Numerical attractors for rough differential equations

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Abstract

We study the explicit Euler scheme to approximate the solutions of rough differential equations under a bounded or linear diffusion term, where the drift term satisfies a local Lipschitz continuity and an one-sided linear growth condition. The Euler scheme is then proved to converge for a given solution, where the convergence rate is independent of the initial condition. For a dissipative drift term with linear growth condition and a bounded diffusion term, the numerical solution under a regular grid generates a random dynamical system which admits a random pullback attractor. We also prove that for bounded drift and diffusion terms and under a centered Gaussian noise with stationary increments, the numerical pullback attractor then converges upper semi-continuously to the continuous-time pullback attractor as the time step goes to zero.

Keywords: rough differential equations (SDE), rough path theory, rough integrals, random dynamical systems, random attractors, Euler numerical scheme.

1 Introduction

The theory of rough paths proposed by Lyons [19, 20] allows one to formulate and investigate stochastic differential equations of the form

$$dy_t = f(y_t)dt + g(y_t)dX_t, \tag{1.1}$$

where $f : \mathbb{R}^d \to \mathbb{R}^d$, $g : \mathbb{R}^d \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^d)$, $d, m \in \mathbb{N}$ have sufficient regularity and $X_t \in \mathbb{R}^m$ is a stochastic process with stationary increments, such that almost surely all realizations are ν-Hölder continuous for some $\nu \in (\frac{1}{3}, 1)$, e.g., fractional Brownian motions [21] with Hurst indices $H \in (\frac{1}{3}, 1)$). Using this theory one attempts to solve the controlled differential equation

$$dy_t = f(y_t)dt + g(y_t)dx_t \tag{1.2}$$

with the driving path x as a realization of X in the space $C^\nu(\mathbb{R}, \mathbb{R}^m)$ of continuous paths with finite ν-Hölder norm on any finite time interval, such that x can be lifted to a rough path $\mathbf{x} = (x, \mathbb{X})$, where $\mathbb{X}$ and x are related to each other by Chen's relation.

The solution of (1.2) in the sense of either Lyons-Davie [19, 20] or Friz-Victoir [11, 22] does not need rough path integrals to be specified. Alternatively, rough path integrals can be

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defined using fractional calculus, and the solution of (1.2) can be understood in couple with its Levy area, see e.g. [12, 16].

Another approach is to interpret equation (1.2) in the integral equation form

$$y_t = y_0 + \int_0^t f(y_s)ds + \int_0^t g(y_s)dx_s, \tag{1.3}$$

where the second integral is a rough integral for controlled rough paths in the sense of Gubinelli [14]. This approach facilitates the derivation of estimates of solutions and is more convenient for investigating their asymptotical behaviour and their approximation under numerical discretization.

It was recently proved in Duc [8] that the system (1.3) has a unique pathwise solution for a given initial condition under a Lipschitz continuity of the drift, which will be relaxed in this paper (Theorem 2.4 and Corollary 2.5) to a local Lipschitz continuity and an one-sided linear growth condition of the drift. A direct consequence gives an estimate of solution supremum norm, which is then used to prove the convergence of the explicit Euler numerical scheme (Theorem 3.1).

In this paper, we propose an analytic approach to study the numerical attractors of the explicit Euler numerical scheme from the rough differential equations (1.1) and (1.2). To do that, we follow a probabilistic setting in Bailleu et al [2] and Duc [7] for the rough noise to prove that system (1.1), understood in the pathwise sense as (1.2) with pathwise solutions in the Gubinelli sense, generates a continuous-time random dynamical system [1], while the discrete-time Euler scheme generates a discrete-time random dynamical system.

Moreover, we go a step further by proving that under the dissipativity of the drift term and with the bounded or linear diffusion term, there exists not only a pullback attractor for the continuous-time RDS generated from (1.1) (see Duc [7, Theorem 3.1] and Theorem 5.1), but also a pullback attractor for the discrete-time RDS generated from the explicit Euler numerical scheme with the regular grid and a sufficiently small step size, although the latter requires additional conditions on the linear growth of the drift term and the boundedness of the diffusion term (see Theorem 5.2).

Finally, we prove in Theorem 5.5 that, under restricted assumptions that the drift term f and diffusion term g are bounded and globally Lipschitz continuous and the driving noise X is a centered Gaussian process, the numerical pullback attractor converges upper semi-continuously and almost surely to the continuous attractor as the step size tends to zero. The same questions on existence of numerical attractor and its upper semi-continuous convergence in case of a linear diffusion term is still open for future work.

2 Rough path theory and rough differential equations

2.1 Rough paths

Let us briefly present the concept of rough paths in the simplest form, following and Friz & Hairer [10] and Lyons [19].

For any finite dimensional vector space W, denote by $C([a,b],W)$ the space of all continuous paths $y : [a,b] \to W$ equipped with the sup norm $\| \cdot \|_{\infty,[a,b]}$ given by $\|y\|_{\infty,[a,b]} = \sup_{t\in[a,b]} \|y_t\|$, where $\| \cdot \|$ is the norm in W. We write $y_{s,t} := y_t - y_s$. For $p \geq 1$, denote by

$C^{p-\mathrm{var}}([a,b],W) \subset C([a,b],W)$ the space of all continuous paths $y:[a,b]\to W$ of finite p-variation $\|y\|_{p-\mathrm{var},[a,b]} := \Big(\sup_{\Pi([a,b])}\sum_{i=1}^n \|y_{t_i,t_{i+1}}\|^p\Big)^{1/p} < \infty$, where the supremum is taken over the whole class of finite partitions of $[a,b]$.

Also for each $0<\alpha<1$, we denote by $C^\alpha([a,b],W)$ the space of Hölder continuous functions with exponent α on $[a,b]$ equipped with the norm

$$\|y\|_{\alpha,[a,b]} := \|y_a\| + \|\!|y|\!\|_{\alpha,[a,b]}, \quad \text{where} \quad \|\!|y|\!\|_{\alpha,[a,b]} := \sup_{s,t\in[a,b],\ s<t} \frac{\|y_{s,t}\|}{(t-s)^\alpha} < \infty. \tag{2.1}$$

For $\alpha\in(\frac{1}{3},\frac{1}{2})$, a couple $\mathbf{x}=(x,\mathbb{X})\in\mathbb{R}^m\oplus(\mathbb{R}^m\otimes\mathbb{R}^m)$, where $x\in C^\alpha([a,b],\mathbb{R}^m)$ and

$$\mathbb{X}\in C^{2\alpha}([a,b]^2,\mathbb{R}^m\otimes\mathbb{R}^m) := \left\{\mathbb{X}\in C([a,b]^2,\mathbb{R}^m\otimes\mathbb{R}^m): \sup_{s,t\in[a,b],\ s<t}\frac{\|\mathbb{X}_{s,t}\|}{|t-s|^{2\alpha}}<\infty\right\},$$

is called a *rough path* if it satisfies Chen's relation

$$\mathbb{X}_{s,t} - \mathbb{X}_{s,u} - \mathbb{X}_{u,t} = x_{s,u}\otimes x_{u,t}, \qquad \forall a\le s\le u\le t\le b. \tag{2.2}$$

We introduce the rough path semi-norm

$$\|\!|\mathbf{x}|\!\|_{\alpha,[a,b]} := \|\!|x|\!\|_{\alpha,[a,b]} + \|\!|\mathbb{X}|\!\|^{\frac{1}{2}}_{2\alpha,[a,b]^2}, \quad \text{where} \quad \|\!|\mathbb{X}|\!\|_{2\alpha,[a,b]^2} := \sup_{s,t\in[a,b];s<t}\frac{\|\mathbb{X}_{s,t}\|}{|t-s|^{2\alpha}} < \infty. \tag{2.3}$$

Throughout this paper, we will fix parameters $\frac{1}{3}<\alpha<\nu<\frac{1}{2}$ and $p=\frac{1}{\alpha}$ so that $C^\alpha([a,b],W)\subset C^{p-\mathrm{var}}([a,b],W)$. We also set $q=\frac{p}{2}$ and consider the $p-$var semi-norm

$$\begin{aligned}
\|\!|\mathbf{x}|\!\|_{p-\mathrm{var},[a,b]} &:= \Big(\|\!|x|\!\|^p_{p-\mathrm{var},[a,b]} + \|\!|\mathbb{X}|\!\|^q_{q-\mathrm{var},[a,b]^2}\Big)^{\frac{1}{p}},\\
\|\!|\mathbb{X}|\!\|_{q-\mathrm{var},[a,b]^2} &:= \left(\sup_{\Pi([a,b])}\sum_{i=1}^n \|\mathbb{X}_{t_i,t_{i+1}}\|^q\right)^{1/q},
\end{aligned} \tag{2.4}$$

where the supremum is taken over the whole class of finite partitions $\Pi([a,b])$ of $[a,b]$.

2.2 Gubinelli's rough path integrals

Following Gubinelli [14], a rough path integral can be defined for a continuous path $y\in C^\alpha([a,b],W)$ which is *controlled by* $x\in C^\alpha([a,b],\mathbb{R}^m)$ in the sense that, there exists a couple (y',R^y) with $y'\in C^\alpha([a,b],\mathcal{L}(\mathbb{R}^m,W))$, $R^y\in C^{2\alpha}([a,b]^2,W)$ such that

$$y_{s,t} = y'_s x_{s,t} + R^y_{s,t}, \qquad \forall a\le s\le t\le b. \tag{2.5}$$

y' is called the *Gubinelli derivative* of y, which is uniquely defined as long as x is *truly rough* [10, Definition 6.3 & Proposition 6.4], namely there exists a dense set of instants s of $[a,b]$ such that x is "rough at time s", i.e.

$$\forall h^*\in(\mathbb{R}^m)^*\setminus\{0\}: \quad \limsup_{t\downarrow s}\frac{|\langle h^*,x_{s,t}\rangle|}{|t-s|^{2\alpha}} = \infty.$$

For instance, almost all trajectories of a fractional Brownian motion B^H with $H > \frac{1}{3}$ is truly rough [10, Section 6].

Denote by $\mathscr{D}_x^{2\alpha}([a,b])$ the space of all the couples (y,y') controlled by x, then $\mathscr{D}_x^{2\alpha}([a,b])$ is a Banach space equipped with the norm

$$\|(y,y')\|_{x,2\alpha,[a,b]} := \|y_a\| + \|y'_a\| + \left\|\!\left|(y,y')\right|\!\right\|_{x,2\alpha,[a,b]}, \quad \left\|\!\left|(y,y')\right|\!\right\|_{x,2\alpha,[a,b]} := \left\|\!\left|y'\right|\!\right\|_{\alpha,[a,b]} + \left\|\!\left|R^y\right|\!\right\|_{2\alpha,[a,b]^2},$$

Then for a fixed rough path $\mathbf{x} = (x, \mathbb{X})$ and any controlled rough path $(y,y') \in \mathscr{D}_x^{2\alpha}([a,b])$, the integral $\int_s^t y_u dx_u$ can be defined as the limit of the Darboux sum

$$\int_s^t y_u dx_u := \lim_{|\Pi| \to 0} \sum_{[u,v]\in\Pi} \Big(y_u \otimes x_{u,v} + y'_u \mathbb{X}_{u,v} \Big)$$

where the limit is taken on all finite partitions Π of $[a,b]$ with $|\Pi| := \max\limits_{[u,v]\in\Pi} |v-u|$. Moreover, there exists a constant $C_\alpha = C_{\alpha,|b-a|} > 1$, such that

$$\begin{aligned} &\Big\| \int_s^t y_u dx_u - y_s \otimes x_{s,t} - y'_s \mathbb{X}_{s,t} \Big\| \\ \leq & C_\alpha |t-s|^{3\alpha} \Big(\left\|\!\left|x\right|\!\right\|_{\alpha,[s,t]} \left\|\!\left|R^y\right|\!\right\|_{2\alpha,[s,t]^2} + \left\|\!\left|y'\right|\!\right\|_{\alpha,[s,t]} \left\|\!\left|\mathbb{X}\right|\!\right\|_{2\alpha,[s,t]^2} \Big). \end{aligned} \tag{2.6}$$

In our paper, we often use a similar version to (2.6) under $p-$variation semi-norm as follows

$$\begin{aligned} &\Big\| \int_s^t y_u dx_u - y_s \otimes x_{s,t} - y'_s \mathbb{X}_{s,t} \Big\| \\ \leq & C_p \Big(\left\|\!\left|x\right|\!\right\|_{p-\mathrm{var},[s,t]} \left\|\!\left|R^y\right|\!\right\|_{q-\mathrm{var},[s,t]^2} + \left\|\!\left|y'\right|\!\right\|_{p-\mathrm{var},[s,t]} \left\|\!\left|\mathbb{X}\right|\!\right\|_{q-\mathrm{var},[s,t]^2} \Big), \end{aligned} \tag{2.7}$$

with constant $C_p > 1$ independent of $\mathbf{x}$ and (y,y').

2.3 Rough differential equations and solution estimates

The existence and uniqueness theorem for system (1.2) is first proved by Riedel & Scheutzow [22], where the solution is understood in the sense of Friz &Victoir [11]. Using rough path integrals, we interpret the rough differential equation (1.2) by writing it in the integral form

$$y_t = y_a + \int_a^t f(y_s) ds + \int_a^t g(y_s) dx_s, \quad \forall t \in [a,b], \tag{2.8}$$

for any interval $[a,b]$ and an initial value $y_a \in \mathbb{R}^d$. Then we search for a solution in the Gubinelli sense, and solve for $(y,y') \in \mathscr{D}_x^{2\alpha}([a,b],\mathbb{R}^d)$. This is possible because for $g : \mathbb{R}^d \to \mathcal{L}(\mathbb{R}^m,\mathbb{R}^d)$ satisfying $(\mathbf{H}_g^b)$ or $(\mathbf{H}_g^l)$ below, it is easy to check (see e.g., [14]) that

$$\begin{aligned} (y,y') \in \mathscr{D}_x^{2\alpha}([a,b],\mathbb{R}^d) \quad &\Rightarrow \quad (g(y),[g(y)]') \in \mathscr{D}_x^{2\alpha}([a,b],\mathcal{L}(\mathbb{R}^m,\mathbb{R}^d)), \\ &\text{with} \quad [g(y)]'_s = Dg(y_s)y'_s \in \mathcal{L}(\mathbb{R}^m,\mathcal{L}(\mathbb{R}^m,\mathbb{R}^d)), \end{aligned}$$

thus the second integral in (2.8) is well defined.

Throughout the paper, we will assume that.

($\mathbf{H}_f$) f is locally Lipschitz continuous and of one-sided linear growth

$$\exists C > 0 : \langle y, f(y)\rangle \leq C(1 + \|y\|^2), \quad \forall y \in \mathbb{R}^d; \tag{2.9}$$

in addition f is of linear growth in the perpendicular direction, i.e. there exists $C_f > 0$ such that

$$\Big\| f(y) - \frac{\langle f(y), y\rangle}{\|y\|^2} y \Big\| \leq C_f\Big(1 + \|y\|\Big), \quad \forall y \neq 0; \tag{2.10}$$

either

($\mathbf{H}_g^b$) g belongs to $C_b^3(\mathbb{R}^d, \mathcal{L}(\mathbb{R}^m, \mathbb{R}^d))$ such that

$$C_g := \max\Big\{\|g\|_\infty, \|Dg\|_\infty, \|D_g^2\|_\infty, \|D_g^3\|_\infty\Big\} < \infty; \tag{2.11}$$

or

($\mathbf{H}_g^l$) $g(y) = Cy$ for $C \in \mathcal{L}(\mathcal{L}(\mathbb{R}^m, \mathbb{R}^d), \mathbb{R}^d)$ such that

$$C_g := \|C\| < \infty; \tag{2.12}$$

($\mathbf{H}_X$) for a given $\nu \in (\frac{1}{3}, \frac{1}{2})$, x belongs to the space $C^\nu(\mathbb{R}, \mathbb{R}^m)$ of all continuous paths which are of finite ν-Hölder norm on any interval $[s,t]$. In particular, x is a realization of a stochastic process $X_t(\omega)$ with stationary increments, such that x can be lifted into a realized component $\mathbf{x} = (x, \mathbb{X})$ of a stochastic process $(x_\cdot(\omega), \mathbb{X}_{\cdot,\cdot}(\omega))$ with stationary increments, and the estimate

$$E\Big(\|x_{s,t}\|^p + \|\mathbb{X}_{s,t}\|^q\Big) \leq C_{T,\nu}|t-s|^{p\nu}, \forall s,t \in [0,T] \tag{2.13}$$

holds for any $[0,T]$, with $p\nu \geq 1, q = \frac{p}{2}$ and some constant $C_{T,\nu}$.

Assumptions ($\mathbf{H}_f$), ($\mathbf{H}_g^b$) or ($\mathbf{H}_g^l$), ($\mathbf{H}_X$) are sufficient to prove the existence and uniqueness of the solution of (1.2), as well as the continuity of the solution semi-flow and the generation of a continuous random dynamical system, see e.g., Bailleul et al [2] and Riedel & Scheutzow [22, Theorem 4.3].

Here we prove another version of the solution estimate of (1.2), under the definition of solution in the Gubinelli sense, which extends the diffusion coefficient g to both the bounded case ($\mathbf{H}_g^b$) and the linear case ($\mathbf{H}_g^l$). We first modify assumption (2.9) by another equivalent one as below.

Lemma 2.1 *Condition* (2.9) *is equivalent to the following condition*

$$\exists \bar{C} > 0 : \langle y, f(y)\rangle \leq \bar{C}\|y\|(1 + \|y\|), \quad \forall y \in \mathbb{R}^d; \tag{2.14}$$

Proof: Condition (2.14) follows (2.9) automatically due to Cauchy inequality. For the other direction, one can easily show that

$$\langle y, f(y)\rangle \leq (C \vee 1)\|y\|\Big(\sup_{\|y\|\leq 1} \|f(y)\| + 1 + \|y\|\Big), \forall y \in \mathbb{R}^d.$$

Indeed, if $\|y\| \leq 1$, then

$$\langle y, f(y) \rangle \leq \|y\| \sup_{\|y\|\leq 1} \|f(y)\| \leq (C \vee 1)\|y\| \Big(\sup_{\|y\|\leq 1} \|f(y)\| + 1 + \|y\| \Big).$$

On the other hand, if $\|y\| \geq 1$ then by (2.14)

$$\langle y, f(y) \rangle \leq C(1 + \|y\|^2) \leq C(\|y\| + \|y\|^2) \leq (C \vee 1)\|y\| \Big(\sup_{\|y\|\leq 1} \|f(y)\| + 1 + \|y\| \Big).$$

Hence (2.14) is followed by choosing $\bar{C} := (C \vee 1)\Big(\sup_{\|y\|\leq 1} \|f(y)\| + 1 \Big)$.

□

Due to Lemma 2.1, from now on we can work with the following assumption for f.

$(\mathbf{H}'_f)$ f is a locally Lipschitz continuous function which satisfies (2.14) and (2.10).

The techniques to be used are the Doss-Sussmann technique [23] and the so-called *greedy sequence of stopping times* in Cass et al [3]. Namely, for any fixed $\gamma \in (0,1)$ the sequence of greedy times $\{\tau_i(\gamma, \mathbf{x}, I)\}_{i\in\mathbb{N}}$ is defined by

$$\tau_0 = \min I, \quad \tau_{i+1} := \inf \Big\{ t > \tau_i : \|\mathbf{x}\|_{p-\text{var},[\tau_i,t]} = \gamma \Big\} \wedge \max I. \tag{2.15}$$

Define $N(\gamma, \mathbf{x}, I) := \sup\{i \in \mathbb{N} : \tau_i \leq \max I\}$, then it is easy to show a rough estimate

$$N(\gamma, \mathbf{x}, I) \leq 1 + \gamma^{-p} \|\mathbf{x}\|^p_{p-\text{var},I}. \tag{2.16}$$

In fact, it is proved in Cass et al [3] that $e^{N(\gamma,\mathbf{x},I)}$ is integrable. Other studies on continuity and properties of stopping times can also be found in Duc et al [9, Section 4].

Note that from Duc [8, Theorem 3.4], the solution $\phi_\cdot(\mathbf{x}, \phi_a)$ of the *pure* rough differential equation

$$d\phi_u = g(\phi_u)dx_u, \quad u \in [a,b], \phi_a \in \mathbb{R}^d \tag{2.17}$$

is C^1 w.r.t. ϕ_a, and $\frac{\partial \phi}{\partial \phi_a}(\cdot, \mathbf{x}, \phi_a)$ is the solution of the linearized system

$$d\xi_u = Dg(\phi_u(\mathbf{x}, \phi_s))\xi_u dx_u, \quad u \in [a,b], \xi_a = Id, \tag{2.18}$$

where $Id \in \mathbb{R}^{d\times d}$ denotes the identity matrix.

The idea is then to prove the existence and uniqueness of the solution on each small interval $[\tau_k, \tau_{k+1}]$ between two consecutive stopping times, and then concatenate to obtain the conclusion on any interval. The Doss-Sussmann technique used in Duc [8, Theorem 3.7] and Riedel & Scheutzow [22] ensures that, by a transformation $y_t = \phi_t(\mathbf{x}, z_t)$ there is an one-to-one correspondence between a solution y_t of (1.2) on a certain interval $[0,\tau]$ and a solution z_t of the associate ordinary differential equation

$$\dot{z}_t = \Big[\frac{\partial \phi}{\partial z}(t, \mathbf{x}, z_t)\Big]^{-1} f(\phi_t(\mathbf{x}, z_t)), \quad t \in [0,\tau], \; z_0 = y_0. \tag{2.19}$$

To estimate the solution norm growth, assign

$$\gamma_t := y_t - z_t, \quad \text{and} \quad \psi_t := \Big[\frac{\partial \phi}{\partial z}(t, \mathbf{x}, z_t)\Big]^{-1} - Id, \quad \forall t \in [0,\tau],$$

where $\tau > 0$ is chosen such that $16C_pC_g \left\Vert\!\left\vert \mathbf{x} \right\vert\!\right\Vert_{p-\mathrm{var},[0,\tau]} \leq \lambda$ for some $\lambda \in (0,1)$ small enough.

The following result from Duc [7, Proposition 2.1] shows solution norm estimates for equation (2.17).

Proposition 2.2 *Assume that ϕ_t is the solutions of (2.17). Introduce the semi-norm*

$$\left\Vert\!\left\vert \kappa, R^\kappa \right\vert\!\right\Vert_{p-\mathrm{var},[s,t]} := \left\Vert\!\left\vert \kappa \right\vert\!\right\Vert_{p-\mathrm{var},[s,t]} + \left\Vert\!\left\vert R^\kappa \right\vert\!\right\Vert_{q-\mathrm{var},[s,t]^2}.$$

Then for any interval $[a,b]$ such that $16C_pC_g \left\Vert\!\left\vert \mathbf{x} \right\vert\!\right\Vert_{p-\mathrm{var},[a,b]} \leq 1$, the following estimates hold

$$i) \quad \left\Vert\!\left\vert \phi, R^\phi \right\vert\!\right\Vert_{p-\mathrm{var},[a,b]} \leq 8C_pC_g \left\Vert\!\left\vert \mathbf{x} \right\vert\!\right\Vert_{p-\mathrm{var},[a,b]}; \tag{2.20}$$

$$ii) \quad \left\Vert \frac{\partial \phi}{\partial \phi_a}(t,\mathbf{x},\phi_a) - Id \right\Vert, \left\Vert \Big[\frac{\partial \phi}{\partial \phi_a}(t,\mathbf{x},\phi_a)\Big]^{-1} - Id \right\Vert \leq 16C_pC_g \left\Vert\!\left\vert \mathbf{x} \right\vert\!\right\Vert_{p-\mathrm{var},[a,b]}. \tag{2.21}$$

A similar result for the linear case $g(y) = Cy$ is formulated as follows.

Proposition 2.3 *Assume that $\phi(\cdot,\mathbf{x},\phi_a)$ is the solution of the rough differential equation*

$$d\phi_t = C\phi_t dx_t, \quad t \in [a,b], \phi_a \in \mathbb{R}^d. \tag{2.22}$$

Then for any interval $[a,b]$ such that $4C_pC_g \left\Vert\!\left\vert \mathbf{x} \right\vert\!\right\Vert_{p-\mathrm{var},[a,b]} \leq 1$, the following estimates hold

$$i) \quad \left\Vert\!\left\vert \phi, R^\phi \right\vert\!\right\Vert_{p-\mathrm{var},[a,b]} \leq 8C_g \left\Vert\!\left\vert \mathbf{x} \right\vert\!\right\Vert_{p-\mathrm{var},[a,b]} \|\phi_a\|; \tag{2.23}$$

$$ii) \quad \left\Vert \frac{\partial \phi}{\partial \phi_a}(t,\mathbf{x},\phi_a) - Id \right\Vert, \left\Vert \Big[\frac{\partial \phi}{\partial \phi_a}(t,\mathbf{x},\phi_a)\Big]^{-1} - Id \right\Vert \leq 8C_g \left\Vert\!\left\vert \mathbf{x} \right\vert\!\right\Vert_{p-\mathrm{var},[a,b]}. \tag{2.24}$$

Proof: The existence and uniqueness theorem for equation (2.22) is proved in [8]. To estimate the solution norms, one uses (2.7) to obtain that

$$\begin{aligned}
\|\phi_{s,t}\| \leq& C_g\|\phi_s\|\|x_{s,t}\| + C_g^2\|\phi_s\|\|\mathbb{X}_{s,t}\| \\
&+ C_p\Big\{ \left\Vert\!\left\vert x \right\vert\!\right\Vert_{p-\mathrm{var},[s,t]} \left\Vert\!\left\vert R^{C\phi} \right\vert\!\right\Vert_{q-\mathrm{var},[s,t]} + \left\Vert\!\left\vert \mathbb{X} \right\vert\!\right\Vert_{q-\mathrm{var},[s,t]} \left\Vert\!\left\vert C \otimes C\phi \right\vert\!\right\Vert_{p-\mathrm{var},[s,t]} \Big\} \\
\leq& \Big(C_g \left\Vert\!\left\vert \mathbf{x} \right\vert\!\right\Vert_{p-\mathrm{var},[s,t]} + C_g^2 \left\Vert\!\left\vert \mathbf{x} \right\vert\!\right\Vert^2_{p-\mathrm{var},[s,t]}\Big)\Big[\|\phi_s\| + C_p \left\Vert\!\left\vert \phi, R^\phi \right\vert\!\right\Vert_{p-\mathrm{var},[s,t]}\Big] \\
\leq& 2C_g \left\Vert\!\left\vert \mathbf{x} \right\vert\!\right\Vert_{p-\mathrm{var},[s,t]} \|\phi_s\| + 2C_pC_g \left\Vert\!\left\vert \mathbf{x} \right\vert\!\right\Vert_{p-\mathrm{var},[s,t]} \left\Vert\!\left\vert \phi, R^\phi \right\vert\!\right\Vert_{p-\mathrm{var},[s,t]}
\end{aligned}$$

whenever $C_g \left\Vert\!\left\vert \mathbf{x} \right\vert\!\right\Vert_{p-\mathrm{var},[s,t]} \leq 1$. As a result

$$\left\Vert\!\left\vert \phi \right\vert\!\right\Vert_{p-\mathrm{var},[s,t]} \leq 2C_g \left\Vert\!\left\vert \mathbf{x} \right\vert\!\right\Vert_{p-\mathrm{var},[s,t]} \|\phi_s\| + 2C_pC_g \left\Vert\!\left\vert \mathbf{x} \right\vert\!\right\Vert_{p-\mathrm{var},[s,t]} \left\Vert\!\left\vert \phi, R^\phi \right\vert\!\right\Vert_{p-\mathrm{var},[s,t]}$$

whenever $C_g \left\Vert\!\left\vert \mathbf{x} \right\vert\!\right\Vert_{p-\mathrm{var},[s,t]} \leq 1$. The similar estimate for $\left\Vert\!\left\vert R^\phi \right\vert\!\right\Vert_{p-\mathrm{var},[s,t]}$ is already included in the above estimate, hence

$$\left\Vert\!\left\vert \phi, R^\phi \right\vert\!\right\Vert_{p-\mathrm{var},[s,t]} \leq 4C_g \left\Vert\!\left\vert \mathbf{x} \right\vert\!\right\Vert_{p-\mathrm{var},[s,t]} \|\phi_s\| + 4C_pC_g \left\Vert\!\left\vert \mathbf{x} \right\vert\!\right\Vert_{p-\mathrm{var},[s,t]} \left\Vert\!\left\vert \phi, R^\phi \right\vert\!\right\Vert_{p-\mathrm{var},[s,t]}$$

whenever $C_g \|\mathbf{x}\|_{p-\text{var},[s,t]} \leq 1$. Taking the term $\left\|\!\left| \phi, R^\phi \right|\!\right\|_{p-\text{var},[s,t]}$ from the right hand side to the left hand side, we obtain

$$\left\|\!\left| \phi, R^\phi \right|\!\right\|_{p-\text{var},[s,t]} \leq 8C_g \|\mathbf{x}\|_{p-\text{var},[s,t]} \|\phi_s\|$$

whenever $4C_pC_g \|\mathbf{x}\|_{p-\text{var},[s,t]} \leq \frac{1}{2}$, which proves (2.23).

To prove (2.24), observe that the solution $\phi(t, \mathbf{x}, \phi_a)$ is linear w.r.t. ϕ_a, i.e.

$$\phi(t, \mathbf{x}, \phi_a + h) - \phi(t, \mathbf{x}, \phi_a) = \phi(t, \mathbf{x}, h) = \frac{\partial \phi}{\partial \phi_a}(t, \mathbf{x}, \phi_a)h.$$

Hence one deduces from (2.23) that

$$\|\phi(t, \mathbf{x}, h) - h\| \leq \|\phi(\cdot, \mathbf{x}, h)\|_{p-\text{var},[a,b]} \leq 8C_g \|\mathbf{x}\|_{p-\text{var},[a,b]} \|h\|,$$

which implies that

$$\|\frac{\partial \phi}{\partial \phi_a}(t, \mathbf{x}, \phi_a) - Id\| = \sup_{h \in \mathbb{R}^d} \frac{\|\frac{\partial \phi}{\partial \phi_a}(t, \mathbf{x}, \phi_a)h - h\|}{\|h\|} \leq 8C_g \|\mathbf{x}\|_{p-\text{var},[a,b]}, \quad \forall t \in [a, b].$$

The estimate for $\left\| \left[\frac{\partial \phi}{\partial \phi_a}(t, \mathbf{x}, \phi_a)\right]^{-1} - Id \right\|$ is similar. □

We now state below the existence and uniqueness theorem as well as the solution norm estimate for rough differential equation (1.2) under bounded diffusion coefficient g.

Theorem 2.4 *Under the assumptions ($\boldsymbol{H}'_f$), ($\boldsymbol{H}^b_g$), ($\boldsymbol{H}_X$), there exists a unique solution of* (1.2) *on any interval* $[0, T]$*. In addition, for each* $\lambda \in (0, 1)$ *small enough, there exist some generic constants* $C(\lambda), \delta(\lambda)$ *such that the solution satisfies the following estimates*

$$\|y\|_{\infty,[0,T]} \leq e^{\delta(\lambda)T}\Big(\|y_0\| + C(\lambda)T + \frac{\lambda}{2}N\Big(\frac{\lambda}{16C_pC_g}, \mathbf{x}, [0, T]\Big)\Big) =: R. \tag{2.25}$$

Proof: The existence and uniqueness of the solution of the equations (1.2) as well as (2.19) on some small interval $[0, \tau_{local}]$, thus we only need to prove that the solution can be extended into whole interval $[0, \tau]$. Indeed, with such τ, it then follows from (2.20) and (2.21) that

$$\|\gamma_t\| = \|\phi_t(\mathbf{x}, z_t) - z_t\| \leq \frac{\lambda}{2} \quad \text{and} \quad \|\psi_t\| \leq \lambda, \quad \forall t \in [0, \tau]. \tag{2.26}$$

To estimate $\|z_t\|$, we rewrite (2.19) as

$$\dot{z}_t = (Id + \psi_t)f(z_t + \gamma_t). \tag{2.27}$$

The additional technical condition (2.10) is equivalent to the following: for $y \in \mathbb{R}^d$ and $y \neq 0$, $f(y)$ is decomposed in the unique form

$$f(y) = \frac{\langle f(y), y\rangle}{\|y\|^2}y + \pi_y^\perp(f(y)), \quad \text{where} \quad \pi_y^\perp = 1 - \pi_y \quad \text{and} \quad \|\pi_y^\perp(f(y))\| \leq C_f(1 + \|y\|). \tag{2.28}$$

Consider two cases.

Case 1: $z_t + \gamma_t \neq 0$. From (2.14) and condition (2.28), we can check that

$$\begin{aligned}
\frac{d}{2dt}\|z_t\|^2 &= \Big\langle z_t, (Id+\psi_t)\Big[\frac{\langle z_t+\gamma_t, f(z_t+\gamma_t)\rangle}{\|z_t+\gamma_t\|^2}(z_t+\gamma_t) + \pi^{\perp}_{z_t+\gamma_t}(f(z_t+\gamma_t))\Big]\Big\rangle \\
&= \Big\langle z_t, (Id+\psi_t)\frac{(z_t+\gamma_t)}{\|z_t+\gamma_t\|}\Big\rangle\Big\langle \frac{z_t+\gamma_t}{\|z_t+\gamma_t\|}, f(z_t+\gamma_t)\Big\rangle + \Big\langle z_t+\gamma_t, \pi^{\perp}_{z_t+\gamma_t}(f(z_t+\gamma_t))\Big\rangle \\
&\quad -\Big\langle \gamma_t, \pi^{\perp}_{z_t+\gamma_t}(f(z_t+\gamma_t))\Big\rangle + \Big\langle z_t, \psi_t\pi^{\perp}_{z_t+\gamma_t}(f(z_t+\gamma_t))\Big\rangle \\
&\leq (1+\lambda)\|z_t\|\bar{C}(1+\|z_t\|) + \Big(\|\gamma_t\| + \|\psi_t\|\|z_t\|\Big)\|\pi^{\perp}_{z_t+\gamma_t}(f(z_t+\gamma_t))\| \\
&\leq (1+\lambda)\|z_t\|\bar{C}(1+\|z_t\|) + \lambda(\frac{1}{2}+\|z_t\|)C_f\Big(1+\|z_t\|+\frac{\lambda}{2}\Big). \qquad (2.29)
\end{aligned}$$

Case 2: $z_t + \gamma_t = 0$. Then the same arguments with the Cauchy inequality show that

$$\frac{d}{2dt}\|z_t\|^2 = \langle z_t, (Id+\psi_t)f(0)\rangle \leq (1+\lambda)\|f(0)\|\|z_t\|. \qquad (2.30)$$

By applying the Cauchy inequality to the right hand side of (2.29) and (2.30), we can show that there exist generic constants $C(\lambda)$ and $\delta(\lambda)$ such that

$$\frac{d}{dt}\|z_t\|^2 \leq C(\lambda) + \delta(\lambda)\|z_t\|^2, \quad \forall t \in [0,\tau], \qquad (2.31)$$

which, together with Gronwall lemma, yields

$$\|z_t\| \leq e^{\delta(\lambda)t}\Big(\|z_0\| + \frac{C(\lambda)}{\delta(\lambda)}\Big) - \frac{C(\lambda)}{\delta(\lambda)} \leq e^{\delta(\lambda)\tau}\|y_0\| + \frac{C(\lambda)}{\delta(\lambda)}\Big(e^{\delta(\lambda)\tau} - 1\Big), \quad \forall t \in [0,\tau].$$

In particular

$$\|y\|_{\infty,[0,\tau]} \leq \|z\|_{\infty,[0,\tau]} + \|\gamma\|_{\infty,[0,\tau]} \leq e^{\delta(\lambda)\tau}\|y_0\| + \frac{C(\lambda)}{\delta(\lambda)}\Big(e^{\delta(\lambda)\tau} - 1\Big) + \frac{\lambda}{2}. \qquad (2.32)$$

(2.32) implies that $\|z_t\|$ is bounded as long as $t \in [0,\tau]$, thereby proving the existence and uniqueness of the solution z_t of equation (2.19) on $[0,\tau]$, and so is the solution y_t of (1.2) on $[0,\tau]$.

Next, with such a $\lambda > 0$, construct a greedy sequence of stopping times $\{\tau_i(\frac{\lambda}{16C_pC_g}, \mathbf{x}, [0,T])\}$. On each interval $[\tau_i, \tau_{i+1}]$ it is similar to prove the existence and uniqueness of the solution of the two differential equations (1.2) and (2.19) with the shifted time

$$\begin{aligned}
dy_{t+\tau_i} &= f(y_{t+\tau_i})dt + g(y_{t+\tau_i})dx_{t+\tau_i}, \quad \forall t \in [0, \tau_{i+1} - \tau_i]; \\
\dot{z}_{t+\tau_i} &= \Big[\frac{\partial \phi}{\partial z}(t, \mathbf{x}_{\cdot+\tau_i}, z_{t+\tau_i})\Big]^{-1} f(\phi_t(\mathbf{x}_{\cdot+\tau_i}, z_{t+\tau_i})), \quad \forall t \in [0, \tau_{i+1} - \tau_i], \quad z_{\tau_i} = y_{\tau_i}.
\end{aligned}$$

As a result, the existence and uniqueness of the solution of the two systems (1.2) and (2.19) on $[0,T]$ is proved by concatenation. To estimate the solution norm, observe from (2.32) that

$$\|y_{\tau_{k+1}}\| \leq e^{\delta(\lambda)(\tau_{k+1}-\tau_k)}\|y_{\tau_k}\| + \frac{C(\lambda)}{\delta(\lambda)}\Big(e^{\delta(\lambda)(\tau_{k+1}-\tau_k)} - 1\Big) + \frac{\lambda}{2}, \quad 0 \leq k \leq N-1,$$

which implies that

$$e^{-\delta(\lambda)\tau_{k+1}}\|y_{\tau_{k+1}}\| \le e^{-\delta(\lambda)\tau_k}\|y_{\tau_k}\| + \frac{C(\lambda)}{\delta(\lambda)}\Big(e^{-\delta(\lambda)\tau_k} - e^{-\delta(\lambda)\tau_{k+1}}\Big) + \frac{\lambda}{2}e^{-\delta(\lambda)\tau_{k+1}}.$$

Hence by induction, one can easily show that

$$\begin{aligned} e^{-\delta(\lambda)\tau_{k+1}}\|y_{\tau_{k+1}}\| &\le \|y_0\| + \frac{C(\lambda)}{\delta(\lambda)}\Big(1 - e^{-\delta(\lambda)\tau_{k+1}}\Big) + \frac{\lambda}{2}\sum_{j=1}^{k+1} e^{-\delta(\lambda)\tau_j}, \\ \text{hence} \qquad \|y_{\tau_{k+1}}\| &\le e^{\delta(\lambda)\tau_{k+1}}\|y_0\| + \frac{C(\lambda)}{\delta(\lambda)}\Big(e^{\delta(\lambda)\tau_{k+1}} - 1\Big) + \frac{\lambda}{2}(k+1)e^{\delta(\lambda)\tau_{k+1}} \\ &\le e^{\delta(\lambda)\tau_{k+1}}\Big(\|z_0\| + C(\lambda)\tau_{k+1} + \frac{\lambda}{2}(k+1)\Big), \quad 0 \le k \le N-1. \end{aligned}$$

That together with (2.32) yields

$$\begin{aligned} \|y\|_{\infty,[\tau_k,\tau_{k+1}]} &\le e^{\delta(\lambda)(\tau_{k+1}-\tau_k)}\|y_{\tau_k}\| + \frac{C(\lambda)}{\delta(\lambda)}\Big(e^{\delta(\lambda)(\tau_{k+1}-\tau_k)} - 1\Big) + \frac{\lambda}{2} \\ &\le e^{\delta(\lambda)\tau_{k+1}}\Big(\|z_0\| + C(\lambda)\tau_{k+1} + \frac{\lambda}{2}(k+1)\Big), \quad 0 \le k \le N-1. \end{aligned}$$

By the definition of stopping times (2.15), $\tau_N = T$, which yields (2.25).

□

A similar result for the linear case is formulated as follows.

Corollary 2.5 *Under the assumptions ($\boldsymbol{H}'_f$), ($\boldsymbol{H}^l_g$), ($\boldsymbol{H}_X$), there exists a unique solution of* (1.2) *on any interval* $[0,T]$*. In addition, for each* $\lambda \in (0,1)$ *small enough, there exist some generic constants* $C(\lambda), \delta(\lambda)$ *such that the solution satisfies*

$$\|y\|_{\infty,[0,T]} \le \exp\{\delta(\lambda)T + \lambda N(\frac{\lambda}{16C_pC_g}, \mathbf{x}, [0,T])\}\Big(\|y_0\| + \frac{C(\lambda)}{\delta(\lambda)}\Big) - \frac{C(\lambda)}{\delta(\lambda)} =: R. \qquad (2.33)$$

Proof: The proof follows the proof of Theorem 2.4 line by line, except for a minor change. Specifically, due to Proposition 2.3, (2.26) has the form

$$\|\gamma_t\| = \|\phi_t(\mathbf{x}, z_t) - z_t\| \le \frac{\lambda}{2}\|z_t\| \quad \text{and} \quad \|\psi_t\| \le \lambda, \quad \forall t \in [0,\tau]. \qquad (2.34)$$

This change does not change (2.30) while it modifies the estimate (2.29) to

$$\frac{d}{2dt}\|z_t\|^2 \le (1+\lambda)\|z_t\|\bar{C}(1 + \|z_t\|) + 2\lambda\|z_t\|C_f\Big(1 + (1+\lambda)\|z_t\|\Big). \qquad (2.35)$$

Therefore one can still prove (2.31), which makes (2.32) have the form

$$\begin{aligned} \|y\|_{\infty,[0,\tau]} &\le (1+\lambda)\|z\|_{\infty,[0,\tau]} \\ &\le (1+\lambda)\Big[e^{\delta(\lambda)\tau}\|z_0\| + \frac{C(\lambda)}{\delta(\lambda)}\Big(e^{\delta(\lambda)\tau} - 1\Big)\Big] \\ &\le e^{\delta(\lambda)\tau+\lambda}\|y_0\| + \frac{C(\lambda)}{\delta(\lambda)}\Big(e^{\delta(\lambda)\tau+\lambda} - 1\Big). \end{aligned} \qquad (2.36)$$

Therefore the existence and uniqueness of the solution on each small interval $[\tau_i, \tau_{i+1}]$ is proved and also on the whole interval $[0,T]$ by concatenation. The solution estimate (2.33) is then followed by induction. □

Proposition 2.6 *Assume that* $\|f\|_\infty := \sup_{y\in\mathbb{R}^d} \|f(y)\| < \infty$. *Then under the assumptions* ($\boldsymbol{H}_g^b$) *and* ($\boldsymbol{H}_X$) *there exists a generic constant* $C_1 = C_1(\|f\|_\infty, C_g, \left\|\mathbf{x}\right\|_{\nu,[0,T]}, T)$ *independent of the initial condition, such that any solution* $y.$ *of* (1.2) *satisfies*

$$\left\|y\right\|_{p-\mathrm{var},[a,b]} \leq C_1(b-a)^\nu \quad \text{and} \quad \left\|R^y\right\|_{p-\mathrm{var},[a,b]} \leq C_1(b-a)^{2\nu}, \quad \forall 0\leq a\leq b\leq T. \tag{2.37}$$

If in addition f *is globally Lipschitz continuous w.r.t. a constant* C_f *then there exists a generic constant* $C_2 = C_2(\|f\|_\infty, C_f, C_g, \left\|\mathbf{x}\right\|_{\nu,[0,T]}, T)$ *independent of the initial conditions such that any two solutions* $y^i_\cdot(\mathbf{x}, y^i_0), i = 1,2,$ *of equation* (1.2) *satisfy*

$$\|y^2 - y^1\|_{\infty,[a,b]} \leq C_2\|y^2_a - y^1_a\|, \quad \forall 0\leq a\leq b\leq T. \tag{2.38}$$

Proof: The proof follows similar arguments and estimates in Duc [7, Proposition 2.1] and Duc [8, Proposition 3.1], thus we only sketch the arguments here. First, observe that $y'_s = g(y_s)$ and $[g(y)]'_s = Dg(y_s)g(y_s)$ with

$$\left\|R^{g(y)}\right\|_{q-\mathrm{var},[a,b]^2} \leq C_g \left\|R^y\right\|_{q-\mathrm{var},[a,b]^2} + \frac{1}{2}C_g^2 \left\|y\right\|_{p-\mathrm{var},[a,b]} \left\|x\right\|_{p-\mathrm{var},[a,b]}.$$

It then follows from the fact $\|f\|_\infty, \|g\|_\infty < \infty$ and the estimate (2.7) that

$$\begin{aligned}
\|y_{s,t}\| &\leq \int_s^t \|f(y_u)\|du + \Big\|\int_s^t g(y_u)dx_u\Big\| \\
&\leq \|f\|_\infty(t-s) + \|g\|_\infty\|x_{s,t}\| + \|Dg(y)g(y)\|_\infty\|\mathbb{X}_{s,t}\| \\
&\quad + C_p\Big\{ \left\|Dg(y)g(y)\right\|_{p-\mathrm{var},[a,b]} \left\|\mathbb{X}\right\|_{q-\mathrm{var},[a,b]^2} + \left\|R^{g(y)}\right\|_{q-\mathrm{var},[a,b]^2} \left\|x\right\|_{p-\mathrm{var},[a,b]} \Big\} \\
&\leq \|f\|_\infty(t-s) + C_g \left\|\mathbf{x}\right\|_{p-\mathrm{var},[s,t]} + C_g^2 \left\|\mathbf{x}\right\|^2_{p-\mathrm{var},[s,t]} \\
&\quad + 3C_p\Big\{ C_g^2 \left\|\mathbf{x}\right\|^2_{p-\mathrm{var},[s,t]} \left\|y\right\|_{p-\mathrm{var},[s,t]} + C_g \left\|\mathbf{x}\right\|_{p-\mathrm{var},[s,t]} \left\|R^y\right\|_{q-\mathrm{var},[s,t]^2} \Big\}.
\end{aligned}$$

As a result,

$$\begin{aligned}
\left\|y\right\|_{p-\mathrm{var},[s,t]} \leq & \|f\|_\infty(t-s) + C_g \left\|\mathbf{x}\right\|_{p-\mathrm{var},[s,t]} + C_g^2 \left\|\mathbf{x}\right\|^2_{p-\mathrm{var},[s,t]} \\
& + 3C_p\Big\{ C_g^2 \left\|\mathbf{x}\right\|^2_{p-\mathrm{var},[s,t]} \left\|y\right\|_{p-\mathrm{var},[s,t]} + C_g \left\|\mathbf{x}\right\|_{p-\mathrm{var},[s,t]} \left\|R^y\right\|_{q-\mathrm{var},[s,t]^2} \Big\}.
\end{aligned} \tag{2.39}$$

A similar estimate for R^y then shows that

$$\begin{aligned}
\left\|R^y\right\|_{q-\mathrm{var},[s,t]} \leq & \|f\|_\infty(t-s) + C_g^2 \left\|\mathbf{x}\right\|^2_{p-\mathrm{var},[s,t]} \\
& + 3C_p\Big\{ C_g^2 \left\|\mathbf{x}\right\|^2_{p-\mathrm{var},[s,t]} \left\|y\right\|_{p-\mathrm{var},[s,t]} + C_g \left\|\mathbf{x}\right\|_{p-\mathrm{var},[s,t]} \left\|R^y\right\|_{q-\mathrm{var},[s,t]^2} \Big\}.
\end{aligned} \tag{2.40}$$

Hence, provided that $16C_pC_g \left\|\mathbf{x}\right\|_{p-\mathrm{var},[s,t]} \leq 1$, one takes $3C_pC_g^2 \left\|\mathbf{x}\right\|^2_{p-\mathrm{var},[s,t]} \left\|y\right\|_{p-\mathrm{var},[s,t]}$, which is smaller that $\frac{1}{2}\left\|y\right\|_{p-\mathrm{var},[s,t]}$, from the right hand side to the left hand side of (2.39) to obtain

$$\left\|y\right\|_{p-\mathrm{var},[s,t]} \leq 2\|f\|_\infty(t-s) + 2C_g \left\|\mathbf{x}\right\|_{p-\mathrm{var},[s,t]} + \left\|R^y\right\|_{q-\mathrm{var},[s,t]^2}. \tag{2.41}$$

Replacing (2.41) to the right hand side of (2.40) and then taking all terms of $\left\|R^y\right\|_{q-\mathrm{var},[s,t]^2}$ from the right hand side to the left hand side of (2.40) yields

$$\left\|R^y\right\|_{q-\mathrm{var},[s,t]^2} \leq 3\|f\|_\infty(t-s) + 3C_g^2 \left\|\mathbf{x}\right\|^2_{p-\mathrm{var},[s,t]}. \tag{2.42}$$

Now replacing (2.42) to (2.41), one deduces

$$\left\| y \right\|_{p-\mathrm{var},[s,t]} \leq 5 \| f\|_\infty (t-s) + 5C_g \left\| \mathbf{x} \right\|_{p-\mathrm{var},[s,t]}. \tag{2.43}$$

Note that (2.43) and (2.42) hold whenever $16C_pC_g \left\| \mathbf{x} \right\|_{p-\mathrm{var},[s,t]} \leq 1$. Next, by constructing a greedy sequence of time $\{\tau_i(\frac{1}{16C_pC_g}, \mathbf{x}, [a,b])\}_{i\in\mathbb{N}}$ as in (2.15) and using (2.16), one can easily show that

$$\begin{aligned}
&\left\| y \right\|_{p-\mathrm{var},[a,b]} \\
\leq& N\Big(\frac{1}{16C_pC_g}, \mathbf{x}, [a,b]\Big)^{\frac{p-1}{p}} \sum_{i=0}^{N\left(\frac{1}{16C_pC_g},\mathbf{x},[a,b]\right)-1} \left\| y \right\|_{p-\mathrm{var},[\tau_i,\tau_{i+1}]} \\
\leq& 5N\Big(\frac{1}{16C_pC_g}, \mathbf{x}, [a,b]\Big)^{\frac{p-1}{p}} \sum_{i=0}^{N\left(\frac{1}{16C_pC_g},\mathbf{x},[a,b]\right)-1} \Big(\|f\|_\infty(\tau_{i+1}-\tau_i) + C_g \left\| \mathbf{x} \right\|_{p-\mathrm{var},[\tau_i,\tau_{i+1}]}\Big) \\
\leq& 5N\Big(\frac{1}{16C_pC_g}, \mathbf{x}, [0,T]\Big)^{\frac{2(p-1)}{p}} \Big(\|f\|_\infty(b-a) + C_g \left\| \mathbf{x} \right\|_{p-\mathrm{var},[a,b]}\Big) \qquad (2.44) \\
\leq& 5N\Big(\frac{1}{16C_pC_g}, \mathbf{x}, [0,T]\Big)^{\frac{2(p-1)}{p}} \Big(\|f\|_\infty(b-a)^{1-\nu} + C_g \left\| \mathbf{x} \right\|_{\nu,[a,b]}\Big)(b-a)^\nu.
\end{aligned}$$

A similar estimate for $\left\| R^y \right\|_{q-\mathrm{var},[a,b]^2}$ shows that

$$\begin{aligned}
\left\| R^y \right\|_{q-\mathrm{var},[a,b]^2} \leq& 3N\Big(\frac{1}{16C_pC_g}, \mathbf{x}, [0,T]\Big)^{\frac{2(p-1)}{p}} \Big(\|f\|_\infty(b-a) + C_g^2 \left\| \mathbf{x} \right\|^2_{p-\mathrm{var},[a,b]}\Big) \qquad (2.45) \\
\leq& 3N\Big(\frac{1}{16C_pC_g}, \mathbf{x}, [0,T]\Big)^{\frac{2(p-1)}{p}} \Big(\|f\|_\infty(b-a)^{1-2\nu} + C_g^2 \left\| \mathbf{x} \right\|^2_{\nu,[a,b]}\Big)(b-a)^{2\nu}.
\end{aligned}$$

Therefore, (2.37) is proved by choosing

$$C_1 := 5N\Big(\frac{1}{16C_pC_g}, \mathbf{x}, [0,T]\Big)^{\frac{2(p-1)}{p}} \Big[\|f\|_\infty\Big(T^{1-2\nu} \vee 1\Big) + C_g \left\| \mathbf{x} \right\|_{\nu,[0,T]} \vee \Big(C_g \left\| \mathbf{x} \right\|_{\nu,[0,T]}\Big)^2\Big]. \tag{2.46}$$

Finally, take any two solution $y^i_\cdot(\mathbf{x}, y^i_0)$, write $z_t := y^2_t - y^1_t$ on $[0,T]$ and use the semi-norm as in Proposition 2.2. If f is globally Lipschitz continuous w.r.t. constant C_f, one can apply the following estimate in Duc [8, Theorem 3.9]

$$\begin{aligned}
\left\| z, R^z \right\|_{p-\mathrm{var},[s,t]} \leq& 2\int_s^t C_f \|z_u\| du + 4C_p\Big\{C_g \left\| \mathbf{x} \right\|_{p-\mathrm{var},[s,t]} \vee C_g^2 \left\| \mathbf{x} \right\|^2_{p-\mathrm{var},[s,t]}\Big\} \qquad (2.47) \\
&\times \Big(1 + \left\| y^1, R^{y^1} \right\|_{p-\mathrm{var},[s,t]} + \left\| y^2, R^{y^2} \right\|_{p-\mathrm{var},[s,t]}\Big)\Big(\|z_s\| + \left\| z, R^z \right\|_{p-\mathrm{var},[s,t]}\Big).
\end{aligned}$$

By (2.42) and (2.43), the term $\Big(1 + \left\| y^1, R^{y^1} \right\|_{p-\mathrm{var},[s,t]} + \left\| y^2, R^{y^2} \right\|_{p-\mathrm{var},[s,t]}\Big)$ are bounded by

$$1 + 16\|f\|_\infty(t-s) + 16C_g \left\| \mathbf{x} \right\|_{p-\mathrm{var},[s,t]} \leq 2 + 16T\|f\|_\infty$$

whenever $16C_pC_g \|\mathbf{x}\|_{p-\mathrm{var},[s,t]} \leq 1$. This and (2.47) leads to

$$\begin{aligned}\|z_s\| + \|z,R^z\|_{p-\mathrm{var},[s,t]} &\leq \|z_s\| + 2\int_s^t C_f\|z_u\|du \\ &+8C_pC_g \|\mathbf{x}\|_{p-\mathrm{var},[s,t]}(1+8T\|f\|_\infty)\Big(\|z_s\| + \|z,R^z\|_{p-\mathrm{var},[s,t]}\Big)\end{aligned}$$

whenever $16C_pC_g \|\mathbf{x}\|_{p-\mathrm{var},[s,t]} \leq 1$. Hence, as long as $8C_pC_g \|\mathbf{x}\|_{p-\mathrm{var},[s,t]}(1+8T\|f\|_\infty) \leq \frac{1}{2}$, by taking the term $\|z_s\| + \|z,R^z\|_{p-\mathrm{var},[s,t]}$ from the right hand side to the left hand side, one obtains

$$\|z_t\| \leq \|z_s\| + \|z,R^z\|_{p-\mathrm{var},[s,t]} \leq 2\|z_s\| + 4\int_s^t C_f\|z_u\|du. \tag{2.48}$$

One can now apply Gronwall lemma for (2.48) to conclude that

$$\|z_t\| \leq 2\|z_s\|e^{4C_f(t-s)} \quad \text{whenever} \quad 16C_pC_g \|\mathbf{x}\|_{p-\mathrm{var},[s,t]}(1+8T\|f\|_\infty) \leq 1.$$

By the same arguments as in [8, Theorem 3.9] together with a construction of the greedy sequence of time $\{\tau_i(\frac{1}{16C_pC_g(1+8T\|f\|_\infty)},\mathbf{x},[a,b])\}_{i\in\mathbb{N}}$, one can show that

$$\|z\|_{\infty,[a,b]} \leq \|z_a\| \exp\Big\{4C_fT + N\Big(\frac{1}{16C_pC_g(1+8T\|f\|_\infty)},\mathbf{x},[0,T]\Big)\log 2\Big\}.$$

This proves (2.38) by choosing

$$C_2 = \exp\Big\{4C_fT + N\Big(\frac{1}{16C_pC_g(1+8T\|f\|_\infty)},\mathbf{x},[0,T]\Big)\Big\}. \tag{2.49}$$

□

3 Explicit Euler scheme for rough differential equations

For any finite partition $\Pi := \{0 = t_0 < t_1 < t_2 < \ldots < t_{m-1} < t_m = T\}$ such that $|\Pi| = \sup_k |t_{k+1}-t_k|$, we consider the explicit Euler scheme of equation (1.2) to approximate the fixed solution $y(\cdot,0,y_0)$, i.e.,

$$\begin{aligned} y_0^\Pi &= y_0; \\ y_{k+1}^\Pi &= y_k^\Pi + f(y_k^\Pi)(t_{k+1}-t_k) + g(y_k^\Pi)x_{t_k,t_{k+1}} + Dg(y_k^\Pi)g(y_k^\Pi)\mathbb{X}_{t_k,t_{k+1}},\ 0\leq k\leq m-1. \end{aligned} \tag{3.1}$$

Since the solution $y(t,0,y_0)$ is fixed on $[0,T]$, its supremum norm is bounded by a fixed number R following (2.25).

Our main result below in this section shows that the error between the continuous solution y and the discrete solution of the explicit Euler numerical scheme (3.1) is small on the whole interval $[0,T]$.

Theorem 3.1 *Assume that $y(t,0,y_0)$ is a solution of the rough differential equation* (1.2) *on* $[0,T]$, *under assumption* ($\boldsymbol{H}'_f$) *for f, assumption* ($\boldsymbol{H}^b_g$) *or* ($\boldsymbol{H}^l_g$) *for g, and assumption* ($\boldsymbol{H}_X$) *for x. Then there exists a generic constant*

$$C = C(f,g,\|\mathbf{x}\|_{\nu,[0,T]},T,\|y_0\|) > 0$$

for R defined in (2.25) *such that for* $|\Pi| < \delta$ *small enough*

$$\sup_{0\leq k\leq m} \|y(t_k,0,y_0) - y_k\| \leq C|\Pi|^{3\nu-1}. \tag{3.2}$$

Proof: We first prove the conclusion for bounded g under assumption ($\mathbf{H}_g^b$). To begin, we follow Garrido-Attienza & Schmalfuss [13] to introduce a cutoff functions f_R such that

- $f_R(y) = f(y)$ for all $y \in B(0,R+1)$ and $f_R(y) = f(0)$ for all $\|y\| \geq R+2$;
- f_R is globally Lipschitz continuous w.r.t. constant C_{f_R} and is bounded by a constant $\|f_R\|_\infty$ on $\mathbb{R}^d$.

Specifically, $f_R(y) := f(\zeta_R(y))$ for all $y \in \mathbb{R}^d$, where $\zeta_R \in C^2(\mathbb{R}^d,\mathbb{R}^d)$ is constructed with $\zeta_R(y) = y$ if $\|y\| \leq R+1$ and $\zeta_R(y) = 0$ if $\|y\| \geq R+2$, such that ζ_R is bounded by $R+1$ and $\|D\zeta_R\|_\infty, \|D^2\zeta_R\|_\infty < \infty$.

Consider the truncated rough differential equation

$$dy_t = f_R(y_t)dt + g(y_t)dx(t),\ t \in [0,T]. \tag{3.3}$$

It is easy to check that equation (3.3) also satisfies the existence and uniqueness theorem. To differentiate the solutions of (1.2) and (3.3), one denotes by $y_R(t,s,\xi)$ the solution of (3.3) that starts at time s at point $\xi \in \mathbb{R}^d$. Since $\|y\|_{\infty,[0,T]} \leq R$, the solution y_t lies entirely in the ball $B(0,R+1)$ and $f(y_t) = f_R(y_t)$ for all $t \in [0,T]$, which implies that $y_\cdot = y_\cdot(\mathbf{x},y_0)$ is also the unique solution of (3.3) starting from y_0, i.e. $y_R(\cdot,0,y_0) \equiv y_\cdot(\mathbf{x},y_0)$ on $[0,T]$. Since f_R and g are bounded, by Proposition 2.6 there exist generic constants $C_1(\|f_R\|_\infty, C_g, \left\|\!\left|\mathbf{x}\right|\!\right\|_{\nu,[0,T]}, T), C_2(\|f_R\|_\infty, C_{f_R}, C_g, \left\|\!\left|\mathbf{x}\right|\!\right\|_{\nu,[0,T]}, T)$ which are independent of the initial conditions, such that any two solutions of (3.3) satisfy (2.37) and (2.38).

Next, define for the finite partition $\Pi := \{0 = t_0 < t_1 < t_2 < \ldots < t_{m-1} < t_m = T\}$ the explicit Euler scheme for the truncated rough differential equation (3.3) as follows

$$\begin{aligned} y_0^* &= y_0; \\ y_{k+1}^* &= y_k^* + f_R(y_k^*)(t_{k+1}-t_k) + g(y_k^*)x_{t_k,t_{k+1}} + Dg(y_k^*)g(y_k^*)\mathbb{X}_{t_k,t_{k+1}}, 0 \leq k \leq m-1. \end{aligned} \tag{3.4}$$

The proof applies traditional arguments from Friz & Victoir [11, Theorem 10.30]. Namely, denote by z_k the solution to (3.3) at time T that starts from y_k^* at time t_k, i.e. $z_k := y_R(T,t_k,y_k^*)$. Then $z_0 = y_R(T,0,y_0^*) = y(T,0,y_0)$ and $z_m = y_R(T,t_m,y_m^*) = y_m^*$. Using (2.38), we obtain

$$\begin{aligned} \|y_R(T,0,y_0^*) - y_m^*\| &\leq \sum_{k=0}^{m-1} \|z_k - z_{k+1}\| \\ &\leq \sum_{k=0}^{m-1} \|y_R(T,t_k,y_k^*) - y_R(T,t_{k+1},y_{k+1}^*)\| \\ &\leq \sum_{k=0}^{m-1} \|y_R(T,t_{k+1},y_R(t_{k+1},t_k,y_k^*)) - y_R(T,t_{k+1},y_{k+1}^*)\| \\ &\leq C_2(\|f_R\|_\infty, C_{f_R}, C_g, \left\|\!\left|\mathbf{x}\right|\!\right\|_{\nu,[0,T]}, T) \sum_{k=0}^{m-1} \|y_R(t_{k+1},t_k,y_k^*) - y_{k+1}^*\|. \end{aligned} \tag{3.5}$$

On the other hand, from the definition of $y_R(t_{k+1},t_k,y_k^*)$ and y_{k+1}^*, we apply (2.7) and (2.37) to obtain, up to a generic constant

$$\begin{aligned}
&\|y_R(t_{k+1},t_k,y_k^*)-y_{k+1}^*\| \\
\leq &\Big|\int_{t_k}^{t_{k+1}}[f_R(y_R(u))-f_R(y_k^*)]du+\int_{t_k}^{t_{k+1}}[g(y_R(u))-g(y_k^*)]dx_u\Big| \\
\leq &C_{f_R}(t_{k+1}-t_k)\left\|y_R(\cdot,t_k,y_k^*)\right\|_{p-\mathrm{var},[t_k,t_{k+1}]} \\
&+C_p\Big(\left\|x\right\|_{p-\mathrm{var},[t_k,t_{k+1}]}\left\|R^{y_R}\right\|_{q-\mathrm{var},[t_k,t_{k+1}]^2}+\left\|\mathbb{X}\right\|_{q-\mathrm{var},[t_k,t_{k+1}]^2}\left\|y_R(\cdot,t_k,y_k^*)\right\|_{p-\mathrm{var},[\tau_k,\tau_{k+1}]}\Big) \\
\leq &C_1(\|f_R\|_\infty,C_g,\left\|\mathbf{x}\right\|_{\nu,[0,T]},T)\Big[C_{f_R}T^{1-2\nu}+C_p\Big(\left\|\mathbf{x}\right\|_{\nu,[0,T]}+\left\|\mathbf{x}\right\|_{\nu,[0,T]}^2\Big)\Big](t_{k+1}-t_k)^{3\nu}.
\end{aligned}$$

Therefore, one can estimate (3.5) with a constant

$$C_3=C_2C_1\Big[C_{f_R}T^{1-2\nu}+C_p\Big(\left\|\mathbf{x}\right\|_{\nu,[0,T]}+\left\|\mathbf{x}\right\|_{\nu,[0,T]}^2\Big)\Big] \tag{3.6}$$

as follows

$$\|y_R(T,0,y_0^*)-y_m^*\|\leq C_3\sum_{k=0}^{m-1}(t_{k+1}-t_k)^{3\nu}\leq C_3|\Pi|^{3\nu-1}\sum_{k=0}^{m-1}(t_{k+1}-t_k)=C_3T|\Pi|^{3\nu-1}. \tag{3.7}$$

The right hand side of (3.7) converges to zero as $|\Pi|\to 0$. Similar arguments also hold if we replace $t_m=T$ above by any t_i and define $z_k:=y_R(t_i,t_k,y_k^*)$ for all $0\leq k\leq i$. Hence one obtains (3.2) for the Euler numerical scheme of the truncated equation (3.3) by assigning

$$C(f,g,\left\|\mathbf{x}\right\|_{\nu,[0,T]},T,\|y_0\|):=C_3T=C_2C_1\Big[C_{f_R}T^{1-2\nu}+C_p\Big(\left\|\mathbf{x}\right\|_{\nu,[0,T]}+\left\|\mathbf{x}\right\|_{\nu,[0,T]}^2\Big)\Big]T, \tag{3.8}$$

which depends on $f,g,\mathbf{x}$ and R, thus also on $\|y_0\|$ due to (2.25). Note that $y_R(t_k,0,y_0)=y(t_k,0,y_0)$, thus by choosing $|\Pi|<\delta$ for $\delta=\delta(\|f_R\|_\infty,C_{f_R},C_g,\left\|\mathbf{x}\right\|_{\nu,[0,T]},T)$ small enough, we deduce that

$$\sup_{0\leq k\leq m}\|y_k^*\|\leq\|y\|_{\infty,[0,T]}+C\delta^{3\nu-1}\leq R+1.$$

As a result, $f_R(y_k^*)=f(y_k^*)$ for $0\leq k\leq m$, hence the Euler scheme (3.4) for the truncated equation (3.3) coincides with the actual Euler scheme (3.1) in the ball $B(0,R+1)$, which proves (3.2).

The conclusion still holds for the linear diffusion function g under assumption $(\mathbf{H}_g^l)$, since one can introduce a bounded function g_R in a similar way to f_R, where R is given by (2.33). Since similar arguments are invovled, we skip the proof for this case here. □

4 Generation of random dynamical systems

4.1 Probabilistic settings

The generation of a random dynamical system from rough differential equations (1.1) and (1.2) is proved in Bailleul et al [2], where the solution of rough equation is understood in the Lyons-Davie as well as the Friz-Victoir sense. In this section we follow Duc [7] to present a similar version for Hölder spaces, where the solution is understood in the Gubinelli sense.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a so-called measurable *metric dynamical system* $\theta : \mathbb{R} \times \Omega \to \Omega$ such that $\theta_t : \Omega \to \Omega$ is $\mathbb{P}-$ preserving, i.e., $\mathbb{P}(B) = \mathbb{P}(\theta_t^{-1}(B))$ for all $B \in \mathcal{F}, t \in \mathbb{R}$, and $\theta_{t+s} = \theta_t \circ \theta_s$ for all $t, s \in \mathbb{R}$. Recall that a continuous *random dynamical system* $\varphi : \mathbb{R} \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d$, $(t, \omega, y_0) \mapsto \varphi(t, \omega) y_0$ is defined as a measurable mapping which is also continuous in t and y_0 such that the cocycle property

$$\varphi(t+s, \omega) y_0 = \varphi(t, \theta_s \omega) \circ \varphi(s, \omega) y_0, \quad \forall t, s \in \mathbb{R}, \omega \in \Omega, y_0 \in \mathbb{R}^d \tag{4.1}$$

is satisfied, see Arnold [1].

In our setting, denote by $T_1^2(\mathbb{R}^m) = 1 \oplus \mathbb{R}^m \oplus (\mathbb{R}^m \otimes \mathbb{R}^m)$ the set with the tensor product

$$(1, g^1, g^2) \otimes (1, h^1, h^2) = (1, g^1 + h^1, g^1 \otimes h^1 + g^2 + h^2),$$

for all $\mathbf{g} = (1, g^1, g^2), \mathbf{h} = (1, h^1, h^2) \in T_1^2(\mathbb{R}^m)$. Then it can be shown that $(T_1^2(\mathbb{R}^m), \otimes)$ is a topological group with unit element $\mathbf{1} = (1, 0, 0)$ and $\mathbf{g}^{-1} = (1, -g^1, g^1 \otimes g^1 - g^2)$.

Given $\alpha \in (\frac{1}{3}, \nu)$, denote by $\mathscr{C}^{0,\alpha}(I, T_1^2(\mathbb{R}^m))$ the closure of $\mathscr{C}^\infty(I, T_1^2(\mathbb{R}^m))$ in the Hölder space $\mathscr{C}^\alpha(I, T_1^2(\mathbb{R}^m))$, and by $\mathscr{C}_0^{0,\alpha}(\mathbb{R}, T_1^2(\mathbb{R}^m))$ the space of all paths $\mathbf{g} : \mathbb{R} \to T_1^2(\mathbb{R}^m))$ such that $\mathbf{g}|_I \in \mathscr{C}^{0,\alpha}(I, T_1^2(\mathbb{R}^m))$ for each compact interval $I \subset \mathbb{R}$ containing 0. Then $\mathscr{C}_0^{0,\alpha}(\mathbb{R}, T_1^2(\mathbb{R}^m))$ is equipped with the compact open topology given by the α-Hölder norm (2.1), i.e the topology generated by the metric

$$d_\alpha(\mathbf{g}, \mathbf{h}) := \sum_{k \geq 1} \frac{1}{2^k} (\|\mathbf{g} - \mathbf{h}\|_{\alpha, [-k,k]} \wedge 1).$$

As a result, it is separable and thus a Polish space.

Let us consider a stochastic process $\bar{\mathbf{X}}$ defined on a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ with realizations in $(\mathscr{C}_0^{0,\alpha}(\mathbb{R}, T_1^2(\mathbb{R}^m)), \mathcal{F})$. Assume further that $\bar{\mathbf{X}}$ has stationary increments. Assign $\Omega := \mathscr{C}_0^{0,\alpha}(\mathbb{R}, T_1^2(\mathbb{R}^m))$ and equip it with the Borel σ-algebra $\mathcal{F}$ and let $\mathbb{P}$ be the law of $\bar{\mathbf{X}}$. Denote by θ the *Wiener-type shift*

$$(\theta_t \omega)_\cdot = \omega_t^{-1} \otimes \omega_{t+\cdot}, \forall t \in \mathbb{R}, \omega \in \mathscr{C}_0^{0,\alpha}(\mathbb{R}, T_1^2(\mathbb{R}^m)), \tag{4.2}$$

and define the so-called *diagonal process* $\mathbf{X} : \mathbb{R} \times \Omega \to T_1^2(\mathbb{R}^m), \mathbf{X}_t(\omega) = \omega_t$ for all $t \in \mathbb{R}, \omega \in \Omega$. Due to the stationarity of $\bar{\mathbf{X}}$, it can be proved that θ is invariant under $\mathbb{P}$, then forming a continuous (and thus measurable) dynamical system on $(\Omega, \mathcal{F}, \mathbb{P})$ [2, Theorem 5]. Moreover, $\mathbf{X}$ forms an *α-rough path cocycle*, namely, $\mathbf{X}_\cdot(\omega) \in \mathscr{C}_0^{0,\alpha}(\mathbb{R}, T_1^2(\mathbb{R}^m))$ for every $\omega \in \Omega$, which satisfies the *cocyle relation*:

$$\mathbf{X}_{t+s}(\omega) = \mathbf{X}_s(\omega) \otimes \mathbf{X}_t(\theta_s \omega), \forall \omega \in \Omega, t, s \in \mathbb{R},$$

in the sense that $\mathbf{X}_{s,s+t} = \mathbf{X}_t(\theta_s \omega)$ with the increment notation $\mathbf{X}_{s,s+t} := \mathbf{X}_s^{-1} \otimes \mathbf{X}_{s+t}$. It is important to note that the two-parameter flow property

$$\mathbf{X}_{s,u} \otimes \mathbf{X}_{u,t} = \mathbf{X}_{s,t}, \forall s, t \in \mathbb{R}$$

is equivalent to the fact that $\mathbf{X}_t(\omega) = (1, \mathbf{x}_t(\omega)) = (1, x_t(\omega), \mathbb{X}_{0,t}(\omega))$, where $x_\cdot(\omega) : \mathbb{R} \to \mathbb{R}^m$ and $\mathbb{X}_{\cdot,\cdot}(\omega) : I^2 \to \mathbb{R}^m \otimes \mathbb{R}^m$ are random funtions satisfying Chen's relation relation (2.2).

To fulfill the Hölder continuity of almost all realizations, it follows from condition (2.13) and the Kolmogorov criterion for rough paths [11, Appendix A.3] that for any $\alpha \in (\frac{1}{3}, \nu)$ and

$p = \frac{1}{\alpha}$, there exists a version of ω-wise $(x, \mathbb{X})$ and random variables $K_\beta \in L^p, \mathbb{K}_\beta \in L^q$, such that speaking ω-wise and with an abuse of notation, $\|x_{s,t}\| \leq K_\alpha |t-s|^\alpha, \quad \|\mathbb{X}_{s,t}\| \leq \mathbb{K}_\alpha |t-s|^{2\alpha}$, for all $s, t \in [0, T]$, so that $\mathbf{x} = (x, \mathbb{X}) \in \mathscr{C}^\alpha(I)$. Moreover, we could modify α such that

$$x \in C^{0,\alpha}(I) := \{x \in C^\alpha(I) : \lim_{\Delta \to 0} \sup_{0<t-s<\Delta} \frac{\|x_{s,t}\|}{|t-s|^\alpha} = 0\},$$
$$\mathbb{X} \in C^{0,2\alpha}(I^2) := \{\mathbb{X} \in C^{2\alpha}(I^2) : \lim_{\Delta \to 0} \sup_{0<t-s<\Delta} \frac{\|\mathbb{X}_{s,t}\|}{|t-s|^{2\alpha}} = 0\},$$

thus $\mathscr{C}^\alpha(I) \subset C^{0,\alpha}(I) \oplus C^{0,2\alpha}(I^2)$ is separable due to the separability of $C^{0,\alpha}(I)$ and $C^{0,2\alpha}(I^2)$.

As pointed out in [7, Remark 1] and due to [2, Corollary 9], the above construction is possible for X_t to be a continuous, centered Gaussian process with stationary increments and independent components, satisfying: there exists for any $T > 0$ a constant C_T such that for all $p \geq \frac{1}{\nu}$, $\mathbb{E}\|X_t - X_s\|^p \leq C_T |t-s|^{p\nu}$ for all $s, t \in [0, T]$. Then $\mathbf{X}$ can be chosen to be the natural lift of X in the sense of Friz-Victoir [11, Chapter 15] with sample paths in the space $C_0^{0,\alpha}(\mathbb{R}, T_1^2(\mathbb{R}^m))$, for a certain $\alpha \in (0, \nu)$. In particular, the Wiener shift (4.2) implies that

$$\|\mathbf{x}(\theta_h \omega)\|_{p-\text{var},[s,t]} = \|\mathbf{x}(\omega)\|_{p-\text{var},[s+h,t+h]}, \quad N_{[s,t]}(\mathbf{x}(\theta_h\omega)) = N_{[s+h,t+h]}(\mathbf{x}(\omega)). \tag{4.3}$$

4.2 Continuous flows

Given the setting in Subsection 4.1, we are going to generate a random dynamical system for stochastic rough differential equation (1.1). The first step is to study the properties of rough path integrals. Given each realization ω of the diagonal process $\mathbf{X}_t(\omega) = \omega_t = (1, \mathbf{x}_t(\omega)) = (1, x_t(\omega), \mathbb{X}_{0,t}(\omega))$, we can define the stochastic integral in the pathwise sense as a rough path integral introduced in Subsection 2.2, i.e.

$$\int_a^b y_u d\omega_u := \lim_{|\Pi| \to 0} \sum_{\Pi} \Big(y_u \otimes x_{u,v}(\omega) + y'_u \mathbb{X}_{u,v}(\omega) \Big).$$

The expression of the Darboux sum in the right hand side can be rewritten as

$$y_u \otimes x_{u,v}(\omega) + y'_u \mathbb{X}_{u,v}(\omega) =: (y_u, y'_u) \bar{\otimes} \Big(1, x_{u,v}(\omega), \mathbb{X}_{u,v}(\omega)\Big), \tag{4.4}$$

where the operator $\bar{\otimes}$ in the right hand side of (4.4) is defined as the left hand side. Since ω is the realization of $\mathbf{X}$, it follows from Chen's relation (2.2) that

$$\Big(1, x_{s,u}(\omega), \mathbb{X}_{s,u}(\omega)\Big) \otimes \Big(1, x_{u,v}(\omega), \mathbb{X}_{u,v}(\omega)\Big) = \Big(1, x_{s,v}(\omega), \mathbb{X}_{s,v}(\omega)\Big)$$

hence the shift property (4.2) yields

$$\Big(1, x_{u,v}(\omega), \mathbb{X}_{u,v}(\omega)\Big) = \omega_u^{-1} \otimes \omega_v = (\theta_u \omega)_{v-u}, \quad \forall 0 \leq s \leq t. \tag{4.5}$$

We therefore can rewrite the definition of the above rough integral as

$$\int_a^b y_u d\omega_u := \lim_{|\Pi| \to 0} \sum_{\Pi} (y_u, y'_u) \bar{\otimes} (\theta_u \omega)_{v-u}. \tag{4.6}$$

Since $\theta_{u+r}\omega = \theta_u \circ \theta_r\omega$, it is easy to check that the rough integral in (4.6) satisfies the additivity and the shift properties, i.e.

$$\int_a^c y_u d\omega_u = \int_a^b y_u d\omega_u + \int_b^c y_u d\omega_u, \quad \forall a \leq b \leq c; \tag{4.7}$$

$$\int_{a+r}^{b+r} y_u d\omega_u = \int_a^b y_{u+r} d(\theta_r\omega)_u, \quad \forall a \leq b, r \in \mathbb{R}. \tag{4.8}$$

These two properties (4.7), (4.8) and the existence and uniqueness theorem 2.4 then suffice to prove the cocycle property (4.1) of the generated random dynamical system from stochastic rough differential equation (1.1). We quote a result from Duc [7, Proposition 2] as follows.

Proposition 4.1 *Given the measurable metric dynamical system $(\Omega, \mathscr{F}, \mathbb{P}, \theta)$ and the p-rough cocycle $\mathbf{X} : \mathbb{R} \times \Omega \to T_1^2(\mathbb{R}^m)$ as above, the system (1.1) generates a continuous random dynamical system φ over $(\Omega, \mathscr{F}, \mathbb{P}, \theta)$, such that for any $[0,T]$ and all $\omega \in \Omega$, $\varphi(t,\omega)y_0$ is the unique solution (in the Gubinelli sense) of (1.2), which is understood in the pathwise integral form (2.8) on [0,T], where $\mathbf{x} = (x, \mathbb{X})$ is the projection of $\mathbf{X}_\cdot(\omega)$ on $\mathbb{R}^m \oplus (\mathbb{R}^m \otimes \mathbb{R}^m)$.*

4.3 Discrete flows

Given the probabilistic setting in Subsection 4.2, for a realization $\omega_t = (1, x_t(\omega), \mathbb{X}_{0,t}(\omega))$ of the diagonal process $\mathbf{X}_t(\omega)$, we consider the explicit Euler scheme for the regular grid with step size $h > 0$, i.e. $\Pi = \{kh\}_{k\in\mathbb{N}}$ and

$$\begin{aligned} &y_0^h \in \mathbb{R}^d, \\ &y_{k+1}^h = y_k^h + f(y_k^h)h + g(y_k^h)x_{kh,(k+1)h}(\omega) + Dg(y_k^h)g(y_k^h)\mathbb{X}_{kh,(k+1)h}(\omega), \quad k \in \mathbb{N}. \end{aligned} \tag{4.9}$$

Such a scheme is well defined. Using (4.4) and (4.5), we rewrite (4.9) as

$$\begin{aligned} y_{k+1}^h &= \underbrace{\Big(y_k^h + f(y_k^h)h\Big)}_{=:F(h,y_k^h)} + \Big\langle \underbrace{\Big(g(y_k^h), Dg(y_k^h)g(y_k^h)\Big)}_{=:G(y_k^h)}, \Big(x_{kh,(k+1)h}(\omega), \mathbb{X}_{kh,(k+1)h}(\omega)\Big)\Big\rangle \\ &= F(h, y_k^h) + G(y_k^h)\bar{\otimes}\Big(1, x_{kh,(k+1)h}(\omega), \mathbb{X}_{kh,(k+1)h}(\omega)\Big) \\ &= F(h, y_k^h) + G(y_k^h)\bar{\otimes}\big(\theta_{kh}\omega\big)_h = H(h, \theta_{kh}\omega)y_k^h, \end{aligned} \tag{4.10}$$

where we introduce the generator function

$$H(h,\omega)y := F(y) + G(y)\bar{\otimes}\omega_h. \tag{4.11}$$

Hence similar to Proposition 4.1, we can easily prove that the Euler numerical scheme (4.10) generates a discrete-time random dynamical system $\varphi^h : \mathbb{N}^h \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ over $\mathbb{N}^h := \{kh\}_{k\in\mathbb{N}}$ and $(\Omega, \mathscr{F}, \mathbb{P}, \theta)$ such that for any $\omega \in \Omega$ and $y_0^h \in \mathbb{R}^d$, $\varphi^h(k,\omega)y_0^h$ is defined from (4.10) by

$$\begin{aligned} &\varphi^h(0,\omega)y_0^h \equiv y_0^h, \\ &\varphi^h(kh,\omega)y_0^h := y_k^h = H(h, \theta_{(k-1)h}\omega) \circ \ldots \circ H(h,\omega)y_0^h, \quad \forall k \geq 1. \end{aligned} \tag{4.12}$$

5 Random pullback attractors

Given the random dynamical systems φ and φ^h on the phase space $\mathbb{R}^d$, we follow Crauel & Flandoli [5] (see also Arnold [1, Chapter 9] and Crauel & Kloeden[6] and the references therein) to briefly present the notion of random pullback attractors.

In the probabilistic setting, recall that a set $\hat{M} := \{M(\omega)\}_{\omega\in\Omega}$ is called a *random set*, if $\omega \mapsto d(y|M(\omega)) := \inf\{d(y,z)|z \in M(\omega)\}$ is $\mathscr{F}$-measurable for each $y \in \mathbb{R}^d$. A *universe* $\mathscr{D}$ is a family of random sets which is closed w.r.t. inclusions (i.e. if $\hat{D}_1 \in \mathscr{D}$ and $\hat{D}_2 \subset \hat{D}_1$ then $\hat{D}_2 \in \mathscr{D}$). In our situation, we define the universe $\mathscr{D}$ to be a family of *tempered* random sets $D(\omega)$, that is, $D(\omega)$ is contained in a ball $B(0,\rho(\omega))$ a.s., where the radius $\rho(\omega) > 0$ is a tempered random variable (i.e. $\lim\limits_{t\to\pm\infty} \frac{1}{t}\log^+ \rho(\theta_t\omega) = 0$ a.s., see e.g., Arnold [1, pp. 164, 386] and Imkeller & Schmalfuss [17, p. 220]).

A random set A is said to be *invariant* if $\varphi(t,\omega)A(\omega) = A(\theta_t\omega)$ for all $t \in \mathbb{R}$, $\omega \in \Omega$. An invariant random compact set $\mathcal{A} \in \mathscr{D}$ is called a *pullback attractor* in $\mathscr{D}$, if $\mathcal{A}$ attracts any closed random set $\hat{D} \in \mathscr{D}$ in the pullback sense, i.e.

$$\lim_{t\to\infty} d_H(\varphi(t,\theta_{-t}\omega)\hat{D}(\theta_{-t}\omega)|\mathcal{A}(\omega)) = 0, \tag{5.1}$$

where $d_H(\cdot|\cdot)$ is the Hausdorff semi-distance, i.e. $d_H(D|A) := \sup_{d\in D}\inf_{a\in A}\|d-a\|$.

The existence of a pullback attractor follows from the existence of a pullback absorbing set (see [6, Theorem 3]), namely a random set $\mathcal{B} \in \mathscr{D}$ is called *pullback absorbing* in the universe $\mathscr{D}$ if $\mathcal{B}$ absorbs all closed random sets in $\mathscr{D}$, i.e. for any closed random set $\hat{D} \in \mathscr{D}$, there exists a time $t_0 = t_0(\omega,\hat{D})$ such that

$$\varphi(t,\theta_{-t}\omega)\hat{D}(\theta_{-t}\omega) \subset \mathcal{B}(\omega), \text{ for all } t \geq t_0. \tag{5.2}$$

Then given the universe $\mathscr{D}$ and a compact pullback absorbing set $\mathcal{B} \in \mathscr{D}$, there exists a unique pullback attractor $\mathcal{A}(\omega)$ in $\mathscr{D}$, given by

$$\mathcal{A}(\omega) = \bigcap_{t\geq 0}\overline{\bigcup_{s\geq t}\varphi(s,\theta_{-s}\omega)\mathcal{B}(\theta_{-s}\omega)}. \tag{5.3}$$

As proved in Duc [7, Theorem 3.1], under the assumptions ($\mathbf{H}'_f$), ($\mathbf{H}^b_g$), ($\mathbf{H}_X$) and the dissipativity condition

$$\exists D_1 \geq 0, D_2 > 0: \quad \langle y, f(y)\rangle \leq \|y\|(D_1 - D_2\|y\|), \quad \forall y \in \mathbb{R}^d, \tag{5.4}$$

there exists a pullback attractor $\mathcal{A}(\omega)$ for the generated random dynamical system of the stochastic system (1.1) such that $|\mathcal{A}(\cdot)| \in \mathcal{L}^\rho$ for any $\rho \geq 1$. It is important to note that assumption (5.4) is equivalent to the dissipativity condition: there exist constants $d_1 \geq 0, d_2 > 0$ such that

$$\langle y, f(y)\rangle \leq d_1 - d_2\|y\|^2, \quad \forall y \in \mathbb{R}^d; \tag{5.5}$$

see Duc [7, Lemma 1.1].

We show below the same conclusion for the linear g, but require further that the stochastic process X is Gaussian and $C_g = \|C\|$ is small enough.

Theorem 5.1 *Let the assumptions ($\boldsymbol{H}'_f$), ($\boldsymbol{H}^l_g$), ($\boldsymbol{H}_X$) hold and further that X is a centered Gaussian process. Then for sufficiently small C_g, there exists a pullback attractor $\mathcal{A}(\omega)$ for the generated random dynamical system of the stochastic system (1.1) such that $|\mathcal{A}(\cdot)| \in \mathcal{L}^\rho$ for any $\rho \geq 1$.*

Proof: We will follow the arguments in Duc [7, Theorem 2.1] line by line to prove that for any $\lambda > 0$ small enough, there exist constants $\delta_\lambda, C_\lambda > 0$ such that the following estimates hold

$$\|y_t\| \leq \exp\Big\{\lambda N\Big(\frac{\lambda}{16C_pC_g}, \mathbf{x}, [0,t]\Big)\Big\}\Big[\|y_0\|e^{-\delta_\lambda t} + C_\lambda N\Big(\frac{\lambda}{16C_pC_g}, \mathbf{x}, [0,t]\Big)\Big], \quad \forall t \in [0,T]. \tag{5.6}$$

To do this, the main task is prove that there exists constants $\bar{C}_\lambda, \delta_\lambda > 0$ such that

$$\frac{1}{2}\frac{d}{dt}\|z_t\|^2 \leq \bar{C}_\lambda - \delta_\lambda\|z_t\|^2. \tag{5.7}$$

To prove (5.7), one follows the notation in (2.34) and (2.28) to estimate

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|z_t\|^2 =& \langle z_t, (Id+\psi_t)f(z_t+\gamma_t)\rangle \\
=& \Big\langle z_t, (Id+\psi_t)\Big[\frac{\langle z_t+\gamma_t, f(z_t+\gamma_t)\rangle}{\|z_t+\gamma_t\|^2}(z_t+\gamma_t) + \pi^\perp_{z_t+\gamma_t}(f(z_t+\gamma_t))\Big]\Big\rangle \\
=& \underbrace{\Big\langle z_t, (Id+\psi_t)\frac{(z_t+\gamma_t)}{\|z_t+\gamma_t\|}\Big\rangle}_{=:M_1}\underbrace{\Big\langle \frac{z_t+\gamma_t}{\|z_t+\gamma_t\|}, f(z_t+\gamma_t)\Big\rangle}_{=:M_2} \\
&+ \underbrace{\Big\langle z_t, (Id+\psi_t)\pi^\perp_{z_t+\gamma_t}(f(z_t+\gamma_t))\Big\rangle}_{=:M_3}.
\end{aligned} \tag{5.8}$$

The estimates for M_1 and M_2 look the same as in the proof of [7, Theorem 2.1], thus there exists a generic constant $\bar{C}_\lambda$ such that

$$M_1M_2 \leq \bar{C}_\lambda - \frac{D_2}{2}(1-\lambda)\|z_t\|^2.$$

With g satisfying ($\mathbf{H}^l_g$), there is a small change with M_3, which according to (2.35) looks like

$$M_3 \leq 2\lambda\|z_t\|C_f\Big(1+(1+\lambda)\|z_t\|\Big) = 2C_f\lambda(1+\lambda)\|z_t\|^2 + C_f\lambda(\|z_t\|^2+1).$$

The coefficient of $\|z_t\|^2$ in M_3 is then can be controlled by choosing sufficiently small $\lambda \in (0,1)$. As a result, one can always find generic constants $\bar{C}_\lambda, \delta_\lambda$ such that $\delta_0 > 0$ and

$$\frac{1}{2}\frac{d}{dt}\|z_t\|^2 = M_1M_2 + M_3 \leq \bar{C}_\lambda - \delta_\lambda\|z_t\|^2, \quad \forall t \in [0,\tau].$$

Hence by Gronwall lemma and Cauchy inequality, one obtains

$$\|z_\tau\| \leq \|z_0\|e^{-\delta_\lambda\tau} + \frac{\bar{C}_\lambda}{\delta_\lambda},$$

which deduces that

$$\|y_\tau\| \le (1+\lambda)\|z_\tau\| \le \|y_0\| e^{\lambda-\delta_\lambda \tau} + (1+\lambda)\frac{\bar{C}_\lambda}{\delta_\lambda}.$$

Assign $C_\lambda := (1+\lambda)\frac{\bar{C}_\lambda}{\delta_\lambda}$, then for each $t\in[0,1]$, by constructing the sequence of greedy times $\{\tau(\frac{\lambda}{16C_pC_g},\mathbf{x}),[0,t]\}$, one can easily prove by induction that

$$\|y_{\tau_i}\| \le \|y_0\| e^{i\lambda-\delta\tau_i} + iC_\lambda e^{i\lambda}, \quad i=0,\dots,N(\frac{\lambda}{16C_pC_g},\mathbf{x},[0,t]).$$

In particular, (5.6) holds. For $t=1$, one obtains

$$\begin{aligned}\|y_1\| \le & \|y_0\| \underbrace{\exp\{\lambda N(\frac{\lambda}{16C_pC_g},\mathbf{x},[0,1]) - \delta_\lambda\}}_{=:\eta(\mathbf{x}(\omega),[0,1])} \\ & + \underbrace{C_\lambda N(\frac{\lambda}{16C_pC_g},\mathbf{x},[0,1]) \exp\{\lambda N(\frac{\lambda}{16C_pC_g},\mathbf{x},[0,1])\}}_{=:\xi(\mathbf{x}(\omega),[0,1])}.\end{aligned} \tag{5.9}$$

By replacing ω by $\theta_{-n}\omega$ and applying the discrete Gronwall lemma [4, Lemma 5.4], one can prove that

$$\begin{aligned}\|y_n(\theta_{-n}\omega,y_0)\| \le & \|y_0\| \prod_{i=0}^{n-1}\eta(\mathbf{x}(\theta_{-n}\omega),[i,i+1]) \\ & + \sum_{i=0}^{n}\xi(\mathbf{x}(\theta_{-n}\omega),[i,i+1]) \prod_{j=i+1}^{n-1}\eta(\mathbf{x}(\theta_{-n}\omega),[j,j+1]).\end{aligned}$$

Since X is Gaussian, it can be lifted to a Gaussian rough path, from which one can prove the integrability of $\exp\{\lambda N(\frac{\lambda}{16C_pC_g},\mathbf{x},[0,1])\}$ (see Cass et al [3]). Hence $\xi(\mathbf{x}(\omega),[0,1])$ is an integrable random variable and tempered. On the other hand, by using (4.3) and applying Birkhorf's ergodic theorem, one can show that

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\log\eta(\mathbf{x}(\theta_{-n}\omega),[i,i+1]) &= \limsup_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\log\eta(\mathbf{x}(\theta_{-i}\omega),[0,1]) \\ &= \mathbb{E}\lambda N(\frac{\lambda}{16C_pC_g},\mathbf{x}(\cdot),[0,1]) - \delta_\lambda.\end{aligned} \tag{5.10}$$

Similar to the arguments in Duc [7, Theorem 3.3], one can choose $\lambda := C_g$ for sufficiently small C_g so that the right hand side of (5.10) is negative. One then follows the arguments in Cong et al [4, Theorem 4.5 & Lemma 5.2] to conclude that there exists a random pullback attractor. □

5.1 Random pullback attractors for the explicit Euler scheme

We now consider a similar result on the existence of a pullback attractor for the discrete-time RDS generated by the Euler numerical scheme (4.10) for sufficiently small step size $h > 0$. The difference is that condition (2.10) is not enough, thus we need f to be of linear growth and g is bounded. We formulate the result as follows.

Theorem 5.2 *Under the hypotheses ($\boldsymbol{H}'_f$),($\boldsymbol{H}^b_g$) and ($\boldsymbol{H}_X$), assume further that f is dissipative with* (5.4) *and of linear growth, i.e. there exists C_f such that*

$$\|f(y)\| \leq C_f(1 + \|y\|). \tag{5.11}$$

Then there exists a $h_0 > 0$ such that for all $h < h_0$, the generated discrete-time random dynamical system φ^h from (4.10) *admits a global random pullback attractor $\mathscr{A}^h(\omega)$.*

Proof: It suffices to prove that there exists an absorbing set for the generated RDS φ^h. Consider the Lyapunov function $\|y^h_k\|$ then by applying Cauchy inequality and using assumptions ($\mathbf{H}_f$), ($\mathbf{H}_g$) and (5.11) we obtain

$$\begin{aligned}
\|y^h_{k+1}\|^2 =& \|y^h_k + f(y^h_k)h + g(y^h_k)x_{kh,(k+1)h} + Dg(y^h_k)g(y^h_k)\mathbb{X}_{kh,(k+1)h}\|^2 \\
\leq& \|y^h_k\|^2 + 2\langle y^h_k, f(y^h_k)\rangle h + \|f(y^h_k)\|^2h^2 \\
&+ 2\Big\langle y^h_k + f(y^h_k)h, g(y^h_k)x_{kh,(k+1)h} + Dg(y^h_k)g(y^h_k)\mathbb{X}_{kh,(k+1)h}\Big\rangle \\
&+ 2\|g(y^h_k)x_{kh,(k+1)h}\|^2 + 2\|Dg(y^h_k)g(y^h_k)\mathbb{X}_{kh,(k+1)h}\|^2 \\
\leq& \|y^h_k\|^2 + 2h\|y^h_k\|(D_1 - D_2\|y^h_k\|) + 2C_f^2h^2(1 + \|y^h_k\|^2) \\
&+ \chi h\Big(\|y^h_k\| + C_f h(1 + \|y^h_k\|)\Big)^2 + \frac{1}{\chi h}(C_g\|x_{kh,(k+1)h}\| + C_g^2\|\mathbb{X}_{kh,(k+1)h}\|)^2 \\
&+ 2C_g^2\|x_{kh,(k+1)h}\|^2 + 2C_g^4\|\mathbb{X}_{kh,(k+1)h}\|^2 \\
\leq& \|y^h_k\|^2\Big(1 - D_2h + 2C_f^2h^2\Big) + \frac{D_1^2}{D_2}h + 2C_f^2h^2 + 4\chi h(1 + C_f^2h^2)\|y^h_k\|^2 + 2\chi C_f^2h^3 \\
&+ 2(1 + \frac{1}{\chi h})\Big(C_g^2\|x_{kh,(k+1)h}\|^2 + C_g^4\|\mathbb{X}_{kh,(k+1)h}\|^2\Big) \\
\leq& \|y^h_k\|^2\Big(1 - D_2h + 2C_f^2h^2 + 4\chi h(1 + C_f^2h^2)\Big) + \xi^h_0\Big(\;\|\!|\mathbf{x}(\omega)|\!\|_{p-\mathrm{var},[kh,(k+1)h]}\Big)
\end{aligned} \tag{5.12}$$

where

$$\xi^h_0(A) := \frac{D_1^2}{D_2}h + 2C_f^2h^2 + 2\chi C_f^2h^3 + 2(1 + \frac{1}{\chi h})\Big(C_g^2A^2 + C_g^4A^4\Big), \tag{5.13}$$

and we can choose $\chi := \frac{D_2}{8}$ so that

$$1 - D_2h + 2C_f^2h^2 + 4\chi h(1 + C_f^2h^2) = 1 - \frac{D_2}{2}h + 2C_f^2h^2 + \frac{D_2}{2}C_f^2h^3 < 1 - \frac{D_2}{4}h \tag{5.14}$$

whenever

$$h < \frac{D_2}{4C_f^2(2+D_2)} \wedge 1 =: h_0. \tag{5.15}$$

Note that from (4.3), $\xi_0^h(\|\mathbf{x}(\omega)\|_{p-\mathrm{var},[kh,(k+1)h]}) = \xi_0^h(\|\mathbf{x}(\theta_{kh}\omega)\|_{p-\mathrm{var},[0,h]})$ which can be written as $\xi_0^h(\theta_{kh}\omega)$, where $\xi_0^h \in L^1$ is an integrable random variable. Replacing (5.14) into (5.12) we show that for $h < h_0$ in (5.15)

$$\|y_{k+1}^h\|^2 \le \Big(1 - \frac{D_2}{4}h\Big)\|y_k^h\|^2 + \xi_0^h(\theta_{kh}\omega) < e^{-\frac{D_2}{4}h}\|y_k^h\|^2 + \xi_0^h(\theta_{kh}\omega).$$

Hence by induction we can prove that

$$\|y_k^h\|^2 \le e^{-\frac{D_2}{4}hk}\|y_0^h\|^2 + \sum_{i=0}^{k-1} e^{-\frac{D_2}{4}ih}\xi_0^h(\theta_{k-i}\omega), \quad \forall k \ge 1.$$

By applying similar arguments to the ones in [7, Theorem 3.3] we conclude that there exists a random pullback absorbing set $B^h(\omega) = B(0, R^h(\omega))$ where

$$R^h(\omega) := \sum_{k=0}^{\infty} e^{-\frac{D_2}{4}kh}\xi_0^h(\theta_{-kh}\omega). \tag{5.16}$$

Since $\xi_0^h \in L^1$, so is $\log^+ \xi_0^h$, which implies that $\frac{1}{t}\log^+ \xi_0^h(\theta_{-t}\omega) \to 0$ as $t \to \infty$. Hence ξ_0^h is tempered, and it follows from Cong et al [4, Lemma 5.2] that $R^h(\omega)$ is finite and also tempered a.s. This proves the existence of a random pullback attractor $\mathscr{A}^h(\omega)$ defined by (5.3). □

Remark 5.3 When g is linear, the question on existence of a numerical attractor for the discrete RDS φ^h generated by the Euler scheme is still open.

Remark 5.4 Although ξ_1^h is integrable random variable, it follows from (5.13) and (5.16) that

$$\mathbb{E}R^h = \frac{1}{1 - e^{-\frac{D_2}{4}h}}\mathbb{E}\xi_1^h \approx \frac{4}{D_2 h}\mathbb{E}\xi_1^h$$

which diverges to infinity as h tends to zero, since $\xi_0^h(\omega)$ contains element $\frac{1}{h}\|\mathbf{x}(\omega)\|_{p-\mathrm{var},[0,h]}^2 \approx h^{2\nu-1}\|\mathbf{x}(\omega)\|_{\nu,[0,1]}^2$. This implies that the absorbing set B^h might blow up as h tends to zero, which makes it difficult to prove the upper semi-continuous convergence of the numerical attractor in the next section.

5.2 Upper semi-continuous convergence of the numerical attractor

The upper semi-continuous convergence of the numerical attractor to the attractor of an autonomous ordinary differential equation is now a classical result in numerical dynamics, see e.g., Han & Kloeden [15]. Similar results have been established for many other types of differential equations including random ordinary differential equations [18]. It is well known that the stronger continuous convergence in the Hausdorff metric holds only in very special cases.

For the rough differential equation (1.3) in the sense of Gubinelli, where the stochastic process X with stationary increments, we can only prove an analogous result for bounded f and g, i.e., that $\mathscr{A}^h \to \mathscr{A}$ in the Hausdorff semi-distance as $h \to 0^+$, i.e., converges upper semi-continuously. We formulate the result as follows.

Theorem 5.5 *Assume ($\boldsymbol{H}_g^b$) and ($\boldsymbol{H}_X$) with a centered Gaussian process X. Assume further that f is globally Lipschitz continuous and bounded, such that the dissipativity condition* (5.4) *is satisfied. Then φ^h admits a numerical pullback attractor $\mathcal{A}^h$ which converges to the attractor $\mathcal{A}$ a.s. in the Hausdorf semi-distance, i.e.,*

$$\lim_{h\to 0} d_H(\mathcal{A}^h|\mathcal{A}) = 0 \quad a.s. \tag{5.17}$$

Proof: For any time step $h < \frac{1}{2}$, assign $l := \lfloor \frac{1}{h} \rfloor \in \mathbb{N}$. Then $h \in (\frac{1}{l+1}, \frac{1}{l}]$ with $1 - h < lh \leq 1$. It implies from the proof of Theorem (3.1) that in case f is bounded, we obtain from (2.37) and (3.7) for $T := 1$ that there exist constants

$$\begin{aligned}
C_3(\omega) = C(\omega) =& C_2 C_1 \Big[C_f + C_p \Big(\vert\!\vert\!\vert \mathbf{x} \vert\!\vert\!\vert_{\nu,[0,1]} + \vert\!\vert\!\vert \mathbf{x} \vert\!\vert\!\vert^2_{\nu,[0,1]} \Big) \Big] \\
=& 5N\Big(\frac{1}{16C_pC_g}, \mathbf{x}, [0,1]\Big)^{\frac{2(p-1)}{p}} \Big[\|f\|_\infty + C_g \vert\!\vert\!\vert \mathbf{x} \vert\!\vert\!\vert_{\nu,[0,1]} \vee \Big(C_g \vert\!\vert\!\vert \mathbf{x} \vert\!\vert\!\vert_{\nu,[0,1]} \Big)^2 \Big] \\
& \times \exp\Big\{ 4C_f + N\Big(\frac{1}{16C_pC_g(1+8\|f\|_\infty)}, \mathbf{x}, [0,1] \Big) \Big\} \\
& \times \Big[C_f + C_p \Big(\vert\!\vert\!\vert \mathbf{x} \vert\!\vert\!\vert_{\nu,[0,1]} + \vert\!\vert\!\vert \mathbf{x} \vert\!\vert\!\vert^2_{\nu,[0,1]} \Big) \Big]
\end{aligned} \tag{5.18}$$

as in (2.46), (2.49) and (3.8) such that

$$\begin{aligned}
\|\varphi^h(lh,\omega)y_0 - \varphi(1,\omega)y_0\| \leq& \|\varphi^h(lh,\omega)y_0 - \varphi(lh,\omega)y_0\| + \|\varphi(lh,\omega)y_0 - \varphi(1,\omega)y_0\| \\
\leq& \sup_{0\leq i\leq l} \|\varphi^h(ih,\omega)y_0 - \varphi(ih,\omega)y_0\| + \vert\!\vert\!\vert \varphi(\cdot,\omega)y_0 \vert\!\vert\!\vert_{p-\mathrm{var},[lh,1]} \\
\leq& C(\omega)h^{3\nu-1} + C_3(\omega)(1-lh)^\nu \\
\leq& C(\omega)(h^{3\nu-1} + h^\nu) \\
\leq& C(\omega)h^{3\nu-1}.
\end{aligned}$$

As a result, there exists a constant $C(\omega)$ independent of the initial condition y_0 such that

$$\|\varphi^h(lh,\omega)y_0\| \leq \|\varphi(1,\omega)y_0\| + C(\omega)h^{3\nu-1}, \quad \forall y_0 \in \mathbb{R}^d. \tag{5.19}$$

Observe that in the last formula in (5.18), $C(\omega)$ is the product of 3 terms, where the first and third terms are integrable due to the fact that $\vert\!\vert\!\vert \mathbf{x} \vert\!\vert\!\vert_{\nu,[0,1]}$ is integrable of any order, and so is the second term due to Cass et al [3, Theorem 6.3]. Hence by Cauchy inequality, $C(\omega)$ is also integrable. Now applying Duc [7, Theorem 3.3] for dissipative function f, there exists a constant $\eta \in (0,1)$ and an integrable random variable $\xi_0(\omega) = \xi_0(\vert\!\vert\!\vert \mathbf{x}(\omega) \vert\!\vert\!\vert_{\nu,[0,1]})$ such that

$$\|\varphi(1,\omega)y_0\| \leq \eta\|y_0\| + \xi_0(\omega), \quad \forall y_0 \in \mathbb{R}^d.$$

Hence

$$\|\varphi^h(lh,\omega)y_0\| \leq \eta\|y_0\| + \xi_0(\omega) + C(\omega)h^{3\nu-1},$$

which, by similar arguments to [7, Theorem 3.3] proves the existence of a pullback absorbing set $B^h(\omega) = B(0, R^h(\omega))$, where

$$R^h(\omega) = \sum_{k=0}^\infty \eta^k \Big(\xi_0(\theta_{-klh}\omega) + C(\theta_{-klh}\omega)h^{3\nu-1} \Big).$$

We are going to find an upper bound for R^h. To do that, we use (4.3) to rewrite R^h as

$$\begin{aligned}R^h(\omega) &= \sum_{k=0}^{\infty}\eta^k\Big[\xi_0(\|\mathbf{x}(\theta_{-klh}\omega)\|_{\nu,[0,1]}) + C(\|\mathbf{x}(\theta_{-klh}\omega)\|_{\nu,[0,1]})h^{3\nu-1}\Big]\\ &= \sum_{k=0}^{\infty}\eta^k\Big[\xi_0(\|\mathbf{x}(\omega)\|_{\nu,[-klh,-klh+1]}) + C(\|\mathbf{x}(\omega)\|_{\nu,[-klh,-klh+1]})h^{3\nu-1}\Big].\end{aligned}$$

It is easy to check that $\lfloor klh\rfloor \geq -klh > -(\lfloor klh\rfloor + 1)$, and

$$[-klh,-klh+1] \subset \Big[-\lfloor klh\rfloor - 1, -\lfloor klh\rfloor + 1\Big], \quad \forall k \in \mathbb{N}.$$

On the other hand, it follows from $lh > \frac{1}{2}$ that

$$\lfloor klh\rfloor \leq \lfloor (k+1)lh\rfloor \leq \lfloor klh\rfloor + 1, \quad \text{and} \quad \lfloor klh\rfloor < \lfloor (k+2)lh\rfloor. \tag{5.20}$$

Hence (5.20) implies that the sequence $\{\lfloor klh\rfloor\}_{k\in\mathbb{N}}$ covers the set $\mathbb{N}$ of natural numbers and every number in the sequence only appears at most twice. By writing $j := \lfloor klh\rfloor$ we can easily prove that

$$\begin{aligned}R^h(\omega) &\leq \sum_{k=0}^{\infty}\eta^{\lfloor klh\rfloor}\Big[\xi_0\Big(\|\mathbf{x}(\omega)\|_{\nu,[-\lfloor klh\rfloor-1,-\lfloor klh\rfloor+1]}\Big) + C\Big(\|\mathbf{x}(\omega)\|_{\nu,[-\lfloor klh\rfloor-1,-\lfloor klh\rfloor+1]}\Big)h^{3\nu-1}\Big]\\ &\leq 2\sum_{j=0}^{\infty}\eta^j\Big[\xi_0\Big(\|\mathbf{x}(\omega)\|_{\nu,[-j-1,-j+1]}\Big) + C\Big(\|\mathbf{x}(\omega)\|_{\nu,[-j-1,-j+1]}\Big)h^{3\nu-1}\Big]\\ &\leq 2\sum_{j=0}^{\infty}\eta^j\Big[\xi_0\Big(\|\mathbf{x}(\theta_{-j}\omega)\|_{\nu,[-1,1]}\Big) + C\Big(\|\mathbf{x}(\theta_{-j}\omega)\|_{\nu,[-1,1]}\Big)\Big] =: 2\bar{R}(\omega).\end{aligned} \tag{5.21}$$

Similar to the argument in proof of Theorem 5.2, since $\xi_0(\|\mathbf{x}(\omega)\|_{\nu,[-1,1]})$ and $C(\|\mathbf{x}(\omega)\|_{\nu,[-1,1]})$ are integrable, it follows that $\log^+\xi_0(\|\mathbf{x}(\omega)\|_{\nu,[-1,1]})$ and $\log^+ C(\|\mathbf{x}(\omega)\|_{\nu,[-1,1]})$ are also integrable. Thus $\xi_0(\|\mathbf{x}(\omega)\|_{\nu,[-1,1]}), C(\|\mathbf{x}(\omega)\|_{\nu,[-1,1]})$ are tempered random variables which, together with [4, Lemma 5.2], shows that $\bar{R}(\omega)$ is well defined and also tempered. That means $A^h(\omega) \subset B^h(\omega) = B(0, R^h(\omega))$ for $h < \frac{1}{2}$ are entirely contained in a tempered set $B(0, 2\bar{R}(\omega))$, hence they are uniformly attracted to $\mathscr{A}(\omega)$ in the pullback sense under the flow φ. Hence for any $\epsilon > 0$ small enough, there exists a $M(\epsilon,\omega)$ such that

$$d_H\Big(\varphi(k,\theta_{-k}\omega)\mathscr{A}^h(\theta_{-k}\omega)|\mathscr{A}(\omega)\Big) < \epsilon, \quad \forall k \geq M(\epsilon,\omega), \quad \forall h < \frac{1}{2}. \tag{5.22}$$

With such fixed $M(\epsilon,\omega)$, there exists a constant $C(\omega, M)$ such that for all $h < \delta(\omega, M) \wedge \frac{1}{2}$ and all $y^h \in \mathscr{A}^h$ in the ω-wise sense

$$\|\varphi^h(M,\theta_{-M}\omega)y^h(\theta_{-M}\omega) - \varphi(M,\theta_{-M}\omega)y^h(\theta_{-M}\omega)\| \leq C(\omega,M)h^{3\nu-1} < \epsilon.$$

Since the above inequality holds for all $y^h \in \mathscr{A}^h$, it yields

$$d_H\Big(\varphi^h(M,\theta_{-M}\omega)\mathscr{A}^h(\theta_{-M}\omega)|\varphi(M,\theta_{-M}\omega)\mathscr{A}^h(\theta_{-M}\omega)\Big) \leq \epsilon. \tag{5.23}$$

From (5.22) and (5.23), it follows from the invariance of $\mathscr{A}^h$ under φ^h and the triangular inequality that

$$d_H(\mathscr{A}^h(\omega)|\mathscr{A}(\omega)) < 2\epsilon, \quad \forall h < \delta(\omega, M) \wedge \frac{1}{2}.$$

This proves that (5.17) hold almost surely. □

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