mat}_{r+1}(\overline{K}[t]),$$

and $\Phi'_{\mathfrak{s},\mathfrak{Q}} \in \mathrm{Mat}_r(\overline{K}[t])$ is the upper left $r \times r$ sub-matrix of $\Phi_{\mathfrak{s},\mathfrak{Q}}$.

Throughout this paper, we work with the Carlitz period $\widetilde{\pi}$ which is a fundamental period of the Carlitz module (see [18, 31]). We fix a choice of $(q-1)$st root of $(-\theta)$ and set

$$\Omega(t) := (-\theta)^{-q/(q-1)} \prod_{i\geq 1} \left(1 - \frac{t}{\theta^{q^i}}\right) \in \mathbb{T}^\times$$

so that

$$\Omega^{(-1)} = (t-\theta)\Omega \quad \text{and} \quad \frac{1}{\Omega(\theta)} = \widetilde{\pi}.$$

Given $(\mathfrak{s}, \mathfrak{Q})$ as above, Chang introduced the following series (see [9, Lemma 5.3.1] and also [10, Eq. (2.3.2)])

$$\mathfrak{L}(\mathfrak{s}; \mathfrak{Q}) = \mathfrak{L}(s_1, \dots, s_r; Q_1, \dots, Q_r) := \sum_{i_1 > \dots > i_r \geq 0} (\Omega^{s_r} Q_r)^{(i_r)} \dots (\Omega^{s_1} Q_1)^{(i_1)}. \tag{2.2}$$

If we denote $\mathcal{E}$ the ring of series $\sum_{n\geq 0} a_n t^n \in \overline{K}[[t]]$ such that $\lim_{n\to+\infty} \sqrt[n]{|a_n|_\infty} = 0$ and $[K_\infty(a_0, a_1, \dots) : K_\infty] < \infty$, then any $f \in \mathcal{E}$ is an entire function. It is proved that $\mathfrak{L}(\mathfrak{s}, \mathfrak{Q}) \in \mathcal{E}$ (see [9, Lemma 5.3.1]). In the sequel, we will use the following crucial property of this series (see [9, Lemma 5.3.5] and [10, Proposition 2.3.3]): for all $j \in \mathbb{Z}^{\geq 0}$, we have

$$\mathfrak{L}(\mathfrak{s}; \mathfrak{Q}) \left(\theta^{q^j}\right) = \left(\mathfrak{L}(\mathfrak{s}; \mathfrak{Q})(\theta)\right)^{q^j}. \tag{2.3}$$

Then the matrix given by

$$\Psi_{\mathfrak{s},\mathfrak{Q}} = \begin{pmatrix} \Omega^{s_1+\dots+s_r} & 0 & 0 & \dots & 0 \\ \mathfrak{L}(s_1;Q_1)\Omega^{s_2+\dots+s_r} & \Omega^{s_2+\dots+s_r} & 0 & \dots & 0 \\ \vdots & \mathfrak{L}(s_2;Q_2)\Omega^{s_3+\dots+s_r} & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \mathfrak{L}(s_1,\dots,s_{r-1};Q_1,\dots,Q_{r-1})\Omega^{s_r} & \mathfrak{L}(s_2,\dots,s_{r-1};Q_2,\dots,Q_{r-1})\Omega^{s_r} & \dots & \Omega^{s_r} & 0 \\ \mathfrak{L}(s_1,\dots,s_r;Q_1,\dots,Q_r) & \mathfrak{L}(s_2,\dots,s_r;Q_2,\dots,Q_r) & \dots & \mathfrak{L}(s_r;Q_r) & 1 \end{pmatrix}$$

$$\in \mathrm{GL}_{r+1}(\mathbb{T})$$

satisfies

$$\Psi_{\mathfrak{s},\mathfrak{Q}}^{(-1)} = \Phi_{\mathfrak{s},\mathfrak{Q}} \Psi_{\mathfrak{s},\mathfrak{Q}}.$$

Thus $\Psi_{\mathfrak{s},\mathfrak{Q}}$ is a rigid analytic trivialization associated to the dual t-motive $\mathcal{M}_{\mathfrak{s},\mathfrak{Q}}$.

We also denote by $\Psi'_{\mathfrak{s},\mathfrak{Q}}$ the upper $r \times r$ sub-matrix of $\Psi_{\mathfrak{s},\mathfrak{Q}}$. It is clear that $\Psi'_{\mathfrak{s}}$ is a rigid analytic trivialization associated to the dual t-motive $\mathcal{M}'_{\mathfrak{s},\mathfrak{Q}}$.

Further, combined with Eq. (2.3), the above construction of dual t-motives implies that $\widetilde{\pi}^w \mathfrak{L}(\mathfrak{s}; \mathfrak{Q})(\theta)$ where $w = s_1 + \dots + s_r$ has the MZ (multizeta) property in the sense of [9, Definition 3.4.1]. By [9, Proposition 4.3.1], we get

Proposition 2.3. *Let $(\mathfrak{s}_i; \mathfrak{Q}_i)$ as before for $1 \leq i \leq m$. We suppose that all the tuples of positive integers $\mathfrak{s}_i$ have the same weight, says w. Then the following assertions are equivalent:*

i) $\mathfrak{L}(\mathfrak{s}_1; \mathfrak{Q}_1)(\theta), \dots, \mathfrak{L}(\mathfrak{s}_m; \mathfrak{Q}_m)(\theta)$ are K-linearly independent.
ii) $\mathfrak{L}(\mathfrak{s}_1; \mathfrak{Q}_1)(\theta), \dots, \mathfrak{L}(\mathfrak{s}_m; \mathfrak{Q}_m)(\theta)$ are $\overline{K}$-linearly independent.

We end this section by mentionning that Chang [9] also proved analogue of Goncharov's conjecture in this setting.

2.2.2. *Dual t-motives connected to MZV's and AMZV's.*

Following Anderson and Thakur [4] we introduce dual t-motives connected to MZV's and AMZV's. We briefly review Anderson-Thakur polynomials introduced in [3]. For $k \geq 0$, we set $[k] := \theta^{q^k} - \theta$ and $D_k := \prod_{\ell=1}^{k} [\ell]^{q^{k-\ell}}$. For $n \in \mathbb{N}$ we write $n-1 = \sum_{j \geq 0} n_j q^j$ with $0 \leq n_j \leq q-1$ and define

$$\Gamma_n := \prod_{j \geq 0} D_j^{n_j}.$$

We set $\gamma_0(t) := 1$ and $\gamma_j(t) := \prod_{\ell=1}^{j} (\theta^{q^j} - t^{q^\ell})$ for $j \geq 1$. Then Anderson-Thakur polynomials $\alpha_n(t) \in A[t]$ are given by the generating series

$$\sum_{n \geq 1} \frac{\alpha_n(t)}{\Gamma_n} x^n := x \left(1 - \sum_{j \geq 0} \frac{\gamma_j(t)}{D_j} x^{q^j} \right)^{-1}.$$

Finally, we define $H_n(t)$ by switching θ and t

$$H_n(t) = \alpha_n(t) \big|_{t=\theta,\, \theta=t}.$$

By [3, Eq. (3.7.3)] we get

$$\deg_\theta H_n \leq \frac{(n-1)q}{q-1} < \frac{nq}{q-1}. \tag{2.4}$$

Let $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ be a tuple and $\epsilon = (\epsilon_1, \dots, \epsilon_r) \in (\mathbb{F}_q^\times)^r$. For all $1 \leq i \leq r$ we fix a fixed $(q-1)$-th root γ_i of $\epsilon_i \in \mathbb{F}_q^\times$ and set $Q_{s_i, \epsilon_i} := \gamma_i H_{s_i}$. Then we set $\mathfrak{Q}_{\mathfrak{s}, \boldsymbol{\epsilon}} := (Q_{s_1, \epsilon_1}, \dots, Q_{s_r, \epsilon_r})$ and put $\mathfrak{L}(\mathfrak{s}; \boldsymbol{\epsilon}) := \mathfrak{L}(\mathfrak{s}; \mathfrak{Q}_{\mathfrak{s}, \boldsymbol{\epsilon}})$. By (2.4) we know that $\|H_n\|_\infty < |\theta|_\infty^{\frac{nq}{q-1}}$ for all $n \in \mathbb{N}$, thus $\mathfrak{Q}_{\mathfrak{s}, \boldsymbol{\epsilon}}$ satisfies Condition (2.1). Thus we can define the dual t-motives $\mathcal{M}_{\mathfrak{s}, \boldsymbol{\epsilon}} = \mathcal{M}_{\mathfrak{s}, \mathfrak{Q}_{\mathfrak{s}, \boldsymbol{\epsilon}}}$ and $\mathcal{M}'_{\mathfrak{s}, \boldsymbol{\epsilon}} = \mathcal{M}'_{\mathfrak{s}, \mathfrak{Q}_{\mathfrak{s}, \boldsymbol{\epsilon}}}$ attached to $\mathfrak{s}$ whose matrices and rigid analytic trivializations will be denoted by $(\Phi_{\mathfrak{s}, \boldsymbol{\epsilon}}, \Psi_{\mathfrak{s}, \boldsymbol{\epsilon}})$ and $(\Phi'_{\mathfrak{s}, \boldsymbol{\epsilon}}, \Psi'_{\mathfrak{s}, \boldsymbol{\epsilon}})$, respectively. These dual t-motives are connected to MZV's and AMZV's by the following result (see [13, Proposition 2.12] for more details):

$$\mathfrak{L}(\mathfrak{s}; \boldsymbol{\epsilon})(\theta) = \frac{\gamma_1 \dots \gamma_r \Gamma_{s_1} \dots \Gamma_{s_r} \zeta_A \begin{pmatrix} \boldsymbol{\epsilon} \\ \mathfrak{s} \end{pmatrix}}{\widetilde{\pi}^{w(\mathfrak{s})}}. \tag{2.5}$$

By a result of Thakur [33], one can show (see [19, Theorem 2.1]) that $\zeta_A \begin{pmatrix} \boldsymbol{\epsilon} \\ \mathfrak{s} \end{pmatrix} \neq 0$. Thus $\mathfrak{L}(\mathfrak{s}; \boldsymbol{\epsilon})(\theta) \neq 0$.

2.2.3. *Dual t-motives connected to CMPL's and ACMPL's.*

We keep the notation as above. Let $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ be a tuple and $\boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_r) \in (\mathbb{F}_q^\times)^r$. For all $1 \leq i \leq r$ we have a fixed $(q-1)$-th root γ_i of $\epsilon_i \in \mathbb{F}_q^\times$ and set $Q'_{s_i, \epsilon_i} := \gamma_i$. Then we set $\mathfrak{Q}'_{\mathfrak{s}, \boldsymbol{\epsilon}} := (Q'_{s_1, \epsilon_1}, \dots, Q'_{s_r, \epsilon_r})$ and put

$$\mathfrak{Li}(\mathfrak{s}; \boldsymbol{\epsilon}) = \mathfrak{L}(\mathfrak{s}; \mathfrak{Q}'_{\mathfrak{s}, \boldsymbol{\epsilon}}) = \sum_{i_1 > \dots > i_r \geq 0} (\gamma_{i_r} \Omega^{s_r})^{(i_r)} \dots (\gamma_{i_1} \Omega^{s_1})^{(i_1)}. \tag{2.6}$$

Thus we can define the dual t-motives $\mathcal{N}_{\mathfrak{s}, \boldsymbol{\epsilon}} = \mathcal{N}_{\mathfrak{s}, \mathfrak{Q}'_{\mathfrak{s}, \boldsymbol{\epsilon}}}$ and $\mathcal{N}'_{\mathfrak{s}, \boldsymbol{\epsilon}} = \mathcal{N}'_{\mathfrak{s}, \mathfrak{Q}'_{\mathfrak{s}, \boldsymbol{\epsilon}}}$ attached to $(\mathfrak{s}, \boldsymbol{\epsilon})$. These dual t-motives are connected to CMPL's and ACMPL's by the following result (see [9, Lemma 5.3.5] and [10, Prop. 2.3.3]):

$$\mathfrak{Li}(\mathfrak{s};\boldsymbol{\epsilon})(\theta) = \frac{\gamma_1 \dots \gamma_r \operatorname{Li}\begin{pmatrix}\boldsymbol{\epsilon}\\ \mathfrak{s}\end{pmatrix}}{\widetilde{\pi}^{w(\mathfrak{s})}}. \tag{2.7}$$

2.3. A result for linear independence.

2.3.1. *Setup.*

Let $w \in \mathbb{N}$ be a positive integer. Let $\{(\mathfrak{s}_i;\mathfrak{Q}_i)\}_{1\leq i\leq n}$ be a collection of pairs satisfying Condition (2.1) such that $\mathfrak{s}_i$ always has weight w. We write $\mathfrak{s}_i = (s_{i1},\dots,s_{i\ell_i}) \in \mathbb{N}^{\ell_i}$ and $\mathfrak{Q}_i = (Q_{i1},\dots,Q_{i\ell_i}) \in (\mathbb{F}_q^\times)^{\ell_i}$ so that $s_{i1}+\dots+s_{i\ell_i} = w$. We introduce the set of tuples

$$I(\mathfrak{s}_i;\mathfrak{Q}_i) := \{\emptyset, (s_{i1};Q_{i1}),\dots,(s_{i1},\dots,s_{i(\ell_i-1)};Q_{i1},\dots,Q_{i(\ell_i-1)})\},$$

and set

$$I := \cup_i I(\mathfrak{s}_i;\mathfrak{Q}_i).$$

For all $(\mathfrak{t};\mathfrak{Q}) \in I$, we set

$$f_{\mathfrak{t},\mathfrak{Q}} := \sum_i a_i(t)\mathfrak{L}(s_{i(k+1)},\dots,s_{i\ell_i};Q_{i(k+1)},\dots,Q_{i\ell_i}), \tag{2.8}$$

where the sum runs through the set of indices i such that $(\mathfrak{t};\mathfrak{Q}) = (s_{i1},\dots,s_{ik};Q_{i1},\dots,Q_{ik})$ for some $0\leq k\leq \ell_i-1$. In particular, $f_\emptyset = \sum_i a_i(t)\mathfrak{L}(\mathfrak{s}_i;\mathfrak{Q}_i)$.

2.3.2. *Linear independence.*

We are now ready to state the main result of this section.

Theorem 2.4. *We keep the above notation. We suppose further that $\{(\mathfrak{s}_i;\mathfrak{Q}_i)\}_{1\leq i\leq n}$ satisfies the following conditions:*

- *(LW) For any weight $w' < w$, the values $\mathfrak{L}(\mathfrak{t};\mathfrak{Q})(\theta)$ with $(\mathfrak{t};\mathfrak{Q}) \in I$ and $w(\mathfrak{t}) = w'$ are all K-linearly independent. In particular, $\mathfrak{L}(\mathfrak{t};\mathfrak{Q})(\theta)$ is always nonzero.*
- *(LD) There exist $a \in A$ and $a_i \in A$ for $1\leq i\leq n$ which are not all zero such that*

$$a + \sum_{i=1}^n a_i \mathfrak{L}(\mathfrak{s}_i;\mathfrak{Q}_i)(\theta) = 0.$$

Then for all $(\mathfrak{t};\mathfrak{Q}) \in I$, $f_{\mathfrak{t},\mathfrak{Q}}(\theta)$ belongs to K where $f_{\mathfrak{t},\mathfrak{Q}}$ is given as in (2.8).

Remark 2.5. 1) Here we note that LW stands for Lower Weights and LD for Linear Dependence.

2) In fact, we improve [27, Theorem B] in two directions. First, we lift the restriction on the Anderson-Thakur polynomials and tuples $\mathfrak{s}_i$. Second and more important, we allow an extra term a which is crucial in the sequel. More precisely, in the case of MZV's, while [27, Theorem B] investigates linear relations amongs MZV's of weight w, Theorem 2.4 studies linear relations amongs MZV's of weight w and suitable powers of the Carlitz period $\widetilde{\pi}^w$.

Proof. The proof will be divided into two steps.

Step 1. We first construct a dual t-motive to which we will apply the Anderson-Brownawell-Papanikolas criterion. In what follows we set $a_i(t) := a_i|_{\theta=t} \in \mathbb{F}_q[t]$.

For each pair $(\mathfrak{s}_i;\mathfrak{Q}_i)$ we have attached to it a matrix $\Phi_{\mathfrak{s}_i,\mathfrak{Q}_i}$. For $\mathfrak{s}_i=(s_{i1},\dots,s_{i\ell_i})\in\mathbb{N}^{\ell_i}$ and $\mathfrak{Q}_i=(Q_{i1},\dots,Q_{i\ell_i})\in(\mathbb{F}_q^\times)^{\ell_i}$ we recall

$$I(\mathfrak{s}_i;\mathfrak{Q}_i)=\{\emptyset,(s_{i1};Q_{i1}),\dots,(s_{i1},\dots,s_{i(\ell_i-1)};Q_{i1},\dots,Q_{(\ell_i-1)})\},$$

and $I:=\cup_i I(\mathfrak{s}_i;\mathfrak{Q}_i)$.

We now construct a new matrix Φ' by merging the same rows of $\Phi'_{\mathfrak{s}_1,\mathfrak{Q}_1},\dots,\Phi'_{\mathfrak{s}_n,\mathfrak{Q}_n}$ as follows. Then the matrix Φ' will be a matrix indexed by elements of I, says $\Phi'=\left(\Phi'_{(\mathfrak{t};\mathfrak{Q}),(\mathfrak{t}';\mathfrak{Q}')}\right)_{(\mathfrak{t};\mathfrak{Q}),(\mathfrak{t}';\mathfrak{Q}')\in I}\in\operatorname{Mat}_{|I|}(\overline{K}[t])$. For the row which corresponds to the empty pair $\emptyset$ we put

$$\Phi'_{\emptyset,(\mathfrak{t}';\mathfrak{Q}')}=\begin{cases}(t-\theta)^w & \text{if } (\mathfrak{t}';\mathfrak{Q}')=\emptyset,\\ 0 & \text{otherwise.}\end{cases}$$

For the row indexed by $(\mathfrak{t};\mathfrak{Q})=(s_{i1},\dots,s_{ij};Q_{i1},\dots,Q_{ij})$ for some i and $1\le j\le\ell_i-1$ we put

$$\Phi'_{(\mathfrak{t};\mathfrak{Q}),(\mathfrak{t}';\mathfrak{Q}')}=\begin{cases}(t-\theta)^{w-w(\mathfrak{t}')} & \text{if } (\mathfrak{t}';\mathfrak{Q}')=(\mathfrak{t};\mathfrak{Q}),\\ Q_{ij}^{(-1)}(t-\theta)^{w-w(\mathfrak{t}')} & \text{if } (\mathfrak{t}';\mathfrak{Q}')=(s_{i1},\dots,s_{i(j-1)};Q_{i1},\dots,Q_{i(j-1)}),\\ 0 & \text{otherwise.}\end{cases}$$

Note that $\Phi'_{\mathfrak{s}_i,\mathfrak{Q}_i}=\left(\Phi'_{(\mathfrak{t};\mathfrak{Q}),(\mathfrak{t}';\mathfrak{Q}')}\right)_{(\mathfrak{t};\mathfrak{Q}),(\mathfrak{t}';\mathfrak{Q}')\in I(\mathfrak{s}_i;\mathfrak{Q}_i)}$ for all i.

We define $\Phi\in\operatorname{Mat}_{|I|+1}(\overline{K}[t])$ by

$$\Phi=\begin{pmatrix}\Phi' & 0\\ \mathbf{v} & 1\end{pmatrix}\in\operatorname{Mat}_{|I|+1}(\overline{K}[t]),\quad \mathbf{v}=(v_{\mathfrak{t},\mathfrak{Q}})_{(\mathfrak{t};\mathfrak{Q})\in I}\in\operatorname{Mat}_{1\times|I|}(\overline{K}[t]),$$

where

$$v_{\mathfrak{t},\mathfrak{Q}}=\begin{cases}a_i(t)Q_{i\ell_i}^{(-1)}(t-\theta)^{w-w(\mathfrak{t})} & \text{if } (\mathfrak{t};\mathfrak{Q})=(s_{i1},\dots,s_{i(\ell_i-1)};Q_{i1},\dots,Q_{i(\ell_i-1)}),\\ 0 & \text{otherwise.}\end{cases}$$

We now introduce a rigid analytic trivialization matrix Ψ for Φ. We define $\Psi'=\left(\Psi'_{(\mathfrak{t};\mathfrak{Q}),(\mathfrak{t}';\mathfrak{Q}')}\right)_{(\mathfrak{t};\mathfrak{Q}),(\mathfrak{t}';\mathfrak{Q}')\in I}\in\operatorname{GL}_{|I|}(\mathbb{T})$ as follows. For the row which corresponds to the empty pair $\emptyset$ we define

$$\Psi'_{\emptyset,(\mathfrak{t}';\mathfrak{Q}')}=\begin{cases}\Omega^w & \text{if } (\mathfrak{t}';\mathfrak{Q}')=\emptyset,\\ 0 & \text{otherwise.}\end{cases}$$

For the row indexed by $(\mathfrak{t};\mathfrak{Q})=(s_{i1},\dots,s_{ij};Q_{i1},\dots,Q_{ij})$ for some i and $1\le j\le\ell_i-1$ we put

$$\Psi'_{(\mathfrak{t};\mathfrak{Q}),(\mathfrak{t}';\mathfrak{Q}')}=$$
$$\begin{cases}\mathfrak{L}(\mathfrak{t};\mathfrak{Q})\Omega^{w-w(\mathfrak{t})} & \text{if } (\mathfrak{t}';\mathfrak{Q}')=\emptyset,\\ \mathfrak{L}(s_{i(k+1)},\dots,s_{ij};Q_{i(k+1)},\dots,Q_{ij})\Omega^{w-w(\mathfrak{t})} & \text{if } (\mathfrak{t}';\mathfrak{Q}')=(s_{i1},\dots,s_{ik};Q_{i1},\dots,Q_{ik}) \text{ for some } 1\le k\le j,\\ 0 & \text{otherwise.}\end{cases}$$

Note that $\Psi'_{\mathfrak{s}_i,\mathfrak{Q}_i}=\left(\Psi'_{(\mathfrak{t};\mathfrak{Q}),(\mathfrak{t}';\mathfrak{Q}')}\right)_{(\mathfrak{t};\mathfrak{Q}),(\mathfrak{t}';\mathfrak{Q}')\in I(\mathfrak{s}_i;\mathfrak{Q}_i)}$ for all i.

We define $\Psi \in \mathrm{GL}_{|I|+1}(\mathbb{T})$ by

$$\Psi = \begin{pmatrix} \Psi' & 0 \\ \mathbf{f} & 1 \end{pmatrix} \in \mathrm{GL}_{|I|+1}(\mathbb{T}), \quad \mathbf{f} = (f_{\mathfrak{t},\mathfrak{Q}})_{\mathfrak{t} \in I} \in \mathrm{Mat}_{1\times|I|}(\mathbb{T}).$$

Here we recall (see Eq. (2.8))

$$f_{\mathfrak{t},\mathfrak{Q}} = \sum_i a_i(t)\mathfrak{L}(s_{i(k+1)}, \dots, s_{i\ell_i}; Q_{i(k+1)}, \dots, Q_{i\ell_i})$$

where the sum runs through the set of indices i such that $(\mathfrak{t};\mathfrak{Q}) = (s_{i1}, \dots, s_{ik}; Q_{i1}, \dots, Q_{ik})$ for some $0 \le k \le \ell_i - 1$. In particular, $f_\emptyset = \sum_i a_i(t)\mathfrak{L}(\mathfrak{s}_i;\mathfrak{Q}_i)$.

By construction and by §2.2, we get $\Psi^{(-1)} = \Phi\Psi$, that means Ψ is a rigid analytic trivialization for Φ.

Step 2. Next we apply the Anderson-Brownawell-Papanikolas criterion (see Theorem 2.2) to prove Theorem 2.4.

In fact, we define

$$\widetilde{\Phi} = \begin{pmatrix} 1 & 0 \\ 0 & \Phi \end{pmatrix} \in \mathrm{Mat}_{|I|+2}(\overline{K}[t])$$

and consider the vector constructed from the first column vector of Ψ

$$\widetilde{\psi} = \begin{pmatrix} 1 \\ \Psi'_{(\mathfrak{t};\mathfrak{Q}),\emptyset} \\ f_\emptyset \end{pmatrix}_{(\mathfrak{t};\mathfrak{Q})\in I}.$$

Then we have $\widetilde{\psi}^{(-1)} = \widetilde{\Phi}\widetilde{\psi}$.

We also observe that for all $(\mathfrak{t};\mathfrak{Q}) \in I$ we have $\Psi'_{(\mathfrak{t};\mathfrak{Q}),\emptyset} = \mathfrak{L}(\mathfrak{t};\mathfrak{Q})\Omega^{w-w(\mathfrak{t})}$. Further,

$$a + f_\emptyset(\theta) = a + \sum_i a_i \mathfrak{L}(\mathfrak{s}_i;\mathfrak{Q}_i)(\theta) = 0.$$

By Theorem 2.2 with $\rho = (a, 0, \dots, 0, 1)$ we deduce that there exists $\mathbf{h} = (g_0, g_{\mathfrak{t},\mathfrak{Q}}, g) \in \mathrm{Mat}_{1\times(|I|+2)}(\overline{K}[t])$ such that $\mathbf{h}\psi = 0$, and that $g_{\mathfrak{t},\mathfrak{Q}}(\theta) = 0$ for $(\mathfrak{t},\mathfrak{Q}) \in I$, $g_0(\theta) = a$ and $g(\theta) = 1 \neq 0$. If we put $\mathbf{g} := (1/g)\mathbf{h} \in \mathrm{Mat}_{1\times(|I|+2)}(\overline{K}(t))$, then all the entries of $\mathbf{g}$ are regular at $t = \theta$.

Now we have

$$(\mathbf{g} - \mathbf{g}^{(-1)}\widetilde{\Phi})\widetilde{\psi} = \mathbf{g}\widetilde{\psi} - (\mathbf{g}\widetilde{\psi})^{(-1)} = 0. \tag{2.9}$$

We write $\mathbf{g} - \mathbf{g}^{(-1)}\widetilde{\Phi} = (B_0, B_\mathfrak{t}, 0)_{\mathfrak{t}\in I}$. We claim that $B_0 = 0$ and $B_{\mathfrak{t},\mathfrak{Q}} = 0$ for all $(\mathfrak{t};\mathfrak{Q}) \in I$. In fact, expanding (2.9) we obtain

$$B_0 + \sum_{\mathfrak{t}\in I} B_{\mathfrak{t},\mathfrak{Q}}\mathfrak{L}(\mathfrak{t};\mathfrak{Q})\Omega^{w-w(\mathfrak{t})} = 0. \tag{2.10}$$

By (2.3) we see that for $(\mathfrak{t};\mathfrak{Q}) \in I$ and $j \in \mathbb{N}$,

$$\mathfrak{L}(\mathfrak{t};\mathfrak{Q})(\theta^{q^j}) = (\mathfrak{L}(\mathfrak{t};\mathfrak{Q})(\theta))^{q^j}$$

which is nonzero by Condition (LW).

First, as the function Ω has a simple zero at $t = \theta^{q^k}$ for $k \in \mathbb{N}$, specializing (2.10) at $t = \theta^{q^j}$ yields $B_0(\theta^{q^j}) = 0$ for $j \ge 1$. Since B_0 belongs to $\overline{K}(t)$, it follows that $B_0 = 0$.

Next, we put $w_0 := \max_{(\mathfrak{t};\mathfrak{Q})\in I} w(\mathfrak{t})$ and denote by $I(w_0)$ the set of $(\mathfrak{t};\mathfrak{Q}) \in I$ such that $w(\mathfrak{t}) = w_0$. Then dividing (2.10) by Ω^{w-w_0} yields

$$\sum_{(\mathfrak{t};\mathfrak{Q})\in I} B_{\mathfrak{t},\mathfrak{Q}}\mathfrak{L}(\mathfrak{t};\mathfrak{Q})\Omega^{w_0-w(\mathfrak{t})} = \sum_{(\mathfrak{t};\mathfrak{Q})\in I(w_0)} B_{\mathfrak{t},\mathfrak{Q}}\mathfrak{L}(\mathfrak{t};\mathfrak{Q}) + \sum_{(\mathfrak{t};\mathfrak{Q})\in I\setminus I(w_0)} B_{\mathfrak{t},\mathfrak{Q}}\mathfrak{L}(\mathfrak{t};\mathfrak{Q})\Omega^{w_0-w(\mathfrak{t})} = 0. \tag{2.11}$$

Since each $B_{\mathfrak{t},\mathfrak{Q}}$ belongs to $\overline{K}(t)$, they are defined at $t = \theta^{q^j}$ for $j \gg 1$. Note that the function Ω has a simple zero at $t = \theta^{q^k}$ for $k \in \mathbb{N}$. Specializing (2.11) at $t = \theta^{q^j}$ and using (2.3) yields

$$\sum_{(\mathfrak{t};\mathfrak{Q})\in I(w_0)} B_{\mathfrak{t},\mathfrak{Q}}(\theta^{q^j})(\mathfrak{L}(\mathfrak{t};\mathfrak{Q})(\theta))^{q^j} = 0$$

for $j \gg 1$.

We claim that $B_{\mathfrak{t},\mathfrak{Q}}(\theta^{q^j}) = 0$ for $j \gg 1$ and for all $(\mathfrak{t};\mathfrak{Q}) \in I(w_0)$. Otherwise, we get a non trivial $\overline{K}$-linear relation among $\mathfrak{L}(\mathfrak{t};\mathfrak{Q})(\theta)$ with $(\mathfrak{t};\mathfrak{Q}) \in I$ of weight w_0. By Proposition 2.3 we deduce a non trivial K-linear relation among $\mathfrak{L}(\mathfrak{t};\mathfrak{Q})(\theta)$ with $(\mathfrak{t};\mathfrak{Q}) \in I(w_0)$, which contradicts with Condition (LW). Now we know that $B_{\mathfrak{t},\mathfrak{Q}}(\theta^{q^j}) = 0$ for $j \gg 1$ and for all $(\mathfrak{t};\mathfrak{Q}) \in I(w_0)$. Since each $B_{\mathfrak{t},\mathfrak{Q}}$ belongs to $\overline{K}(t)$, it follows that $B_{\mathfrak{t},\mathfrak{Q}} = 0$ for all $(\mathfrak{t};\mathfrak{Q}) \in I(w_0)$.

Next, we put $w_1 := \max_{(\mathfrak{t};\mathfrak{Q})\in I\setminus I(w_0)} w(\mathfrak{t})$ and denote by $I(w_1)$ the set of $(\mathfrak{t};\mathfrak{Q}) \in I$ such that $w(\mathfrak{t}) = w_1$. Dividing (2.10) by Ω^{w-w_1} and specializing at $t = \theta^{q^j}$ yields

$$\sum_{(\mathfrak{t};\mathfrak{Q})\in I(w_1)} B_{\mathfrak{t},\mathfrak{Q}}(\theta^{q^j})(\mathfrak{L}(\mathfrak{t};\mathfrak{Q})(\theta))^{q^j} = 0$$

for $j \gg 1$. Since $w_1 < w$, by Proposition 2.3 and Condition (LW) again we deduce that $B_{\mathfrak{t},\mathfrak{Q}}(\theta^{q^j}) = 0$ for $j \gg 1$ and for all $(\mathfrak{t};\mathfrak{Q}) \in I(w_1)$. Since each $B_{\mathfrak{t},\mathfrak{Q}}$ belongs to $\overline{K}(t)$, it follows that $B_{\mathfrak{t},\mathfrak{Q}} = 0$ for all $(\mathfrak{t};\mathfrak{Q}) \in I(w_1)$. Repeating the previous arguments we deduce that $B_{\mathfrak{t},\mathfrak{Q}} = 0$ for all $(\mathfrak{t};\mathfrak{Q}) \in I$ as required.

We have proved that $\mathbf{g} - \mathbf{g}^{(-1)}\widetilde{\Phi} = 0$. Thus

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{Id} & 0 \\ g_0/g & (g_{\mathfrak{t},\mathfrak{Q}}/g)_{(\mathfrak{t};\mathfrak{Q})\in I} & 1 \end{pmatrix}^{(-1)} \begin{pmatrix} 1 & 0 \\ 0 & \Phi \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \Phi' & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{Id} & 0 \\ g_0/g & (g_{\mathfrak{t},\mathfrak{Q}}/g)_{(\mathfrak{t};\mathfrak{Q})\in I} & 1 \end{pmatrix}.$$

By [10, Prop. 2.2.1] we see that the common denominator b of g_0/g and $g_{\mathfrak{t},\mathfrak{Q}}/g$ for $(\mathfrak{t},\mathfrak{Q}) \in I$ belongs to $\mathbb{F}_q[t] \setminus \{0\}$. If we put $\delta_0 = bg_0/g$ and $\delta_{\mathfrak{t},\mathfrak{Q}} = bg_{\mathfrak{t},\mathfrak{Q}}/g$ for $(\mathfrak{t},\mathfrak{Q}) \in I$ which belong to $\overline{K}[t]$ and $\delta := (\delta_{\mathfrak{t},\mathfrak{Q}})_{\mathfrak{t}\in I} \in \mathrm{Mat}_{1\times|I|}(\overline{K}[t])$, then $\delta_0^{(-1)} = \delta_0$ and

$$\begin{pmatrix} \mathrm{Id} & 0 \\ \delta & 1 \end{pmatrix}^{(-1)} \begin{pmatrix} \Phi' & 0 \\ b\mathbf{v} & 1 \end{pmatrix} = \begin{pmatrix} \Phi' & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{Id} & 0 \\ \delta & 1 \end{pmatrix}. \tag{2.12}$$

If we put $X := \begin{pmatrix} \mathrm{Id} & 0 \\ \delta & 1 \end{pmatrix} \begin{pmatrix} \Psi' & 0 \\ b\mathbf{f} & 1 \end{pmatrix}$, then $X^{(-1)} = \begin{pmatrix} \Phi' & 0 \\ 0 & 1 \end{pmatrix} X$. By [28, §4.1.6] there exist $\nu_{\mathfrak{t},\mathfrak{Q}} \in \mathbb{F}_q(t)$ for $(\mathfrak{t},\mathfrak{Q}) \in I$ such that if we set $\nu = (\nu_{\mathfrak{t},\mathfrak{Q}})_{(\mathfrak{t},\mathfrak{Q})\in I} \in \mathrm{Mat}_{1\times|I|}(\mathbb{F}_q(t))$,

$$X = \begin{pmatrix} \Psi' & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{Id} & 0 \\ \nu & 1 \end{pmatrix}.$$

Thus the equation $\begin{pmatrix} \mathrm{Id} & 0 \\ \delta & 1 \end{pmatrix} \begin{pmatrix} \Psi' & 0 \\ b\mathbf{f} & 1 \end{pmatrix} = \begin{pmatrix} \Psi' & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{Id} & 0 \\ \nu & 1 \end{pmatrix}$ implies

$$\delta\Psi' + b\mathbf{f} = \nu. \tag{2.13}$$

The left-hand side belongs to $\mathbb{T}$, so does the right-hand side. Thus $\nu = (\nu_{\mathfrak{t},\mathfrak{Q}})_{(\mathfrak{t},\mathfrak{Q})\in I} \in \mathrm{Mat}_{1\times|I|}(\mathbb{F}_q[t])$. For any $j \in \mathbb{N}$, by specializing (2.13) at $t = \theta^{q^j}$ and using (2.3) and the fact that Ω has a simple zero at $t = \theta^{q^j}$ we deduce that

$$\mathbf{f}(\theta) = \nu(\theta)/b(\theta).$$

Thus for all $(\mathfrak{t}, \mathfrak{Q}) \in I$, $f_{\mathfrak{t},\mathfrak{Q}}(\theta)$ given as in (2.8) belongs to K. The proof is complete. ☐

3. Linear relations among ACMPL's

In this section we use freely the notation of §1 and §2.2.3.

3.1. Preliminaries.

We begin this section by proving several auxiliary lemmas which will be useful in the sequel.

Lemma 3.1. *Let $\epsilon_i \in \mathbb{F}_q^\times$ be different elements. We denote by $\gamma_i \in \overline{\mathbb{F}}_q$ a $(q-1)$-th root of ϵ_i. Then γ_i are all $\mathbb{F}_q$-linearly independent.*

Proof. We know that $\mathbb{F}_q^\times$ is cyclic as a multiplicative group. Let ϵ be a generating element of $\mathbb{F}_q^\times$ so that $\mathbb{F}_q^\times = \langle\epsilon\rangle$. Let γ be the associated $(q-1)$-th root of ϵ. Then for all $1 \le i \le q-1$ it follows that γ^i is a $(q-1)$-th root of ϵ^i. Thus it suffices to show that the polynomial $P(X) = X^{q-1} - \epsilon$ is irreducible in $\mathbb{F}_q[X]$. Suppose that this is not the case, write $P(X) = P_1(X)P_2(X)$ with $1 \le \deg P_1 < q-1$. Since the roots of $P(X)$ are of the form $\alpha\gamma$ with $\alpha \in \mathbb{F}_q^\times$, those of $P_1(X)$ are also of this form. Looking at the constant term of $P_1(X)$, we deduce that $\gamma^{\deg P_1} \in \mathbb{F}_q^\times$. If we put $m = \mathrm{pgcd}(\deg P_1, q-1)$, then $1 \le m < q-1$ and $\gamma^m \in \mathbb{F}_q^\times$. Letting $\beta := \gamma^m \in \mathbb{F}_q^\times$, we get $\beta^{\frac{q-1}{m}} = \gamma^{q-1} = \epsilon$. Since $1 \le m < q-1$, we get a contradiction with the fact that $\mathbb{F}_q^\times = \langle\epsilon\rangle$. The proof is finished. ☐

Lemma 3.2. *Let $\mathrm{Li}\begin{pmatrix} \boldsymbol{\epsilon}_i \\ \mathfrak{s}_i \end{pmatrix} \in \mathcal{AL}_w$ and $a_i \in K$ satisfying*

$$\sum_i a_i \mathfrak{Li}(\mathfrak{s}_i; \boldsymbol{\epsilon}_i)(\theta) = 0.$$

For $\epsilon \in \mathbb{F}_q^\times$ we denote by $I(\epsilon) = \{i : \chi(\epsilon_i) = \epsilon\}$ the set of pairs such that the corresponding character equals ϵ. Then for all $\epsilon \in \mathbb{F}_q^\times$,

$$\sum_{i\in I(\epsilon)} a_i \mathfrak{Li}(\mathfrak{s}_i; \boldsymbol{\epsilon}_i)(\theta) = 0.$$

Proof. We keep the notation of Lemma 3.1. Suppose that we have a relation

$$\sum_i \gamma_i a_i = 0$$

with $a_i \in K_\infty$. By Lemma 3.1 and the fact that $K_\infty = \mathbb{F}_q((1/\theta))$, we deduce that $a_i = 0$ for all i.

By (2.5) the relation $\sum_i a_i \mathfrak{Li}(\mathfrak{s}_i; \boldsymbol{\epsilon}_i)(\theta) = 0$ is equivalent to the following one

$$\sum_i a_i \gamma_{i1} \dots \gamma_{i\ell_i} \operatorname{Li} \begin{pmatrix} \boldsymbol{\epsilon}_i \\ \mathfrak{s}_i \end{pmatrix} = 0.$$

By the previous discussion, for all $\epsilon \in \mathbb{F}_q^\times$,

$$\sum_{i \in I(\epsilon)} a_i \gamma_{i1} \dots \gamma_{i\ell_i} \operatorname{Li} \begin{pmatrix} \boldsymbol{\epsilon}_i \\ \mathfrak{s}_i \end{pmatrix} = 0.$$

By (2.5) again we deduce the desired relation

$$\sum_{i \in I(\epsilon)} a_i \mathfrak{Li}(\mathfrak{s}_i; \boldsymbol{\epsilon}_i)(\theta) = 0.$$

□

Lemma 3.3. *Let $m \in \mathbb{N}$, $\varepsilon \in \mathbb{F}_q^\times$, $\delta \in \overline{K}[t]$ and $F(t,\theta) \in \overline{\mathbb{F}}_q[t,\theta]$ (resp. $F(t,\theta) \in \mathbb{F}_q[t,\theta]$) satisfying*

$$\varepsilon\delta = \delta^{(-1)}(t-\theta)^m + F^{(-1)}(t,\theta).$$

Then $\delta \in \overline{\mathbb{F}}_q[t,\theta]$ (resp. $\delta \in \mathbb{F}_q[t,\theta]$) and

$$\deg_\theta \delta \leq \max\left\{\frac{qm}{q-1}, \frac{\deg_\theta F(t,\theta)}{q}\right\}.$$

Proof. The proof follows the same line as that of [23, Theorem 2] where it is shown that if $F(t,\theta) \in \mathbb{F}_q[t,\theta]$ and $\varepsilon = 1$, then $\delta \in \mathbb{F}_q[t,\theta]$. We write down the proof for the case $F(t,\theta) \in \overline{\mathbb{F}}_q[t,\theta]$ for the convenience of the reader.

By twisting once the equality $\varepsilon\delta = \delta^{(-1)}(t-\theta)^m + F^{(-1)}(t,\theta)$ and the fact that $\varepsilon^q = \varepsilon$, we get

$$\varepsilon\delta^{(1)} = \delta(t-\theta^q)^m + F(t,\theta).$$

We put $n = \deg_t \delta$ and express

$$\delta = a_n t^n + \dots + a_1 t + a_0 \in \overline{K}[t]$$

with $a_0, \dots, a_n \in \overline{K}$. For $i < 0$ we put $a_i = 0$.

Since $\deg_t \delta^{(1)} = \deg_t \delta = n < \delta(t-\theta^q)^m = n+m$, it follows that $\deg_t F(t,\theta) = n+m$. Thus we write $F(t,\theta) = b_{n+m}t^{n+m} + \dots + b_1 t + b_0$ with $b_0, \dots, b_{n+m} \in \overline{\mathbb{F}}_q[\theta]$. Plugging into the previous equation, we obtain

$$\varepsilon(a_n^q t^n + \dots + a_0^q) = (a_n t^n + \dots + a_0)(t-\theta^q)^m + b_{n+m}t^{n+m} + \dots + b_0.$$

Comparing the coefficients t^j for $n+1 \leq j \leq n+m$ yields

$$a_{j-m} + \sum_{i=j-m+1}^{n} \binom{m}{j-i} (-\theta^q)^{m-j+i} a_i + b_j = 0.$$

Since $b_j \in \overline{\mathbb{F}}_q[\theta]$ for all $n+1 \leq j \leq n+m$, we can show by descending induction that $a_j \in \overline{\mathbb{F}}_q[\theta]$ for all $n+1-m \leq j \leq n$.

If $n+1-m \leq 0$, then we are done. Otherwise, comparing the coefficients t^j for $m \leq j \leq n$ yields

$$a_{j-m} + \sum_{i=j-m+1}^{n} \binom{m}{j-i} (-\theta^q)^{m-j+i} a_i + b_j - \varepsilon a_j^q = 0.$$

Since $b_j \in \overline{\mathbb{F}}_q[\theta]$ for all $m \leq j \leq n$ and $a_j \in \overline{\mathbb{F}}_q[\theta]$ for all $n+1-m \leq j \leq n$, we can show by descending induction that $a_j \in \overline{\mathbb{F}}_q[\theta]$ for all $0 \leq j \leq n-m$. We conclude that $\delta \in \overline{\mathbb{F}}_q[t,\theta]$.

We now show that $\deg_\theta \delta \leq \max\{\frac{qm}{q-1}, \frac{\deg_\theta F(t,\theta)}{q}\}$. Otherwise, suppose that $\deg_\theta \delta > \max\{\frac{qm}{q-1}, \frac{\deg_\theta F(t,\theta)}{q}\}$. Then $\deg_\theta \delta^{(1)} = q \deg_\theta \delta$. It implies that $\deg_\theta \delta^{(1)} > \deg_\theta(\delta(t-\theta^q)^m) = \deg_\theta \delta + qm$ and $\deg_\theta \delta^{(1)} > \deg_\theta F(t,\theta)$. Hence we get

$$\deg_\theta(\varepsilon \delta^{(1)}) = \deg_\theta \delta^{(1)} > \deg_\theta(\delta(t-\theta^q)^m + F(t,\theta)),$$

which is a contradiction. □

3.2. Linear relations: statement of the main result.

Theorem 3.4. *Let $w \in \mathbb{N}$. We recall that the set $\mathcal{J}'_w$ consists of positive tuples $\mathfrak{s} = (s_1, \dots, s_n)$ of weight w such that $q \nmid s_i$ for all i. Suppose that we have a non trivial relation*

$$a + \sum_{\mathfrak{s}_i \in \mathcal{J}'_w} a_i \mathfrak{Li}(\mathfrak{s}_i; \boldsymbol{\epsilon}_i)(\theta) = 0, \quad \text{for } a, a_i \in K.$$

Then $q-1 \mid w$ and $a \neq 0$.

Further, if $q-1 \mid w$, then there is a unique relation

$$1 + \sum_{\mathfrak{s}_i \in \mathcal{J}'_w} a_i \mathfrak{Li}(\mathfrak{s}_i; \boldsymbol{\epsilon}_i)(\theta) = 0, \quad \text{for } a_i \in K.$$

In particular, the ACMPL's in $\mathcal{AS}_w$ are linearly independent over K.

Remark 3.5. We stress that although Theorem 3.4 is purely a transcendental result, it is crucial that we do need the full strength of the algebraic theory for ACMPL's (i.e., Theorem 1.10) to conclude (see the last step of the proof).

As a direct consequence of Theorem 3.4, we obtain:

Theorem 3.6. *Let $w \in \mathbb{N}$. Then the ACMPL's in $\mathcal{AS}_w$ form a basis for $\mathcal{AL}_w$. In particular,*

$$\dim_K \mathcal{AL}_w = s(w).$$

Proof. By Theorem 3.4 the ACMPL's in $\mathcal{AS}_w$ are all linearly independent over K. Then by Theorem 1.10 we deduce that the ACMPL's in $\mathcal{AS}_w$ form a basis for $\mathcal{AL}_w$. Hence $\dim_K \mathcal{AL}_w = |\mathcal{AL}_w| = s(w)$ as required. □

3.3. Proof of Theorem 3.4.

It is clear that if $q-1 \nmid w$, then any linear relation

$$a + \sum_{\mathfrak{s}_i \in \mathcal{J}'_w} a_i \mathfrak{Li}(\mathfrak{s}_i; \boldsymbol{\epsilon}_i)(\theta) = 0$$

with $a, a_i \in K$ implies that $a = 0$.

The proof is by induction on the weight $w \in \mathbb{N}$. For $w = 1$, by the previous remark it suffices to show that if

$$a + \sum_i a_i \mathfrak{Li}(1; \epsilon_i)(\theta) = 0,$$

then $a_i = 0$ for all i. In fact, it follows immediately from Lemma 3.2. Suppose that Theorem 3.4 holds for all $w' < w$. We now prove that it holds for w. Suppose that we have a linear relation

$$a + \sum_i a_i \mathfrak{Li}(\mathfrak{s}_i; \boldsymbol{\epsilon}_i)(\theta) = 0. \tag{3.1}$$

By Lemma 3.2 we can suppose further that $\boldsymbol{\epsilon}_i$ has the same character, i.e., there exists $\epsilon \in \mathbb{F}_q^\times$ such that for all i,

$$\chi(\boldsymbol{\epsilon}_i) = \epsilon_{i1} \dots \epsilon_{i\ell_i} = \epsilon. \tag{3.2}$$

We now apply Theorem 2.4 to our setting of ACMPL's. We know that the hypothesis are verified:

(LW) By the induction hypothesis, for any weight $w' < w$, the values $\mathfrak{Li}(\mathfrak{t}; \boldsymbol{\epsilon})(\theta)$ with $(\mathfrak{t}; \boldsymbol{\epsilon}) \in I$ and $w(\mathfrak{t}) = w'$ are all K-linearly independent.

(LD) By (3.1), there exist $a \in A$ and $a_i \in A$ for $1 \leq i \leq n$ which are not all zero such that
$$a + \sum_{i=1}^n a_i \mathfrak{Li}(\mathfrak{s}_i; \boldsymbol{\epsilon}_i)(\theta) = 0.$$

Thus Theorem 2.4 implies that for all $(\mathfrak{t}; \boldsymbol{\epsilon}) \in I$, $f_{\mathfrak{t},\boldsymbol{\epsilon}}(\theta)$ belongs to K where $f_{\mathfrak{t},\boldsymbol{\epsilon}}$ is given by

$$f_{\mathfrak{t};\boldsymbol{\epsilon}} := \sum_i a_i(t) \mathfrak{Li}(s_{i(k+1)}, \dots, s_{i\ell_i}; \epsilon_{i(k+1)}, \dots, \epsilon_{i\ell_i}).$$

Here, the sum runs through the set of indices i such that $(\mathfrak{t}; \boldsymbol{\epsilon}) = (s_{i1}, \dots, s_{ik}; \epsilon_{i1}, \dots, \epsilon_{ik})$ for some $0 \leq k \leq \ell_i - 1$.

We derive a direct consequence of the previous rationality result. Let $(\mathfrak{t}; \boldsymbol{\epsilon}) \in I$ and $\mathfrak{t} \neq \emptyset$. Then $(\mathfrak{t}; \boldsymbol{\epsilon}) = (s_{i1}, \dots, s_{ik}; \epsilon_{i1}, \dots, \epsilon_{ik})$ for some i and $1 \leq k \leq \ell_i - 1$. We denote by $J(\mathfrak{t}; \boldsymbol{\epsilon})$ the set of all such i. We know that there exists $b \in K$ such that

$$b + f_{\mathfrak{t};\boldsymbol{\epsilon}} = 0,$$

or equivalently,

$$b + \sum_{i \in J(\mathfrak{t};\boldsymbol{\epsilon})} a_i(t) \mathfrak{Li}(s_{i(k+1)}, \dots, s_{i\ell_i}; \epsilon_{i(k+1)}, \dots, \epsilon_{i\ell_i}) = 0.$$

The ACMPL's appearing in the above equality belong to $\mathcal{AS}_{w-w(\mathfrak{t})}$. By the induction hypothesis, we can suppose that $\epsilon_{i(k+1)} = \dots = \epsilon_{i\ell_i} = 1$. Further, if $q - 1 \nmid w - w(\mathfrak{t})$, then $a_i(t) = 0$ for all $i \in J(\mathfrak{t}; \boldsymbol{\epsilon})$. Therefore, letting $\mathfrak{s}_i = (s_{i1}, \dots, s_{i\ell_i}; \epsilon_{i1}, \dots, \epsilon_{i\ell_i})$ we can suppose that $s_{i2}, \dots, s_{i\ell_i}$ are all divisible by $q - 1$ and $\epsilon_{i2} = \dots = \epsilon_{i\ell_i} = 1$. In particular, for all i, $\epsilon_{i1} = \chi(\boldsymbol{\epsilon}_i) = \epsilon$.

Now we want to solve (2.12) and we can assume that $b = 1$. We define

$$J := I \cup \{(\mathfrak{s}_i; \boldsymbol{\epsilon}_i)\}$$

For $(\mathfrak{t}; \boldsymbol{\epsilon}) \in J$ we denote by $J_0(\mathfrak{t}; \boldsymbol{\epsilon})$ consisting of $(\mathfrak{t}'; \boldsymbol{\epsilon}') \in I$ such that there exist i and $0 \leq j < \ell_i$ so that $(\mathfrak{t}; \boldsymbol{\epsilon}) = (s_{i1}, s_{i2}, \dots, s_{ij}; \epsilon, 1, \dots, 1)$ and $\mathfrak{t}' = (s_{i1}, s_{i2}, \dots, s_{i(j+1)}; \epsilon, 1, \dots, 1)$. In particular, for $(\mathfrak{t}; \boldsymbol{\epsilon}) = (\mathfrak{s}_i; \boldsymbol{\epsilon}_i)$, $J_0(\mathfrak{t}; \boldsymbol{\epsilon})$ is the empty set. For $(\mathfrak{t}; \boldsymbol{\epsilon}) \in J \setminus \emptyset$, we also put

$$m_{\mathfrak{t}} := \frac{w - w(\mathfrak{t})}{q - 1} \in \mathbb{Z}^{\geq 0}.$$

Then it is clear that (2.12) is equivalent finding $(\delta_{\mathfrak{t},\epsilon})_{(\mathfrak{t};\epsilon)\in J} \in \mathrm{Mat}_{1\times|J|}(\overline{K}[t])$ such that

(3.3)
$$\delta_{\mathfrak{t},\epsilon} = \delta_{\mathfrak{t},\epsilon}^{(-1)}(t-\theta)^{w-w(\mathfrak{t})} + \sum_{(\mathfrak{t}',\epsilon')\in J_0(\mathfrak{t},\epsilon)} \delta_{\mathfrak{t}',\epsilon'}^{(-1)}(t-\theta)^{w-w(\mathfrak{t})}, \quad \text{for all } (\mathfrak{t},\epsilon)\in J\setminus\emptyset,$$

and

$$\delta_{\mathfrak{t},\epsilon} = \delta_{\mathfrak{t},\epsilon}^{(-1)}(t-\theta)^{w-w(\mathfrak{t})} + \sum_{(\mathfrak{t}',\epsilon')\in J_0(\mathfrak{t},\epsilon)} \delta_{\mathfrak{t}',\epsilon'}^{(-1)}\gamma^{(-1)}(t-\theta)^{w-w(\mathfrak{t})}, \quad \text{for } (\mathfrak{t},\epsilon)=\emptyset. \tag{3.4}$$

Here $\gamma^{q-1}=\epsilon$. In fact, for $(\mathfrak{t},\epsilon)=(\mathfrak{s}_i,\epsilon_i)$, the corresponding equation becomes $\delta_{\mathfrak{s}_i,\epsilon_i}=\delta_{\mathfrak{s}_i,\epsilon_i}^{(-1)}$. Thus $\delta_{\mathfrak{s}_i,\epsilon_i}=a_i(t)\in\mathbb{F}_q[t]$.

Letting y be a variable, we denote by v_y the valuation associated to the place y of the field $\mathbb{F}_q(y)$. We put

$$T := t - t^q, \quad X := t^q - \theta^q.$$

We claim that

1) For all $(\mathfrak{t},\epsilon)\in J\setminus\emptyset$, the polynomial $\delta_{\mathfrak{t},\epsilon}$ is of the form
$$\delta_{\mathfrak{t},\epsilon} = f_{\mathfrak{t}}\left(X^{m_{\mathfrak{t}}} + \sum_{i=0}^{m_{\mathfrak{t}}-1} P_{\mathfrak{t},i}(T)X^i\right)$$
where
   - $f_{\mathfrak{t}}\in\mathbb{F}_q[t]$,
   - for all $0\le i\le m_{\mathfrak{t}}-1$, $P_{\mathfrak{t},i}(y)$ belongs to $\mathbb{F}_q(y)$ with $v_y(P_{\mathfrak{t},i})\ge 1$.
2) For all $\mathfrak{t}\in J\setminus\emptyset$ and all $\mathfrak{t}'\in J_0(\mathfrak{t})$, there exists $P_{\mathfrak{t},\mathfrak{t}'}\in\mathbb{F}_q(y)$ such that
$$f_{\mathfrak{t}'} = f_{\mathfrak{t}}P_{\mathfrak{t},\mathfrak{t}'}(T).$$
In particular, if $f_{\mathfrak{t}}=0$, then $f_{\mathfrak{t}'}=0$.

The proof is by induction on $m_{\mathfrak{t}}$. We start with $m_{\mathfrak{t}}=0$. Then $\mathfrak{t}=\mathfrak{s}_i$ and $\epsilon=\epsilon_i$ for some i. We have observed that $\delta_{\mathfrak{s}_i,\epsilon_i}=a_i(t)\in\mathbb{F}_q[t]$. Thus we are done.

Suppose that the claim holds for all $(\mathfrak{t},\epsilon)\in J\setminus\emptyset$ with $m_{\mathfrak{t}}<m$. We now prove the claim for all $(\mathfrak{t},\epsilon)\in J\setminus\emptyset$ with $m_{\mathfrak{t}}=m$. In fact, we fix such $\mathfrak{t}$ and want to find $\delta_{\mathfrak{t},\epsilon}\in\overline{K}[t]$ such that

$$\delta_{\mathfrak{t},\epsilon} = \delta_{\mathfrak{t},\epsilon}^{(-1)}(t-\theta)^{(q-1)m} + \sum_{(\mathfrak{t}',\epsilon')\in J_0(\mathfrak{t};\epsilon)} \delta_{\mathfrak{t}',\epsilon'}^{(-1)}(t-\theta)^{(q-1)m}. \tag{3.5}$$

By the induction hypothesis, for all $(\mathfrak{t}',\epsilon')\in J_0(\mathfrak{t};\epsilon)$, we know that

$$\delta_{\mathfrak{t}',\epsilon'} = f_{\mathfrak{t}'}\left(X^{m_{\mathfrak{t}'}} + \sum_{i=0}^{m_{\mathfrak{t}'}-1} P_{\mathfrak{t}',i}(T)X^i\right)$$

where

- $f_{\mathfrak{t}'}\in\mathbb{F}_q[t]$,
- for all $0\le i\le m_{\mathfrak{t}'}-1$, $P_{\mathfrak{t}',i}(y)\in\mathbb{F}_q(y)$ with $v_y(P_{\mathfrak{t},i})\ge 1$.

For $(\mathfrak{t}',\boldsymbol{\epsilon}') \in J_0(\mathfrak{t};\boldsymbol{\epsilon})$, we write $\mathfrak{t}' = (\mathfrak{t},(m-k)(q-1))$ with $0 \leq k < m$ and $k \not\equiv m \pmod q$, in particular $m_{\mathfrak{t}'} = k$. We put $f_k = f_{\mathfrak{t}'}$ and $P_{\mathfrak{t}',i} = P_{k,i}$ so that

$$\delta_{\mathfrak{t}',\boldsymbol{\epsilon}'} = f_k \left(X^k + \sum_{i=0}^{k-1} P_{k,i}(T) X^i \right) \in \mathbb{F}_q[t,\theta^q]. \tag{3.6}$$

By Lemma 3.3, $\delta_{\mathfrak{t},\boldsymbol{\epsilon}}$ belongs to $K[t]$, and $\deg_\theta \delta_{\mathfrak{t},\boldsymbol{\epsilon}} \leq mq$. Further, since $\delta_{\mathfrak{t},\boldsymbol{\epsilon}}$ is divisible by $(t-\theta)^{(q-1)m}$, we write $\delta_{\mathfrak{t},\boldsymbol{\epsilon}} = F(t-\theta)^{(q-1)m}$ with $F \in K[t]$ and $\deg_\theta F \leq m$. Dividing (3.5) by $(t-\theta)^{(q-1)m}$ and twisting once yields

$$F^{(1)} = F(t-\theta)^{(q-1)m} + \sum_{(\mathfrak{t}',\boldsymbol{\epsilon}') \in J_0(\mathfrak{t},\boldsymbol{\epsilon})} \delta_{\mathfrak{t}',\boldsymbol{\epsilon}'}. \tag{3.7}$$

As $\delta_{\mathfrak{t}',\boldsymbol{\epsilon}'} \in \mathbb{F}_q[t,\theta^q]$ for all $(\mathfrak{t}',\boldsymbol{\epsilon}') \in J_0(\mathfrak{t};\boldsymbol{\epsilon})$, it follows that $F(t-\theta)^{(q-1)m} \in \mathbb{F}_q[t,\theta^q]$. As $\deg_\theta F \leq m$, we get

$$F = \sum_{0 \leq i \leq m/q} f_{m-iq}(t-\theta)^{m-iq}, \quad \text{for } f_{m-iq} \in \mathbb{F}_q[t].$$

Thus

$$\begin{aligned} F(t-\theta)^{(q-1)m} &= \sum_{0 \leq i \leq m/q} f_{m-iq}(t-\theta)^{mq-iq} = \sum_{0 \leq i \leq m/q} f_{m-iq} X^{m-i}, \\ F^{(1)} &= \sum_{0 \leq i \leq m/q} f_{m-iq}(t-\theta^q)^{m-iq} = \sum_{0 \leq i \leq m/q} f_{m-iq}(T+X)^{m-iq}. \end{aligned}$$

Putting these and (3.6) into (3.7) gets

$$\begin{aligned} &\sum_{0 \leq i \leq m/q} f_{m-iq}(T+X)^{m-iq} \\ &= \sum_{0 \leq i \leq m/q} f_{m-iq} X^{m-i} + \sum_{\substack{0 \leq k < m \\ k \not\equiv m \pmod q}} f_k \left(X^k + \sum_{i=0}^{k-1} P_{k,i}(T) X^i \right). \end{aligned}$$

Comparing the coefficients of powers of X yields the following linear system in the variables $f_0, \dots, f_{m-1}$:

$$B_{\big| y=T} \begin{pmatrix} f_{m-1} \\ \vdots \\ f_0 \end{pmatrix} = f_m \begin{pmatrix} Q_{m-1} \\ \vdots \\ Q_0 \end{pmatrix}_{\big| y=T}.$$

Here for $0 \leq i \leq m-1$, $Q_i = \binom{m}{i} y^{m-i} \in y\mathbb{F}_q[y]$ and $B = (B_{ij})_{0 \leq i,j \leq m-1} \in \mathrm{Mat}_m(\mathbb{F}_q(y))$ such that

- $v_y(B_{ij}) \geq 1$ if $i > j$,
- $v_y(B_{ij}) \geq 0$ if $i < j$,
- $v_y(B_{ii}) = 0$ as $B_{ii} = \pm 1$.

The above properties follow from the fact that $P_{k,i} \in \mathbb{F}_q(y)$ and $v_y(P_{k,i}) \geq 1$. Thus $v_y(\det B) = 0$ so that $\det B \neq 0$. It follows that for all $0 \leq i \leq m-1$, $f_i = f_m P_i(T)$ with $P_i \in \mathbb{F}_q(y)$ and $v_y(P_i) \geq 1$ and we are done.

To conclude, we have to solve (3.3) for $(\mathfrak{t};\boldsymbol{\epsilon}) = \emptyset$. We have some extra work as we have a factor $\gamma^{(-1)}$ on the right hand side of (3.4). We use $\gamma^{(-1)} = \gamma/\epsilon$ and put $\delta := \delta_{\emptyset,\emptyset}/\gamma \in \overline{K}[t]$. Then we have to solve

$$\epsilon\delta = \delta^{(-1)}(t-\theta)^w + \sum_{(\mathfrak{t}',\boldsymbol{\epsilon}')\in J_0(\emptyset)} \delta^{(-1)}_{\mathfrak{t}',\boldsymbol{\epsilon}'}(t-\theta)^w. \tag{3.8}$$

We distinguish two cases.

3.3.1. *Case 1: $q-1 \nmid w$, says $w = m(q-1)+r$ with $0 < r < q-1$.*

We know that for all $(\mathfrak{t}',\boldsymbol{\epsilon}') \in J_0(\emptyset)$, says $\mathfrak{t}' = ((m-k)(q-1)+r)$ with $0 \le k \le m$ and $k \not\equiv m-r \pmod q$,

$$\delta_{\mathfrak{t}',\boldsymbol{\epsilon}'} = f_k\left(X^k + \sum_{i=0}^{k-1} P_{k,i}(T)X^i\right) \in \mathbb{F}_q[t,\theta^q] \tag{3.9}$$

where

- $f_k \in \mathbb{F}_q[t]$,
- for all $0 \le i \le k-1$, $P_{k,i}(y)$ belongs to $\mathbb{F}_q(y)$ with $v_y(P_{k,i}) \ge 1$.

By Lemma 3.3, δ belongs to $K[t]$. We claim that $\deg_\theta \delta \le mq$. Otherwise, we have $\deg_\theta \delta_\emptyset > mq$. Twisting (3.8) once gets

$$\epsilon\delta^{(1)} = \delta(t-\theta^q)^w + \sum_{(\mathfrak{t}';\boldsymbol{\epsilon}')\in J_0(\emptyset)} \delta_{\mathfrak{t}',\boldsymbol{\epsilon}'}(t-\theta^q)^w.$$

As $\deg_\theta \delta > mq$, we compare the degrees of θ on both sides and obtain

$$q\deg_\theta \delta = \deg_\theta \delta + wq.$$

Thus $q-1 \mid w$, which is a contradiction. We conclude that $\deg_\theta \delta \le mq$.

From (3.8) we see that δ is divisible by $(t-\theta)^w$. Thus we write $\delta = F(t-\theta)^w$ with $F \in K[t]$ and $\deg_\theta F \le mq - w = m - r$. Dividing (3.8) by $(t-\theta)^w$ and twisting once yields

$$\epsilon F^{(1)} = F(t-\theta)^w + \sum_{(\mathfrak{t}';\boldsymbol{\epsilon}')\in J_0(\emptyset)} \delta_{\mathfrak{t}'}. \tag{3.10}$$

Since $\delta_{\mathfrak{t}';\boldsymbol{\epsilon}'} \in \mathbb{F}_q[t,\theta^q]$ for all $(\mathfrak{t}';\boldsymbol{\epsilon}') \in J_0(\emptyset)$, it follows that $F(t-\theta)^w \in \mathbb{F}_q[t,\theta^q]$. As $\deg_\theta F \le m-r$, we write

$$F = \sum_{0\le i\le (m-r)/q} f_{m-r-iq}(t-\theta)^{m-r-iq}, \quad \text{for } f_{m-r-iq} \in \mathbb{F}_q[t].$$

It follows that

$$F(t-\theta)^w = \sum_{0\le i\le (m-r)/q} f_{m-r-iq}(t-\theta)^{mq-iq} = \sum_{0\le i\le (m-r)/q} f_{m-r-iq}X^{m-i},$$

$$F^{(1)} = \sum_{0\le i\le (m-r)/q} f_{m-r-iq}(t-\theta^q)^{m-r-iq} = \sum_{0\le i\le (m-r)/q} f_{m-r-iq}(T+X)^{m-r-iq}.$$

Putting these and (3.9) into (3.10) yields

$$\epsilon \sum_{0\le i\le (m-r)/q} f_{m-r-iq}(T+X)^{m-r-iq}$$

$$= \sum_{0\le i\le (m-r)/q} f_{m-r-iq} X^{m-i} + \sum_{\substack{0\le k\le m \\ k\not\equiv m-r \pmod q}} f_k \left(X^k + \sum_{i=0}^{k-1} P_{k,i}(T) X^i \right).$$

Comparing the coefficients of powers of X yields the following linear system in the variables $f_0, \dots, f_m$:

$$B_{\big|y=T} \begin{pmatrix} f_m \\ \vdots \\ f_0 \end{pmatrix} = 0.$$

Here $B = (B_{ij})_{0\le i,j\le m} \in \mathrm{Mat}_{m+1}(\mathbb{F}_q(y))$ such that

- $v_y(B_{ij}) \ge 1$ if $i > j$,
- $v_y(B_{ij}) \ge 0$ if $i < j$,
- $v_y(B_{ii}) = 0$ as $B_{ii} \in \mathbb{F}_q^\times$.

The above properties follow from the fact that $P_{k,i} \in \mathbb{F}_q(y)$ and $v_y(P_{k,i}) \ge 1$. Thus $v_y(\det B) = 0$. Hence $f_0 = \dots = f_m = 0$. It follows that $\delta_{\emptyset;\emptyset} = 0$ as $\delta = 0$ and $\delta_{\mathfrak{t}',\boldsymbol{\epsilon}'} = 0$ for all $(\mathfrak{t}';\boldsymbol{\epsilon}') \in J_0(\emptyset)$. We conclude that $\delta_{\mathfrak{t},\boldsymbol{\epsilon}} = 0$ for all $(\mathfrak{t},\boldsymbol{\epsilon}) \in J$. In particular, for all i, $a_i(t) = \delta_{\mathfrak{s}_i,\boldsymbol{\epsilon}_i} = 0$, which is a contradiction. Thus this case can never happen.

3.3.2. *Case 2: $q-1 \mid w$, says $w = m(q-1)$.*

By similar arguments as above, we show that $\delta = F(t-\theta)^{(q-1)m}$ with $F \in K[t]$ of the form

$$F = \sum_{0\le i\le m/q} f_{m-iq}(t-\theta)^{m-iq}, \quad \text{for } f_{m-iq} \in \mathbb{F}_q[t].$$

Thus

$$F(t-\theta)^{(q-1)m} = \sum_{0\le i\le m/q} f_{m-iq}(t-\theta)^{mq-iq} = \sum_{0\le i\le m/q} f_{m-iq} X^{m-i},$$

$$F^{(1)} = \sum_{0\le i\le m/q} f_{m-iq}(t-\theta^q)^{m-iq} = \sum_{0\le i\le m/q} f_{m-iq}(T+X)^{m-iq}.$$

Putting these and (3.6) into (3.8) gets

$$\epsilon \sum_{0\le i\le m/q} f_{m-iq}(T+X)^{m-iq}$$
$$= \sum_{0\le i\le m/q} f_{m-iq} X^{m-i} + \sum_{\substack{0\le k< m \\ k\not\equiv m \pmod q}} f_k \left(X^k + \sum_{i=0}^{k-1} P_{k,i}(T) X^i \right).$$

Comparing the coefficients of powers of X yields

$$\epsilon f_m = f_m$$

and the following linear system in the variables $f_0, \dots, f_{m-1}$:

$$B_{\big|y=T} \begin{pmatrix} f_{m-1} \\ \vdots \\ f_0 \end{pmatrix} = f_m \begin{pmatrix} Q_{m-1} \\ \vdots \\ Q_0 \end{pmatrix}_{\big|y=T}.$$

Here for $0 \le i \le m-1$, $Q_i = \binom{m}{i} y^{m-i} \in y\mathbb{F}_q[y]$ and $B = (B_{ij})_{0 \le i,j \le m-1} \in \mathrm{Mat}_m(\mathbb{F}_q(y))$ such that

- $v_y(B_{ij}) \ge 1$ if $i > j$,
- $v_y(B_{ij}) \ge 0$ if $i < j$,
- $v_y(B_{ii}) = 0$ as $B_{ii} \in \mathbb{F}_q^\times$.

The above properties follow from the fact that $P_{k,i} \in \mathbb{F}_q(y)$ and $v_y(P_{k,i}) \ge 1$. Thus $v_y(\det B) = 0$ so that $\det B \neq 0$.

We distinguish two subcases.

Subcase 1: $\epsilon \neq 1$**.**

It follows that $f_m = 0$. Then $f_0 = \cdots = f_{m-1} = 0$. Thus $\delta_{\mathfrak{t},\boldsymbol{\epsilon}} = 0$ for all $(\mathfrak{t}, \boldsymbol{\epsilon}) \in J$. In particular, for all i, $a_i(t) = \delta_{\mathfrak{s}_i,\boldsymbol{\epsilon}_i} = 0$. This is a contradiction and we conclude that this case can never happen.

Subcase 2: $\epsilon = 1$**.**

It follows that $\gamma \in \mathbb{F}_q^\times$ and thus

1) The polynomial $\delta_{\emptyset;\emptyset} = \delta\gamma$ is of the form

$$\delta_{\emptyset;\emptyset} = f_\emptyset \left(X^m + \sum_{i=0}^{m-1} P_{\emptyset,i}(T) X^i \right)$$

with
   - $f_\emptyset \in \mathbb{F}_q[t]$,
   - for all $0 \le i \le m-1$, $P_{\emptyset,i}(y) \in \mathbb{F}_q(y)$ with $v_y(P_{\emptyset,i}) \ge 1$.

2) For all $(\mathfrak{t}', \boldsymbol{\epsilon}') \in J_0(\emptyset)$, there exists $P_{\emptyset,\mathfrak{t}'} \in \mathbb{F}_q(y)$ such that

$$f_{\mathfrak{t}'} = f_\emptyset P_{\emptyset,\mathfrak{t}'}(T).$$

Hence there exists a unique solution $(\delta_{\mathfrak{t},\boldsymbol{\epsilon}})_{(\mathfrak{t},\boldsymbol{\epsilon}) \in J} \in \mathrm{Mat}_{1 \times |J|}(K[t])$ of (3.3) up to a factor in $\mathbb{F}_q(t)$. Recall that for all i, $a_i(t) = \delta_{\mathfrak{s}_i,\boldsymbol{\epsilon}_i}$. Therefore, up to a scalar in $K^\times$, there exists at most one non trivial relation

$$a\widetilde{\pi}^w + \sum_i a_i \operatorname{Li} \begin{pmatrix} \boldsymbol{\varepsilon}_i \\ \mathfrak{s}_i \end{pmatrix} = 0$$

with $a_i \in K$ and $\operatorname{Li} \begin{pmatrix} \boldsymbol{\varepsilon}_i \\ \mathfrak{s}_i \end{pmatrix} \in \mathcal{AS}_w$. Further, we must have $\boldsymbol{\varepsilon}_i = (1, \dots, 1)$ for all i.

To conclude, it suffices to exhibit such a relation with $a \neq 0$. In fact, we recall $w = (q-1)m$ and then express $\operatorname{Li}(q-1)^m = \operatorname{Li} \begin{pmatrix} 1 \\ q-1 \end{pmatrix}^m$ as a K-linear combination of ACMPL's of weight w. By Theorem 1.9, we can write

$$\operatorname{Li}(q-1)^m = \operatorname{Li} \begin{pmatrix} 1 \\ q-1 \end{pmatrix}^m = \sum_i a_i \operatorname{Li} \begin{pmatrix} \boldsymbol{\varepsilon}_i \\ \mathfrak{s}_i \end{pmatrix} = 0, \quad \text{where } a_i \in K, \operatorname{Li} \begin{pmatrix} \boldsymbol{\varepsilon}_i \\ \mathfrak{s}_i \end{pmatrix} \in \mathcal{AS}_w.$$

We note that $\operatorname{Li}(q-1) = \zeta_A(q-1) = -D_1^{-1} \widetilde{\pi}^{q-1}$. Thus

$$(-D_1)^m \widetilde{\pi}^w - \sum_i a_i \operatorname{Li} \begin{pmatrix} \boldsymbol{\varepsilon}_i \\ \mathfrak{s}_i \end{pmatrix} = 0,$$

which is the desired relation. Hence we are done.

4. Applications on AMZV's and Zagier-Hoffman's conjectures in positive characteristic

In this section we give two applications of the study of ACMPL's.

First we apply Theorem 3.6 to prove Theorem A which calculates the dimensions of the vector space $\mathcal{AZ}_w$ of alternating multiple zeta values in positive characteristic (AMZV's) of fixed weight introduced by Harada [19]. Consequently we determine all linear relations for AMZV's. To do so we develop an algebraic theory to obtain a weak version of Brown's theorem for AMZV's. Then we deduce that $\mathcal{AZ}_w$ and $\mathcal{AL}_w$ are the same and conclude. Contrary to the setting of MZV's, although the results are neat, we are unable to obtain neither sharp upper bounds nor sharp lower bounds for $\mathcal{AZ}_w$ for general w without the theory of ACMPL's.

Second we restrict our attention to MZV's and determine all linear relations among MZV's. In particular, we obtain a proof of Zagier-Hoffman's conjectures in positive characteristic in full generality (i.e., Theorem B) and generalize the work of one of the authors [27].

4.1. **Linear relations among AMZV's.**

4.1.1. *Preliminaries.* For $d \in \mathbb{Z}$ and for $\mathfrak{s} = (s_1, \dots, s_n) \in \mathbb{N}^n$, recalling $S_d(\mathfrak{s})$ and $S_{<d}(\mathfrak{s})$ given in §1.1.2, and further letting $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} = \begin{pmatrix} \varepsilon_1 & \dots & \varepsilon_n \\ s_1 & \dots & s_n \end{pmatrix}$ be a positive array, we recall (see §1.1.2)

$$S_d \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} = \sum_{\substack{a_1, \dots, a_n \in A_+ \\ d = \deg a_1 > \dots > \deg a_n \geq 0}} \frac{\varepsilon_1^{\deg a_1} \dots \varepsilon_n^{\deg a_n}}{a_1^{s_1} \dots a_n^{s_n}} \in K$$

and

$$S_{<d} \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} = \sum_{\substack{a_1, \dots, a_n \in A_+ \\ d > \deg a_1 > \dots > \deg a_n \geq 0}} \frac{\varepsilon_1^{\deg a_1} \dots \varepsilon_n^{\deg a_n}}{a_1^{s_1} \dots a_n^{s_n}} \in K.$$

One verifies easily the following formulas:

$$\begin{aligned} S_d \begin{pmatrix} \varepsilon \\ s \end{pmatrix} &= \varepsilon^d S_d(s), \\ S_d \begin{pmatrix} 1 & \dots & 1 \\ s_1 & \dots & s_n \end{pmatrix} &= S_d(s_1, \dots, s_n), \\ S_{<d} \begin{pmatrix} 1 & \dots & 1 \\ s_1 & \dots & s_n \end{pmatrix} &= S_{<d}(s_1, \dots, s_n), \\ S_d \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} &= S_d \begin{pmatrix} \varepsilon_1 \\ s_1 \end{pmatrix} S_{<d} \begin{pmatrix} \boldsymbol{\varepsilon}_- \\ \mathfrak{s}_- \end{pmatrix}. \end{aligned}$$

Harada [19] introduced the alternating multiple zeta value (AMZV) as follows;

$$\zeta_A \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} = \sum_{d \geq 0} S_d \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} = \sum_{\substack{a_1, \dots, a_n \in A_+ \\ \deg a_1 > \dots > \deg a_n \geq 0}} \frac{\varepsilon_1^{\deg a_1} \dots \varepsilon_n^{\deg a_n}}{a_1^{s_1} \dots a_n^{s_n}} \in K_\infty.$$

Using Chen's formula (see [12]), Harada proved that for $s,t \in \mathbb{N}$ and $\varepsilon, \epsilon \in \mathbb{F}_q^\times$, we have

$$S_d\begin{pmatrix}\varepsilon\\ s\end{pmatrix} S_d\begin{pmatrix}\epsilon\\ t\end{pmatrix} = S_d\begin{pmatrix}\varepsilon\epsilon\\ s+t\end{pmatrix} + \sum_i \Delta^i_{s,t} S_d\begin{pmatrix}\varepsilon\epsilon & 1\\ s+t-i & i\end{pmatrix},$$

where

$$\Delta^i_{s,t} = \begin{cases} (-1)^{s-1}\binom{i-1}{s-1} + (-1)^{t-1}\binom{i-1}{t-1} & \text{if } q-1 \mid i \text{ and } 0<i<s+t,\\ 0 & \text{otherwise.}\end{cases} \tag{4.1}$$

He then proved similar results for products of AMZV's (see [19]):

Proposition 4.1. *Let* $\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix}$, $\begin{pmatrix}\boldsymbol{\epsilon}\\ \mathfrak{t}\end{pmatrix}$ *be two positive arrays. Then*

(1) There exist $f_i \in \mathbb{F}_p$ *and arrays* $\begin{pmatrix}\boldsymbol{\mu}_i\\ \mathfrak{u}_i\end{pmatrix}$ *with* $\begin{pmatrix}\boldsymbol{\mu}_i\\ \mathfrak{u}_i\end{pmatrix} \le \begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix} + \begin{pmatrix}\boldsymbol{\epsilon}\\ \mathfrak{t}\end{pmatrix}$ *and* $\operatorname{depth}(\mathfrak{u}_i) \le \operatorname{depth}(\mathfrak{s}) + \operatorname{depth}(\mathfrak{t})$ *for all i such that*

$$S_d\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix} S_d\begin{pmatrix}\boldsymbol{\epsilon}\\ \mathfrak{t}\end{pmatrix} = \sum_i f_i S_d\begin{pmatrix}\boldsymbol{\mu}_i\\ \mathfrak{u}_i\end{pmatrix} \quad \text{for all } d \in \mathbb{Z}.$$

(2) There exist $f'_i \in \mathbb{F}_p$ *and arrays* $\begin{pmatrix}\boldsymbol{\mu}'_i\\ \mathfrak{u}'_i\end{pmatrix}$ *with* $\begin{pmatrix}\boldsymbol{\mu}'_i\\ \mathfrak{u}'_i\end{pmatrix} \le \begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix} + \begin{pmatrix}\boldsymbol{\epsilon}\\ \mathfrak{t}\end{pmatrix}$ *and* $\operatorname{depth}(\mathfrak{u}'_i) \le \operatorname{depth}(\mathfrak{s}) + \operatorname{depth}(\mathfrak{t})$ *for all i such that*

$$S_{<d}\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix} S_{<d}\begin{pmatrix}\boldsymbol{\epsilon}\\ \mathfrak{t}\end{pmatrix} = \sum_i f'_i S_{<d}\begin{pmatrix}\boldsymbol{\mu}'_i\\ \mathfrak{u}'_i\end{pmatrix} \quad \text{for all } d \in \mathbb{Z}.$$

(3) There exist $f''_i \in \mathbb{F}_p$ *and arrays* $\begin{pmatrix}\boldsymbol{\mu}''_i\\ \mathfrak{u}''_i\end{pmatrix}$ *with* $\begin{pmatrix}\boldsymbol{\mu}''_i\\ \mathfrak{u}''_i\end{pmatrix} \le \begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix} + \begin{pmatrix}\boldsymbol{\epsilon}\\ \mathfrak{t}\end{pmatrix}$ *and* $\operatorname{depth}(\mathfrak{u}''_i) \le \operatorname{depth}(\mathfrak{s}) + \operatorname{depth}(\mathfrak{t})$ *for all i such that*

$$S_d\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix} S_{<d}\begin{pmatrix}\boldsymbol{\epsilon}\\ \mathfrak{t}\end{pmatrix} = \sum_i f''_i S_d\begin{pmatrix}\boldsymbol{\mu}''_i\\ \mathfrak{u}''_i\end{pmatrix} \quad \text{for all } d \in \mathbb{Z}.$$

We denote by $\mathcal{AZ}$ the K-vector space generated by the AMZV's and $\mathcal{AZ}_w$ the K-vector space generated by the AMZV's of weight w. It follows from Proposition 4.1 that $\mathcal{AZ}$ is a K-algebra under the multiplication of K_∞.

4.1.2. *Proof of Theorem A.* We can extend an algebraic theory for AMZV's which follow the same line as that in §1. In particular, we can for Todd's operations $\mathcal{B}^*_{\Sigma,V}$ and $\mathcal{C}_{\Sigma,V}$ for the setting of AMZV's.

Consequently, we obtain a weak version of Brown's theorem for AMZV's as follows.

Proposition 4.2. *The set of all elements* $\zeta_A\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix}$ *such that* $\zeta_A\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix} \in \mathcal{AT}_w$ *forms a set of generators for* $\mathcal{AZ}_w$*. Here we recall that* $\mathcal{AT}_w$ *is the set of all AMZV's* $\zeta_A\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix} = \operatorname{Li}\begin{pmatrix}\varepsilon_1 & \dots & \varepsilon_n\\ s_1 & \dots & s_n\end{pmatrix}$ *of weight w such that $s_1, \dots, s_{n-1} \le q$ and $s_n < q$ introduced in the paragraph preceding Proposition 1.9.*

As a direct consequence, we get

Theorem 4.3. *The K-vector space $\mathcal{AZ}_w$ of AMZV's of weight w and the K-vector space $\mathcal{AL}_w$ of ACMPL's of weight w are the same.*

By this identification we apply Theorem 3.6 to obtain Theorem A.

4.1.3. *Sharp bounds without ACMPL's.* To end this section we mention that without the theory of ACMPL's it seems very hard to obtain for arbitrary weight w

- either the sharp upper bound $\dim_K \mathcal{AZ}_w \leq s(w)$,
- or the sharp lower bound $\dim_K \mathcal{AZ}_w \geq s(w)$.

We are only able to do so for small weights by ad hoc arguments. We collect the results below, sketch some ideas of the proofs and omit long computations.

Proposition 4.4. *Let $w \leq 2q-2$. Then $\dim_K \mathcal{AZ}_w \leq s(w)$.*

Proof. We denote by $\mathcal{AT}^1_w$ the subset of AMZV's $\zeta_A \begin{pmatrix} \boldsymbol{\epsilon} \\ \mathfrak{s} \end{pmatrix}$ of $\mathcal{AT}_w$ such that $\epsilon_i = 1$ whenever $s_i = q$ (i.e., the character corresponding to q is always 1) and by $\langle \mathcal{AT}^1_w \rangle$ the K-vector space spanned by the AMZV's in $\mathcal{AT}^1_w$. We see that $|\mathcal{AT}^1_w| = s(w)$. Thus it suffices to prove that $\langle \mathcal{AT}^1_w \rangle = \mathcal{AZ}_w$.

Recall that for any $\epsilon \in \mathbb{F}_q^\times$, we recall the relation

$$R_\varepsilon\colon \quad S_d \begin{pmatrix} \varepsilon \\ q \end{pmatrix} + \varepsilon^{-1} D_1 S_{d+1} \begin{pmatrix} \varepsilon & 1 \\ 1 & q-1 \end{pmatrix} = 0,$$

where $D_1 = \theta^q - \theta$.

We recall that the coefficients $\Delta^i_{s,t}$ are given in (4.1). Let $U = (u_1, \dots, u_n)$ and $W = (w_1, \dots, w_r)$ be tuples of positive integers such that $u_n \leq q-1$ and $w_1, \dots, w_r \leq q$. Let $\boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_n) \in (\mathbb{F}_q^\times)^n$ and $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_r) \in (\mathbb{F}_q^\times)^r$. By direct calculations, we show that $\mathcal{B}_{\boldsymbol{\epsilon},U} \mathcal{C}_{\boldsymbol{\lambda},W}(R_\epsilon)$ can be written as

$$\begin{aligned}
& S_d \begin{pmatrix} \boldsymbol{\epsilon} & \epsilon & \boldsymbol{\lambda} \\ U & q & W \end{pmatrix} + S_d \begin{pmatrix} \boldsymbol{\epsilon} & \epsilon\lambda_1 & \boldsymbol{\lambda}_- \\ U & q+w_1 & W_- \end{pmatrix} \\
&+ \Delta^{q-1}_{q,w_1} S_d \begin{pmatrix} \boldsymbol{\epsilon} & \epsilon\lambda_1 & 1 & \boldsymbol{\lambda}_- \\ U & w_1+1 & q-1 & W_- \end{pmatrix} \\
&+ \Delta^{q-1}_{q,w_1} S_d \begin{pmatrix} \boldsymbol{\epsilon} & \epsilon\lambda_1 & \lambda_2 & \lambda_3 & \dots & \lambda_r \\ U & w_1+1 & q+w_2-1 & w_3 & \dots & w_r \end{pmatrix} \\
&+ \Delta^{q-1}_{q,w_1} \sum_{i=2}^{r-1} \prod_{j=2}^{i} (\Delta^{q-1}_{q-1,w_j}+1) S_d \begin{pmatrix} \boldsymbol{\epsilon} & \epsilon\lambda_1 & \lambda_2 \dots \lambda_i & \lambda_{i+1} & \lambda_{i+2} \dots \lambda_r \\ U & w_1+1 & w_2 \dots w_i & q+w_{i+1}-1 & w_{i+2} \dots w_r \end{pmatrix} \\
&+ \Delta^{q-1}_{q,w_1} \sum_{i=2}^{r} \prod_{j=2}^{i} (\Delta^{q-1}_{q-1,w_j}+1) S_d \begin{pmatrix} \boldsymbol{\epsilon} & \epsilon\lambda_1 & \lambda_2 & \dots & \lambda_i & 1 & \lambda_{i+1} & \dots & \lambda_r \\ U & w_1+1 & w_2 & \dots & w_i & q-1 & w_{i+1} & \dots & w_r \end{pmatrix} \\
&+ \epsilon^{-1} D_1 S_d \begin{pmatrix} \boldsymbol{\epsilon} & \epsilon & 1 & \boldsymbol{\lambda} \\ U & 1 & q-1 & W \end{pmatrix} + \epsilon^{-1} D_1 S_d \begin{pmatrix} \boldsymbol{\epsilon} & \epsilon & \lambda_1 & \boldsymbol{\lambda}_- \\ U & 1 & q+w_1-1 & W_- \end{pmatrix} \\
&+ \epsilon^{-1} D_1 S_d \begin{pmatrix} \epsilon_1 & \dots & \epsilon_{n-1} & \epsilon_n\epsilon & 1 & \boldsymbol{\lambda} \\ u_1 & \dots & u_{n-1} & u_n+1 & q-1 & W \end{pmatrix} \\
&+ \epsilon^{-1} D_1 S_d \begin{pmatrix} \epsilon_1 & \dots & \epsilon_{n-1} & \epsilon_n\epsilon & \lambda_1 & \boldsymbol{\lambda}_- \\ u_1 & \dots & u_{n-1} & u_n+1 & q+w_1-1 & W_- \end{pmatrix}
\end{aligned}$$

$$+\epsilon^{-1}D_1\sum_{i=1}^{r-1}\prod_{j=1}^{i}(\Delta^{q-1}_{q-1,w_j}+1)S_d\begin{pmatrix}\boldsymbol{\epsilon} & \epsilon & \lambda_1 & \dots & \lambda_i & \lambda_{i+1} & \lambda_{i+2} & \dots & \lambda_r\\ U & 1 & w_1 & \dots & w_i & q+w_{i+1}-1 & w_{i+2} & \dots & w_r\end{pmatrix}$$
$$+\epsilon^{-1}D_1\sum_{i=1}^{r}\prod_{j=1}^{i}(\Delta^{q-1}_{q-1,w_j}+1)S_d\begin{pmatrix}\boldsymbol{\epsilon} & \epsilon & \lambda_1\dots\lambda_i & 1 & \lambda_{i+1}\dots\lambda_r\\ U & 1 & w_1\dots w_i & q-1 & w_{i+1}\dots w_r\end{pmatrix}$$
$$+\epsilon^{-1}D_1\sum_{i=1}^{r}\prod_{j=1}^{i}(\Delta^{q-1}_{q-1,w_j}+1)S_d\begin{pmatrix}\epsilon_1\dots\epsilon_{n-1} & \epsilon_n\epsilon & \lambda_1\dots\lambda_i & 1 & \lambda_{i+1}\dots\lambda_r\\ u_1\dots u_{n-1} & u_n+1 & w_1\dots w_i & q-1 & w_{i+1}\dots w_r\end{pmatrix}$$
$$+\epsilon^{-1}D_1\sum_{i=1}^{r-1}\prod_{j=1}^{i}(\Delta^{q-1}_{q-1,w_j}+1)S_d\begin{pmatrix}\epsilon_1\dots\epsilon_{n-1} & \epsilon_n\epsilon & \lambda_1\dots\lambda_i & \lambda_{i+1} & \lambda_{i+2}\dots\lambda_r\\ u_1\dots u_{n-1} & u_n+1 & w_1\dots w_i & q+w_{i+1}-1 & w_{i+2}\dots w_r\end{pmatrix}$$
$$=0.$$

We denote by $(*)$ this formula.

We denote by H_r the following claim: for any tuples of positive integers U and $W=(w_1,\dots,w_r)$ of depth r, $\boldsymbol{\epsilon}\in(\mathbb{F}_q^\times)^{\operatorname{depth}U}$ of any depth, $\boldsymbol{\lambda}=(\lambda_1,\dots,\lambda_r)\in(\mathbb{F}_q^\times)^r$, and $\epsilon\in\mathbb{F}_q^\times$ such that $w(U)+w(W)+q=w$, the AMZV's $\zeta_A\begin{pmatrix}\boldsymbol{\epsilon} & \epsilon & \boldsymbol{\lambda}\\ U & q & W\end{pmatrix}$ and $\zeta_A\begin{pmatrix}\boldsymbol{\epsilon} & \epsilon & \boldsymbol{\lambda}_-\\ U & q+w_1 & W_-\end{pmatrix}$ belong to $\langle\mathcal{A}\mathcal{T}^1_w\rangle$.

We claim that H_r holds for any $r\geq 0$. The proof is by induction on r. For $r=0$, we know that $W=\emptyset$. Letting $U=(u_1,\dots,u_n)$ and $\boldsymbol{\epsilon}=(\epsilon_1,\dots,\epsilon_n)$ we apply the formula $(*)$ to get an explicit expression for $\mathcal{B}_{\boldsymbol{\epsilon},U}(R_\epsilon)$ given by

$$S_d\begin{pmatrix}\boldsymbol{\epsilon} & \epsilon\\ U & q\end{pmatrix}+\epsilon^{-1}D_1S_d\begin{pmatrix}\boldsymbol{\epsilon} & \epsilon & 1\\ U & 1 & q-1\end{pmatrix}$$
$$+\epsilon^{-1}D_1S_d\begin{pmatrix}\epsilon_1 & \dots & \epsilon_{n-1} & \epsilon_n\epsilon & 1\\ u_1 & \dots & u_{n-1} & u_n+1 & q-1\end{pmatrix}=0.$$

Since $u_i\leq w(U)=w-q\leq q-2$, we deduce that $\zeta_A\begin{pmatrix}\boldsymbol{\epsilon} & \epsilon\\ U & q\end{pmatrix}\in\langle\mathcal{A}\mathcal{T}^1_w\rangle$ as required.

Suppose that $H_{r'}$ holds for any $r'<r$. We now show that H_r holds. We proceed again by induction on w_1. For $w_1=1$, letting $U=(u_1,\dots,u_n)$ and $\boldsymbol{\epsilon}=(\epsilon_1,\dots,\epsilon_n)$ we apply the formula $(*)$ to get an explicit expression for $\mathcal{B}_{\boldsymbol{\epsilon},U}\mathcal{C}_{\boldsymbol{\lambda},W}(R_\epsilon)$. As $w(U)+w(W)=w-q\leq q-2$, by induction we deduce that all the terms except the first two ones in this expression belong to $\langle\mathcal{A}\mathcal{T}^1_w\rangle$. Thus for any $\epsilon\in\mathbb{F}_q^\times$,

$$\zeta_A\begin{pmatrix}\boldsymbol{\epsilon} & \epsilon & \boldsymbol{\lambda}\\ U & q & W\end{pmatrix}+\zeta_A\begin{pmatrix}\boldsymbol{\epsilon} & \epsilon\lambda_1 & \boldsymbol{\lambda}_-\\ U & q+1 & W_-\end{pmatrix}\in\langle\mathcal{A}\mathcal{T}^1_w\rangle. \tag{4.2}$$

We take $\epsilon=1$. As the first term lies in $\mathcal{A}\mathcal{T}^1_w$ by definition, we deduce that

$$\zeta_A\begin{pmatrix}\boldsymbol{\epsilon} & \lambda_1 & \boldsymbol{\lambda}_-\\ U & q+1 & W_-\end{pmatrix}\in\langle\mathcal{A}\mathcal{T}^1_w\rangle.$$

Thus in (4.2) we now know that the second term lies in $\langle\mathcal{A}\mathcal{T}^1_w\rangle$, which implies that

$$\zeta_A\begin{pmatrix}\boldsymbol{\epsilon} & \epsilon & \boldsymbol{\lambda}\\ U & q & W\end{pmatrix}\in\langle\mathcal{A}\mathcal{T}^1_w\rangle.$$

We suppose that H_r holds for all $W'=(w'_1,\dots,w'_r)$ such that $w'_1<w_1$. We have to show that H_r holds for all $W=(w_1,\dots,w_r)$. The proof follows the same line as

before. Letting $U=(u_1,\dots,u_n)$ and $\boldsymbol{\epsilon}=(\epsilon_1,\dots,\epsilon_n)$ we apply the formula $(*)$ to get an explicit expression for $\mathcal{B}_{\boldsymbol{\epsilon},U}\mathcal{C}_{\boldsymbol{\lambda},W}(R_\epsilon)$. As $w(U)+w(W)=w-q\le q-2$, by induction we deduce that all the terms except the first two ones in this expression belong to $\langle \mathcal{AT}^1_w\rangle$. Thus for any $\epsilon\in\mathbb{F}_q^\times$,

$$\zeta_A\begin{pmatrix}\boldsymbol{\epsilon} & \epsilon & \boldsymbol{\lambda}\\ U & q & W\end{pmatrix}+\zeta_A\begin{pmatrix}\boldsymbol{\epsilon} & \epsilon\lambda_1 & \boldsymbol{\lambda}_-\\ U & q+w_1 & W_-\end{pmatrix}\in\langle\mathcal{AT}^1_w\rangle. \tag{4.3}$$

We take $\epsilon=1$ and deduce

$$\zeta_A\begin{pmatrix}\boldsymbol{\epsilon} & \lambda_1 & \boldsymbol{\lambda}_-\\ U & q+w_1 & W_-\end{pmatrix}\in\langle\mathcal{AT}^1_w\rangle.$$

Thus in (4.3) the second term lies in $\langle\mathcal{AT}^1_w\rangle$, which implies

$$\zeta_A\begin{pmatrix}\boldsymbol{\epsilon} & \epsilon & \boldsymbol{\lambda}\\ U & q & W\end{pmatrix}\in\langle\mathcal{AT}^1_w\rangle.$$

The proof is complete. ☐

Remark 4.5. The condition $w\le 2q-2$ is essential in the previous proof as it allows us to significantly simplify the expression of $\mathcal{B}_{\boldsymbol{\epsilon},U}\mathcal{C}_{\boldsymbol{\lambda},W}(R_\epsilon)$ (see Eq. (4.3)). For $w=2q-1$ the situation is already complicated but we can manage to prove Proposition 4.4. Unfortunately, we are not be able to extend it to $w=2q$.

Proposition 4.6. *Let $w\le 3q-2$. Then $\dim_K\mathcal{AZ}_w\ge s(w)$.*

Proof. We denote by $\mathcal{AT}'_w$ the subset of AMZV's as follows.

- For $1\le w\le 2q-2$, $\mathcal{AT}'_w$ consists of $\zeta_A\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix}$ of weight w such that if we write $\mathfrak{s}=(s_1,\dots,s_r)$, then $q-1\nmid s_i$.
- For $2q-1\le w\le 3q-2$, we consider I_w the set of tuples $\mathfrak{s}=(s_1,\dots,s_r)$ of weight w where $s_i\ne q,2q-1,2q,3q-2$ for all i satisfying
  - if $s_{i+1}=q$ or $2q-1$ for some $i=1,2,\dots,r-1$, then $q-1|s_i$,
  - $\mathfrak{s}_i\ne(q-1,q-1,q)$ when $w=3q-2$,

  and define
  $$\mathcal{AT}'_w:=\left\{\zeta_A\begin{pmatrix}\boldsymbol{\epsilon}\\ \mathfrak{s}\end{pmatrix}:\mathfrak{s}\in I_w,\ \text{and } \epsilon_i=1 \text{ whenever } s_i=q\right\}.$$

Then for $w\le 3q-2$, one shows that

$$|\mathcal{AT}'_w|=s(w).$$

Further, by construction, for any $(\mathfrak{s};\boldsymbol{\epsilon})=(s_1,\dots,s_r;\epsilon_1,\dots,\epsilon_r)\in\mathbb{N}^r\times(\mathbb{F}_q^\times)^r$, if $\zeta_A\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix}\in\mathcal{AT}'_w$, then $\zeta_A\begin{pmatrix}s_1 & \dots & s_{r-1}\\ \epsilon_1 & \dots & \epsilon_{r-1}\end{pmatrix}$ belongs to $\mathcal{AT}'_{w-s_r}$. This property allows us to apply Theorem 2.4 and show by induction on $w\le 3q-2$ that the AMZV's in $\mathcal{AT}'_w$ are all linearly independent over K. The proof is similar to that of Theorem 3.4. We apply Theorem 2.4 and reduce to solve a system of σ-linear equations. By direct but complicated calculations, we show that there does not exist non trivial solutions and we are done. ☐

Remark 4.7. 1) We note that the MZV's $\zeta_A(1,2q-2)$ and $\zeta_A(2q-1)$ (resp. $\zeta_A(1,3q-3)$ and $\zeta_A(3q-2)$) are linearly dependent over K. This explains the above ad hoc construction of $\mathcal{AT}'_w$.

2) Despite extensive numerical experiments, we cannot find a suitable basis $\mathcal{AT}'_w$ for the case $w = 3q - 1$.

Consequently, by combining the previous results we obtain

- for $w \leq 2q - 1$ a basis $\mathcal{AT}^1_w$ of $\mathcal{AZ}_w$ consisting of AMZV's of weight w (see Remark 4.5);
- for $w \leq 3q-2$ another basis $\mathcal{AT}'_w$ of $\mathcal{AZ}_w$ consisting of AMZV's of weight w;
- a direct proof of Theorem A for $w \leq 2q - 1$, which generalizes [27, Theorem D].

4.2. **Zagier-Hoffman's conjectures in positive characteristic.**

4.2.1. *Known results.* We use freely the notation introduced in §0.2.1. We recall that for $w \in \mathbb{N}$, $\mathcal{Z}_w$ denotes the K-vector space spanned by the MZV's of weight w and $\mathcal{T}_w$ denotes the set of $\zeta_A(\mathfrak{s})$ where $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ of weight w with $1 \leq s_i \leq q$ for $1 \leq i \leq r-1$ and $s_r < q$.

Recall that the main results of [27] states that

- For all $w \in \mathbb{N}$ we always have $\dim_K \mathcal{Z}_w \leq d(w)$ (see [27, Theorem A]).
- For $w \leq 2q - 2$ we have $\dim_K \mathcal{Z}_w \geq d(w)$ (see [27, Theorem B]). In particular, Conjecture 0.7 holds for $w \leq 2q - 2$ (see [27, Theorem D]).

However, as stated in [27, Remark 6.3] it would be very difficult to extend the method of [27] for general weights.

As an application of our main results, we present a proof of Theorem B which settles both Conjectures 0.6 and 0.7.

4.2.2. *Proof of Theorem B.* As we have already known the sharp upper bound for $\mathcal{Z}_w$ (see [27, Theorem A]), Theorem B follows immediately from the following proposition.

Proposition 4.8. *For all $w \in \mathbb{N}$ we have $\dim_K \mathcal{Z}_w \geq d(w)$.*

Proof. We denote by $\mathcal{S}_w$ the set of MZV's consisting of $\mathrm{Li}(s_1, \dots, s_r)$ of weight w with $q \nmid s_i$ for all i. Then $\mathcal{S}_w$ is a subset of $\mathcal{AS}_w$ and belong to $\mathcal{Z}_w$ by Theorem 4.3. Further, by §1.4.1, $|\mathcal{S}_w| = d(w)$. By Theorem 3.4 we deduce that elements in $\mathcal{S}_w$ are all linearly independent over K. Therefore, $\dim_K \mathcal{Z}_w \geq |\mathcal{S}_w| = d(w)$. □

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