Meromorphic nearby cycle functors and monodromies of meromorphic functions (with Appendix by T. Saito) *

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Abstract

We introduce meromorphic nearby cycle functors and study their functorial properties. Moreover we apply them to monodromies of meromorphic functions in various situations. Combinatorial descriptions of their reduced Hodge spectra and Jordan normal forms will be obtained.

1 Introduction

In [15] Gusein-Zade, Luengo and Melle-Hernández generalized Milnor's fibration theorem to meromorphic functions and defined their Milnor fibers. Moreover they obtained a formula for their monodromy zeta functions. Since then many authors studied Milnor fibers of meromorphic functions (see e.g. [3], [4], [12], [24], [32], [38], [41] [46], [50] etc.). However, in contrast to Milnor fibers of holomorphic functions, the geometric structures of those of meromorphic functions look much more complicated. For example, Milnor [31] proved that if a holomorphic function has an isolated singular point then the Milnor fiber at it has the homotopy type of a bouquet of some spheres. This implies that its reduced cohomology groups are concentrated in the middle dimension. To the best of our knowledge, we do not know so far such a nice structure theorem for Milnor fibers of meromorphic functions. For this reason, we cannot know any property of each Milnor monodromy operator of a meromorphic function even if we have a formula for its monodromy zeta function. We have also a serious obstruction that the theory of nearby and vanishing cycle functors for meromorphic functions is not fully developed yet. Indeed, nowadays the corresponding functors for holomorphic functions are not only indispensable for the study of Milnor monodromies but also very useful in many fields of mathematics.

In this paper, we overcome the above-mentioned problems partially by laying a foundation of the theory of nearby cycle functors for meromorphic functions. In particular, we prove that they preserve the perversity as in the holomorphic case. Then we apply our new algebraic machineries to Milnor monodromies of meromorphic functions. In this

way, we obtain various new results on them especially for the eigenvalues $\lambda \neq 1$. In order to describe our results more precisely, from now we prepare some notations. Let X be a complex manifold and $P(x), Q(x)$ holomorphic functions on it. Assume that $Q(x)$ is not identically zero on each connected component of X. Then we define a meromorphic function $f(x)$ on X by

$$f(x) = \frac{P(x)}{Q(x)} \qquad (x \in X). \tag{1.1}$$

Let us set $I(f) = P^{-1}(0) \cap Q^{-1}(0) \subset X$. If P and Q are coprime in the local ring $\mathcal{O}_{X,x}$ at a point $x \in X$, then $I(f)$ is nothing but the set of the indeterminacy points of f on a neighborhood of x. Note that the set $I(f)$ depends on the pair $(P(x), Q(x))$ of holomorphic functions representing $f(x)$. For example, if we take a holomorphic function $R(x)$ on X (which is not identically zero on each connected component of X) and set

$$g(x) = \frac{P(x)R(x)}{Q(x)R(x)} \qquad (x \in X), \tag{1.2}$$

then the set $I(g) = I(f) \cup R^{-1}(0)$ might be bigger than $I(f)$. In this way, we distinguish $f(x) = \frac{P(x)}{Q(x)}$ from $g(x) = \frac{P(x)R(x)}{Q(x)R(x)}$ even if their values coincide over an open dense subset of X. This is the convention due to Gusein-Zade, Luengo and Melle-Hernández [15] etc. Now we recall the following fundamental theorem due to [15].

Theorem 1.1. *(Gusein-Zade, Luengo and Melle-Hernández [15]) For any point $x \in P^{-1}(0)$ there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ and the open ball $B(x; \varepsilon) \subset X$ of radius $\varepsilon > 0$ with center at x (in a local chart of X) the restriction*

$$B(x; \varepsilon) \setminus Q^{-1}(0) \longrightarrow \mathbb{C} \tag{1.3}$$

of $f : X \setminus Q^{-1}(0) \longrightarrow \mathbb{C}$ is a locally trivial fibration over a sufficiently small punctured disk in $\mathbb{C}$ with center at the origin $0 \in \mathbb{C}$

We call the fiber in this theorem the Milnor fiber of the meromorphic function $f(x) = \frac{P(x)}{Q(x)}$ at $x \in P^{-1}(0)$ and denote it by F_x. As in the holomorphic case, we obtain also its Milnor monodromy operators

$$\Phi_{j,x} : H^j(F_x; \mathbb{C}) \xrightarrow{\sim} H^j(F_x; \mathbb{C}) \qquad (j \in \mathbb{Z}). \tag{1.4}$$

Then we define the monodromy zeta function $\zeta_{f,x}(t) \in \mathbb{C}(t)$ of f at $x \in P^{-1}(0)$ by

$$\zeta_{f,x}(t) = \prod_{j \in \mathbb{Z}} \left\{ \det(\mathrm{id} - t\Phi_{j,x}) \right\}^{(-1)^j} \in \mathbb{C}(t). \tag{1.5}$$

In [15] Gusein-Zade, Luengo and Melle-Hernández obtained a formula which expresses $\zeta_{f,x}(t) \in \mathbb{C}(t)$ in terms of the Newton polyhedra of P and Q at x (for the details, see Theorem 3.9 below). However it is not possible to deduce any property of each monodromy operator $\Phi_{j,x}$ from it. From now, we shall explain how we can overcome this problem for the eigenvalues $\lambda \neq 1$ of $\Phi_{j,x}$. First we extend the classical notion of nearby cycle functors to meromorphic functions as follows (see also Raibaut [41] for a similar but sligthly different approach to them). Denote by $\mathbf{D}^b(X)$ the derived category whose objects

are bounded complexes of sheaves of $\mathbb{C}_X$-modules on X. For the meromorphic function $f(x) = \frac{P(x)}{Q(x)}$ let

$$i_f : X \setminus Q^{-1}(0) \hookrightarrow X \times \mathbb{C}_t \qquad (1.6)$$

be the (not necessarily) closed embedding defined by $x \mapsto (x, f(x))$. Let $t : X \times \mathbb{C} \to \mathbb{C}$ be the second projection. Then for $\mathcal{F} \in \mathbf{D}^b(X)$ we set

$$\psi_f^{\mathrm{mero}}(\mathcal{F}) := \psi_t(Ri_{f*}(\mathcal{F}|_{X \setminus Q^{-1}(0)})) \in \mathbf{D}^b(X). \qquad (1.7)$$

We call $\psi_f^{\mathrm{mero}}(\mathcal{F})$ the meromorphic nearby cycle sheaf of $\mathcal{F}$ along f. Then as in the holomorphic case, for any point $x \in P^{-1}(0)$ and $j \in \mathbb{Z}$ we have an isomorphism (see Lemma 2.1)

$$H^j \psi_f^{\mathrm{mero}}(\mathcal{F})_x \simeq H^j(F_x; \mathcal{F}). \qquad (1.8)$$

Moreover we will show (see Theorem 2.2) that the functor

$$\psi_f^{\mathrm{mero}}(\cdot) : \mathbf{D}^b(X) \longrightarrow \mathbf{D}^b(X) \qquad (1.9)$$

preserves the constructibility and the perversity (up to some shift). Then thanks to the perversity, we obtain the following theorem. The problem being local, we may assume that $X = \mathbb{C}^n$ and $0 \in I(f) = P^{-1}(0) \cap Q^{-1}(0)$. For $j \in \mathbb{Z}$ and $\lambda \in \mathbb{C}$ we denote by

$$H^j(F_0; \mathbb{C})_\lambda \subset H^j(F_0; \mathbb{C}) \qquad (1.10)$$

the generalized eigenspace of $\Phi_{j,0}$ for the eigenvalue λ.

Theorem 1.2. *Assume that the hypersurfaces $P^{-1}(0)$ and $Q^{-1}(0)$ of $X = \mathbb{C}^n$ have an isolated singular point at the origin $0 \in X = \mathbb{C}^n$ and intersect transversally on $X \setminus \{0\}$. Then for any $\lambda \neq 1$ we have the concentration*

$$H^j(F_0; \mathbb{C})_\lambda \simeq 0 \qquad (j \neq n-1). \qquad (1.11)$$

Combining the formula for $\zeta_{f,0}(t) \in \mathbb{C}(t)$ in [15] (see Theorem 3.9 below) with our Theorem 1.2, we obtain a formula for the multiplicities of the eigenvalues $\lambda \neq 1$ in $\Phi_{n-1,0}$. It seems that there is some geometric background on F_0 (like Milnor's celebrated bouquet decomposition theorem in [31]) for Theorem 1.2 to hold. It would be an interesting problem to know it and reprove Theorem 1.2 in a purely geometric manner. From now on, we assume also that $f(x) = \frac{P(x)}{Q(x)}$ is a rational function. To obtain also a formula for the Jordan normal form of its monodromy $\Phi_{n-1,0}$ as in Matsui-Takeuchi [30], Stapledon [47] and Saito [43], we have to assume moreover that f is polynomial-like in the following sense.

Definition 1.3. We say that the rational function $f(x) = \frac{P(x)}{Q(x)}$ is polynomial-like if there exists a resolution $\pi_0 : \widetilde{X} \to X = \mathbb{C}^n$ of singularities of $P^{-1}(0) \cup Q^{-1}(0)$ which induces an isomorphism $\widetilde{X} \setminus \pi_0^{-1}(\{0\}) \xrightarrow{\sim} X \setminus \{0\}$ such that for any irreducible component D_i of the (exceptional) normal crossing divisor $D = \pi_0^{-1}(\{0\})$ we have the condition

$$\mathrm{ord}_{D_i}(P \circ \pi_0) > \mathrm{ord}_{D_i}(Q \circ \pi_0). \qquad (1.12)$$

For typical examples of polynomial-like $f(x) = \frac{P(x)}{Q(x)}$, see Definition 5.13. If $f(x)$ is non-degenerate and satisfies the condition in it, then by a toric modification $\pi_0 \colon \widetilde{X} \to X = \mathbb{C}^n$ of $X = \mathbb{C}^n$ we can check that it is polynomial-like in the above sense. Note that in the study of fibrations of mixed functions (of type $f\bar{g}$) recently Oka [37] introduced a similar condition and called it the multiplicity condition. Then we obtain the following result.

Theorem 1.4. *In the situation of Theorem 1.2, assume also that f is polynomial-like. Then for any $\lambda \neq 1$ the weight filtration of the mixed Hodge structure of $H^{n-1}(F_0; \mathbb{C})_\lambda$ is the monodromy weight filtration of the Milnor monodromy $\Phi_{n-1,0} \colon H^{n-1}(F_0; \mathbb{C})_\lambda \xrightarrow{\sim} H^{n-1}(F_0; \mathbb{C})_\lambda$ centered at $n-1$.*

By this theorem, forgetting the eigenvalue 1 parts of the Milnor monodromies $\Phi_{j,0}$ we can define (see Definition 5.11) the reduced Hodge spectrum $\widetilde{\mathrm{sp}}_{f,0}(t)$ of the Milnor fiber F_0 which satisfies the symmetry

$$\widetilde{\mathrm{sp}}_{f,0}(t) = t^n \cdot \widetilde{\mathrm{sp}}_{f,0}\Big(\frac{1}{t}\Big) \tag{1.13}$$

centered at $\frac{n}{2}$. Moreover, we define the motivic zeta function of the rational function f and the motivic Milnor fiber as its limit. In fact, in [41] Raibaut has defined the same objects earlier, but our proofs are different from his ones. In the last half of Section 5, we also give more explicit formulas in terms of Newton polyhedrons. See Section 5 for the details. Then, assuming also that f is non-degenerate at the origin $0 \in X = \mathbb{C}^n$ and $P(x), Q(x)$ are convenient, we obtain combinatorial descriptions of the reduced Hodge spectrum $\widetilde{\mathrm{sp}}_{f,0}(t)$ of F_0 and the Jordan normal forms of $\Phi_{n-1,0}$ for the eigenvalues $\lambda \neq 1$ as in Esterov-Takeuchi [10], Matsui-Takeuchi [30], Stapledon [47] and Saito [43]. See Section 6 for the details. We can also globalize these results and obtain similar formulas for monodromies at infinity of the rational function $f(x) = \frac{P(x)}{Q(x)}$. They are natural generalizations of the results in Libgober-Sperber [25], Matsui-Takeuchi [28], [29], Stapledon [47] and Takeuchi-Tibăr [49]. See Section 7 for the details.

2   Meromorphic nearby cycle functors

In this section, we introduce meromorphic nearby cycle functors and study their functorial properties. In this paper we essentially follow the terminology of [9], [18] and [19]. Let k

be an arbitrary field and for a topological space X denote by $\mathbf{D}^{\mathrm{b}}(X)$ the derived category whose objects are bounded complexes of sheaves of k_X-modules on X. If X is a complex manifold, we denote by $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(X)$ the full subcategory of $\mathbf{D}^{\mathrm{b}}(X)$ consisting of constructible objects. Let X, $P(x)$, $Q(x)$ and $f(x) = \frac{P(x)}{Q(x)}$ etc. be as in Section 1 and for $\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(X)$ define the meromorphic nearby cycle sheaf $\psi_f^{\mathrm{mero}}(\mathcal{F}) \in \mathbf{D}^{\mathrm{b}}(X)$ of $\mathcal{F}$ along f as before. Then we have the following results.

Lemma 2.1. (i) *The support of $\psi_f^{\mathrm{mero}}(\mathcal{F})$ is contained in $P^{-1}(0)$.*

(ii) *There exists an isomorphism*

$$\psi_f^{\mathrm{mero}}(\mathcal{F}) \xrightarrow{\sim} \psi_f^{\mathrm{mero}}(R\Gamma_{X \setminus (P^{-1}(0) \cup Q^{-1}(0))}(\mathcal{F})). \tag{2.1}$$

(iii) *For any point $x \in P^{-1}(0)$ and $j \in \mathbb{Z}$ we have an isomorphism*

$$H^j \psi_f^{\mathrm{mero}}(\mathcal{F})_x \simeq H^j(F_x; \mathcal{F}) \tag{2.2}$$

compatible with the monodromy automorphisms on the both sides.

Proof. The assertion (i) is trivial. For the proof of (ii) it suffices to show

$$\psi_f^{\mathrm{mero}}(R\Gamma_{P^{-1}(0) \cup Q^{-1}(0)}(\mathcal{F})) \simeq 0. \tag{2.3}$$

But this follows from (iii). Let us prove (iii). The problem being local, we may assume that $X = \mathbb{C}^n$ and x is the origin $0 \in P^{-1}(0) \subset X = \mathbb{C}^n$. We denote by $F_{t,0}$ the (usual) Milnor fiber of the projection $t \colon X \times \mathbb{C}_t \to \mathbb{C}_t$ at $0 \in X = \mathbb{C}^n$. We will see later that $Ri_{f*}(\mathcal{F}|_{X \setminus Q^{-1}(0)})$ is constructible (see the proof of Theorem 2.2). Therefore, by the basic fact for the nearby cycle functors (see e.g. [9, Proposition 4.2.2]), we have an isomorphism

$$H^j \psi_t(Ri_{f*}(\mathcal{F}|_{X \setminus Q^{-1}(0)}))_0 \simeq H^j(F_{t,0}; Ri_{f*}(\mathcal{F}|_{X \setminus Q^{-1}(0)})) \tag{2.4}$$

which is compatible with the monodromy automorphisms on the both sides. Consider a Whitney stratification of $X \times \mathbb{C}$ adapted to $Ri_{f*}(\mathcal{F}|_{X \setminus Q^{-1}(0)})$ and refining the partition $X \times \mathbb{C} = i_f(X \setminus Q^{-1}(0)) \sqcup (X \times \mathbb{C}) \setminus i_f(X \setminus Q^{-1}(0))$. Then, for any $t \in \mathbb{C}^*$ such that $0 < |t| \ll 1$ the hyperplane $H_t = X \times \{t\} \subset X \times \mathbb{C}$ of $X \times \mathbb{C}$ intersects the strata in it transversally on a neighborhood of $F_{t,0} \subset H_t$ (see e.g. [26, Proposition 1.3]). This implies that for the inclusion map

$$i_{f,t} \colon i_f^{-1}(H_t) \simeq i_f(X \setminus Q^{-1}(0)) \cap H_t \hookrightarrow H_t \tag{2.5}$$

we have an isomorphism

$$\{Ri_{f*}(\mathcal{F}|_{X \setminus Q^{-1}(0)})\}|_{H_t} \simeq R(i_{f,t})_*(\mathcal{F}|_{i_f^{-1}(H_t)}) \tag{2.6}$$

on the open subset $F_{t,0} \subset H_t$ of H_t. Since we have $i_{f,t}^{-1}(F_{t,0}) = F_0$, the cohomology group $H^j(F_{t,0}; Ri_{f*}(\mathcal{F}|_{X \setminus Q^{-1}(0)}))$ is isomorphic to $H^j(F_0; \mathcal{F})$. $\qquad \square$

From now on, we shall prove that the functor

$$\psi_f^{\mathrm{mero}}(\cdot) : \mathbf{D}^{\mathrm{b}}(X) \longrightarrow \mathbf{D}^{\mathrm{b}}(X) \tag{2.7}$$

thus defined preserves the constructibility and the perversity (up to some shift). For the closed embedding

$$k_f : X \setminus Q^{-1}(0) \hookrightarrow (X \setminus Q^{-1}(0)) \times \mathbb{C}_t \tag{2.8}$$

defined by $x \mapsto (x, f(x))$ and the inclusion map $j_f : (X \setminus Q^{-1}(0)) \times \mathbb{C}_t \hookrightarrow X \times \mathbb{C}_t$ we have $i_f = j_f \circ k_f$ and hence an isomorphism

$$Ri_{f*}(\mathcal{F}|_{X \setminus Q^{-1}(0)}) \simeq Rj_{f*}(Rk_{f*}(\mathcal{F}|_{X \setminus Q^{-1}(0)})). \tag{2.9}$$

Moreover, if $\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(X)$ is constructible (resp. perverse), then $Rk_{f*}(\mathcal{F}|_{X \setminus Q^{-1}(0)}) \in \mathbf{D}^{\mathrm{b}}((X \setminus Q^{-1}(0)) \times \mathbb{C}_t)$ is constructible (resp. perverse). However the functor

$$Rj_{f*} : \mathbf{D}^{\mathrm{b}}((X \setminus Q^{-1}(0)) \times \mathbb{C}_t) \longrightarrow \mathbf{D}^{\mathrm{b}}(X \times \mathbb{C}_t) \tag{2.10}$$

does not preserve the constructibility (resp. perversity) in general. Nevertheless, we can overcome this difficulty as follows.

Theorem 2.2. (i) *If $\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(X)$ is constructible, then $\psi_f^{\mathrm{mero}}(\mathcal{F}) \in \mathbf{D}^{\mathrm{b}}(X)$ is also constructible.*

(ii) *If $\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(X)$ is perverse, then $\psi_f^{\mathrm{mero}}(\mathcal{F})[-1] \in \mathbf{D}^{\mathrm{b}}(X)$ is also perverse.*

Proof. Assume that $\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(X)$ is constructible. Define a hypersurface W of $X \times \mathbb{C}_t$ by

$$W = \{(x, t) \in X \times \mathbb{C} \mid P(x) - tQ(x) = 0\} \tag{2.11}$$

and let $\rho : W \to X$ be the restriction of the first projection $X \times \mathbb{C}_t \to X$ to it. Then ρ induces an isomorphism

$$\rho^{-1}(X \setminus Q^{-1}(0)) \xrightarrow{\sim} X \setminus Q^{-1}(0) \tag{2.12}$$

and $\rho^{-1}(X \setminus Q^{-1}(0))$ is nothing but the graph

$$\{(x, f(x)) \in (X \setminus Q^{-1}(0)) \times \mathbb{C} \mid x \in X \setminus Q^{-1}(0)\} \tag{2.13}$$

of $f : X \setminus Q^{-1}(0) \to \mathbb{C}$. In this way, we identify $X \setminus Q^{-1}(0)$ and the open subset $\rho^{-1}(X \setminus Q^{-1}(0))$ of W. Let

$$\iota_f : X \setminus Q^{-1}(0) \simeq \rho^{-1}(X \setminus Q^{-1}(0)) \hookrightarrow W \tag{2.14}$$

and $i_W : W \hookrightarrow X \times \mathbb{C}_t$ be the inclusion maps. Then for the constructible sheaf $\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(X)$ we have an isomorphism

$$Ri_{f*}(\mathcal{F}|_{X \setminus Q^{-1}(0)}) \simeq i_{W*}(R\iota_{f*}(\mathcal{F}|_{X \setminus Q^{-1}(0)})). \tag{2.15}$$

Moreover, by the Cartesian diagram

$$\begin{array}{ccc}
X \setminus Q^{-1}(0) & \xrightarrow{\;\iota_f\;} & W \\
{\scriptstyle k_f}\downarrow & & \downarrow{\scriptstyle i_W} \\
(X \setminus Q^{-1}(0)) \times \mathbb{C} & \xrightarrow{\;j_f\;} & X \times \mathbb{C}
\end{array} \tag{2.16}$$

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we obtain isomorphisms

$$Rj_{f*}j_f^{-1}(i_{W*}\rho^{-1}\mathcal{F}) \simeq Rj_{f*}k_{f*}\iota_f^{-1}\rho^{-1}\mathcal{F}$$
$$\simeq i_{W*}(R\iota_{f*}(\mathcal{F}|_{X\setminus Q^{-1}(0)})). \tag{2.17}$$

Then we obtain the constructibility of

$$Ri_{f*}(\mathcal{F}|_{X\setminus Q^{-1}(0)}) \simeq Rj_{f*}j_f^{-1}(i_{W*}\rho^{-1}\mathcal{F})$$
$$\simeq R\mathcal{H}om_{\mathbb{C}_{X\times\mathbb{C}}}(\mathbb{C}_{(X\setminus Q^{-1}(0))\times\mathbb{C}}, i_{W*}\rho^{-1}\mathcal{F}) \tag{2.18}$$

by [19, Theorem 8.5.7 (ii)]. If moreover $\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(X)$ is perverse, then the perversity of $Rj_{f*}j_f^{-1}(i_{W*}\rho^{-1}\mathcal{F}) \in \mathbf{D}_{\mathrm{c}}^{\mathrm{b}}(X \times \mathbb{C}_t)$ follows from [19, Proposition 10.3.17 (i)] on the Stein map $j_f : (X \setminus Q^{-1}(0)) \times \mathbb{C}_t \hookrightarrow X \times \mathbb{C}_t$ (see also the paragraph below [9, Corollary 5.2.17] and Schürmann [45, page 410]). Finally, by applying the t-exact functor $\psi_t(\cdot)[-1] : \mathbf{D}_{\mathrm{c}}^{\mathrm{b}}(X \times \mathbb{C}_t) \longrightarrow \mathbf{D}_{\mathrm{c}}^{\mathrm{b}}(X)$ to it, we obtain the assertions. $\square$

By this theorem we obtain a functor

$$\psi_f^{\mathrm{mero}}(\cdot) : \mathbf{D}_{\mathrm{c}}^{\mathrm{b}}(X) \longrightarrow \mathbf{D}_{\mathrm{c}}^{\mathrm{b}}(X). \tag{2.19}$$

Moreover by its proof, for $\mathcal{F} \in \mathbf{D}_{\mathrm{c}}^{\mathrm{b}}(X)$ there exists a natural morphism

$$\mathcal{F}_{P^{-1}(0)} \longrightarrow \psi_f^{\mathrm{mero}}(\mathcal{F}). \tag{2.20}$$

REMARK 2.3. Assume that the meromorphic function $f(x) = \frac{P(x)}{Q(x)}$ is holomorphic on a neighborhood of a point $x \in X$ i.e. there exists a holomorphic function $g(x)$ defined on a neighborhood of $x \in X$ such that $P(x) = Q(x) \cdot g(x)$ on it. Then by identifying $f(x)$ with the holomorphic function $g(x)$, we have an isomorphism

$$\psi_f^{\mathrm{mero}}(\mathcal{F})_x \simeq \psi_f(R\Gamma_{X\setminus Q^{-1}(0)}(\mathcal{F}))_x \tag{2.21}$$

for the classical (holomorphic) nearby cycle functor $\psi_f(\cdot)$. This implies that even if $f(x)$ is holomorphic on a neighborhood of $x \in X$ we do not have an isomorphism

$$\psi_f^{\mathrm{mero}}(\mathcal{F})_x \simeq \psi_f(\mathcal{F})_x \tag{2.22}$$

in general.

The following useful result is an analogue for $\psi_f^{\mathrm{mero}}(\cdot)$ of the classical one for $\psi_f(\cdot)$ (see e.g. [9, Proposition 4.2.11] and [19, Exercise VIII.15] etc.).

Proposition 2.4. *Let $\pi : Y \longrightarrow X$ be a proper morphism of complex manifolds and $f \circ \pi$ a meromorphic function on Y defined by*

$$f \circ \pi = \frac{P \circ \pi}{Q \circ \pi}. \tag{2.23}$$

Then for $\mathcal{G} \in \mathbf{D}^{\mathrm{b}}(Y)$ there exists an isomorphism

$$\psi_f^{\mathrm{mero}}(R\pi_*\mathcal{G}) \simeq R\pi_*\psi_{f\circ\pi}^{\mathrm{mero}}(\mathcal{G}). \tag{2.24}$$

If moreover π induces an isomorphism

$$Y \setminus \pi^{-1}(P^{-1}(0) \cup Q^{-1}(0)) \xrightarrow{\sim} X \setminus (P^{-1}(0) \cup Q^{-1}(0)), \tag{2.25}$$

then for $\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(X)$ there exists an isomorphism

$$\psi_f^{\mathrm{mero}}(\mathcal{F}) \simeq R\pi_*\psi_{f\circ\pi}^{\mathrm{mero}}(\pi^{-1}\mathcal{F}). \tag{2.26}$$

Proof. Let $\pi^\circ : Y \setminus (Q \circ \pi)^{-1}(0) \to X \setminus Q^{-1}(0)$ be the restriction of π to $Y \setminus (Q \circ \pi)^{-1}(0)$. Then by the commutative diagram

$$
\begin{array}{ccc}
Y \setminus (Q \circ \pi)^{-1}(0) & \xrightarrow{\ i_{f \circ \pi}\ } & Y \times \mathbb{C}_t \\[2pt]
{\scriptstyle \pi^\circ}\Big\downarrow & & \Big\downarrow{\scriptstyle \pi \times \mathrm{id}_{\mathbb{C}_t}} \\[2pt]
X \setminus Q^{-1}(0) & \xrightarrow[\ i_f\ ]{} & X \times \mathbb{C}_t
\end{array}
\tag{2.27}
$$

for $\mathcal{G} \in \mathbf{D}^{\mathrm{b}}(Y)$ there exist isomorphisms

$$
\begin{aligned}
Ri_{f*}(R\pi_* \mathcal{G}|_{X \setminus Q^{-1}(0)}) &\simeq Ri_{f*}(R\pi_*^\circ(\mathcal{G}|_{Y \setminus (Q \circ \pi)^{-1}(0)})) \\
&\simeq R(\pi \times \mathrm{id}_{\mathbb{C}_t})_* Ri_{f \circ \pi *}(\mathcal{G}|_{Y \setminus (Q \circ \pi)^{-1}(0)}).
\end{aligned}
\tag{2.28}
$$

Then by [9, Proposition 4.2.11] and [19, Exercise VIII.15] we obtain isomorphisms

$$
\begin{aligned}
\psi_f^{\mathrm{mero}}(R\pi_* \mathcal{G}) &= \psi_t(Ri_{f*}(R\pi_* \mathcal{G}|_{X \setminus Q^{-1}(0)})) \\
&\simeq \psi_t(R(\pi \times \mathrm{id}_{\mathbb{C}_t})_* Ri_{f \circ \pi *}(\mathcal{G}|_{Y \setminus (Q \circ \pi)^{-1}(0)})) \\
&\simeq R\pi_* \psi_t(Ri_{f \circ \pi *}(\mathcal{G}|_{Y \setminus (Q \circ \pi)^{-1}(0)})) = R\pi_* \psi_{f \circ \pi}^{\mathrm{mero}}(\mathcal{G}).
\end{aligned}
\tag{2.29}
$$

Assume now that π induces an isomorphism

$$
Y \setminus \pi^{-1}(P^{-1}(0) \cup Q^{-1}(0)) \xrightarrow{\ \sim\ } X \setminus (P^{-1}(0) \cup Q^{-1}(0)).
\tag{2.30}
$$

Then for $\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(X)$ we have an isomorphism

$$
R\Gamma_{X \setminus (P^{-1}(0) \cup Q^{-1}(0))}(\mathcal{F}) \xrightarrow{\ \sim\ } R\Gamma_{X \setminus (P^{-1}(0) \cup Q^{-1}(0))}(R\pi_* \pi^{-1}\mathcal{F}).
\tag{2.31}
$$

By Lemma 2.1 (ii) we thus obtain isomorphisms

$$
\begin{aligned}
\psi_f^{\mathrm{mero}}(\mathcal{F}) &\simeq \psi_f^{\mathrm{mero}}(R\Gamma_{X \setminus (P^{-1}(0) \cup Q^{-1}(0))}(\mathcal{F})) \\
&\simeq \psi_f^{\mathrm{mero}}(R\pi_* \pi^{-1}\mathcal{F}) \simeq R\pi_* \psi_{f \circ \pi}^{\mathrm{mero}}(\pi^{-1}\mathcal{F}).
\end{aligned}
\tag{2.32}
$$

This completes the proof. $\qquad\square$

For the meromorphic function $f(x) = \frac{P(x)}{Q(x)}$ and $\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(X)$ we set also

$$
\psi_f^{\mathrm{mero,c}}(\mathcal{F}) := \psi_t(Ri_{f!}(\mathcal{F}|_{X \setminus Q^{-1}(0)})) \in \mathbf{D}^{\mathrm{b}}(X).
\tag{2.33}
$$

(see Raibaut [41]). We call it the meromorphic nearby cycle sheaf with compact support of $\mathcal{F}$ along f. Then we obtain a functor

$$
\psi_f^{\mathrm{mero,c}}(\cdot) : \mathbf{D}^{\mathrm{b}}(X) \longrightarrow \mathbf{D}^{\mathrm{b}}(X)
\tag{2.34}
$$

which satisfies the properties similar to the ones in Lemma 2.1 (i) (ii), Theorem 2.2 and Proposition 2.4. Moreover if f is holomorphic on a neighborhood of a point $x \in X$, then we have an isomorphism

$$
\psi_f^{\mathrm{mero,c}}(\mathcal{F})_x \simeq \psi_f(\mathcal{F}_{X \setminus Q^{-1}(0)})_x
\tag{2.35}
$$

for the classical (holomorphic) nearby cycle functor $\psi_f(\cdot)$. However the isomorphism in Lemma 2.1 (iii) does not hold for $\psi_f^{\mathrm{mero,c}}(\mathcal{F})$. This implies that the natural morphism

$$\psi_f^{\mathrm{mero,c}}(\mathcal{F}) \longrightarrow \psi_f^{\mathrm{mero}}(\mathcal{F}) \tag{2.36}$$

is not an isomorphism in general. In fact, in [41] Raibaut introduced a functor which is identical to our $\psi_f^{\mathrm{mero,c}}(\cdot)$.

From now on, we restrict ourselves to the case $k = \mathbb{C}$. For a point $x \in X$ and $\mathcal{F} \in \mathbf{D}_c^b(X)$ let

$$\Phi(\mathcal{F})_{j,x} : H^j \psi_f^{\mathrm{mero}}(\mathcal{F})_x \xrightarrow{\sim} H^j \psi_f^{\mathrm{mero}}(\mathcal{F})_x \qquad (j \in \mathbb{Z}) \tag{2.37}$$

be the monodromy automorphisms of $H^j \psi_f^{\mathrm{mero}}(\mathcal{F})$ at x. We define the monodromy zeta function $\zeta(\mathcal{F})_{f,x}(t) \in \mathbb{C}(t)$ of f for $\mathcal{F}$ at $x \in X$ by

$$\zeta(\mathcal{F})_{f,x}(t) = \prod_{j\in\mathbb{Z}}\Big\{\det(\mathrm{id} - t\Phi(\mathcal{F})_{j,x})\Big\}^{(-1)^j} \in \mathbb{C}(t). \tag{2.38}$$

Then we obtain the following sheaf-theoretical reformulation of [15, Theorem 1], which is also an analogue for meromorphic functions of [27, Propositions 5.2 and 5.3].

Proposition 2.5. *(cf. [15, Theorem 1]) Assume that $X = \mathbb{C}^n$, the hypersurface $P^{-1}(0) \cup Q^{-1}(0)$ of X is a normal crossing divisor $\{x \in X \mid x_1 x_2 \cdots x_r = 0\}$ for some $1 \le r \le n$ and $f(x) = x_1^{m_1} x_2^{m_2} \cdots x_r^{m_r}$ ($m_i \in \mathbb{Z}$). By the inclusion map $j : X \backslash (P^{-1}(0) \cup Q^{-1}(0)) \hookrightarrow X$ set*

$$\mathcal{F} = Rj_*(\mathbb{C}_{X\backslash(P^{-1}(0)\cup Q^{-1}(0))}) \in \mathbf{D}_c^b(X). \tag{2.39}$$

Then, if $r = 1$ and $m_1 > 0$ we have $\zeta(\mathcal{F})_{f,0}(t) = 1 - t^{m_1}$. Otherwise, we have $\zeta(\mathcal{F})_{f,0}(t) = 1$.

Proof. As in the proof of [28, Theorem 3.6] we construct towers of blow-ups of X over the normal crossing divisor $P^{-1}(0) \cup Q^{-1}(0)$ to eliminate the points of indeterminacy of f. Then we obtain a proper morphism $\pi : Y \longrightarrow X$ of complex manifolds which induces an isomorphism

$$Y \backslash \pi^{-1}(P^{-1}(0) \cup Q^{-1}(0)) \xrightarrow{\sim} X \backslash (P^{-1}(0) \cup Q^{-1}(0)). \tag{2.40}$$

If $r \ge 2$, then by calculating $\zeta(\pi^{-1}\mathcal{F})_{f\circ\pi,y}(t) \in \mathbb{C}(t)$ at each point y of $\pi^{-1}(0)$ the assertion follows from [9, page 170-173] (see also [45] and [28, Proposition 2.9]) and Proposition 2.4. $\square$

For a (shifted) perverse sheaf $\mathcal{F} \in \mathbf{D}_c^b(X)$ on X and $\lambda \in \mathbb{C}$, let

$$\Phi(\mathcal{F}) : \psi_f^{\mathrm{mero}}(\mathcal{F}) \xrightarrow{\sim} \psi_f^{\mathrm{mero}}(\mathcal{F}) \tag{2.41}$$

be the monodromy automorphism of $\psi_f^{\mathrm{mero}}(\mathcal{F})$ and by taking $N \gg 0$ set

$$\psi_{f,\lambda}^{\mathrm{mero}}(\mathcal{F}) := \mathrm{Ker}\Big[\big(\lambda \cdot \mathrm{id} - \Phi(\mathcal{F})\big)^N : \psi_f^{\mathrm{mero}}(\mathcal{F}) \longrightarrow \psi_f^{\mathrm{mero}}(\mathcal{F})\Big] \in \mathbf{D}_c^b(X), \tag{2.42}$$

where the right hand side is the kernel in the abelian category of (shifted) perverse sheaves. We can define $\psi_{f,\lambda}^{\mathrm{mero}}(\mathcal{F})$ also when $\mathcal{F}$ is not perverse but constructible. For a more precise

account, see e.g. [9, Remark 4.2.5]. We call $\psi_{f,\lambda}^{\mathrm{mero}}(\mathcal{F})$ the generalized eigenspace of $\psi_f^{\mathrm{mero}}(\mathcal{F})$ for the eigenvalue λ. Then we have a decomposition

$$\psi_f^{\mathrm{mero}}(\mathcal{F}) \simeq \bigoplus_{\lambda \in \mathbb{C}} \psi_{f,\lambda}^{\mathrm{mero}}(\mathcal{F}). \tag{2.43}$$

Similarly we can define also $\psi_{f,\lambda}^{\mathrm{mero,c}}(\mathcal{F})$. The following result will be used in the proof of Proposition 4.2.

Proposition 2.6. *Assume that $X = \mathbb{C}^n$ and the meromorphic function $f(x) = \frac{P(x)}{Q(x)}$ is defined by $P(x) = x_1^{m_1} x_2^{m_2} \cdots x_k^{m_k}$ ($m_i \in \mathbb{Z}$, $m_i > 0$) for some $1 \leq k \leq n-1$ and $Q(x) = x_n$. Then for any $\lambda \neq 1$ the natural morphism*

$$\psi_{f,\lambda}^{\mathrm{mero,c}}(\mathbb{C}_X)_0 \longrightarrow \psi_{f,\lambda}^{\mathrm{mero}}(\mathbb{C}_X)_0 \tag{2.44}$$

is an isomorphism.

Proof. Note that the hypersurface $P^{-1}(0) \cup Q^{-1}(0)$ of X is a normal crossing divisor $\{x \in X \mid x_1 x_2 \cdots x_k x_n = 0\}$. As in the proof of [28, Theorem 3.6] we construct towers of blow-ups of X over the normal crossing divisor $P^{-1}(0) \cup Q^{-1}(0)$ to eliminate the points of indeterminacy of f. Then we obtain a proper morphism $\pi : Y \longrightarrow X$ of complex manifolds which induces an isomorphism

$$Y \setminus \pi^{-1}(P^{-1}(0) \cup Q^{-1}(0)) \overset{\sim}{\longrightarrow} X \setminus (P^{-1}(0) \cup Q^{-1}(0)). \tag{2.45}$$

By Proposition 2.4 and its analogue for $\psi_f^{\mathrm{mero,c}}(\cdot)$ we obtain isomorphisms

$$\psi_{f,\lambda}^{\mathrm{mero,c}}(\mathbb{C}_X) \simeq R\pi_* \psi_{f\circ\pi,\lambda}^{\mathrm{mero,c}}(\mathbb{C}_Y), \tag{2.46}$$

$$\psi_{f,\lambda}^{\mathrm{mero}}(\mathbb{C}_X) \simeq R\pi_* \psi_{f\circ\pi,\lambda}^{\mathrm{mero}}(\mathbb{C}_Y) \tag{2.47}$$

for any $\lambda \in \mathbb{C}$. Set $D = \pi^{-1}(P^{-1}(0) \cup Q^{-1}(0))$ and let $j : Y \setminus D \hookrightarrow Y$ be the inclusion map. Then by Remark 2.3 and its analogue for $\psi_f^{\mathrm{mero,c}}(\cdot)$ we have isomorphisms

$$\psi_{f\circ\pi,\lambda}^{\mathrm{mero,c}}(\mathbb{C}_Y) \simeq \psi_{f\circ\pi,\lambda}(j_! \mathbb{C}_{Y\setminus D}), \tag{2.48}$$

$$\psi_{f\circ\pi,\lambda}^{\mathrm{mero}}(\mathbb{C}_Y) \simeq \psi_{f\circ\pi,\lambda}(Rj_* \mathbb{C}_{Y\setminus D}) \tag{2.49}$$

for any $\lambda \in \mathbb{C}$. Hence it suffices to prove that the natural morphism

$$R\Gamma(\pi^{-1}(0); \psi_{f\circ\pi,\lambda}(j_! \mathbb{C}_{Y\setminus D})) \longrightarrow R\Gamma(\pi^{-1}(0); \psi_{f\circ\pi,\lambda}(Rj_* \mathbb{C}_{Y\setminus D})) \tag{2.50}$$

is an isomorphism for any $\lambda \neq 1$. By the distinguished triangle

$$j_! \mathbb{C}_{Y\setminus D} \longrightarrow Rj_* \mathbb{C}_{Y\setminus D} \longrightarrow (Rj_* \mathbb{C}_{Y\setminus D})_D \overset{+1}{\longrightarrow} \tag{2.51}$$

we have only to show the vanishing

$$R\Gamma(\pi^{-1}(0); \psi_{f\circ\pi,\lambda}((Rj_* \mathbb{C}_{Y\setminus D})_D)) \simeq 0 \tag{2.52}$$

for any $\lambda \neq 1$. From now on, assume that $\lambda \neq 1$. First let us treat the simplest case $k = 1$. In this case, the normal crossing divisor $D = \pi^{-1}(P^{-1}(0) \cup Q^{-1}(0))$ in Y has

$m_1 + 2$ irreducible components D_i $(-1 \leq i \leq m_1)$ such that we have $\mathrm{ord}_{D_i}(f \circ \pi) = i$. See the proof of [28, Theorem 3.6] for the details. Note that D_{-1} is noting but the proper transform of the pole set $Q^{-1}(0) = \{x \in X \mid x_n = 0\}$ of f in Y. Since the support of $(Rj_*\mathbb{C}_{Y \setminus D})_D$ is contained in D, that of its nearby cycle $\psi_{f \circ \pi}((Rj_*\mathbb{C}_{Y \setminus D})_D)$ is contained in $D_1 \cap D_0$. But by our assumption $\lambda \neq 1$ we have $\psi_{f \circ \pi, \lambda}((Rj_*\mathbb{C}_{Y \setminus D})_D) \simeq 0$ also on $D_1 \cap D_0$. Next consider the case $k = 2$. Let $\pi_1 : Y_1 \longrightarrow X$ be the blow-up of X over the set $\{x \in X \mid x_1 = x_n = 0\} \subset I(f)$ in the first step of the construction of $\pi : Y \longrightarrow X$. We define divisors $D_i \subset Y_1$ $(-1 \leq i \leq m_1)$ as in the case $k = 1$. Now let $K \subset Y_1$ be the proper transform of $\{x \in X \mid x_2 = 0\} \subset X$ in Y_1. Then $f \circ \pi_1$ still has some points of indeterminacy in the set $D_{-1} \cap K$. So we construct a tower of blow-ups over it until we get a morphism $\pi_2 : Y \longrightarrow Y_1$ such that $\pi = \pi_1 \circ \pi_2$. See the proof of [28, Theorem 3.6] for the details. Let $E_i \subset Y$ $(-1 \leq i \leq m_2)$ be the exceptional divisors of π_2 such that $\mathrm{ord}_{E_i}(f \circ \pi) = i$. Denote the proper transform of $D_0 \subset Y_1$ in Y by H. Then (on a neighborhood of $E = \pi_2^{-1}(D_{-1} \cap K)$ in Y) we have

$$(f \circ \pi)^{-1}(0) = E_1 \cup E_2 \cup \cdots \cup E_{m_2} \tag{2.53}$$

and the support of $(Rj_*\mathbb{C}_{Y \setminus D})_D$ is contained in $E_{-1} \cup \cdots \cup E_{m_2} \cup H$. For $\lambda \neq 1$ this implies that the support of $\psi_{f \circ \pi, \lambda}((Rj_*\mathbb{C}_{Y \setminus D})_D)$ is contained in $(E_1 \cup \cdots \cup E_{m_2}) \cap H$. Moreover by truncation functors, it suffices to prove the vanishing

$$R\Gamma(\pi^{-1}(0); \psi_{f \circ \pi, \lambda}(\mathbb{C}_H)) \simeq 0. \tag{2.54}$$

But this follows from the primitive decompositons (of the graded pieces w.r.t. the weight filtration) of the nearby cycle sheaf $\psi_{f \circ \pi, \lambda}(\mathbb{C}_H)$ (see e.g. [6] etc. for the details). Indeed, for any $1 \leq i \leq m_2 - 1$ the restriction of $\psi_{f \circ \pi, \lambda}(\mathbb{C}_H)$ to the subset

$$\pi^{-1}(0) \cap \{E_i \setminus (E_{i-1} \cup E_{i+1})\} \simeq \mathbb{C}^* \tag{2.55}$$

of $\pi^{-1}(0)$ is zero or a non-trivial local system of rank one. Moreover by our assumption $\lambda \neq 1$ we never have the condition $\lambda^i = \lambda^{i+1} = 1$. This implies that the restriction of $\psi_{f \circ \pi, \lambda}(\mathbb{C}_H)$ to $E_i \cap E_{i+1}$ is zero. Similarly, we can prove the assertion for any $1 \leq k \leq n-1$. This completes the proof. $\qquad\square$

3 Milnor monodromies of meromorphic functions

In this section, by using the meromorphic nearby cycle functors introduced in Section 2 we study Milnor monodromies of meromorphic functions. Let us consider the meromorphic function $f(x) = \frac{P(x)}{Q(x)}$ in Section 2. The problem being local, we may assume that $X = \mathbb{C}^n$ and $0 \in I(f) = P^{-1}(0) \cap Q^{-1}(0)$. Let F_0 be the Milnor fiber of f at the origin $0 \in X = \mathbb{C}^n$ and

$$\Phi_{j,0} : H^j(F_0; \mathbb{C}) \xrightarrow{\sim} H^j(F_0; \mathbb{C}) \qquad (j \in \mathbb{Z}) \tag{3.1}$$

its Milnor monodromy operators. Then we define the monodromy zeta function $\zeta_{f,0}(t) \in \mathbb{C}(t)$ of f at the origin $0 \in X = \mathbb{C}^n$ by

$$\zeta_{f,0}(t) = \prod_{j \in \mathbb{Z}} \left\{ \det(\mathrm{id} - t\Phi_{j,0}) \right\}^{(-1)^j} \in \mathbb{C}(t). \tag{3.2}$$

By Propositions 2.4 and 2.5 we can reprove the following result of Gusein-Zade, Luengo and Melle-Hernández [15, Theorem 1], [17, Theorem 1.19], which is an analogue for meromorphic functions of A'Campo's formula in [2].

Theorem 3.1. *(Gusein-Zade, Luengo and Melle-Hernández [15, Theorem 1], [17, Theorem 1.19]) Let $\pi : Y \to X = \mathbb{C}^n$ be a resolution of singularities of $P^{-1}(0) \cup Q^{-1}(0)$ which induces an isomorphism*

$$Y \setminus \pi^{-1}(P^{-1}(0) \cup Q^{-1}(0)) \xrightarrow{\sim} X \setminus (P^{-1}(0) \cup Q^{-1}(0)) \tag{3.3}$$

such that $\pi^{-1}(0)$ and $D = \pi^{-1}(P^{-1}(0) \cup Q^{-1}(0))$ are strict normal crossing divisors in Y. Let $\cup_{i=1}^k D_i$ be the irreducible decomposition of $D = \pi^{-1}(P^{-1}(0) \cup Q^{-1}(0))$ such that $\pi^{-1}(0) = \cup_{i=1}^r D_i$ for some $1 \leq r \leq k$. For $1 \leq i \leq r$ set

$$D_i^\circ = D_i \setminus (\cup_{j \neq i} D_j) \tag{3.4}$$

and

$$m_i = \mathrm{ord}_{D_i}(P \circ \pi) - \mathrm{ord}_{D_i}(Q \circ \pi) \in \mathbb{Z}. \tag{3.5}$$

Then we have

$$\zeta_{f,0}(t) = \prod_{i:m_i>0} (1 - t^{m_i})^{\chi(D_i^\circ)}. \tag{3.6}$$

For $m \geq 1$ we define the Lefschetz number $\Lambda(m)_{f,0} \in \mathbb{Z}$ of the meromorphic function f at the origin $0 \in X = \mathbb{C}^n$ by

$$\Lambda(m)_{f,0} = \sum_{j \in \mathbb{Z}} (-1)^j \mathrm{tr} \big\{ \Phi_{j,0}^m : H^j(F_0; \mathbb{C}) \xrightarrow{\sim} H^j(F_0; \mathbb{C}) \big\}. \tag{3.7}$$

Then as in [1] we obtain the following corollary (see also e.g. [28, Remark 3.2]).

Corollary 3.2. *In the situation of Theorem 3.1, for any $m \geq 1$ we have*

$$\Lambda(m)_{f,0} = \sum_{i:m_i>0, m_i | m} \chi(D_i^\circ) \cdot m_i. \tag{3.8}$$

From now, let us prove Theorem 1.2.

Proof. By Theorem 2.2 $\psi_f^{\mathrm{mero}}(\mathbb{C}_X[n])[-1] \in \mathbf{D}^b(X)$ is a perverse sheaf. The same is true also for its λ-part $\psi_{f,\lambda}^{\mathrm{mero}}(\mathbb{C}_X[n])[-1] \in \mathbf{D}^b(X)$. Let $\pi_0 : \widetilde{X} \to X = \mathbb{C}^n$ be a resolution of singularities of $P^{-1}(0)$ and $Q^{-1}(0)$ which induces an isomorphism $\widetilde{X} \setminus \pi_0^{-1}(\{0\}) \xrightarrow{\sim} X \setminus \{0\}$. Let $\widetilde{P^{-1}(0)}$ and $\widetilde{Q^{-1}(0)}$ be the (smooth) proper transforms of $P^{-1}(0)$ and $Q^{-1}(0)$ in $\widetilde{X}$ respectively. We may assume that they intersect transversally. Now let $\pi_1 : Y \to \widetilde{X}$ be the blow-up of $\widetilde{X}$ along $\widetilde{P^{-1}(0)} \cap \widetilde{Q^{-1}(0)}$ and set $\pi := \pi_0 \circ \pi_1 : Y \to X$. Then by Propostion 2.4 there exists an isomorphism

$$\psi_{f,\lambda}^{\mathrm{mero}}(\mathbb{C}_X[n])[-1] \simeq R\pi_* \psi_{f\circ\pi,\lambda}^{\mathrm{mero}}(\mathbb{C}_Y[n])[-1]. \tag{3.9}$$

By the construction of π, we can show that for $\lambda \neq 1$ the support of $\psi_{f\circ\pi,\lambda}^{\mathrm{mero}}(\mathbb{C}_Y[n])[-1] \in \mathbf{D}^b(Y)$ is contained in $\pi^{-1}(\{0\}) \subset Y$. Indeed, let $E \subset Y$ be the exceptional divisor of the

blow-up $\pi_1 : Y \to \widetilde{X}$ and denote the (smooth) proper transform of $\widetilde{P^{-1}(0)}$ (resp. $\widetilde{Q^{-1}(0)}$) in Y by D_P (resp. D_Q). Then we have $D_P \cap D_Q = \emptyset$ and $D_P \cup D_Q \cup E$ is a normal crossing divisor in Y. Moreover on $Y \setminus \pi^{-1}(\{0\})$ (i.e. outside $\pi^{-1}(\{0\})$) the meromorphic function $f \circ \pi$ on Y takes the value 0 only on D_P. This implies that on $Y \setminus \pi^{-1}(\{0\})$ the support of the perverse sheaf $\psi_{f \circ \pi}^{\mathrm{mero}}(\mathbb{C}_Y[n])[-1] \in \mathbf{D}^{\mathrm{b}}(Y)$ is contained in D_P. Since the divisor $D_P \cup D_Q \cup E$ is normal crossing, we can easily see also that there exists an isomorphism

$$\psi_{f \circ \pi}^{\mathrm{mero}}(\mathbb{C}_Y[n])[-1] \simeq \psi_{f \circ \pi, 1}^{\mathrm{mero}}(\mathbb{C}_Y[n])[-1] \qquad (3.10)$$

on $Y \setminus \pi^{-1}(\{0\})$. Hence the support of the perverse sheaf $\psi_{f, \lambda}^{\mathrm{mero}}(\mathbb{C}_X[n-1]) \in \mathbf{D}^{\mathrm{b}}(X)$ is contained in the origin $\{0\} \subset X$. Then by [18, Proposition 8.1.22] we obtain the concentration

$$H^j \psi_{f, \lambda}^{\mathrm{mero}}(\mathbb{C}_X[n-1])_0 \simeq H^{j+n-1}(F_0; \mathbb{C})_\lambda \simeq 0 \qquad (j \neq 0). \qquad (3.11)$$

$\square$

By Theorems 3.1 and 1.2 we obtain the following result.

Corollary 3.3. *Assume that the meromorphic function $f(x) = \frac{P(x)}{Q(x)}$ satisfies the conditions in Theorem 1.2. Then in the notations of Theorem 3.1, for any $\lambda \neq 1$ the multiplicity of the eigenvalue λ in $\Phi_{n-1,0}$ is equal to that of the factor $t - \lambda$ in the rational function*

$$\prod_{i : m_i > 0} (t^{m_i} - 1)^{(-1)^{n-1} \chi(D_i^\circ)} \in \mathbb{C}(t). \qquad (3.12)$$

Definition 3.4. Let $g(x) = \sum_{v \in \mathbb{Z}^n} c_v x^v$ $(c_v \in \mathbb{C})$ be a Laurent polynomial on the algebraic torus $T = (\mathbb{C}^*)^n$.

(i) We call the convex hull of $\mathrm{supp}(g) := \{v \in \mathbb{Z}^n \mid c_v \neq 0\} \subset \mathbb{Z}^n \subset \mathbb{R}^n$ in $\mathbb{R}^n$ the Newton polytope of g and denote it by $NP(g)$.

(ii) If g is a polynomial, we call the convex hull of $\cup_{v \in \mathrm{supp}(g)}(v + \mathbb{R}_+^n)$ in $\mathbb{R}_+^n$ the Newton polyhedron of g at the origin $0 \in \mathbb{C}^n$ and denote it by $\Gamma_+(g)$.

(iii) For a face $\gamma \prec NP(g)$ of $NP(g)$, we define the γ-part g^γ of g by $g^\gamma(x) := \sum_{v \in \gamma} c_v x^v$.

Let $\Gamma_+(P), \Gamma_+(Q) \subset \mathbb{R}^n$ be the Newton polyhedra of P and Q at the origin $0 \in \mathbb{C}^n$ and

$$\Gamma_+(f) = \Gamma_+(P) + \Gamma_+(Q) \qquad (3.13)$$

their Minkowski sum. From now, we recall Bernstein-Khovanskii-Kushnirenko's theorem [22]. Let $\Delta \subset \mathbb{R}^n$ be a lattice polytope in $\mathbb{R}^n$. For an element $u \in \mathbb{R}^n$ of (the dual vector space of) $\mathbb{R}^n$ we define the supporting face $\gamma_u \prec \Delta$ of u in Δ by

$$\gamma_u = \left\{ v \in \Delta \mid \langle u, v \rangle = \min_{w \in \Delta} \langle u, w \rangle \right\}, \qquad (3.14)$$

where for $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ we set $\langle u, v \rangle = \sum_{i=1}^n u_i v_i$. For a face γ of Δ set

$$\sigma(\gamma) = \overline{\{u \in \mathbb{R}^n \mid \gamma_u = \gamma\}} \subset \mathbb{R}^n. \qquad (3.15)$$

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Then $\sigma(\gamma)$ is an $(n - \dim\gamma)$-dimensional rational convex polyhedral cone in $\mathbb{R}^n$. Moreover the family $\{\sigma(\gamma) \mid \gamma \prec \Delta\}$ of cones in $\mathbb{R}^n$ thus obtained is a subdivision of $\mathbb{R}^n$. We call it the dual subdivision of $\mathbb{R}^n$ by Δ. If $\dim\Delta = n$ it satisfies the axiom of fans (see [11] and [35] etc.). We call it the dual fan of Δ. More generally, let $\Delta_1, \ldots, \Delta_p \subset \mathbb{R}^n$ be lattice polytopes in $\mathbb{R}^n$ and $\Delta = \Delta_1 + \cdots + \Delta_p \subset \mathbb{R}^n$ their Minkowski sum. Then for a face $\gamma \prec \Delta$ of Δ, by taking a point $u \in \mathbb{R}^n$ in the relative interior of its dual cone $\sigma(\gamma)$ we define the supporting face $\gamma_i \prec \Delta_i$ of u in Δ_i so that we have $\gamma = \gamma_1 + \cdots + \gamma_p$.

Definition 3.5. (see [36] etc.) Let $g_1, g_2, \ldots, g_p$ be Laurent polynomials on $T = (\mathbb{C}^*)^n$. Set $\Delta_i = NP(g_i)$ $(i = 1, \ldots, p)$ and $\Delta = \Delta_1 + \cdots + \Delta_p$. Then we say that the subvariety $Z = \{x \in T = (\mathbb{C}^*)^n \mid g_1(x) = g_2(x) = \cdots = g_p(x) = 0\}$ of $T = (\mathbb{C}^*)^n$ is a non-degenerate complete intersection if for any face $\gamma \prec \Delta$ of Δ the p-form $dg_1^{\gamma_1} \wedge dg_2^{\gamma_2} \wedge \cdots \wedge dg_p^{\gamma_p}$ does not vanish on $\{x \in T = (\mathbb{C}^*)^n \mid g_1^{\gamma_1}(x) = \cdots = g_p^{\gamma_p}(x) = 0\}$.

Definition 3.6. Let $\Delta_1, \ldots, \Delta_n$ be lattice polytopes in $\mathbb{R}^n$. Then their normalized n-dimensional mixed volume $\mathrm{Vol}_{\mathbb{Z}}(\Delta_1, \ldots, \Delta_n) \in \mathbb{Z}$ is defined by the formula

$$\mathrm{Vol}_{\mathbb{Z}}(\Delta_1, \ldots, \Delta_n) = \frac{1}{n!} \sum_{k=1}^n (-1)^{n-k} \sum_{\substack{I \subset \{1, \ldots, n\} \\ |I| = k}} \mathrm{Vol}_{\mathbb{Z}} \left(\sum_{i \in I} \Delta_i \right) \tag{3.16}$$

where $\mathrm{Vol}_{\mathbb{Z}}(\,\cdot\,) = n! \mathrm{Vol}(\,\cdot\,) \in \mathbb{Z}$ is the normalized n-dimensional volume with respect to the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$.

Theorem 3.7. *(Bernstein-Khovanskii-Kushnirenko's theorem [22]) Let $g_1, g_2, \ldots, g_p$ be Laurent polynomials on $T = (\mathbb{C}^*)^n$. Assume that the subvariety $Z = \{x \in T = (\mathbb{C}^*)^n \mid g_1(x) = g_2(x) = \cdots = g_p(x) = 0\}$ of $T = (\mathbb{C}^*)^n$ is a non-degenerate complete intersection. Set $\Delta_i = NP(g_i)$ $(i = 1, \ldots, p)$. Then we have*

$$\chi(Z) = (-1)^{n-p} \sum_{\substack{m_1, \ldots, m_p \geq 1 \\ m_1 + \cdots + m_p = n}} \mathrm{Vol}_{\mathbb{Z}}(\underbrace{\Delta_1, \ldots, \Delta_1}_{m_1\text{-}times}, \ldots, \underbrace{\Delta_p, \ldots, \Delta_p}_{m_p\text{-}times}), \tag{3.17}$$

where $\mathrm{Vol}_{\mathbb{Z}}(\underbrace{\Delta_1, \ldots, \Delta_1}_{m_1\text{-}times}, \ldots, \underbrace{\Delta_p, \ldots, \Delta_p}_{m_p\text{-}times}) \in \mathbb{Z}$ is the normalized n-dimensional mixed volume with respect to the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$.

Now, let Σ_f, Σ_P and Σ_Q be the dual fans of $\Gamma_+(f)$, $\Gamma_+(P)$ and $\Gamma_+(Q)$ in $\mathbb{R}^n_+$ respectively. Then the dual fan Σ_f of the Minkowski sum $\Gamma_+(f) = \Gamma_+(P) + \Gamma_+(Q)$ is the coarsest common subdivision of Σ_P and Σ_Q. This implies that for each face $\gamma \prec \Gamma_+(f)$ we have the corresponding faces

$$\gamma(P) \prec \Gamma_+(P), \qquad \gamma(Q) \prec \Gamma_+(Q) \tag{3.18}$$

such that

$$\gamma = \gamma(P) + \gamma(Q). \tag{3.19}$$

Definition 3.8. We say that the meromorphic function $f(x) = \frac{P(x)}{Q(x)}$ is non-degenerate at the origin $0 \in X = \mathbb{C}^n$ if for any compact face γ of $\Gamma_+(f)$ the complex hypersurfaces $\{x \in T = (\mathbb{C}^*)^n \mid P^{\gamma(P)}(x) = 0\}$ and $\{x \in T = (\mathbb{C}^*)^n \mid Q^{\gamma(Q)}(x) = 0\}$ are smooth and reduced and intersect transversally in $T = (\mathbb{C}^*)^n$.

For a subset $S \subset \{1, 2, \ldots, n\}$ we set

$$\mathbb{R}^S = \{v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n \mid v_i = 0 \ (i \notin S)\} \simeq \mathbb{R}^{|S|} \tag{3.20}$$

and

$$\Gamma_+(f)^S = \Gamma_+(f) \cap \mathbb{R}^S. \tag{3.21}$$

Similarly, we define $\Gamma_+(P)^S, \Gamma_+(Q)^S \subset \mathbb{R}^S$ so that we have

$$\Gamma_+(f)^S = \Gamma_+(P)^S + \Gamma_+(Q)^S. \tag{3.22}$$

Let $\gamma_1^S, \gamma_2^S, \ldots, \gamma_{n(S)}^S$ be the compact facets of $\Gamma_+(f)^S$ and for each γ_i^S $(1 \leq i \leq n(S))$ consider the corresponding faces

$$\gamma_i^S(P) \prec \Gamma_+(P)^S, \qquad \gamma_i^S(Q) \prec \Gamma_+(Q)^S \tag{3.23}$$

such that

$$\gamma_i^S = \gamma_i^S(P) + \gamma_i^S(Q). \tag{3.24}$$

By using the primitive inner conormal vector $\alpha_i^S \in \mathbb{Z}_+^S \setminus \{0\}$ of the facet $\gamma_i^S \prec \Gamma_+(f)^S$ we define the lattice distance $d_i^S(P) > 0$ (resp. $d_i^S(Q) > 0$) of $\gamma_i^S(P)$ (resp. $\gamma_i^S(Q)$) from the origin $0 \in \mathbb{R}^S$ to be the (unique) value of α_i^S on $\gamma_i^S(P)$ (resp. $\gamma_i^S(Q)$) and set

$$d_i^S = d_i^S(P) - d_i^S(Q) \in \mathbb{Z}. \tag{3.25}$$

Finally by using the normalized $(|S| - 1)$-dimensional volume $\mathrm{Vol}_\mathbb{Z}(\, \cdot \,)$ we set

$$v_i^S = \sum_{k=0}^{|S|-1} \mathrm{Vol}_\mathbb{Z}(\underbrace{\gamma_i^S(P), \ldots, \gamma_i^S(P)}_{k\text{-times}}, \underbrace{\gamma_i^S(Q), \ldots, \gamma_i^S(Q)}_{(|S| - 1 - k)\text{-times}}) \in \mathbb{Z}. \tag{3.26}$$

Then we have the following celebrated theorem of Gusein-Zade, Luengo and Melle-Hernández [15].

Theorem 3.9. *(Gusein-Zade, Luengo and Melle-Hernández [15]) Assume that the meromorphic function $f(x) = \frac{P(x)}{Q(x)}$ is non-degenerate at the origin $0 \in X = \mathbb{C}^n$. Then we have*

$$\zeta_{f,0}(t) = \prod_{S \neq \emptyset} \left\{ \prod_{i: d_i^S > 0} (1 - t^{d_i^S})^{(-1)^{|S|-1} v_i^S} \right\}. \tag{3.27}$$

Decomposing $X = \mathbb{C}^n$ into some tori $(\mathbb{C}^*)^k$ as in the proof of [27, Theorem 3.12], we can reprove this theorem by Propositions 2.4 and 2.5 and Theorem 3.7 (see also Varchenko [51] and Oka [36]). By Theorems 3.9 and 1.2 we obtain the following result.

Corollary 3.10. *Assume that $f(x) = \frac{P(x)}{Q(x)}$ is non-degenerate at the origin $0 \in X = \mathbb{C}^n$ and $P(x), Q(x)$ are convenient. Then in the notations of Theorem 3.9, for any $\lambda \neq 1$ the multiplicity of the eigenvalue λ in $\Phi_{n-1,0}$ is equal to that of the factor $t - \lambda$ in the rational function*

$$\prod_{S \neq \emptyset} \left\{ \prod_{i: d_i^S > 0} (t^{d_i^S} - 1)^{(-1)^{n-|S|} v_i^S} \right\} \in \mathbb{C}(t). \tag{3.28}$$

4 Mixed Hodge structures of Milnor fibers of rational functions

In this section, by using the meromorphic nearby cycle functors introduced in Section 2 we study the mixed Hodge structures of Milnor fibers of rational functions and apply them to the Jordan normal forms of their monodromies. Let us consider a rational function $f(x) = \frac{P(x)}{Q(x)}$ on $X = \mathbb{C}^n$ such that $0 \in I(f) = P^{-1}(0) \cap Q^{-1}(0)$. In order to obtain also a formula for the Jordan normal form of its monodromy $\Phi_{n-1,0}$ as in Matsui-Takeuchi [30], Stapledon [47] and Saito [43], we need the following condition.

Definition 4.1. We say that the rational function $f(x) = \frac{P(x)}{Q(x)}$ is polynomial-like if there exists a resolution $\pi_0 : \widetilde{X} \to X = \mathbb{C}^n$ of singularities of $P^{-1}(0)$ and $Q^{-1}(0)$ which induces an isomorphism $\widetilde{X} \setminus \pi_0^{-1}(\{0\}) \xrightarrow{\sim} X \setminus \{0\}$ such that for any irreducible component D_i of the (exceptional) normal crossing divisor $D = \pi_0^{-1}(\{0\})$ we have the condition

$$\mathrm{ord}_{D_i}(P \circ \pi_0) > \mathrm{ord}_{D_i}(Q \circ \pi_0). \tag{4.1}$$

Proposition 4.2. *Assume that the rational function $f(x) = \frac{P(x)}{Q(x)}$ is polynomial-like and satisfies the conditions in Theorem 1.2. Then for any $\lambda \neq 1$ the natural morphism*

$$\psi_{f,\lambda}^{\mathrm{mero,c}}(\mathbb{C}_X[n])[-1] \longrightarrow \psi_{f,\lambda}^{\mathrm{mero}}(\mathbb{C}_X[n])[-1] \tag{4.2}$$

is an isomorphism. Morevoer, the supports of the both sides of it are contained in the origin $\{0\} \subset X = \mathbb{C}^n$.

Proof. Let $\pi_0 : \widetilde{X} \to X = \mathbb{C}^n$ be a resolution of singularities of $P^{-1}(0)$ and $Q^{-1}(0)$ which induces an isomorphism $\widetilde{X} \setminus \pi_0^{-1}(\{0\}) \xrightarrow{\sim} X \setminus \{0\}$ such that for any irreducible component D_i of the (exceptional) normal crossing divisor $D = \pi_0^{-1}(\{0\})$ we have the condition

$$\mathrm{ord}_{D_i}(P \circ \pi_0) > \mathrm{ord}_{D_i}(Q \circ \pi_0). \tag{4.3}$$

Then by Proposition 2.6 we can show that for any $\lambda \neq 1$ the natural morphism

$$\psi_{f \circ \pi_0, \lambda}^{\mathrm{mero,c}}(\mathbb{C}_{\widetilde{X}}[n])[-1] \longrightarrow \psi_{f \circ \pi_0, \lambda}^{\mathrm{mero}}(\mathbb{C}_{\widetilde{X}}[n])[-1] \tag{4.4}$$

is an isomorphism and the supports of the both sides of it are contained in $\pi_0^{-1}(\{0\})$. Indeed, let $\widetilde{P^{-1}(0)}$ (resp. $\widetilde{Q^{-1}(0)}$) be the (smooth) proper transform of $P^{-1}(0)$ (resp. $Q^{-1}(0)$) in $\widetilde{X}$. Then the divisor $\widetilde{P^{-1}(0)} \cup \widetilde{Q^{-1}(0)} \cup D$ in $\widetilde{X}$ is normal crossing, and the meromorphic function $f \circ \pi_0$ on $\widetilde{X}$ takes the value 0 on $(\widetilde{P^{-1}(0)} \cup D) \setminus \widetilde{Q^{-1}(0)}$ and has a pole of order 1 along $\widetilde{Q^{-1}(0)}$. First, by Remark 2.3 and its analogue for meromorphic nearby cycle functor with compact support, on $(\widetilde{P^{-1}(0)} \cup D) \setminus \widetilde{Q^{-1}(0)}$ we have an isomorphism

$$\psi_{f \circ \pi_0}^{\mathrm{mero,c}}(\mathbb{C}_{\widetilde{X}}[n])[-1] \xrightarrow{\sim} \psi_{f \circ \pi_0}^{\mathrm{mero}}(\mathbb{C}_{\widetilde{X}}[n])[-1]. \tag{4.5}$$

Next, with the help of Lemma 2.1 (ii) and its analogue for meromorphic nearby cycle functor with compact support, by reducing the problem to the situation in Proposition 2.6, we can check that for $\lambda \neq 1$ the stalk of the morphism (4.4) at each point of $(\widetilde{P^{-1}(0)} \cup$

$D) \cap \widetilde{Q^{-1}(0)}$ is an isomorphism. Moreover, since $\widetilde{P^{-1}(0)} \cup \widetilde{Q^{-1}(0)}$ is normal crossing in $\widetilde{X} \setminus \pi_0^{-1}(\{0\}) \simeq X \setminus \{0\}$, by the proof of Theorem 1.2 if $\lambda \neq 1$ we have

$$\psi_{f \circ \pi_0, \lambda}^{\mathrm{mero,c}}(\mathbb{C}_{\widetilde{X}}[n])[-1] \xrightarrow{\sim} \psi_{f \circ \pi_0, \lambda}^{\mathrm{mero}}(\mathbb{C}_{\widetilde{X}}[n])[-1] \simeq 0 \tag{4.6}$$

on $\widetilde{X} \setminus \pi_0^{-1}(\{0\}) \simeq X \setminus \{0\}$. Now by Proposition 2.4 and its analogue for meromorphic nearby cycle functors with compact support, all the assertions immediately follow. $\qquad\square$

Now we shall introduce natural mixed Hodge structures on the cohomology groups $H^j(F_0; \mathbb{C})_\lambda$ ($\lambda \in \mathbb{C}$) of the Milnor fiber F_0. For a variety Z over $\mathbb{C}$ we denote by MHM_Z the abelian category of mixed Hodge modules on Z (see e.g. [18, Section 8.3] etc.). First we regard $\psi_{f, \lambda}^{\mathrm{mero}}(\mathbb{C}_X[n])[-1]$ (resp. $\psi_{f, \lambda}^{\mathrm{mero,c}}(\mathbb{C}_X[n])[-1]$) as the underlying perverse sheaf of the mixed Hodge module $\psi_{t, \lambda}^H(i_{f_*}(\mathbb{C}_X^H[n]|_{X \setminus Q^{-1}(0)}))$ (resp. $\psi_{t, \lambda}^H(i_{f_!}(\mathbb{C}_X^H[n]|_{X \setminus Q^{-1}(0)}))) \in \mathrm{MHM}_X$, where $\psi_{t, \lambda}^H$ (resp. i_{f_*}, $i_{f_!}$) is a functor between the categories of mixed Hodge modules corresponding to the functor $\psi_{t, \lambda}[-1]$ (resp. Ri_{f_*}, $Ri_{f_!}$) and $\mathbb{C}_X^H[n] \in \mathrm{MHM}_X$ is the mixed Hodge module whose underlying perverse sheaf is $\mathbb{C}_X[n]$. For the inclusion map $j_0: \{0\} \hookrightarrow X$, we consider the pullback

$$j_0^* \psi_{t, \lambda}^H(i_{f_*}(\mathbb{C}_X^H[n]|_{X \setminus Q^{-1}(0)})) \in \mathbf{D}^{\mathrm{b}}(\mathrm{MHM}_{\{0\}}) \tag{4.7}$$

by j_0, whose underlying constructible sheaf is $j_0^{-1}(\psi_{f, \lambda}^{\mathrm{mero}}(\mathbb{C}_X[n])[-1])$. Since the $(j-n+1)$-th cohomology group of $j_0^{-1}(\psi_{f, \lambda}^{\mathrm{mero}}(\mathbb{C}_X[n])[-1])$ is $H^j(F_0; \mathbb{C})_\lambda$, we thus obtain a natural mixed Hodge structure of $H^j(F_0; \mathbb{C})_\lambda$. In the following, we focus our attention on its weight filtration $W_\bullet H^j(F_x; \mathbb{C})_\lambda$. Recall that the weight filtration of the (middle dimensional) cohomology group of the Milnor fiber of an isolated hypersurface singular point is the monodromy weight filtration (for the definition, see e.g. [29, Section A.2] etc.) of the Milnor monodromy. We will show that if the rational function $f(x) = \frac{P(x)}{Q(x)}$ satisfies the conditions of Proposition 4.2 the cohomology groups of the Milnor fiber F_0 of f also have a similar property. By Proposition 4.2, we thus obtain the same nice property also for the corresponding stalk of the perverse sheaf $\psi_{f, \lambda}^{\mathrm{mero,c}}(\mathbb{C}_X[n])[-1]$ (see Theorem 5.9 for its description by a motivic Milnor fiber). First, we need the following lemma.

Lemma 4.3. *Let g be a holomorphic function on a complex manifold Z. Moreover, let M, M' be mixed Hodge modules on Z and $M \to M'$ a morphism in the category of mixed Hodge modules. Assume that M (resp. M') has weights $\leq l$ (resp. $\geq l$) for some integer l, i.e. we have $\mathrm{Gr}_k^W M = 0$ ($k > l$) (resp. $\mathrm{Gr}_k^W M' = 0$ ($k < l$)). Then, for $\lambda \neq 1$ if the natural morphism $\psi_{g, \lambda}^H(M) \to \psi_{g, \lambda}^H(M')$ is an isomorphism, the weight filtration of $\psi_{g, \lambda}^H(M')$ is the monodromy weight filtration centered at $l - 1$.*

Proof. See Appendix A. $\qquad\square$

In the situation of Proposition 4.2, we set

$$M := i_{f_!}(\mathbb{C}_X^H[n]|_{X \setminus Q^{-1}(0)}), \quad M' := i_{f_*}(\mathbb{C}_X^H[n]|_{X \setminus Q^{-1}(0)}) \quad \in \mathbf{D}^{\mathrm{b}}(\mathrm{MHM}_{X \times \mathbb{C}}). \tag{4.8}$$

Recall that $\mathbb{C}_X^H[n]$ has a pure weight n. Therefore, by the basic properties of the functor i_{f_*} (resp. $i_{f_!}$) (see [18, Section 8.3] etc.) the mixed Hodge module M (resp. M') has weights $\leq n$ (resp. $\geq n$). Then we can apply Lemma 4.3 to obtain the following theorem.

Theorem 4.4. *In the situation of Proposition 4.2, for any $\lambda \neq 1$ the weight filtration of $H^{n-1}(F_0; \mathbb{C})_\lambda$ is the monodromy weight filtration of the Milnor monodromy $\Phi_{n-1,0} \colon H^{n-1}(F_0; \mathbb{C})_\lambda \xrightarrow{\sim} H^{n-1}(F_0; \mathbb{C})_\lambda$ centered at $n-1$.*

Proof. Since by Theorem 1.2 we have

$$H^j(F_0; \mathbb{C})_\lambda = 0 \qquad (j \neq n-1), \tag{4.9}$$

the complex $j_0^* \psi_{t,\lambda}^H(i_{f_*}(\mathbb{C}_X^H[n]|_{X \setminus Q^{-1}(0)})) \in \mathbf{D}^b(\mathrm{MHM}_{\{0\}})$ is quasi-isomorphic to $H^0 j_0^* \psi_{t,\lambda}^H(i_{f_*}(\mathbb{C}_X^H[n]|_{X \setminus Q^{-1}(0)}))$ and its underlying perverse sheaf is $H^{n-1}(F_0; \mathbb{C})_\lambda$. By Lemma 4.3 the weight filtration of $\psi_{t,\lambda}^H(i_{f_*}(\mathbb{C}_X^H[n]|_{X \setminus Q^{-1}(0)}))$ is the monodromy weight filtration centered at $n-1$. Moreover, in this situation, the support of $\psi_{t,\lambda}^H(i_{f_*}(\mathbb{C}_X^H[n]|_{X \setminus Q^{-1}(0)}))$ is contained in $\{0\}$. This implies that $\psi_{t,\lambda}^H(i_{f_*}(\mathbb{C}_X^H[n]|_{X \setminus Q^{-1}(0)}))$ is the zero extension of $H^0 j_0^* \psi_{t,\lambda}^H(i_{f_*}(\mathbb{C}_X^H[n]|_{X \setminus Q^{-1}(0)}))$. Hence we can identify the weight filtration of $H^0 j_0^* \psi_{t,\lambda}^H(i_{f_*}(\mathbb{C}_X^H[n]|_{X \setminus Q^{-1}(0)}))$ with that of $\psi_{t,\lambda}^H(i_{f_*}(\mathbb{C}_X^H[n]|_{X \setminus Q^{-1}(0)}))$. This completes the proof. $\square$

REMARK 4.5. For $k \in \mathbb{Z}_{>0}$ and $\lambda \in \mathbb{C}$ we denote by $J_{k,\lambda}$ the number of the Jordan blocks in $\Phi_{n-1,0}$ with size k for the eigenvalue λ. Then by Theorem 4.4 for $\lambda \neq 1$ we can describe $J_{k,\lambda}$ in terms of the weight filtration of $H^{n-1}(F_0; \mathbb{C})_\lambda$ as follows:

$$J_{k,\lambda} = \dim \mathrm{Gr}_{n-k}^W H^{n-1}(F_0; \mathbb{C})_\lambda - \dim \mathrm{Gr}_{n-k-2}^W H^{n-1}(F_0; \mathbb{C})_\lambda. \tag{4.10}$$

Moreover the number of the Jordan blocks in $\Phi_{n-1,0}$ with size $\geq k$ for the eigenvalue λ is equal to

$$\dim \mathrm{Gr}_{n-2+k}^W H^{n-1}(F_0; \mathbb{C})_\lambda + \dim \mathrm{Gr}_{n-1+k}^W H^{n-1}(F_0; \mathbb{C})_\lambda. \tag{4.11}$$

5 Motivic Milnor fibers of of rational functions

Following Denef-Loeser [6], [7], [8], Guibert-Loeser-Merle [14] and Raibaut [40], [41], we shall define and study the motivic reincarnations of the Milnor fibers of rational functions. More precisely, we will define the notion of motivic Milnor fiber of rational functions and give their relation with the topological properties of the Milnor fiber. Such work was studied by Raibaut [41] where the author considered elements in the Grothendieck ring under the action of $\mathbb{G}_m$. In this section, we use another construction for the motivic Milnor fiber, namely, we consider the Grothendieck ring under the action of $\hat{\mu}$ and define the notion of motivic Milnor fiber as an element in $\mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$. In our opinion, in this section we give, in some sense, another proof for the results in [41].

Let $f(x) = \frac{P(x)}{Q(x)}$ be a rational function on $X = \mathbb{C}^n$ and set $X_0 := P^{-1}(0)$. Assume that $0 \in P^{-1}(0) \cap Q^{-1}(0)$. For two positive integers $m, r > 0$ we set

$$\mathcal{X}_{m,1}^r := \{\varphi \in \mathcal{L}_m(X) \mid \mathrm{ord}Q(\varphi(t)) \leq rm, f(\varphi(t)) = t^m \bmod t^{m+1}\} \tag{5.1}$$

and

$$\mathcal{X}_{m,1,0}^r := \{\varphi \in \mathcal{X}_{m,1}^r \mid \pi_0^m(\varphi) = 0\} \tag{5.2}$$

where $\mathcal{L}_m(X)$ is the space of order m arcs on X and π_0^m is the truncation morphism $\mathcal{L}_m(X) \to X$. They are locally closed subvarieties of $\mathcal{L}_m(X)$ (see, e.g. [8, Section 1] for

the notions of arc spaces). Since $m \geq 1$, we see that $\pi_0^m(\mathcal{X}_{m,1}^r) \subset X_0$. Then $\mathcal{X}_{m,1}^r$ is an X_0-variety. Now for $m \in \mathbb{Z}_{>0}$, let $\mu_m \simeq \mathbb{Z}/\mathbb{Z}m$ be the multiplicative group consisting of the m-roots in $\mathbb{C}$. We denote by $\hat{\mu}$ the projective limit $\varprojlim_m \mu_m$ of the projective system $\{\mu_i\}_{i \geq 1}$ with morphisms $\mu_{im} \longrightarrow \mu_i$ given by $t \longmapsto t^m$. Then $\mathcal{X}_{m,1}^r$ and $\mathcal{X}_{m,1,0}^r$ are endowed with a good action of μ_m (and hence of $\hat{\mu}$) defined by $\lambda \cdot \varphi(t) = \varphi(\lambda t)$.

Following the notations of [7], for a variety S over $\mathbb{C}$ we denote by $\mathcal{M}_S^{\hat{\mu}}$ the ring obtained from the Grothendieck ring $K_0^{\hat{\mu}}(\mathrm{Var}_S)$ of varieties over S with good $\hat{\mu}$-actions by inverting the Lefschetz motive $\mathbb{L}$ which is the class of $\mathbb{A}_{\mathbb{C}}^1 \times S \in K_0^{\hat{\mu}}(\mathrm{Var}_S)$ with trivial action of $\hat{\mu}$. Recall also that for an S-variety E with good $\hat{\mu}$-action, we denote its class in $K_0^{\hat{\mu}}(\mathrm{Var}_S)$ by $[E, \hat{\mu}]$ (also denoted by $[E/S, \hat{\mu}]$, or simply by $[E]$). When S consists of only one geometric point, i.e. $S = \mathrm{Spec}(\mathbb{C})$, we will write $K_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})$ instead of $K_0^{\hat{\mu}}(\mathrm{Var}_S)$. For any $s \in S(\mathbb{C})$, there are natural maps $i_s^{-1} : K_0^{\hat{\mu}}(\mathrm{Var}_S) \to K_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})$ and $i_s^{-1} : \mathcal{M}_S^{\hat{\mu}} \to \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$, defined by $[E, \hat{\mu}] \mapsto [E_s, \hat{\mu}]$ where E_s is the fiber at s of $E \to S$. Thus we obtain elements $[\mathcal{X}_{m,1}^r/X_0, \hat{\mu}]$ (or $[\mathcal{X}_{m,1}^r, \hat{\mu}]$, or $[\mathcal{X}_{m,1}^r]$) of $\mathcal{M}_{X_0}^{\hat{\mu}}$ and $[\mathcal{X}_{m,1,0}^r]$ of $\mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$. Note that $[\mathcal{X}_{m,1,0}^r] = i_0^{-1}([\mathcal{X}_{m,1}^r])$.

Definition 5.1. For a positive integer $r > 0$ and the rational function $f = \frac{P}{Q} : X \to \mathbb{A}_{\mathbb{C}}^1$ we define a power series $Z_f^r(T)$ of T over $\mathcal{M}_{X_0}^{\hat{\mu}}$ by

$$Z_f^r(T) = \sum_{m \geq 1} \left([\mathcal{X}_{m,1}^r/X_0, \hat{\mu}] \cdot \mathbb{L}^{-mn} \right) T^m. \tag{5.3}$$

We call it a motivic zeta function of f.

We recall the notion of rational series.

Definition 5.2. Let $\mathcal{A}$ be one of the following rings

$$\mathbb{Z}[\mathbb{L}, \mathbb{L}^{-1}], \quad \mathbb{Z}\left[\mathbb{L}, \mathbb{L}^{-1}, \frac{1}{1 - \mathbb{L}^{-i}}\right]_{i>0}, \quad \text{and} \quad \mathcal{M}_S^{\hat{\mu}}. \tag{5.4}$$

Let $\mathcal{A}[[T]]_{\mathrm{sr}}$ be the $\mathcal{A}$-submodule of $\mathcal{A}[[T]]$ generated by 1 and by finite products of elements of the form $\frac{\mathbb{L}^a T^b}{1 - \mathbb{L}^a T^b}$ with a in $\mathbb{Z}$ and b in $\mathbb{Z}_{>0}$.

By [6], there is a unique $\mathcal{A}$-linear homomorphism

$$\lim_{T \to \infty} : \mathcal{A}[[T]]_{\mathrm{sr}} \to \mathcal{A} \tag{5.5}$$

such that

$$\lim_{T \to \infty} \frac{\mathbb{L}^a T^b}{1 - \mathbb{L}^a T^b} = -1. \tag{5.6}$$

For a finite set I, we consider rational polyhedral convex cones in $\mathbb{R}_{>0}^I$. By this, we mean a convex subset of $\mathbb{R}_{>0}^I$ defined by a finite number of integral linear inequalities of type $l \geq 0$ or $l > 0$ and stable by multiplication by $\mathbb{R}_{>0}$. Let Δ be a rational polyhedral convex cone in $\mathbb{R}_{>0}^I$ and let $\overline{\Delta}$ denote its closure in $\mathbb{R}_{\geq 0}^I$. Let l and v be two integral linear forms on $\mathbb{Z}^I$ positive on $\overline{\Delta} \setminus \{(0, \ldots, 0)\}$. Let us consider the series

$$S_{\Delta,l,v}(T) := \sum_{a \in \Delta \cap \mathbb{N}_{>0}^I} \mathbb{L}^{-v(a)} T^{l(a)} \tag{5.7}$$

in $\mathbb{Z}[\mathbb{L}, \mathbb{L}^{-1}][[T]]$. Then we have the following lemma (see [13, Lemma 2.1.5] and [14, Section 2.9]).

Lemma 5.3. *With the above notations, the series $S_{\Delta,l,v}(T)$ lies in $\mathbb{Z}[\mathbb{L}, \mathbb{L}^{-1}][[T]]_{sr}$ and the limit $\lim_{T\to\infty} S_{\Delta,l,v}(T)$ is equal to $\chi_c(\Delta)$ i.e. the Euler characteristic with compact support of Δ. In particular, if Δ is a rational polyhedral convex cone in $\mathbb{R}^I_{>0}$ defined by*

$$\sum_{i\in K} a_i x_i \leq \sum_{i\in I\setminus K} a_i x_i \tag{5.8}$$

with $a_i \in \mathbb{Z}_+$, $a_i > 0$ for $i \in K$, K and $I\setminus K$ non-empty, then we have $\lim_{T\to\infty} S_{\Delta,l,v}(T) = 0$.

We shall describe the motivic zeta function of f in terms of log-resolutions. For this purpose, assume that f satisfies the conditions in Theorem 1.2. Then there exists a resolution of singularities $\pi_0 : \widetilde{X} \to X = \mathbb{C}^n$ of $P^{-1}(0)$ and $Q^{-1}(0)$ which induces an isomorphism $\widetilde{X} \setminus \pi_0^{-1}(\{0\}) \xrightarrow{\sim} X \setminus \{0\}$ such that $\pi_0^{-1}(\{0\})$ and $\pi_0^{-1}(P^{-1}(0) \cup Q^{-1}(0))$ are strict normal crossing divisors in $\widetilde{X}$. Denote by D_i $(1 \leq i \leq m)$, the irreducible components of the normal crossing divisor $D = \pi_0^{-1}(\{0\})$, let $D_P = \widetilde{P^{-1}(0)}$ and $D_Q = \widetilde{Q^{-1}(0)}$ be the (smooth) proper transforms of $P^{-1}(0)$ and $Q^{-1}(0)$ in $\widetilde{X}$ respectively. Note that they intersect transversally. Then the rational function $f \circ \pi_0$ on $\widetilde{X}$ has some points of indeterminacy. As in the proof of [28, Theorem 3.6] we construct towers of blow-ups of $\widetilde{X}$ over it to eliminate the points of indeterminacy of $f \circ \pi_0$. Then we obtain a proper morphism $\pi_1 : Y \longrightarrow \widetilde{X}$ of smooth complex varieties. Set $\pi = \pi_0 \circ \pi_1 : Y \longrightarrow X$ and let $\pi^{-1}(0) = \cup_{i=1}^k E_i$ be the irreducible decomposition of the normal crossing divisor $\pi^{-1}(0)$ in Y. Let E_P and E_Q be the (smooth) proper transforms of $D_P = \widetilde{P^{-1}(0)}$ and $D_Q = \widetilde{Q^{-1}(0)}$ in Y respectively. By our construction of Y, the divisor $\pi^{-1}(0) \cup E_P \cup E_Q$ in Y is strict normal crossing. Denote by G the union of its irreducible components along which the order of the rational function $g = f \circ \pi$ is ≤ 0 so that we have $E_Q \subset G$. Now we define an open subset Ω of Y by $\Omega = Y \setminus G$ and set

$$U = \pi^{-1}(0) \setminus G = \pi^{-1}(0) \cap \Omega \subset \pi^{-1}(0). \tag{5.9}$$

Then we have an isomorphism

$$\psi_f^{\mathrm{mero,c}}(\mathbb{C}_X)_0 \simeq R\Gamma_c(U; \psi_{f\circ\pi}(\mathbb{C}_Y)). \tag{5.10}$$

For each $i \in \{1,\ldots,k\}\cup\{P\}$, let $N_i(P), N_i(Q) > 0$ be the orders of the zeros of $P\circ\pi, Q\circ\pi$ along E_i and $b_i := N_i(P) - N_i(Q) \in \mathbb{Z}$ that of $g = f \circ \pi$ along E_i. Set

$$C := \{i \in \{1,\ldots,k\} : b_i > 0\} \cup \{P\}. \tag{5.11}$$

For each $i \in \{1,\ldots,k\}\cup\{P\}$, we also denote by $\nu_i - 1$ the multiplicity of E_i in the divisor of $\pi^* dx$, where dx is a local non-vanishing volume form at 0, i.e. a local generator of the sheaf of differential forms of maximal degree at 0. Note that we have $\nu_i > 0$.

For a non-empty subset $I \subset \{1, 2, \ldots, k, P\}$, set $E_I = \bigcap_{i\in I} E_i$,

$$E_I^\circ = E_I \setminus \left\{ \left(\bigcup_{i\notin I} E_i \right) \cup E_Q \right\} \subset Y \tag{5.12}$$

and $d_I = \gcd(b_i)_{i\in I} > 0$. Then, as in [8, Section 2.3], we can construct an unramified Galois covering $\widetilde{E_I^\circ} \longrightarrow E_I^\circ$ of E_I° as follows. First, for a point $p \in E_I^\circ$ we take an Zariski

affine open neighborhood W of p in Y on which there exist regular functions ξ_i $(i \in I)$ such that $E_i \cap W = \{\xi_i = 0\}$ for any $i \in I$. Then on W we have $g = g_{1,W}(g_{2,W})^{d_I}$, where we set $g_{1,W} = g \prod_{i \in I} \xi_i^{-b_i}$ and $g_{2,W} = \prod_{i \in I} \xi_i^{\frac{b_i}{d_I}}$. Note that $g_{1,W}$ is a unit on W and $g_{2,W} \colon W \longrightarrow \mathbb{C}$ is a regular function. It is easy to see that E_I° is covered by such affine open subsets W. Then as in [8, Section 2.3] by gluing the varieties

$$\widetilde{E_{I,W}^\circ} := \{(t,z) \in \mathbb{C}^* \times (E_I^\circ \cap W) \mid t^{d_I} = (g_{1,W})^{-1}(z)\} \tag{5.13}$$

together in an obvious way, we obtain the variety $\widetilde{E_I^\circ}$ over E_I° as an X_0-variety. The unramified Galois covering $\widetilde{E_I^\circ}$ of E_I° admits a natural μ_{d_I}-action defined by assigning the automorphism $(t,z) \longmapsto (\zeta_{d_I} t, z)$ of $\widetilde{E_I^\circ}$ to the generator $\zeta_{d_I} := \exp(2\pi\sqrt{-1}/d_I) \in \mu_{d_I}$. Namely the variety $\widetilde{E_I^\circ}$ is equipped with a good $\hat{\mu}$-action in the sense of [7, Section 2.4]. Then $[\widetilde{E_I^\circ}, \hat{\mu}]$ is an element in $\mathrm{K}_0^{\hat{\mu}}(\mathrm{Var}_{X_0})$.

By the same argument as in proof of [8, Theorem 2.4] we obtain the following.

Lemma 5.4. *With the previous notations, for any positive integers $r, m > 0$, we have the following equality in $\mathcal{M}_{X_0}^{\hat{\mu}}$*

$$[\mathcal{X}_{m,1}^r] = \mathbb{L}^{mn} \sum_{\substack{I \subset \{1,\ldots,k\} \cup \{P\} \\ I \neq \emptyset}} (\mathbb{L}-1)^{|I|-1} [\widetilde{E_I^\circ}] \left(\sum_{\substack{k_i \geq 1, i \in I \\ \sum k_i b_i = m, \sum k_i N_i(Q) \leq r \sum k_i b_i}} \mathbb{L}^{-\sum_{i \in I} k_i \nu_i} \right). \tag{5.14}$$

The Euler characteristic of complex constructible sets can be regarded as a ring homomorphism

$$\chi \colon \mathrm{K}_0^{\hat{\mu}}(\mathrm{Var}_\mathbb{C}) \longrightarrow \mathbb{Z}. \tag{5.15}$$

Since we have $\chi(\mathbb{L}) = 1$, it extends uniquely to a ring homomorphism

$$\chi \colon \mathcal{M}_\mathbb{C}^{\hat{\mu}} \longrightarrow \mathbb{Z}. \tag{5.16}$$

The following result is an analogue for rational function of [8, Theorem 1.1] and is also obtained in [41] under a different construction.

Corollary 5.5. *There exists $r_0 \gg 0$ such that for any $r > r_0$ and $m \geq 1$, the Lefschetz number $\Lambda(m)_{f,0}$ of f at the origin $0 \in X$ is equal to $\chi(\mathcal{X}_{m,1,0}^r)$.*

Proof. With the previous notations, we have $\chi((\mathbb{L}-1)^{|I|-1}) = 0$ for any I with $|I| > 1$. Then, it follows from Lemma 5.4 that

$$\chi(\mathcal{X}_{m,1,0}^r) = \sum_{\substack{i \in \{1,\ldots,k\}, b_i > 0 \\ b_i \mid m, N_i(Q) \leq r b_i}} b_i \chi(E_i^\circ). \tag{5.17}$$

Therefore, if $r > r_0 := \sup_{\{i : b_i > 0\}} \frac{N_i(Q)}{b_i}$, we get

$$\chi(\mathcal{X}_{m,1,0}^r) = \sum_{\substack{i \in \{1,\ldots,k\}, b_i > 0 \\ b_i \mid m}} b_i \chi(E_i^\circ). \tag{5.18}$$

Combining this with Corollary 3.2 we obtain

$$\Lambda(m)_{f,0} = \chi(\mathfrak{X}^r_{m,1,0}). \tag{5.19}$$

$\square$

The following results and definitions are inspired by [7, Section 3.5], [8, Section 2], [14, Section 3.8], [40, Section 1.4] and [41, Section 4.1]. The result below could be implied from [41, Theorem 6], though we provide here a different proof.

Theorem 5.6. *There exists $r_0 \gg 0$ such that for any $r > r_0$ the series $Z^r_f(T)$ is rational, and it is independent of $r > r_0$. Moreover for $r > r_0$ the limit $\lim_{T\to\infty} Z^r_f(T) \in \mathcal{M}^{\hat{\mu}}_{X_0}$ exists. For $r > r_0$ we set*

$$\mathcal{S}^{\mathrm{mero,c}}_f := -\lim_{T\to\infty} Z^r_f(T) \in \mathcal{M}^{\hat{\mu}}_{X_0}. \tag{5.20}$$

Then we have

$$\mathcal{S}^{\mathrm{mero,c}}_f = \sum_{I \subset C, I \neq \emptyset} (1 - \mathbb{L})^{|I|-1} [\widetilde{E^\circ_I}, \hat{\mu}]. \tag{5.21}$$

Proof. It follows from Lemma 5.4 that

$$Z^r_f(T) = \sum_{\substack{I \subset \{1,\dots,k\}\cup\{P\} \\ I \neq \emptyset}} (\mathbb{L} - 1)^{|I|-1} [\widetilde{E^\circ_I}] \sum_{m \geq 1} \left(\sum_{\substack{k_i \geq 1, i \in I, m \geq 1 \\ \sum k_i b_i = m, \sum k_i N_i(Q) \leq r \sum k_i b_i}} \mathbb{L}^{-\sum_{i \in I} k_i \nu_i} T^m \right). \tag{5.22}$$

For each nonempty subset I of the set $\{1, \dots, k\} \cup \{P\}$, we consider the cone

$$\Delta^r_I := \left\{ (m, (k_i)) \in \mathbb{R}_{>0} \times \mathbb{R}^I_{>0} \mid \sum_{i \in I} k_i N_i(Q) \leq r \sum_{i \in I} k_i b_i, \ \sum_{i \in I} k_i b_i = m \right\}. \tag{5.23}$$

Let us consider also the linear forms l, v on $\mathbb{R}_{>0} \times \mathbb{R}^I_{>0}$ defined by

$$l(m, (k_i)) = m, \qquad v(m, (k_i)) = \sum_{i \in I} \nu_i k_i. \tag{5.24}$$

Then we can rewrite the motivic zeta function as follows

$$Z^r_f(T) = \sum_{\substack{I \subset \{1,\dots,k\}\cup\{P\} \\ I \neq \emptyset}} (\mathbb{L} - 1)^{|I|-1} [\widetilde{E^\circ_I}] \cdot S_{\Delta^r_I, l, v}(T). \tag{5.25}$$

It is easy to see that Δ^r_I is a rational polyhedral cone and the integral linear forms l, v are positive on $\overline{\Delta^r_I} \setminus \{0\}$. Then by Lemma 5.3 the series $Z^r_f(T)$ is rational.

First, consider the case where $I \subset C$. Then we have $b_i > 0$ for any $i \in I$. So for $r > r_0 := \sup_{i \in I} N_i(Q)/b_i$ we get

$$S_{\Delta^r_I, l, v}(T) = \prod_{i \in I} \frac{\mathbb{L}^{-\nu_i} T^{b_i}}{1 - \mathbb{L}^{-\nu_i} T^{b_i}} \tag{5.26}$$

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and hence $\lim_{T \to \infty} S_{\Delta_I^r, l, v}(T) = (-1)^{|I|}$. Next, consider the case where $K := I \setminus C \neq \emptyset$. Then we have $\lim_{T \to \infty} S_{\Delta_I^r, l, v}(T) = \chi_c(\Delta_I^r)$. Nevertheless, the cone Δ_I^r is homeomorphic to the following one in $\mathbb{R}_{>0}^I$:

$$\left\{ (k_i) \in \mathbb{R}_{>0}^I \mid \sum_{i \in K} k_i(N_i(Q) - rb_i) \leq \sum_{i \in I \setminus K} k_i(rb_i - N_i(Q)) \right\}. \tag{5.27}$$

By Lemma 5.3 its Euler characteristic with compact support is equal to 0. This completes the proof. $\qquad\square$

Applying the base change morphism $\mathrm{Fiber}_0 : \mathcal{M}_{X_0}^{\hat{\mu}} \to \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$, defined by $[A/X_0, \hat{\mu}] \mapsto [A \times_{X_0} 0, \hat{\mu}]$ we set

$$\mathcal{S}_{f,0}^{\mathrm{mero,c}} := \mathrm{Fiber}_0(\mathcal{S}_f^{\mathrm{mero,c}}) \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}. \tag{5.28}$$

Definition 5.7. We call $\mathcal{S}_{f,0}^{\mathrm{mero,c}} \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$ the motivic Milnor fiber with compact support of f at the origin $0 \in X = \mathbb{C}^n$.

By Theorem 5.6 we obtain the following result. The formula in this result could be obtained from [41, Proposition 4].

Theorem 5.8. *The motivic Milnor fiber with compact support $\mathcal{S}_{f,0}^{\mathrm{mero,c}}$ of f at the origin $0 \in X = \mathbb{C}^n$ is written as*

$$\mathcal{S}_{f,0}^{\mathrm{mero,c}} = \sum_{\substack{I \subset \{1,\ldots,k\} \cap C \\ I \neq \emptyset}} \left\{ (1 - \mathbb{L})^{|I|-1} [\widetilde{E_I^\circ \setminus E_P}] + (1 - \mathbb{L})^{|I|} [E_I^\circ \cap E_P] \right\} \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}, \tag{5.29}$$

where $[E_I^\circ \cap E_P] \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$ is endowed with the trivial action of $\hat{\mu}$.

As in [7, Section 3.1.2 and 3.1.3], we denote by $\mathrm{HS}^{\mathrm{mon}}$ the abelian category of Hodge structures with a quasi-unipotent endomorphism. Then, to the object $\psi_f^{\mathrm{mero,c}}(\mathbb{C}_X)_0 \in \mathbf{D}_c^b(\{0\})$ and the semisimple part of the monodromy automorphism acting on it, we can associate an element

$$[H_{f,0}^{\mathrm{mero,c}}] = \sum_{j \in \mathbb{Z}} (-1)^j [H^j \psi_f^{\mathrm{mero,c}}(\mathbb{C}_X)_0] \in \mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}}) \tag{5.30}$$

as in [6] and [7], where the weight filtration of the limit mixed Hodge structure $[H^j \psi_f^{\mathrm{mero,c}}(\mathbb{C}_X)_0] \in \mathrm{HS}^{\mathrm{mon}}$ is the "relative" monodromy filtration defined by the Milnor monodromy of $\psi_f^{\mathrm{mero,c}}(\mathbb{C}_X)_0$. To describe the element $[H_{f,0}^{\mathrm{mero,c}}] \in \mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$ in terms of $\mathcal{S}_{f,0}^{\mathrm{mero,c}} \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$, let

$$\chi_h \colon \mathcal{M}_{\mathbb{C}}^{\hat{\mu}} \longrightarrow \mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}}) \tag{5.31}$$

be the Hodge characteristic morphism defined in [7] which associates to a variety Z with a good μ_d-action the Hodge structure

$$\chi_h([Z]) = \sum_{j \in \mathbb{Z}} (-1)^j [H_c^j(Z; \mathbb{Q})] \in \mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}}) \tag{5.32}$$

with the actions induced by the one $z \longmapsto \exp(2\pi\sqrt{-1}/d)z$ $(z \in Z)$ on Z. Then as in [30, Theorem 4.4] and [39], by applying [6, Theorem 4.2.1] and [14, Section 3.16] to our situation (5.10), we obtain the following result. This result could be also implied from [41, Theorem 8].

Theorem 5.9. *In the Grothendieck group* $\mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$, *we have the equality*

$$[H^{\mathrm{mero,c}}_{f,0}] = \chi_h(\mathcal{S}^{\mathrm{mero,c}}_{f,0}). \tag{5.33}$$

From now on, we assume that f is polynomial-like and satisfies the conditions in Theorem 1.2. For an element $[V] \in \mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$, $V \in \mathrm{HS}^{\mathrm{mon}}$ with a quasi-unipotent endomorphism $\Theta \colon V \xrightarrow{\sim} V$, $p, q \geq 0$ and $\lambda \in \mathbb{C}$ denote by $e^{p,q}([V])_\lambda$ the dimension of the λ-eigenspace of the morphism $V^{p,q} \xrightarrow{\sim} V^{p,q}$ induced by Θ on the (p,q)-part $V^{p,q}$ of V. Then by Proposition 4.2 and Theorem 4.4 we obtain the following result.

Corollary 5.10. *Assume that* $\lambda \neq 1$. *Then we have* $e^{p,q}([H^{\mathrm{mero,c}}_{f,0}])_\lambda = 0$ *for* $(p,q) \notin [0, n-1] \times [0, n-1]$. *Moreover for any* $(p,q) \in [0, n-1] \times [0, n-1]$ *we have the Hodge symmetry*

$$e^{p,q}([H^{\mathrm{mero,c}}_{f,0}])_\lambda = e^{n-1-q,n-1-p}([H^{\mathrm{mero,c}}_{f,0}])_\lambda. \tag{5.34}$$

Definition 5.11. We define a Puiseux series $\widetilde{\mathrm{sp}}_{f,0}(t)$ with coefficients in $\mathbb{Z}$ by

$$\widetilde{\mathrm{sp}}_{f,0}(t) = \sum_{\alpha \in (\mathbb{Q}\setminus\mathbb{Z})\cap(0,n)} \left(\dim \mathrm{Gr}^{\lfloor\alpha\rfloor}_F H^{n-1}(F_0; \mathbb{C})_{\exp(2\pi\sqrt{-1}\alpha)} \right) t^\alpha. \tag{5.35}$$

We call it the reduced Hodge spectrum of f at the origin $0 \in \mathbb{C}^n$.

By Corollary 5.10 and

$$e^{p,q}([H^{\mathrm{mero,c}}_{f,0}])_\lambda = e^{q,p}([H^{\mathrm{mero,c}}_{f,0}])_{\overline{\lambda}} \tag{5.36}$$

we obtain the symmetry of the reduced Hodge spectrum

$$\widetilde{\mathrm{sp}}_{f,0}(t) = t^n \cdot \widetilde{\mathrm{sp}}_{f,0}\left(\frac{1}{t}\right) \tag{5.37}$$

centered at $\frac{n}{2}$.

We can reduce $\mathcal{S}^{\mathrm{mero,c}}_{f,0} \in \mathcal{M}^{\hat\mu}_{\mathbb{C}}$ as follows. Let $\pi_0 \colon \widetilde{X} \to X = \mathbb{C}^n$ be a resolution of singularities of $P^{-1}(0)$ and $Q^{-1}(0)$ which induces an isomorphism $\widetilde{X}\setminus\pi_0^{-1}(\{0\}) \xrightarrow{\sim} X\setminus\{0\}$ such that for any irreducible component D_i ($1 \leq i \leq m$) of the (exceptional) normal crossing divisor $D = \pi_0^{-1}(\{0\})$ we have the condition

$$\mathrm{ord}_{D_i}(P \circ \pi_0) > \mathrm{ord}_{D_i}(Q \circ \pi_0). \tag{5.38}$$

For $1 \leq i \leq m$ let $a_i > 0$ be the order of $\widetilde{g} = f \circ \pi_0$ along D_i. Namely we set

$$a_i = \mathrm{ord}_{D_i}(P \circ \pi_0) - \mathrm{ord}_{D_i}(Q \circ \pi_0) > 0. \tag{5.39}$$

Let $D_P = \widetilde{P^{-1}(0)}$ and $D_Q = \widetilde{Q^{-1}(0)}$ be the (smooth) proper transforms of $P^{-1}(0)$ and $Q^{-1}(0)$ in $\widetilde{X}$ respectively. For a non-empty subset $I \subset \{1, 2, \ldots, m\}$, set $D_I = \bigcap_{i \in I} D_i$,

$$D^\circ_I = D_I \setminus \left\{ \left(\bigcup_{i \notin I} D_i \right) \cup D_Q \right\} \subset \widetilde{X} \tag{5.40}$$

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and $e_I = \gcd(a_i)_{i \in I} > 0$. Then we can construct an unramified Galois covering $\widetilde{D_I^\circ \setminus D_P}$ of $D_I^\circ \setminus D_P$ with a natural μ_{e_I}-action as above. Let $[\widetilde{D_I^\circ \setminus D_P}]$ be the element of the ring $\mathcal{M}_\mathbb{C}^{\hat\mu}$ which corresponds to $\widetilde{D_I^\circ \setminus D_P}$. Then as in the proof of [29, Theorem 4.7] we obtain the following result. Define an element $\mathcal{R}_{f,0}^{\mathrm{mero,c}} \in \mathcal{M}_\mathbb{C}^{\hat\mu}$ by

$$\mathcal{R}_{f,0}^{\mathrm{mero,c}} = \sum_{I \neq \emptyset} \left\{ (1 - \mathbb{L})^{|I|-1} [\widetilde{D_I^\circ \setminus D_P}] + (1 - \mathbb{L})^{|I|} [D_I^\circ \cap D_P] \right\} \in \mathcal{M}_\mathbb{C}^{\hat\mu}, \qquad (5.41)$$

where $[D_I^\circ \cap D_P] \in \mathcal{M}_\mathbb{C}^{\hat\mu}$ is endowed with the trivial action of $\hat\mu$.

Theorem 5.12. *In the Grothendieck group* $\mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$, *we have the equality*

$$\chi_h(\mathcal{S}_{f,0}^{\mathrm{mero,c}}) = \chi_h(\mathcal{R}_{f,0}^{\mathrm{mero,c}}). \qquad (5.42)$$

From now on, we shall rewrite our formula for $\mathcal{R}_{f,0}^{\mathrm{mero,c}} \in \mathcal{M}_\mathbb{C}^{\hat\mu}$ more explicitly by using the Newton polyhedron $\Gamma_+(f)$ of f. For this purpose, we assume that the rational function $f(x) = \frac{P(x)}{Q(x)}$ is non-degenerate at the origin $0 \in X = \mathbb{C}^n$ and $P(x), Q(x)$ are convenient. For f to be polynomial-like, we assume moreover that $\Gamma_+(P)$ is properly contained in $\Gamma_+(Q)$ in the following sense.

Definition 5.13. We say that the Newton polyhedron $\Gamma_+(P)$ is properly contained in the one $\Gamma_+(Q)$ if for any vector $u \in \mathrm{Int}(\mathbb{R}_+^n)$ in the interior $\mathrm{Int}(\mathbb{R}_+^n)$ of $\mathbb{R}_+^n$ we have

$$\min_{v \in \Gamma_+(P)} \langle u, v \rangle > \min_{v \in \Gamma_+(Q)} \langle u, v \rangle. \qquad (5.43)$$

In this case, we write $\Gamma_+(P) \subset\subset \Gamma_+(Q)$.

We use the notations in Section 3. For a compact face $\gamma \prec \Gamma_+(f)$ of $\Gamma_+(f)$ let

$$\gamma(P) \prec \Gamma_+(P), \qquad \gamma(Q) \prec \Gamma_+(Q) \qquad (5.44)$$

be the corresponding faces such that

$$\gamma = \gamma(P) + \gamma(Q). \qquad (5.45)$$

Let $\square_\gamma \subset \mathbb{R}^n$ be the convex hull of $\gamma(P)$ and $\gamma(Q)$. We define the Cayley polyhedron $\Gamma_+(P) * \Gamma_+(Q) \subset \mathbb{R}^{n+1}$ to be the convex hull of

$$(\Gamma_+(P) \times \{0\}) \cup (\Gamma_+(Q) \times \{1\}) \qquad (5.46)$$

in $\mathbb{R}^{n+1}$.

EXAMPLE 5.14. Let $P(x,y) = x^4 + y^4$, $Q(x,y) = x^2 + xy + y^3$. Then, one can check that the rational function $f(x,y) = \frac{P(x,y)}{Q(x,y)}$ is non-degenerate at the origin $0 \in X = \mathbb{C}^2$ and $\Gamma_+(P) \subset\subset \Gamma_+(Q)$. The Newton polyhedron of P has one compact facet AB and that of Q has two compact ones CD and DE (see Figure 1). For the face $\gamma = FG$ of $\Gamma_+(f)$ we have

$$\gamma(P) = AB, \qquad \gamma(Q) = CD,$$

and $\square_\gamma$ is the convex hull of the four points A, B, C, D (see Figure 1). The Caley polyhedron $\Gamma_+(P) * \Gamma_+(Q)$ is shown in Figure 2 below.
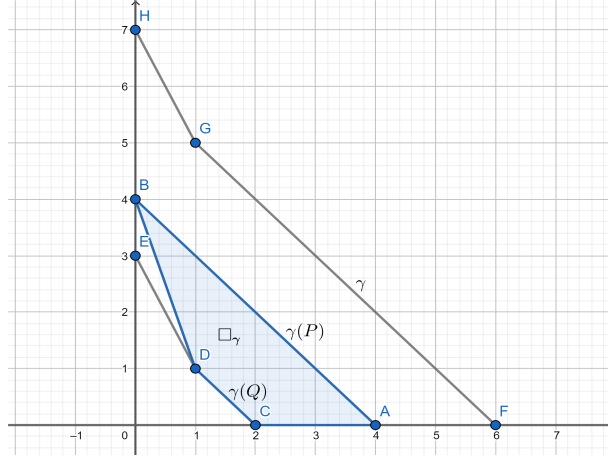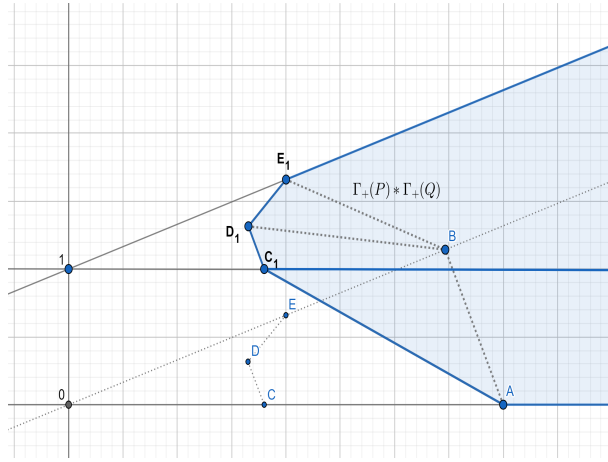
Figure 1: Newton polygons

Figure 2: Caley polyhedron

By our assumption $\Gamma_+(P) \subset\subset \Gamma_+(Q)$, the first projection $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}^1 \to \mathbb{R}^n$ induces an isomorphism of a side face $\tilde{\gamma}$ of $\Gamma_+(P) * \Gamma_+(Q)$ to $\square_\gamma$. Hence we have $\dim\square_\gamma = \dim\gamma + 1$. Denote by $\mathbb{L}(\square_\gamma) \simeq \mathbb{R}^{\dim\gamma+1}$ the linear subspace of $\mathbb{R}^n$ parallel to the affine span $\mathrm{Aff}(\square_\gamma) \simeq \mathbb{R}^{\dim\gamma+1}$ of $\square_\gamma$ in $\mathbb{R}^n$. Let $H(\gamma, P)$ (resp. $H(\gamma, Q)) \subset \mathrm{Aff}(\square_\gamma)$ be the affine hyperplane of $\mathrm{Aff}(\square_\gamma)$ containing $\gamma(P)$ (resp. $\gamma(Q)$) and parallel to $\mathrm{Aff}(\gamma)$. By a suitable choice of a translation isomorphism $\mathrm{Aff}(\square_\gamma) \simeq \mathbb{L}(\square_\gamma)$, we may assume that the image of $H(\gamma, Q) \subset \mathrm{Aff}(\square_\gamma)$ in $\mathbb{L}(\square_\gamma)$ passes through the origin $0 \in \mathbb{L}(\square_\gamma)$. Denote by $L(\gamma, P)$ (resp. $L(\gamma, Q)) \subset \mathbb{L}(\square_\gamma)$ the image of $H(\gamma, P)$ (resp. $H(\gamma, Q)$). Let $M_\gamma = \mathbb{Z}^n \cap \mathbb{L}(\square_\gamma) \simeq \mathbb{Z}^{\dim\gamma+1}$ be the lattice in $\mathbb{L}(\square_\gamma) \simeq \mathbb{R}^{\dim\gamma+1}$. In the dual $\mathbb{L}(\square_\gamma)^* \simeq \mathbb{R}^{\dim\gamma+1}$ of $\mathbb{L}(\square_\gamma)$ consider also its dual lattice $M_\gamma^* \simeq \mathbb{Z}^{\dim\gamma+1}$. We define a one dimensional subspace $L(\gamma, Q)^\perp \simeq \mathbb{R}$ of $\mathbb{L}(\square_\gamma)^*$ by

$$L(\gamma, Q)^\perp = \{u \in \mathbb{L}(\square_\gamma)^* \mid \langle u, v \rangle = 0 \ (v \in L(\gamma, Q))\} \subset \mathbb{L}(\square_\gamma)^*. \qquad (5.47)$$

Let $\alpha_\gamma \in (L(\gamma, Q)^\perp \cap M_\gamma^*) \setminus \{0\} \simeq \mathbb{Z} \setminus \{0\}$ be the primitive vector whose value on $L(\gamma, P) \subset \mathbb{L}(\square_\gamma)$ is a positive integer. We call it the lattice distance of $L(\gamma, P)$ from $L(\gamma, Q)$ and denote it by $d_\gamma > 0$. By using the lattice $M_\gamma = \mathbb{Z}^n \cap \mathbb{L}(\square_\gamma) \simeq \mathbb{Z}^{\dim\gamma+1}$ in

$\mathbb{L}(\square_\gamma) \simeq \mathbb{R}^{\dim\gamma+1}$ we set

$$T_{\square_\gamma} := \mathrm{Spec}(\mathbb{C}[M_\gamma]) \simeq (\mathbb{C}^*)^{\dim\gamma+1}. \tag{5.48}$$

For $v \in M_\gamma$ define their lattice heights $\mathrm{ht}(v, \gamma) \in \mathbb{Z}$ from $L(\gamma, Q)$ in $\mathbb{L}(\square_\gamma)$ by $\mathrm{ht}(v, \gamma) = \langle \alpha_\gamma, v \rangle$. Set $\zeta_{d_\gamma} = \exp(\frac{2\pi\sqrt{-1}}{d_\gamma}) \in \mathbb{C}^*$. Then to the group homomorphism $M_\gamma \longrightarrow \mathbb{C}^*$ defined by $v \longmapsto \zeta_{d_\gamma}^{\mathrm{ht}(v,\gamma)}$ we can naturally associate an element $\tau_\gamma \in T_{\square_\gamma}$. We define a Laurent polynomial $g_\gamma(x) = \sum_{v \in M_\gamma} c_v x^v$ on $T_{\square_\gamma}$ by

$$c_v = \begin{cases} a_v & (v \in \gamma(P)), \\ -b_v & (v \in \gamma(Q)), \\ 0 & (\text{otherwise}), \end{cases} \tag{5.49}$$

where $P(x) = \sum_{v \in \mathbb{Z}_+^n} a_v x^v$ and $Q(x) = \sum_{v \in \mathbb{Z}_+^n} b_v x^v$. Then the Newton polytope $NP(g_\gamma)$ of g_γ is $\square_\gamma$, $\mathrm{supp} g_\gamma \subset \gamma(P) \sqcup \gamma(Q)$ and the hypersurface $Z_{\square_\gamma}^* = \{x \in T_{\square_\gamma} \mid g_\gamma(x) = 0\}$ is non-degenerate by our assumption. Since $Z_{\square_\gamma}^* \subset T_{\square_\gamma}$ is invariant by the multiplication $l_{\tau_\gamma} \colon T_{\square_\gamma} \xrightarrow{\sim} T_{\square_\gamma}$ by τ_γ, $Z_{\square_\gamma}^*$ admits an action of μ_{d_γ}. We thus obtain an element $[Z_{\square_\gamma}^*]$ of $\mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$. For a compact face $\gamma \prec \Gamma_+(f)$ let $s_\gamma > 0$ be the dimension of the minimal coordinate subspace of $\mathbb{R}^n$ containing γ and set $m_\gamma = s_\gamma - \dim\gamma - 1 \geq 0$. Finally, for $\lambda \in \mathbb{C}$ and an element $H \in \mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$ denote by $H_\lambda \in \mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$ the eigenvalue λ-part of H. Then by applying the proof of [30, Theorem 4.3 (i)] to our geometric situation in Theorems 5.9 and 5.12, we obtain the following result.

Theorem 5.15. *Assume that $\lambda \neq 1$. Then we have the equality*

$$[H_{f,0}^{\mathrm{mero,c}}]_\lambda = \chi_h(\mathcal{S}_{f,0}^{\mathrm{mero,c}})_\lambda = \sum_\gamma \chi_h((1-\mathbb{L})^{m_\gamma} \cdot [Z_{\square_\gamma}^*])_\lambda \tag{5.50}$$

in $\mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$, where in the sum $\sum_\gamma$ the face γ of $\Gamma_+(f)$ ranges through the compact ones.

Proof. For a compact face $\gamma \prec \Gamma_+(f)$ of $\Gamma_+(f)$ set $T_\gamma := \mathrm{Spec}(\mathbb{C}[\mathbb{Z}^n \cap L(\gamma, Q)]) \simeq (\mathbb{C}^*)^{\dim\gamma}$. Then we can naturally define a Laurent polynomial $P_\gamma(x)$ (resp. $Q_\gamma(x)$) on it whose Newton polytope is $\gamma(P)$ (resp. $\gamma(Q)$) and the non-degenerate hypersurface $Z_{\square_\gamma}^* \subset T_{\square_\gamma} \simeq (\mathbb{C}^*)^{\dim\gamma+1}$ is isomorphic to

$$\{(x, t) \in T_\gamma \times \mathbb{C}^* \mid P_\gamma(x)t^{d_\gamma} - Q_\gamma(x) = 0\}. \tag{5.51}$$

On the other hand, as in the proof of [30, Theorem 4.3], we can show that the contribution to $\chi_h(\mathcal{S}_{f,0}^{\mathrm{mero,c}})_\lambda = \chi_h(\mathcal{R}_{f,0}^{\mathrm{mero,c}})_\lambda$ for $\lambda \neq 1$ from the compact face γ is equal to

$$\chi_h((1-\mathbb{L})^{m_\gamma} \cdot [Z_{\square_\gamma}^\circ])_\lambda, \tag{5.52}$$

where we set

$$Z_{\square_\gamma}^\circ = \left\{(x, t) \in T_\gamma \times \mathbb{C}^* \mid P_\gamma(x) \cdot Q_\gamma(x) \neq 0, t^{-d_\gamma} = \frac{P_\gamma(x)}{Q_\gamma(x)}\right\}. \tag{5.53}$$

Let us set

$$Z_\gamma = \{(x,t) \in T_\gamma \times \mathbb{C}^* \mid P_\gamma(x) = Q_\gamma(x) = 0\} \subset Z^*_{\square_\gamma}. \tag{5.54}$$

Then we have an equality

$$[Z^\circ_{\square_\gamma}] = [Z^*_{\square_\gamma}] - [Z_\gamma]. \tag{5.55}$$

in $\mathcal{M}^{\hat{\mu}}_{\mathbb{C}}$. Since the restriction of l_{τ_γ} to Z_γ is homotopic to the identity, for $\lambda \neq 1$ we obtain

$$\chi_h((1 - \mathbb{L})^{m_\gamma} \cdot [Z^\circ_{\square_\gamma}])_\lambda = \chi_h((1 - \mathbb{L})^{m_\gamma} \cdot [Z^*_{\square_\gamma}])_\lambda. \tag{5.56}$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Now by Theorems 4.4 and 5.15 and Remark 4.5 (see also the proof of [30, Theorem 4.3 (ii)]), we obtain the following theorem.

Theorem 5.16. *Assume that $\lambda \neq 1$ and $k \geq 1$. Then the number of the Jordan blocks for the eigenvalue λ with sizes $\geq k$ in $\Phi_{n-1,0} \colon H^{n-1}(F_0; \mathbb{C}) \xrightarrow{\sim} H^{n-1}(F_0; \mathbb{C})$ is equal to*

$$(-1)^{n-1} \sum_{p+q=n-2+k,n-1+k} \left\{ \sum_\gamma e^{p,q}(\chi_h((1 - \mathbb{L})^{m_\gamma} \cdot [Z^*_{\square_\gamma}]))_\lambda \right\}, \tag{5.57}$$

where in the sum $\sum_\gamma$ the face γ of $\Gamma_+(f)$ ranges through the comapct ones.

6 Combinatorial descriptions of Jordan normal forms and reduced Hodge spectra

In this section, for the meromorphic function f we give combinatorial descriptions of the Jordan normal forms of its Milnor monodromy $\Phi_{n-1,0}$ for the eigenvalues $\lambda \neq 1$ and its reduced Hodge spectrum as in Matsui-Takeuchi [30], Stapledon [47] and Saito [43].

6.1 Equivariant Ehrhart theory of Katz-Stapledon

First we recall some polynomials in the Equivariant Ehrhart theory of Katz-Stapledon [20] and Stapledon [47]. Throughout this paper, we regard the empty set $\emptyset$ as a (-1)-dimensional polytope, and as a face of any polytope. Let P be a polytope. If a subset $F \subset P$ is a face of P, we write $F \prec P$. For a pair of faces $F \prec F' \prec P$ of P, we denote by $[F, F']$ the face poset $\{F'' \prec P \mid F \prec F'' \prec F'\}$, and by $[F, F']^*$ a poset which is equal to $[F, F']$ as a set with the reversed order.

Definition 6.1. Let B be a poset $[F, F']$ or $[F, F']^*$. We define a polynomial $g(B, t)$ of degree $\leq (\dim F' - \dim F)/2$ as follows. If $F = F'$, we set $g(B; t) = 1$. If $F \neq F'$ and $B = [F, F']$ (resp. $B = [F, F']^*$), we define $g(B; t)$ inductively by

$$t^{\dim F' - \dim F} g(B; t^{-1}) = \sum_{F'' \in [F,F']} (t - 1)^{\dim F' - \dim F''} g([F, F'']; t). \tag{6.1}$$

$$(\text{resp. } t^{\dim F' - \dim F} g(B; t^{-1}) = \sum_{F'' \in [F,F']^*} (t - 1)^{\dim F'' - \dim F} g([F'', F']^*; t).) \tag{6.2}$$

Let us set

$$Z_\gamma = \{(x,t) \in T_\gamma \times \mathbb{C}^* \mid P_\gamma(x) = Q_\gamma(x) = 0\} \subset Z^*_{\square_\gamma}. \tag{5.54}$$

Then we have an equality

$$[Z^\circ_{\square_\gamma}] = [Z^*_{\square_\gamma}] - [Z_\gamma]. \tag{5.55}$$

in $\mathcal{M}^{\hat{\mu}}_{\mathbb{C}}$. Since the restriction of l_{τ_γ} to Z_γ is homotopic to the identity, for $\lambda \neq 1$ we obtain

$$\chi_h((1 - \mathbb{L})^{m_\gamma} \cdot [Z^\circ_{\square_\gamma}])_\lambda = \chi_h((1 - \mathbb{L})^{m_\gamma} \cdot [Z^*_{\square_\gamma}])_\lambda. \tag{5.56}$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Now by Theorems 4.4 and 5.15 and Remark 4.5 (see also the proof of [30, Theorem 4.3 (ii)]), we obtain the following theorem.

Theorem 5.16. *Assume that $\lambda \neq 1$ and $k \geq 1$. Then the number of the Jordan blocks for the eigenvalue λ with sizes $\geq k$ in $\Phi_{n-1,0} \colon H^{n-1}(F_0; \mathbb{C}) \xrightarrow{\sim} H^{n-1}(F_0; \mathbb{C})$ is equal to*

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where in the sum $\sum_\gamma$ the face γ of $\Gamma_+(f)$ ranges through the comapct ones.

6 Combinatorial descriptions of Jordan normal forms and reduced Hodge spectra

In this section, for the meromorphic function f we give combinatorial descriptions of the Jordan normal forms of its Milnor monodromy $\Phi_{n-1,0}$ for the eigenvalues $\lambda \neq 1$ and its reduced Hodge spectrum as in Matsui-Takeuchi [30], Stapledon [47] and Saito [43].

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$$t^{\dim F' - \dim F} g(B; t^{-1}) = \sum_{F'' \in [F,F']} (t - 1)^{\dim F' - \dim F''} g([F, F'']; t). \tag{6.1}$$

$$(\text{resp. } t^{\dim F' - \dim F} g(B; t^{-1}) = \sum_{F'' \in [F,F']^*} (t - 1)^{\dim F'' - \dim F} g([F'', F']^*; t).) \tag{6.2}$$

In what follows, we assume that P is a lattice polytope in $\mathbb{R}^n$. Let S be a subset of $P \cap \mathbb{Z}^n$ containing the vertices of P, and $\omega \colon S \to \mathbb{Z}$ be a function. We denote by UH_ω the convex hull in $\mathbb{R}^n \times \mathbb{R}$ of the set $\{(v, s) \in \mathbb{R}^n \times \mathbb{R} \mid v \in S, s \geq \omega(v)\}$. Then, the set of all the projections of the bounded faces of UH_ω to $\mathbb{R}^n$ defines a lattice polyhedral subdivision $\mathcal{S}$ of P. Here a lattice polyhedral subdivision $\mathcal{S}$ of a polytope P is a set of some polytopes in P such that the intersection of any two polytopes in $\mathcal{S}$ is a face of both and all vertices of any polytope in $\mathcal{S}$ are in $\mathbb{Z}^n$. Moreover, the set of all the bounded faces of UH_ω defines a piecewise $\mathbb{Q}$-affine convex function $\nu \colon P \to \mathbb{R}$. For a cell $F \in \mathcal{S}$, we denote by $\sigma(F)$ the smallest face of P containing F, and $\mathrm{lk}_\mathcal{S}(F)$ the set of all cells of $\mathcal{S}$ containing F. We call $\mathrm{lk}_\mathcal{S}(F)$ the link of F in $\mathcal{S}$. Note that $\sigma(\emptyset) = \emptyset$ and $\mathrm{lk}_\mathcal{S}(\emptyset) = \mathcal{S}$.

Definition 6.2. For a cell $F \in \mathcal{S}$, the h-polynomial $h(\mathrm{lk}_\mathcal{S}(F); t)$ of the link $\mathrm{lk}_\mathcal{S}(F)$ of F is defined by

$$t^{\dim P - \dim F} h(\mathrm{lk}_\mathcal{S}(F); t^{-1}) = \sum_{F' \in \mathrm{lk}_\mathcal{S}(F)} g([F, F']; t)(t - 1)^{\dim P - \dim F'}. \tag{6.3}$$

The local h-polynomial $l_P(\mathcal{S}, F; t)$ of F in $\mathcal{S}$ is defined by

$$l_P(\mathcal{S}, F; t) = \sum_{\sigma(F) \prec Q \prec P} (-1)^{\dim P - \dim Q} h(\mathrm{lk}_{\mathcal{S}|_Q}(F); t) \cdot g([Q, P]^*; t). \tag{6.4}$$

For $\lambda \in \mathbb{C}$ and $v \in mP \cap \mathbb{Z}^n$ ($m \in \mathbb{Z}_+ := \mathbb{Z}_{\geq 0}$) we set

$$w_\lambda(v) = \begin{cases} 1 & \left(\exp\left(2\pi\sqrt{-1} \cdot m\nu(\tfrac{v}{m}) \right) = \lambda \right) \\ \\ 0 & (\text{otherwise}). \end{cases} \tag{6.5}$$

We define the λ-weighted Ehrhart polynomial $\phi_\lambda(P, \nu; m) \in \mathbb{Z}[m]$ of P with respect to $\nu : P \to \mathbb{R}$ by

$$\phi_\lambda(P, \nu; m) := \sum_{v \in mP \cap \mathbb{Z}^n} w_\lambda(v). \tag{6.6}$$

Then $\phi_\lambda(P, \nu; m)$ is a polynomial in m with coefficients $\mathbb{Z}$ whose degree is $\leq \dim P$ (see [47]).

Definition 6.3. ([47])

(i) We define the λ-weighted h^*-polynomial $h_\lambda^*(P, \nu; u) \in \mathbb{Z}[u]$ by

$$\sum_{m \geq 0} \phi_\lambda(P, \nu; m) u^m = \frac{h_\lambda^*(P, \nu; u)}{(1 - u)^{\dim P + 1}}. \tag{6.7}$$

If P is the empty polytope, we set $h_1^*(P, \nu; u) = 1$ and $h_\lambda^*(P, \nu; u) = 0$ ($\lambda \neq 1$).

(ii) We define the λ-local weighted h^*-polynomial $l_\lambda^*(P, \nu; u) \in \mathbb{Z}[u]$ by

$$l_\lambda^*(P, \nu; u) = \sum_{Q \prec P} (-1)^{\dim P - \dim Q} h_\lambda^*(Q, \nu|_Q; u) \cdot g([Q, P]^*; u). \tag{6.8}$$

If P is the empty polytope, we set $l_1^*(P, \nu; u) = 1$ and $l_\lambda^*(P, \nu; u) = 0$ ($\lambda \neq 1$).

Definition 6.4. ([47])

(i) We define the λ-weighted limit mixed h^*-polynomial $h^*_\lambda(P, \nu; u, v) \in \mathbb{Z}[u, v]$ by

$$h^*_\lambda(P, \nu; u, v) := \sum_{F \in \mathcal{S}} v^{\dim F + 1} l^*_\lambda(F, \nu|_F; uv^{-1}) \cdot h(\text{lk}_\mathcal{S}(F); uv). \qquad (6.9)$$

(ii) We define the λ-local weighted limit mixed h^*-polynomial $l^*_\lambda(P, \nu; u, v) \in \mathbb{Z}[u, v]$ by

$$l^*_\lambda(P, \nu; u, v) := \sum_{F \in \mathcal{S}} v^{\dim F + 1} l^*_\lambda(F, \nu|_F; uv^{-1}) \cdot l_P(\mathcal{S}, F; uv). \qquad (6.10)$$

6.2 Jordan normal forms and reduced Hodge spectra

Assume that the meromorphic function $f(x) = \frac{P(x)}{Q(x)}$ is non-degenerate at the origin $0 \in X = \mathbb{C}^n$ and $P(x), Q(x)$ are convenient. For f to be polynomial-like, we assume moreover that $\Gamma_+(P)$ is properly contained in $\Gamma_+(Q)$: $\Gamma_+(P) \subset\subset \Gamma_+(Q)$ (see Definition 5.13). We call the union of the compact faces of $\Gamma_+(P)$ (resp. $\Gamma_+(Q)$) the Newton boundary of P (resp. Q) and denote it by Γ_P (resp. Γ_Q). Denote by K the convex hull of the closure of $\Gamma_+(Q) \setminus \Gamma_+(P)$ in $\mathbb{R}^n$ and define a piecewise $\mathbb{Q}$-affine function ν on K which takes the value 1 (resp. 0) on $\Gamma_Q \subset \mathbb{R}^n$ (resp. on the convex hull of $\Gamma_P \subset \mathbb{R}^n$) such that for any compact face γ of $\Gamma_+(f)$ the restriction of ν to $\square_\gamma$ is an affine function. For $\lambda \in \mathbb{C}$ we define the equivariant Hodge-Deligne polynomial for the eigenvalue λ (of the mixed Hodge structures of the cohomology groups of the Milnor fiber F_0) $E_\lambda(F_0; u, v) \in \mathbb{Z}[u, v]$ by

$$E_\lambda(F_0; u, v) = \sum_{p,q \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} (-1)^j h^{p,q}_\lambda(H^j(F_0; \mathbb{C})) u^p v^q \in \mathbb{Z}[u, v], \qquad (6.11)$$

where $h^{p,q}_\lambda(H^j(F_0; \mathbb{C}))$ is the dimension of $\text{Gr}^p_F \text{Gr}^W_{p+q} H^j(F_0; \mathbb{C})_\lambda$. Then for $\lambda \neq 1$, as in [30], [47] and [43], by Theorem 5.15 we can calculate the λ-part of the Hodge realization of the motivic Milnor fiber $\mathcal{S}^{\text{mero,c}}_{f,0}$ of f and obtain the following formula for $E_\lambda(F_0; u, v)$.

Theorem 6.5. *In the situation as above, for any $\lambda \neq 1$ we have*

$$uv E_\lambda(F_0; u, v) = (-1)^{n-1} l^*_\lambda(K, \nu; u, v). \qquad (6.12)$$

Let $\mathcal{S}_\nu$ be the polyhedral subdivision of the polytope K defined by ν. By the definition of the h^*-polynomial, for $\lambda \neq 1$ we have

$$l^*_\lambda(K, \nu; u, u) = \sum_{\gamma \prec \Gamma_+(f) : \text{compact}} u^{\dim \square_\gamma + 1} l^*_\lambda(\square_\gamma, \nu; 1) \cdot l_K(\mathcal{S}_\nu, \square_\gamma; u^2), \qquad (6.13)$$

where in the sum Σ the face γ ranges through the compact ones of $\Gamma_+(f)$. The polynomial $l_K(\mathcal{S}_\nu, \square_\gamma; t)$ is symmetric and unimodal centered at $(n - \dim \gamma - 1)/2$, i.e. if $a_i \in \mathbb{Z}$ is the coefficient of t^i in $l_K(\mathcal{S}_\nu, \square_\gamma; t)$ we have $a_i = a_{n-\dim\gamma-1-i}$ and $a_i \leq a_j$ for $0 \leq i \leq j \leq (n - \dim \gamma - 1)/2$. Therefore, it can be expressed in the form

$$l_K(\mathcal{S}_\nu, \square_\gamma; t) = \sum_{i=0}^{\lfloor (n-1-\dim\gamma)/2 \rfloor} \widetilde{l}_{\gamma,i}(t^i + t^{i+1} + \cdots + t^{n-1-\dim\gamma-i}), \qquad (6.14)$$

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for some non-negative integers $\widetilde{l}_{\gamma,i} \in \mathbb{Z}_{\geq 0}$. We set

$$\widetilde{l}_K(\mathcal{S}_\nu, \square_\gamma, t) := \sum_{i=0}^{\lfloor (n-1-\dim\gamma)/2 \rfloor} \widetilde{l}_{\gamma,i} t^i. \tag{6.15}$$

For $k \in \mathbb{Z}_{>0}$ and $\lambda \in \mathbb{C}$ we denote by $J_{k,\lambda}$ the number of the Jordan blocks in $\Phi_{n-1,0}$ with size k for the eigenvalue λ. Then we obtain the following formula for them.

Corollary 6.6. *In the situation as above, for any $\lambda \neq 1$ we have*

$$\sum_{0 \leq k \leq n-1} J_{n-k,\lambda} u^{k+2} = \sum_{\gamma \prec \Gamma_+(f):\text{compact}} u^{\dim\square_\gamma + 1} l_\lambda^*(\square_\gamma, \nu; 1) \cdot \widetilde{l}_K(\mathcal{S}_\nu, \square_\gamma; u^2), \tag{6.16}$$

where in the sum Σ of the right hand side the face γ ranges through the compact ones of $\Gamma_+(f)$.

Let $q_1, \ldots, q_l$ (resp. $\gamma_1, \ldots, \gamma_{l'}$) be the 0-dimensional (resp. 1-dimensional) faces of $\Gamma_+(f)$ such that $q_i \in \text{Int}(\mathbb{R}_+^n)$ (resp. the relative interior $\text{rel.int}(\gamma_i)$ of γ_i is contained in $\text{Int}(\mathbb{R}_+^n)$). For each q_i (resp. γ_i), we set $d_i := d_{q_i} > 0$ (resp. $e_i := d_{\gamma_i} > 0$). Moreover, for $\lambda \neq 1$ and $1 \leq i \leq l'$ such that $\lambda^{e_i} = 1$ we set

$$n(\lambda)_i := \sharp\{v \in \mathbb{Z}^n \cap \text{rel.int}(\square_{\gamma_i}) \mid \text{ht}(v)_i = k\}$$
$$+ \sharp\{v \in \mathbb{Z}^n \cap \text{rel.int}(\square_{\gamma_i}) \mid \text{ht}(v)_i = e_i - k\}, \tag{6.17}$$

where k is the minimal positive integer satisfying $\lambda = \zeta_{e_i}^k$ and for $v \in \mathbb{Z}^n \cap \text{rel.int}(\square_{\gamma_i})$ we denote by $\text{ht}(v)_i > 0$ the lattice height of v from the hyperplane $H(\gamma_i, Q) \subset \text{Aff}(\square_{\gamma_i})$. Then we have the following generalization of [30, Theorem 4.4] to polynomial-like rational functions.

Theorem 6.7. *In the situation as above, for any $\lambda \neq 1$ we have*

(i) *The number of the Jordan blocks for the eigenvalue λ with the maximal possible size n in $\Phi_{n-1,0} : H^{n-1}(F_0; \mathbb{C}) \xrightarrow{\sim} H^{n-1}(F_0; \mathbb{C})$ is equal to $\sharp\{q_i \mid \lambda^{d_i} = 1\}$.*

(ii) *The number of the Jordan blocks for the eigenvalue λ with the second maximal possible size $n - 1$ in $\Phi_{n-1,0}$ is equal to $\sum_{i: \lambda^{e_i}=1} n(\lambda)_i$.*

For a compact face $\gamma \prec \Gamma_+(f)$ we define a Puiseux series $h_\gamma(t)$ with coefficients in $\mathbb{Z}$ by

$$h_\gamma(t) = \sum_{\beta \in (\mathbb{Q}\backslash\mathbb{Z}) \cap (0,+\infty)} \left\{ \phi_{\exp(2\pi\sqrt{-1}\beta)}(\square_\gamma, \nu; \lfloor\beta\rfloor + 1) - \phi_{\exp(2\pi\sqrt{-1}\beta)}(\square_\gamma, \nu; \lfloor\beta\rfloor) \right\} t^\beta. \tag{6.18}$$

Then as in [29, Theorem 5.16] (or [43, Corollary 5.3]), by Theorem 5.15 (or Theorem 6.5) we obtain the following result.

Corollary 6.8. *In the situation as above, we have*

$$\widetilde{\text{sp}}_{f,0}(t) = \sum_{\gamma \prec \Gamma_+(f):\text{compact}} (-1)^{n-1-\dim\gamma}(1-t)^{s_\gamma} \cdot h_\gamma(t), \tag{6.19}$$

where in the sum Σ of the right hand side the face γ ranges through the compact ones of $\Gamma_+(f)$.

7 Monodromies at infinity of rational functions

In this section, we consider monodromies at infinity of rational functions. Let X be a smooth and connected algebraic variety over $\mathbb{C}$ and $P(x), Q(x)$ regular functions on it. Assume that $Q(x)$ is not identically zero on X. We define a rational function $f(x)$ on X by

$$f(x) = \frac{P(x)}{Q(x)} \qquad (x \in X). \tag{7.1}$$

Then there exists a fnite subset $B \subset \mathbb{C}$ such that $f : X \setminus Q^{-1}(0) \to \mathbb{C}$ induces a locally trivial fibration

$$f^{-1}(\mathbb{C} \setminus B) \to \mathbb{C} \setminus B. \tag{7.2}$$

The smallest finite subset $B \subset \mathbb{C}$ satisfying this property is called the bifurcation set of f. For the study of such subsets for regular and rational functions, see e.g. [3], [5], [17], [32], [33], [34], [48], [52] etc. For large enough $R \gg 0$ let

$$\Phi_j^\infty : H^j(f^{-1}(R); \mathbb{C}) \xrightarrow{\sim} H^j(f^{-1}(R); \mathbb{C}) \qquad (j \in \mathbb{Z}) \tag{7.3}$$

be the monodromy operators associated to the preceding fibration. Then we define the monodromy zeta function $\zeta_f^\infty(t) \in \mathbb{C}(t)$ at infinity of f by

$$\zeta_f^\infty(t) = \prod_{j \in \mathbb{Z}} \Big\{ \det(\mathrm{id} - t\Phi_j^\infty) \Big\}^{(-1)^j} \in \mathbb{C}(t). \tag{7.4}$$

We have the following global analogue for $\zeta_f^\infty(t)$ of A'Campo's formula in [2], which could be deduced also from the proof of [16, Theorem 1] and the results in [9, Section 6.1]. Here we give a short proof to it for the reader's convenience.

Theorem 7.1. *Assume that the hypersurfaces $P^{-1}(0)$ and $Q^{-1}(0)$ are smooth and intersect transversally in X. Let $\overline{X} \supset X$ be a smooth compactification of X such that for the complement $D := \overline{X} \setminus X$ the union $D \cup P^{-1}(0) \cup Q^{-1}(0) \subset \overline{X}$ is a strict normal crossing divisor in $\overline{X}$. Let $D = \cup_{i=1}^r D_i$ be the irreducible decomposition of D. For $1 \le i \le r$ set*

$$D_i^\circ = D_i \setminus (\cup_{j \neq i} D_j \cup P^{-1}(0) \cup Q^{-1}(0)) \tag{7.5}$$

and

$$l_i = \mathrm{ord}_{D_i}(Q) - \mathrm{ord}_{D_i}(P) \in \mathbb{Z}. \tag{7.6}$$

Then we have

$$\zeta_f^\infty(t) = (1 - t)^{\chi(Q^{-1}(0) \setminus P^{-1}(0))} \cdot \Big\{ \prod_{i : l_i > 0} (1 - t^{l_i})^{\chi(D_i^\circ)} \Big\}. \tag{7.7}$$

Proof. Let $h : \mathbb{P}^1 \to \mathbb{C}$ be a local coordinate at $\infty \in \mathbb{P}$ such that $h(\infty) = 0$. By $f : X \setminus Q^{-1}(0) \to \mathbb{C}$ and the inclusion map $j : \mathbb{C} \hookrightarrow \mathbb{P}^1$ we set

$$\mathcal{F} := j_! Rf_! \mathbb{C}_{X \setminus Q^{-1}(0)} \in \mathbf{D}_c^b(\mathbb{P}). \tag{7.8}$$

Then we have $\zeta_f^\infty(t) = \zeta(\mathcal{F})_{h,\infty}(t)$. Indeed, $\zeta_f^\infty(t)$ is equal to the zeta function associated to the monodromy automorphisms

$$H_j(f^{-1}(R); \mathbb{C}) \xrightarrow{\sim} H_j(f^{-1}(R); \mathbb{C}) \qquad (j \in \mathbb{Z}). \tag{7.9}$$

Moreover we have the Poincaré duality isomorphisms

$$H_j(f^{-1}(R); \mathbb{C}) \simeq H_c^{2n-j-2}(f^{-1}(R); \mathbb{C}) \qquad (j \in \mathbb{Z}). \tag{7.10}$$

Hence the monodromy zeta function at infinity $\zeta_f^\infty(t)$ is equal to the one associated to the monodromy automorphisms

$$H_c^j(f^{-1}(R); \mathbb{C}) \xrightarrow{\sim} H_c^j(f^{-1}(R); \mathbb{C}) \qquad (j \in \mathbb{Z}). \tag{7.11}$$

As in the proof of [28, Theorem 3.6] we construct towers of blow-ups of $\overline{X}$ over the normal crossing divisor $D \cup P^{-1}(0) \cup Q^{-1}(0)$ to eliminate the points of indeterminacy of f. Then we obtain a proper morphism $\pi : Y \longrightarrow \overline{X}$ of complex manifolds which induces an isomorphism

$$Y \setminus \pi^{-1}(D \cup P^{-1}(0) \cup Q^{-1}(0)) \xrightarrow{\sim} \overline{X} \setminus (D \cup P^{-1}(0) \cup Q^{-1}(0)) \tag{7.12}$$

and the assertion follows from the proof of [28, Theorem 3.6]. $\qquad\square$

Definition 7.2. We say that the rational function $f(x) = \frac{P(x)}{Q(x)}$ is polynomial-like at infinity if it satisfies the assumptions of Theorem 7.1 and there exists a smooth compactification $\overline{X} \supset X$ of X (satisfying the condition in Theorem 7.1) such that for any irreducible component D_i of $D = \overline{X} \setminus X$ we have the condition

$$\mathrm{ord}_{D_i}(P) < \mathrm{ord}_{D_i}(Q) \tag{7.13}$$

i.e. f has a pole of order $\mathrm{ord}_{D_i}(Q) - \mathrm{ord}_{D_i}(P) > 0$ along D_i.

For $j \in \mathbb{Z}$ and $\lambda \in \mathbb{C}$ and $R \gg 0$ we denote by

$$H^j(f^{-1}(R); \mathbb{C})_\lambda \subset H^j(f^{-1}(R); \mathbb{C}) \tag{7.14}$$

the generalized eigenspace of $\Phi_j^\infty : H^j(f^{-1}(R); \mathbb{C}) \xrightarrow{\sim} H^j(f^{-1}(R); \mathbb{C})$ for the eigenvalue λ. Then as in Takeuchi-Tibăr [49] we obtain the following result.

Theorem 7.3. *Assume that X is affine and the rational function $f(x) = \frac{P(x)}{Q(x)}$ is polynomial-like at infinity. Then for any $\lambda \neq 1$ we have the concentration*

$$H^j(f^{-1}(R); \mathbb{C})_\lambda \simeq 0 \qquad (j \neq \dim X - 1). \tag{7.15}$$

Moreover the weight filtration on $H^{\dim X - 1}(f^{-1}(R); \mathbb{C})_\lambda$ coincides with the monodromy filtration of Φ_{n-1}^∞.

Combining the formula for $\zeta_f^\infty(t) \in \mathbb{C}(t)$ in Theorem 7.1 with Theorem 7.3 above, we obtain a formula for the multiplicities of the eigenvalues $\lambda \neq 1$ in $\Phi_{\dim X - 1}^\infty$.

From now on, we consider the special case where $X = \mathbb{C}^n$ and $P(x), Q(x)$ are convenient polynomials. Let $\Gamma_\infty(P) \subset \mathbb{R}^n$ (resp. $\Gamma_\infty(Q) \subset \mathbb{R}^n$) be the convex hull of $\{0\} \cup NP(P)$ (resp. $\{0\} \cup NP(Q)$) in $\mathbb{R}^n$ and

$$\Gamma_\infty(f) = \Gamma_\infty(P) + \Gamma_\infty(Q) \tag{7.16}$$

their Minkowski sum. Since $P(x), Q(x)$ are convenient, they are n-dimensional polytopes in $\mathbb{R}^n$. As in the case of $\Gamma_+(f)$, for each face $\gamma \prec \Gamma_\infty(f)$ we have the corresponding faces

$$\gamma(P) \prec \Gamma_\infty(P), \qquad \gamma(Q) \prec \Gamma_\infty(Q) \tag{7.17}$$

such that

$$\gamma = \gamma(P) + \gamma(Q). \tag{7.18}$$

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Definition 7.4. We say that the rational function $f(x) = \frac{P(x)}{Q(x)}$ is non-degenerate at infinity if for any face γ of $\Gamma_\infty(f)$ such that $0 \notin \gamma$ the complex hypersurfaces $\{x \in T = (\mathbb{C}^*)^n \mid P^{\gamma(P)}(x) = 0\}$ and $\{x \in T = (\mathbb{C}^*)^n \mid Q^{\gamma(Q)}(x) = 0\}$ are smooth and reduced and intersect transversally in $T = (\mathbb{C}^*)^n$.

For a subset $S \subset \{1, 2, \ldots, n\}$ we set

$$\Gamma_\infty(f)^S = \Gamma_\infty(f) \cap \mathbb{R}^S. \tag{7.19}$$

Similarly, we define $\Gamma_\infty(P)^S, \Gamma_\infty(Q)^S \subset \mathbb{R}^S$ so that we have

$$\Gamma_\infty(f)^S = \Gamma_\infty(P)^S + \Gamma_\infty(Q)^S. \tag{7.20}$$

Let $\gamma_1^S, \gamma_2^S, \ldots, \gamma_{n(S)}^S$ be the facets of $\Gamma_\infty(f)^S$ such that $0 \notin \gamma_i^S$ and for each γ_i^S ($1 \leq i \leq n(S)$) consider the corresponding faces

$$\gamma_i^S(P) \prec \Gamma_\infty(P)^S, \qquad \gamma_i^S(Q) \prec \Gamma_\infty(Q)^S \tag{7.21}$$

such that

$$\gamma_i^S = \gamma_i^S(P) + \gamma_i^S(Q). \tag{7.22}$$

By using the primitive outer conormal vector $\alpha_i^S \in \mathbb{Z}_+^S \setminus \{0\}$ of the facet $\gamma_i^S \prec \Gamma_\infty(f)^S$ we define the lattice distance $d_i^S(P) > 0$ (resp. $d_i^S(Q) > 0$) of $\gamma_i^S(P)$ (resp. $\gamma_i^S(Q)$) from the origin $0 \in \mathbb{R}^S$ and set

$$d_i^S = d_i^S(P) - d_i^S(Q) \in \mathbb{Z}. \tag{7.23}$$

Finally we define $v_i^S > 0$ as in Section 3. Then we obtain the following result.

Theorem 7.5. *Assume that the rational function $f(x) = \frac{P(x)}{Q(x)}$ is non-degenerate at infinity and the hypersurfaces $P^{-1}(0)$ and $Q^{-1}(0)$ are smooth and intersect transversally in $X = \mathbb{C}^n$. Then we have*

$$\zeta_f^\infty(t) = (1-t)^{\chi(Q^{-1}(0) \setminus P^{-1}(0))} \cdot \prod_{S \neq \emptyset} \left\{ \prod_{i:d_i^S > 0} (1 - t^{d_i^S})^{(-1)^{|S|-1} v_i^S} \right\}. \tag{7.24}$$

Proof. Let Σ_f be the dual fan of $\Gamma_\infty(f)$ in $\mathbb{R}^n$. Since $P(x), Q(x)$ are convenient, any face of $\mathbb{R}_+^n$ is a cone in it. Then we can construct a smooth subdivision Σ of Σ_f without subdividing such cones. In other words, the fan Σ_0 in $\mathbb{R}_+^n$ formed by all the faces of $\mathbb{R}_+^n$ is a subfan of Σ. Denote by X_Σ the smooth toric variety associated to it and containing $X = \mathbb{C}^n$. Then we obtain the assertion just by applying Theorem 7.1 to the smooth compactification $X_\Sigma \supset X$ of $X = \mathbb{C}^n$. $\qquad \square$

If moreover $\Gamma_\infty(Q) \subset\subset \Gamma_\infty(P)$, then the rational function $f(x) = \frac{P(x)}{Q(x)}$ is polynomial-like at infinity in the sense of Definition 7.4 and we obtain also combinatorial descriptions of the Jordan normal forms for the eigenvalues $\lambda \neq 1$ in Φ_{n-1}^∞ and its reduced Hodge spectra at infinity. We leave their precise formulations to the readers.

A Proof of Lemma 4.3 by Takahiro Saito

In this appendix, we prove Lemma 4.3 in the main paper. For $k \in \mathbb{Z}$ we set

$$L_k \psi_{g,\lambda}^H(M) := \psi_{g,\lambda}^H(W_{k+1}M), \qquad L_k \psi_{g,\lambda}^H(M') := \psi_{g,\lambda}^H(W_{k+1}M'). \tag{A.1}$$

Recall that the weight filtration $W_\bullet \psi_{g,\lambda}^H(M)$ of $\psi_{g,\lambda}^H(M)$ is the relative monodromy filtration with respect to the filtration $L_\bullet \psi_{g,\lambda}^H(M)$. Namely for any $k \in \mathbb{Z}$ the filtration on $\mathrm{Gr}_k^L \psi_{g,\lambda}^H(M)$ induced by the weight filtration $W_\bullet \psi_{g,\lambda}^H(M)$ is the monodromy filtration centered at k. Therefore, it is enough to show that

$$\mathrm{Gr}_k^L \psi_{g,\lambda}^H(M) = 0 \tag{A.2}$$

for any $k \neq l-1$. Take a sufficiently large $k_0 (> l-1)$ such that

$$L_{k_0} \psi_{g,\lambda}^H(M) = \psi_{g,\lambda}^H(M), \qquad L_{k_0} \psi_{g,\lambda}^H(M') = \psi_{g,\lambda}^H(M'). \tag{A.3}$$

Since we have $\psi_{g,\lambda}^H(M) \xrightarrow{\sim} \psi_{g,\lambda}^H(M')$, for such k_0 the morphism $\mathrm{Gr}_{k_0}^L \psi_{g,\lambda}^H(M) \to \mathrm{Gr}_{k_0}^L \psi_{g,\lambda}^H(M')$ is an epimorphism. On the other hand, by the exactness of the functor $\psi_{g,\lambda}^H(\cdot)$, it follows from our assumption on the weights of M that we have $\mathrm{Gr}_{k_0}^L \psi_{g,\lambda}^H(M) = \psi_{g,\lambda}^H(\mathrm{Gr}_{k_0+1}^W M) = 0$. Therefore, $\mathrm{Gr}_{k_0}^L \psi_{g,\lambda}^H(M')$ is also zero and hence we obtain

$$\psi_{g,\lambda}^H(M') = L_{k_0} \psi_{g,\lambda}^H(M') = L_{k_0-1} \psi_{g,\lambda}^H(M'). \tag{A.4}$$

Repeating this argument, we get $\mathrm{Gr}_k^L \psi_{g,\lambda}^H(M') = 0$ for any $k > l-1$. Similarly, we can show that $\mathrm{Gr}_k^L \psi_{g,\lambda}^H(M') = 0$ for any $k \neq l-1$. This completes the proof.

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