Upper bound for conjunction probability of two-dimensional Gaussian fields

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Abstract

In this paper we provide an upper bound for the conjunction probability of smooth stationary two-dimensional Gaussian fields. The key ingredient is the record method that is generalized from the problem estimating the distribution of the maximum. The given upper bound also provides a good asymptotic formula.

Key-words: Conjunction probability, Gaussian fields, Kac-Rice formula, record method.

Classifications: 60G15, 60G60, 60G70.

1 Introduction

Let X be a real stationary centered Gaussian field with unit variance and almost surely smooth sample paths. Assume more that it is defined on a compact set $S \subset \mathbb{R}^d$. Consider n independent copies $\{X_i(t); i = 1, 2, \dots, n\}$ of X. For a given level u, the *conjunction probability* can be defined as

$$\mathrm{P}\left(\exists t \in S:\ X_i(t) \geq u,\ \forall i \in \overline{1, n}\right) = \mathrm{P}\left(\max_{t \in S} \min_{i=\overline{1,n}} \{X_i(t)\} \geq u\right). \tag{1}$$

This quantity is proposed by Friston and Worsley [19] with motivation in neurology. In particular, this quantity provides the critical value in statistical application to test whether the functional organization of the brain for language differs according to sex. In this application, the index set is the brain, $n = 2$ stands for two genders male and female, and the value of each field at each point represents the intensity.

The conjunction probability can be seen as a generalization of the tail distribution of the maximum of a smooth Gaussian field. Computing the tail of the maximum of random processes and random fields is a classic topic that has drawn much of interest in probability theory, see three standard monographs [1, 5, 14]. Many practical applications of this problem can be found in spatial statistics, image processing, oceanography, genetics etc ..., see for example Cressie and Wikle [6]. Let us summarize some celebrated methods to deal with this problem: double sum method (by Piterbarg [12] and Hashorva et al [7]), tube method (by Sun [16] and Akimichi and Takemura [17]), Euler characteristic method (by Adler and Taylor [1]), Rice method (by Azaïs and Delmas [2]), direct method (by Azaïs and Wschebor [5]), record method (see Rychlik [15], Mercadier [9], and Azaïs and Pham [3]).

Recently many authors have been interested in the conjunction problem. They have managed to exploit the methods mentioned above. By Euler characteristic method, Friston and Worsley [19] and also Taylor [18] have provided a beautiful formula for the expectation of the Euler characteristic of the conjunction set. By heuristic argument, they have argued that this expectation can be seen as a good approximation for the considering conjunction probability. However, proving the validity of this method is still an open, challenging and interesting (of course) question. By double-sum method, Hashorva et al [8] have given a one-term asymptotic formula, but there appears an unknown constant so-called *the generalized Pickands constants*. In [10], the Rice method has been applied successfully to give an upper bound and also an asymptotic formula for the conjunction probability of smooth processes, but it seems hard to deal with higher dimensions. For direct method, let us mention a recent result by Pham [11]. In this result for the conjunction probability of two-dimensional Gaussian fields, under some conditions, as the level u tends to infinity,

$$\mathrm{P}\left(\max_{t\in S}\min_{1\le i\le n} X_i(t)\ge u\right)=u^{2-n}\varphi^n(u)\left[\frac{\lambda_2(S)}{2\pi}\left(n+\frac{n(n-1)\pi}{4}\right)+o(1)\right], \tag{2}$$

where λ_2 stands for the standard two-dimensional Lebesgue measure and $\varphi(.)$ is the density of a standard normal random variable. In fact, he provided a complicated asymptotic formula for any dimension, and it is interesting that this formula coincides with the leading term of the heuristic approximation given by Euler characteristic method in [19].

In this paper, we are interested in giving an explicit upper bound for the conjunction probability of two-dimensional Gaussian fields. Such an explicit is very important in practical statistics test to guaranty critical value α. Here we rely on the record method. As mentioned above, to estimate the tail distribution of the maximum of Gaussian fields, this method has been introduced for random processes by Rychlik [15] and has been later extended to two-dimensional random fields by Mercadier [9]. In [3], the authors did refine the result of Mercadier by giving an explicit upper bound, and also generalized to three-dimensional fields.

Throughout this paper, we assume the following condition on the considering field.

Assumption A: $\{X(t),\ t\in NS\subset\mathbb{R}^2\}$ is a stationary Gaussian field, defined in a neighborhood NS of S with $\mathcal{C}^4$ paths. Without loss of generality, we can assume more that

$$\mathrm{E}(X(t))=0,\ \mathrm{Var}(X(t))=1,\ \mathrm{Var}(X'(t))=I_2,$$

since these conditions can be obtained by a suitable scaling.

We assume more that $\mathrm{Var}(X''_{11}(t))>1$. This condition is mild in the sense that it is true for a wide class of stationary Gaussian field, see [3].

Our main result is the following.

Theorem 1. *Let X satisfy the Assumption A and suppose that S is the Hausdorff limit of a sequence of connected polygons S_n. Consider n independent copies $\{X_i(t); i=1,2,\dots,n\}$ of X. Then for each given level $u\in\mathbb{R}$,*

$$\mathrm{P}\{M_S\ge u\}\le\overline{\Phi}(u)+\frac{\liminf_n\sigma_1(\partial S_n)\varphi(u)}{2\sqrt{2\pi}}+\frac{\sigma_2(S)}{2\pi}\left[c\varphi(u/c)+u\Phi(u/c)\right]\varphi(u), \tag{3}$$

where $c=\sqrt{\mathrm{Var}(X''_{11})-1}$.

The detailed proof of the main result is provided in Section 2. As mentioned above, here we exploit the record method given in [3] in a suitable definition of the record point. Therefore the readers can see the similarity between our presenting proof and the one in [3]. We would like to emphasis that in conjunction problem, the geometric configuration is more complicated, see Subsubsection 2.1.1.

In comparison with the asymptotic formula in (2), it is clear that our given bound in (3) is also a good asymptotic.

Notation

- For $S \subset \mathbb{R}^2$, ∂S is its boundary; $\overset{\circ}{S}$ is its interior.
- σ_i is the surface measure of dimension i. It can be defined as a Hausdorff measure.
- $\partial_1 X(t), \partial_1 X(t), \partial_\alpha X(t)$ and $\partial^2_{11} X(t)$ are corresponding the derivatives along the first, second ordinate, along the direction α, and the second partial derivatives.
- d_H is the Hausdorff distance between sets, defined by

$$d_H(S,T) = \inf\{\epsilon : \; S \subset T^{+\epsilon},\, T \subset S^{+\epsilon}\}.$$

- $\varphi(x)$ and $\Phi(x)$ are the density and distribution function of a standard normal variable. $\overline{\Phi}(x) = 1 - \Phi(x)$.
- $p_Z(.)$ stands for the density function of a random vector Z.

2 Proof of the main result

Let us first recall the definition of the "emptyable" property of the set S. This definition has been proposed in [3].

Definition 1. *The compact set S is emptyable if there exists a point $O \in S$ which has minimal ordinate, and such that for every point $s \in S$ there exists a continuous path inside S from O to s with non decreasing ordinate.*

The meaning of the terminology "emptyable" is that if we make a small hole at O, then all the water filled in S will empty out because of gravity. We also recall an example of a non-emptyable set, see Figure 1.

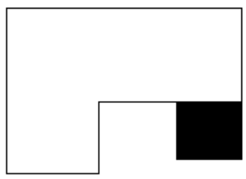

Figure 1: Example of non-emptyable set. The non-emptyable part is displayed in black.

We need the following technical lemma. This is a high dimensional version of Lemma 2 in [10].

Lemma 1. *Let $X_1(t), X_2(t), \dots, X_{d+1}(t)$ be independent Gaussian fields with continuously differentiable sample paths such that they are all defined on a same bounded set $B \subset \mathbb{R}^d$. Then for a given $u_1, \dots, u_{d+1}$,*

$$\mathrm{P}(\exists t \in B : \; X_1(t) = u_1, \dots, = X_{d+1}(t) = u_{d+1}) = 0.$$

Proof. For each positive ϵ, we have

$$\begin{aligned} &\mathrm{P}(\exists t \in S : \; X_1(t) = u_1, \dots, = X_{d+1}(t) = u_{d+1}) \\ \le &\mathrm{P}(\exists t \in B : \; X_1(t) = u_1, \dots, = X_d(t) = u_d \text{ and } |X_{d+1}(t) - u_{d+1}| \le \epsilon) \\ \le &\mathrm{E}\left(\mathrm{card}\{t \in B : \; X_1(t) = u_1, \dots, = X_d(t) = u_d \text{ and } |X_{d+1}(t) - u_{d+1}| \le \epsilon\}\right). \end{aligned}$$

By the Rice formula (see [5]), the above expectation is equal to

$$\begin{aligned}&\int_B \mathrm{E}(|\det(Z'(t))|\mathbb{I}_{\{|X_{d+1}(t)-u_{d+1}|\le\epsilon\}} \mid Z(t)=(u_1,\ldots,u_d))p_{Z(t)}(u_1,\ldots,u_d)dt\\ =&\mathrm{P}(|X_{d+1}(t)-u_{d+1}|\le\epsilon)\int_B \mathrm{E}(|\det(Z'(t))| \mid Z(t)=(u_1,\ldots,u_d))p_{Z(t)}(u_1,\ldots,u_d)dtt,\end{aligned}$$

where at each point $t\in S$, $Z(t)$ stands for the random vector $(X_1(t),X_2(t),\ldots,X_d(t))$, and $p_{Z(t)}(.)$ is its density function.

Let ϵ tend to 0, the result follows. □

We are now ready to prove the main theorem. The proof consists of two steps: at first, we prove for the base case when S is an emptyable polygon, then we generalize to general polygon. In conclusion, by taking the limit of the upper bound for the sequence of polygons S_n, the bound for the conjunction probability on S is direct.

2.1 Step 1: Emptyable polygon

Suppose now that S is an emptyable polygon. We have

$$\begin{aligned}\mathrm{P}\{\max_{t\in S}\min_{i=\overline{1,n}}\{X_i(t)\}\ge u\} &= \mathrm{P}\{\min_{i=\overline{1,n}}\{X_i(O)\}\ge u\}+\mathrm{P}\{\exists i\in\overline{1,n}: X_i(O)<u, \max_{t\in S}\min_{i=\overline{1,n}}\{X_i(t)\}\ge u\}\\ &=\overline{\Phi}^n(u)+\mathrm{P}\{\exists i\in\overline{1,n}: X_i(O)<u, \max_{t\in S}\min_{i=\overline{1,n}}\{X_i(t)\}\ge u\}. \end{aligned}\tag{4}$$

Consider the domain

$$H_u=\{(x_1,\ldots,x_n)\in\mathbb{R}^n:\ x_i\ge u\,\forall i\in\overline{1,n}\}.$$

Then the condition $\{\min_{i=\overline{1,n}}\{X_i(t)\}\ge u\}$ implies that the conjunction set

$$\mathcal{C}(u)=\{t\in S:\ (X_1(t),\ldots,X_n(t))\in H_u\}$$

is not empty. Let us consider the record point T on $\mathcal{C}(u)$ with minimal ordinate. Since the fact $\{\exists i\in\overline{1,n}: X_i(O)<u\}$ means that the point $(X_1(O),\ldots,X_n(O))$ lies outside the domain H_u and the domain S is path-connected with non-decreasing ordinate (emptyable property), then under this condition, at the record point T, the point $(X_1(T),\ldots,X_n(T))$ must be on the boundary of H_u.

By considering two possibilities for the location of the record point T: in the interior or on the boundary of S, we have

$$\mathrm{P}\{\max_{t\in S}\min_{i=\overline{1,n}}\{X_i(t)\}\ge u\}\le\overline{\Phi}^n(u)+\mathrm{P}(T\in\overset{\circ}{S})+\mathrm{P}(T\in\partial S).\tag{5}$$

Now we will deal with the last two terms in the above inequality.

2.1.1 Record point in the interior of S

Suppose now that the record point T is in the interior of S. Since S is two-dimensional, then applying Lemma 1 for the case $d=2$ and $u_i=u$, there are at most two indexes $i\neq j$ such that $X_i(T)=X_j(T)=u$. We have

$$\begin{aligned}\mathrm{P}(T\in\overset{\circ}{S})\le{}&\mathrm{P}(T\in\overset{\circ}{S},\ \exists\,\text{only one index}\,i\in\overline{1,n}:\ X_i(T)=u)\\ &+\mathrm{P}(T\in\overset{\circ}{S},\ \exists\,\text{two indexes}\,i\neq j\in\overline{1,n}:\ X_i(T)=X_j(T)=u).\end{aligned}\tag{6}$$

- Suppose that i is the unique index such that $X_i(T) = u$. Then for $j \neq i$, it must be $X_j(T) > u$. Because of the minimal property of the record point, locally, for index i and all point $s \in S$ with smaller ordinate than T ($s_2 < T_2$, i.e, s is lower than T), $X_i(s) < u$. It implies that

$$\partial_1 X_i(T) = 0,\ \partial_2 X_i(T) \geq 0,\ \partial^2_{11} X_i(T) \leq 0 \text{ and } X_j(T) > u \,\forall j \neq i.$$

Therefore

$$\begin{aligned}
&\mathrm{P}(T \in \overset{\circ}{S}, \exists \text{only one index}\, i \in \overline{1,n} :\ X_i(T) = u) \\
&\leq \sum_{i=\overline{1,n}} \mathrm{P}\left(T \in \overset{\circ}{S} : \partial_1 X_i(T) = 0,\ \partial_2 X_i(T) \geq 0,\ \partial^2_{11} X_i(T) \leq 0 \text{ and } X_j(T) > u \,\forall j \neq i\right) \\
&\leq \sum_{i=\overline{1,n}} \mathrm{E}\left(\mathrm{card}\{t \in \overset{\circ}{S} : \partial_1 X_i(t) = 0,\ \partial_2 X_i(t) \geq 0,\ \partial^2_{11} X_i(t) \leq 0 \text{ and } X_j(t) > u \,\forall j \neq i\}\right) \\
&= \sum_{i=\overline{1,n}} \int_{\overset{\circ}{S}} \mathrm{E}\left(|\det(Z_i'(t))| \mathbb{I}_{\partial_2 X_i(t) \geq 0} \mathbb{I}_{\partial^2_{11} X_i(t) \leq 0} \prod_{j \neq i} \mathbb{I}_{X_j(t) > u} \mid Z_i(t) = (u,0)\right) p_{Z_i(t)}(u,0)\, dt,
\end{aligned}$$

where the third line follows by Markov inequality, and the last line follows by Rice formula applied to the field $Z_i(t) = (X_i(t), \partial_1 X_i(t))$ from $\mathbb{R}^2$ to $\mathbb{R}^2$. Since the fields X_i's are the independent copies of X, we continue as

$$\begin{aligned}
&\mathrm{P}(T \in \overset{\circ}{S}, \exists \text{only one index}\, i \in \overline{1,n} :\ X_i(T) = u) \\
&\leq\ n.\overline{\Phi}^{n-1}(u) \int_{\overset{\circ}{S}} \mathrm{E}\left(|\det(Z'(t))| \mathbb{I}_{\partial_2 X(t) \geq 0} \mathbb{I}_{\partial^2_{11} X(t) \leq 0} \mid Z(t) = (u,0)\right) p_{Z(t)}(u,0)\, dt \\
&=\ n.\overline{\Phi}^{n-1}(u) \sigma_2(S) \frac{\varphi(u)}{\sqrt{2\pi}}\, \mathrm{E}\left([\partial^2_{11} X(t)]^- [\partial_2 X(t)]^+ \mid X(t) = u, \partial_1 X(t) = 0\right) \\
&=\ n.\overline{\Phi}^{n-1}(u) \sigma_2(S) \frac{\varphi(u)}{\sqrt{2\pi}}\, \mathrm{E}\left([\partial_2 X(t)]^+\right)\, \mathrm{E}\left([\partial^2_{11} X(t)]^- \mid X(t) = u, \partial_1 X(t) = 0\right) \\
&=\ n.\overline{\Phi}^{n-1}(u) \sigma_2(S) \frac{\varphi(u)}{2\pi} \left[c\varphi(u/c) + u\Phi(u/c)\right]. \qquad (7)
\end{aligned}$$

where with abuse of notation $Z(t) = (X(t), \partial_1 X(t))$, $c = \sqrt{\mathrm{Var}(\partial^2_{11} X) - 1}$ as introduced in the statement of the main theorem. Here we use the fact that there random variables $(X(t), \partial_1 X(t), \partial_2 X(t))$ are independent and under the condition $\{\partial_1 X(t) = 0\}$,

$$\det(Z'(t)) = -\partial^2_{11} X(t) \partial_2 X(t).$$

- Now we consider the case that there exist two indexes $i \neq j$ such that $X_i(T) = X_j(T) = u$ and $X_k(T) > u$ for all $k \neq i, j$. Fixing these two indexes, by the minimal property of the record point, we can eliminate the following cases

$$\{\partial_1 X_i(T) < 0,\ \partial_1 X_j(T) < 0\},$$

$$\{\partial_1 X_i(T) > 0,\ \partial_1 X_j(T) > 0\}\},$$

and

$$\{\partial_2 X_i(T) < 0,\ \partial_2 X_j(T) < 0\}\}.$$

Note that by Lemma 1, the equality such as $\partial_{1\,(\text{or } 2)} X_i(T) = 0$ almost surely never occurs, since we have already have two equations $X_i(T) = u$, $X_j(T) = u$. The remaining cases will be considered as follows.

Case 1: $\partial_1 X_i(T) > 0, \partial_1 X_j(T) < 0, \partial_2 X_i(T) > 0,\ \partial_2 X_j(T) > 0$. By Rice formula,

$$
\begin{aligned}
&\mathrm{P}(T \in \overset{\circ}{S}, X_i(T) = X_j(T) = u, \partial_1 X_i(T) > 0, \partial_1 X_j(T) < 0, \partial_2 X_i(T) > 0,\ \partial_2 X_j(T) > 0, X_k(T) > u \forall k \neq i,j) \\
\leq &\mathrm{P}(\exists t \in \overset{\circ}{S} :\ X_i(t) = X_j(t) = u, \partial_1 X_i(t) \partial_1 X_i(T) > 0, \partial_1 X_j(t) < 0, \partial_2 X_i(t) > 0,\ \partial_2 X_j(t) > 0, X_k(t) > u \forall k \neq i,j) \\
\leq &\mathrm{E}\left(\mathrm{card}\{t \in \overset{\circ}{S} :\ X_i(t) = X_j(t) = u, \partial_1 X_i(t) > 0, \partial_1 X_j(t) < 0, \partial_2 X_i(t) > 0,\ \partial_2 X_j(t) > 0, X_k(t) > u \forall k \neq i,j\} \right) \\
= &\int_{\overset{\circ}{S}} \mathrm{E}\Big(|\partial_1 X_i . \partial_2 X_j - \partial_2 X_i \partial_1 X_j| \mathbb{I}_{\partial_1 X_i(t) > 0} \mathbb{I}_{\partial_1 X_j(t) < 0} \mathbb{I}_{\partial_2 X_i(t) > 0} \mathbb{I}_{\partial_2 X_j(t) > 0} \\
&\qquad \prod_{k \neq i,j} \mathbb{I}_{X_k(t) > u} \mid (X_i(t), X_j(t)) = (u,u) \Big) p_{X_i(t), X_j(t)}(u,u)\, dt \\
= &\sigma_2(S) \overline{\Phi}^{n-2}(u) \varphi^2(u) \mathrm{E}\left(|\partial_1 X_i . \partial_2 X_j - \partial_2 X_i . \partial_1 X_j| \mathbb{I}_{\partial_1 X_i(t) > 0} \mathbb{I}_{\partial_1 X_j(t) < 0} \mathbb{I}_{\partial_2 X_i(t) > 0} \mathbb{I}_{\partial_2 X_j(t) > 0} \right).
\end{aligned} \tag{8}
$$

Case 2: $\partial_1 X_i(T) < 0, \partial_1 X_j(T) > 0, \partial_2 X_i(T) > 0,\ \partial_2 X_j(T) > 0$. As in Case 1, we get the upper bound

$$
\sigma_2(S) \overline{\Phi}^{n-2}(u) \varphi^2(u) \mathrm{E}\left(|\partial_1 X_i . \partial_2 X_j - \partial_2 X_i . \partial_1 X_j| \mathbb{I}_{\partial_1 X_i(t) < 0} \mathbb{I}_{\partial_1 X_j(t) > 0} \mathbb{I}_{\partial_2 X_i(t) > 0} \mathbb{I}_{\partial_2 X_j(t) > 0} \right). \tag{9}
$$

Case 3: $\partial_1 X_i(T) < 0, \partial_1 X_j(T) > 0, \partial_2 X_i(T) > 0, \partial_2 X_j(T) < 0$. It is clear that there exists a line l_i passing through T such that locally from T, if we go upward then the value of the field X_i is decreasing, and meanwhile if we go downward then the value of the field X_i is increasing. There exists also a line l_j passing through T and reversing the monoticity of the value of the field X_j in comparison with l_i to X_i, see Figure 2.

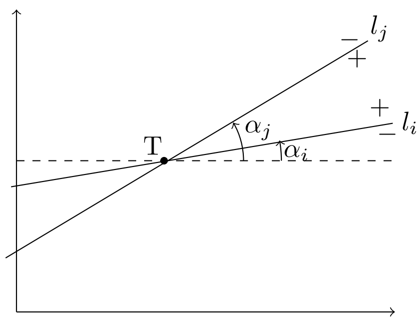

Figure 2: Two lines l_i and l_j in Case 3.

Let α_i be the angle of the line l_i and the horizontal line (x-axis). For the vector $\overrightarrow{e}_i = (\cos\alpha_i, \sin\alpha_i)$, we have

$$
0 = \partial_{e_i} X_i(T) = \cos\alpha_i \partial_1 X_i(T) + \sin\alpha_i \partial_2 X_i(T).
$$

Therefore

$$
\tan\alpha_i = \frac{\sin\alpha_i}{\cos\alpha_i} = \frac{-\partial_1 X_i(T)}{\partial_2 X_i(T)}.
$$

Similarly,

$$
\tan\alpha_j = \frac{\sin\alpha_j}{\cos\alpha_j} = \frac{\partial_1 X_j(T)}{-\partial_2 X_j(T)}.
$$

The minimal property of the record point T implies that $\alpha_j \geq \alpha_i$. Is is equivalent to $\tan(\alpha_j) \geq \tan(\alpha_i)$ and also to

$$
\partial_1 X_i(T) \partial_2 X_j(T) - \partial_2 X_i(T) \partial_1 X_j(T) \leq 0.
$$

As in Case 1, by Rice formula, we get the upper bound

$$\sigma_2(S)\overline{\Phi}^{n-2}(u)\varphi^2(u)\mathrm{E}\left(|\partial_1 X_i.\partial_2 X_j - \partial_2 X_i.\partial_1 X_j|\mathbb{I}_{\partial_1 X_i(t)\partial_2 X_j(t)-\partial_2 X_i(t)\partial_1 X_j(t)\leq 0}\right.$$
$$\left.\mathbb{I}_{\partial_1 X_i(t)<0}\mathbb{I}_{\partial_1 X_j(t)>0}\mathbb{I}_{\partial_2 X_i(t)>0}\mathbb{I}_{\partial_2 X_j(t)<0}\right). \tag{10}$$

Case 4: $\partial_1 X_i(T) < 0, \partial_1 X_j(T) > 0, \partial_2 X_i(T) < 0, \partial_2 X_j(T) > 0$. Define two lines l_i and l_j as in Case 3. By the same argument, we also deduce that

$$\partial_1 X_i(T)\partial_2 X_j(T) - \partial_2 X_i(T)\partial_1 X_j(T) \leq 0.$$

Then, as in Case 3, we get the upper bound

$$\sigma_2(S)\overline{\Phi}^{n-2}(u)\varphi^2(u)\mathrm{E}\left(|\partial_1 X_i.\partial_2 X_j - \partial_2 X_i.\partial_1 X_j|\mathbb{I}_{\partial_1 X_i(t)\partial_2 X_j(t)-\partial_2 X_i(t)\partial_1 X_j(t)\leq 0}\right.$$
$$\left.\mathbb{I}_{\partial_1 X_i(t)<0}\mathbb{I}_{\partial_1 X_j(t)>0}\mathbb{I}_{\partial_2 X_i(t)<0}\mathbb{I}_{\partial_2 X_j(t)>0}\right). \tag{11}$$

Case 5: $\partial_1 X_i(T) > 0, \partial_1 X_j(T) < 0, \partial_2 X_i(T) > 0, \partial_2 X_j(T) < 0$. Similar to Case 4, we get the upper bound

$$\sigma_2(S)\overline{\Phi}^{n-2}(u)\varphi^2(u)\mathrm{E}\left(|\partial_1 X_i.\partial_2 X_j - \partial_2 X_i.\partial_1 X_j|\mathbb{I}_{\partial_1 X_i(t)\partial_2 X_j(t)-\partial_2 X_i(t)\partial_1 X_j(t)\leq 0}\right.$$
$$\left.\mathbb{I}_{\partial_1 X_i(t)>0}\mathbb{I}_{\partial_1 X_j(t)<0}\mathbb{I}_{\partial_2 X_i(t)>0}\mathbb{I}_{\partial_2 X_j(t)<0}\right). \tag{12}$$

Case 6: $\partial_1 X_i(T) > 0, \partial_1 X_j(T) < 0, \partial_2 X_i(T) < 0, \partial_2 X_j(T) > 0$. Similar to Case 5, we get the upper bound

$$\sigma_2(S)\overline{\Phi}^{n-2}(u)\varphi^2(u)\mathrm{E}\left(|\partial_1 X_i.\partial_2 X_j - \partial_2 X_i.\partial_1 X_j|\mathbb{I}_{\partial_1 X_i(t)\partial_2 X_j(t)-\partial_2 X_i(t)\partial_1 X_j(t)\leq 0}\right.$$
$$\left.\mathbb{I}_{\partial_1 X_i(t)>0}\mathbb{I}_{\partial_1 X_j(t)<0}\mathbb{I}_{\partial_2 X_i(t)<0}\mathbb{I}_{\partial_2 X_j(t)>0}\right). \tag{13}$$

Summing up the bounds (8)-(13),

$$\begin{aligned}
&\mathrm{P}(T \in \overset{\circ}{S}, X_i(T) = X_j(T) = u, X_k(T) > u \forall k \neq i,j) \\
\leq &\sigma_2(S)\overline{\Phi}^{n-2}(u)\varphi^2(u)\mathrm{E}\left(|\partial_1 X_i.\partial_2 X_j - \partial_2 X_i.\partial_1 X_j|\right)/4 = \sigma_2(S)\overline{\Phi}^{n-2}(u)\varphi^2(u)/4,
\end{aligned} \tag{14}$$

Here the fact that

$$\mathrm{E}\left(|\partial_1 X_i.\partial_2 X_j - \partial_2 X_i.\partial_1 X_j|\right) = 1,$$

can be deduced from the fact that the four random variables are i.i.d. standard normal variables and in such case

$$\begin{aligned}
\mathrm{E}|XT - YZ| &= \int_{\mathbb{R}^2} \mathrm{E}(|X.t - Y.z| \mid T = t, Z = z)\exp[-(t^2+z^2)/2]dtdz \\
&= \int_{\mathbb{R}^2} \mathrm{E}(|\mathcal{N}(0, t^2+z^2)|)\exp[-(t^2+z^2)/2]dtdz \\
&= \int_{\mathbb{R}^2} \frac{2\sqrt{t^2+z^2}}{\sqrt{2\pi}}\exp[-(t^2+z^2)/2]dtdz = \frac{2}{\sqrt{2\pi}}\mathrm{E}\sqrt{T^2+Z^2} = 1,
\end{aligned}$$

where $\sqrt{T^2+Z^2}$ has Rayleigh distribution with expectation $\sqrt{\pi/2}$.

Note that, one has $\binom{n}{2}$ ways to choose two indexes i, j. Then substituting (7) and (14) into (6) ,

$$\mathrm{P}(T \in \overset{\circ}{S}) \leq \sigma_2(S)\left[n.\overline{\Phi}^{n-1}(u)\frac{\varphi(u)}{2\pi}\left(c\varphi(u/c) + u\Phi(u/c)\right) + \binom{n}{2}\overline{\Phi}^{n-2}(u)\varphi^2(u)/4\right]. \tag{15}$$

2.1.2 Record point on the boundary of S

T is on the boundary of S consisting of the edges $(F_1, \dots, F_m)$. Almost surely, it is not located at a vertex. Suppose that, without loss of generality, it belongs to F_1. Again, by Lemma 1, there is exactly one index i such that $X_i(T) = u$ and $X_j(T) > u$ for $j \neq i$, since F_1 is a one-dimensional edge. By the property of the record point, it is easy to see that

$$X_i(T) = u,\ \partial_\alpha X_i(T) \geq 0, \partial_\beta X_i(T) \leq 0,\ \text{and } X_j(T) > u\,\forall j \neq i,$$

where $\overrightarrow{\alpha}$ is the upward direction on F_1 and $\overrightarrow{\beta}$ is the inward horizontal direction.

Then, apply the Markov inequality and Rice formula in the edge F_1,

$$\begin{aligned}
& \mathrm{P}\{\exists T \in F_1 :\ T \text{is the record point}\} \\
\leq\ & \sum_{i=\overline{1,n}} \mathrm{P}\{\exists t \in F_1 :\ X_i(T) = u,\ \partial_\alpha X_i(T) \geq 0, \partial_\beta X_i(T) \leq 0,\ \text{and } X_j(T) > u\,\forall j \neq i\} \\
\leq\ & \sum_{i=\overline{1,n}} \mathrm{E}\left(\mathrm{card}\{t \in F_1 :\ X_i(T) = u,\ \partial_\alpha X_i(T) \geq 0, \partial_\beta X_i(T) \leq 0,\ \text{and } X_j(T) > u\,\forall j \neq i\}\right) \\
=\ & \sum_{i=\overline{1,n}} \int_{F_1} \mathrm{E}\left(|\partial_\alpha X_i(t)| \mathbb{I}_{\partial_\alpha X_i(t) \geq 0} \mathbb{I}_{\partial_\beta X_i(t) \leq 0} \prod_{j \neq i} \mathbb{I}_{X_j(t) > u} \mid X_i(t) = u \right) p_{X_i(t)}(u)\, dt \\
=\ & n\sigma_1(F_1)\overline{\Phi}^{n-1}(u)\varphi(u)\,\mathrm{E}\left((\partial_\alpha X(t))^+\,\mathbb{I}_{\partial_\beta X(t) \leq 0}\right) = n\sigma_1(F_1)\overline{\Phi}^{n-1}(u)\varphi(u)\frac{1-\cos\theta_1}{2\sqrt{2\pi}},
\end{aligned}$$

where θ_1 is the angle $(\overrightarrow{\alpha}, \overrightarrow{\beta})$. Here the equality

$$\mathrm{E}\left((\partial_\alpha X(t))^+\,\mathbb{I}_{\partial_\beta X(t) \leq 0}\right) = \frac{1-\cos\theta_1}{2\sqrt{2\pi}}$$

can be deduced from the expression $\partial_\beta X(t) = \cos\theta_1\, \partial_\alpha X(t) + \sin\theta_1\, Y$, where Y is a standard normal variable independent with $\partial_\alpha X(t)$.

Summing up all the bounds corresponding to all the edges F_k, the probability that the record point is on the boundary of S is at most equal to

$$n\overline{\Phi}^{n-1}(u)\varphi(u)\sum_{k=1}^{m}\frac{(1-\cos\theta_k)\sigma_1(F_k)}{2\sqrt{2\pi}} = \frac{n\overline{\Phi}^{n-1}(u)\varphi(u)\sigma_1(\partial S)}{2\sqrt{2\pi}}, \tag{16}$$

since $\sum_{i=1}^{n} \sigma_1(F_i)\,\cos\theta_i = 0$.

Hence, summing up (15), (16) and substituting into (5), we obtain the desired upper-bound in our particular case.

2.2 Step 2: General polygon

In a more general case when S is a connected polygon, we will follow the idea as in [**?**]. The idea is to decompose the original polygon S into smaller polygons with S_1 the maximal emptyable subset of S that contains O, and the complement $S \backslash S_1$ consisting of several polygons $S_2^1, \dots, S_2^m$, see Figure 3.

We have

$$\begin{aligned}
\mathrm{P}\{\max_{t \in S}\min_{i=\overline{1,n}}\{X_i(t)\} \geq u\} \leq & \mathrm{P}\{\max \min_{i=\overline{1,n}}\{X_i(O)\} \geq u \geq u\} \\
& + \mathrm{P}\{\max_{t \in S_1}\min_{i=\overline{1,n}}\{X_i(t)\} \geq u,\ max\min_{i=\overline{1,n}}\{X_i(O)\} < u\} \\
& + \sum_{j=1}^{m}\mathrm{P}\{\max_{t \in S_1}\min_{i=\overline{1,n}}\{X_i(t)\} < u,\ \max_{t \in S_2^j}\min_{i=\overline{1,n}}\{X_i(t)\} \geq u\}.
\end{aligned} \tag{17}$$

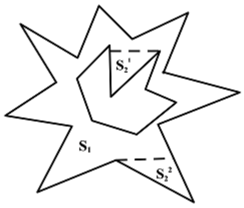

Figure 3: Decomposition of S.

Suppose for the moment that all the $S_2^j,\ j=1,\dots,m$ are emptyable. Then, to give bounds to the event

$$\{\max_{t\in S_1}\min_{i=\overline{1,n}}\{X_i(t)\}<u,\ \max_{t\in S_2^j}\min_{i=\overline{1,n}}\{X_i(t)\}\ge u\},$$

we can apply the reasoning of the preceding part but inverting the direction: in S_2^j, we search points on the conjunction with **maximum ordinate**. We again consider two possibilities.

- The probability that the record point in the interior of S_2^j is at most equal to

$$\sigma_2(S_2^i)\left[n.\overline{\Phi}^{n-1}(u)\frac{\varphi(u)}{2\pi}\left(c\varphi(u/c)+u\Phi(u/c)\right)+\binom{n}{2}\overline{\Phi}^{n-2}(u)\varphi^2(u)/4\right] \tag{18}$$

- T lies on some edges of S_2^i. Denote E_j the common horizontal edge between S_1 and S_2^j. Since $\{\max_{t\in S_1}\min_{i=\overline{1,n}}\{X_i(t)\}<u\}$, then T is not on E. As in above part, we consider the event T is on each edge of S_2^j and sum up the bounds to obtain

$$\begin{aligned}&\mathrm{P}\left(\{\exists T\in\partial S_2^j\}\cap\{\max_{t\in S_1}\min_{i=\overline{1,n}}\{X_i(t)\}<u\}\right)\\ \le\ &\frac{n\overline{\Phi}^{n-1}(u)\varphi(u)\left[\sigma_1(\partial S_2^j)-2\sigma_1(E_j)\right]}{2\sqrt{2\pi}}.\end{aligned} \tag{19}$$

From (18) and (19) we have

$$\mathrm{P}\{M_{S_1}<u,\ M_{S_2^i}\ge u\}\le\frac{\varphi(u)\sigma_2(S_2^i)}{2\pi}\left[c\varphi(u/c)+u\Phi(u/c)\right]+\frac{\varphi(u)[\sigma_1(\partial S_2^i)-2\sigma_1(E)]}{2\sqrt{2\pi}}. \tag{20}$$

Considering all the upper bounds for S_2^j's as in (20), and using the bound for S_1 that is obtained in in Step 1 and substituting into (17), we can deduce the result.

In the general case, when some S_2^i is not emptyable, we can decompose S_2^j as we did for S, and search for the record point as above. Then the result follows.

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