On fixed point approach to equilibrium problem

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Abstract. The equilibrium problem defined by the Nikaidô-Isoda-Fan inequality contains a number of problems such as optimization, variational inequality, Kakutani fixed point, Nash equilibria, and others as special cases. This paper presents a picture for the relationship between the fixed points of the Moreau proximal mapping and the solutions of the equilibrium problem that satisfies some kinds of monotonicity and Lipschitz-type condition.

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1. Introduction

In this paper we are concerned with the equilibrium problem stated as

$$\text{Find } x^* \in C \text{ such that } f(x^*, y) \geq 0 \text{ for all } y \in C, \tag{EP}$$

in which C is a nonempty closed convex subset in a Hilbert space $\mathcal{H}$ endowed with an inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$, and $f : \mathcal{H} \times \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ is a bifunction such that $f(x, x) = 0$ for all $x \in C$ and $f(x, y) < +\infty$ for every $x, y \in C$. The inequality in Problem (EP) was first used in [36] for convex noncooperative game theory. The first result on solution existence of (EP) is due to Fan [16], where this problem was called a minimax inequality. The name equilibria was first used in [33]. After the appearance of the paper by Blum and Oettli [11], the problem (EP) has attracted much attention of many authors and a lot of algorithms have been developed for solving the problem where the bifunction f has monotonic properties. These algorithms are based upon different methods such as penalty and gap functions [7, 8, 9, 23, 24, 25, 29, 33], regularization [3, 20, 28, 34, 35], extragradient methods [10, 19, 26, 37, 39, 40, 41, 44, 46, 47, 48], splitting technique [2, 15, 32]. A comprehensive reference-list on algorithms for the equilibrium problem can be found in the interesting monograph [6].

An interesting of this problem is that, despite its simple formulation, it contains many problems such as optimization, reverse optimization, variational inequality, minimax, saddle point, Kakutani fixed point, Nash equilibrium problems, and some others as special cases (see the interesting monographs [6, 22] and the papers [11, 33]).

In what follows we always suppose that $\phi : C \times C \to \mathbb{R}$ is a bifunction such that $\phi(x, \cdot)$ is convex for any $x \in C$, and $\varphi : C \to \mathbb{R}$ is a convex function on C. For continuity (resp. lower and upper continuity) of ϕ and φ we mean the continuity (reps. lower and upper continuity) with respect to the set C. Then we consider Problem (EP) with $f(x, y) := \phi(x, y) + \varphi(y) - \varphi(x)$. In this case Problem (EP) becomes a mixed equilibrium problem of the form

$$\text{Find } x^* \in C \text{ such that } f(x^*, y) := \phi(x^*, y) + \varphi(y) - \varphi(x^*) \geq 0 \text{ for all } y \in C. \tag{MEP}$$

By considering this mixed form one can employ special structures of each ϕ and φ in subgradient splitting algorithms, where the bifunction $f(x, y)$ can be expressed by the sum of two bifunctions $f_1(x, y) + f_2(x, y)$ and the iterates are defined by taking the proximal mappings of each f_1 and f_2 separately, see [2, 3, 15, 32, 35].

The first fixed point approach to equilibrium problem (EP) was first developed in 1972 by Fan in [16]. There, by using the KKM lemma, it has been proved that if C is compact and $f(x, \cdot)$ is quasiconvex on C, then Problem (EP) admits a solution under a certain continuity property of f. Note that in this result of Fan, it does not require any monotonicity of the bifunction f.

A direct proof using the Kakutani fixed point theorem for the solution existence of Problem (EP) is based upon the mapping K defined by taking, for each $x \in C$,

$$K(x) := \operatorname{argmin}\{f(x, y) : y \in C\}. \tag{P1}$$

Clearly, if $f(x, x) = 0$ for every $x \in C$, then x^* is a solution to (EP) if and only if it is a fixed point of K, i.e., $x^* \in K(x^*)$. Thus, if C is convex, compact, $f(x, \cdot)$ is convex on C and K is upper semicontinuous on C, then by the well known Kakutani fixed point theorem, the mapping K has a fixed point. It can be noticed that the mapping K is set-valued in general.

In order to avoid multivalues of K, an auxiliary principle has been used by defining the proximal mapping

$$B_\lambda(x) := \operatorname{argmin}\left\{\lambda f(x, y) + \frac{1}{2}\langle y - x, G(y - x)\rangle : y \in C\right\}, \tag{P2}$$

where $\lambda > 0$ and G is a self-adjoint positive linear bounded operator from $\mathcal{H}$ into itself. In the sequel, for simplicity of the presentation, we always suppose that G is the identity operator. It is well known [12] that if $f(x, \cdot)$ is convex and subdifferentiable on C, Problem (P2) is uniquely solvable even for the case C is not compact. Moreover, a point $x^* \in C$ is a solution of Problem (EP) if and only if x^* is a fixed point of B_λ for any $\lambda > 0$. So the solution existence of (EP) can be proved by using the Brouwer fixed point theorem whenever C is compact and B_λ is continuous on C.

These results suggest that one can apply the existing algorithms such as ones based on the Scaft pivoting method [27] for computing a fixed point of the mapping B_λ, thereby solving the equilibrium problem (EP). However, the computational results [27, 45] show that the pivoting methods are efficient only for problems with moderate size. Since in the fixed point theory, iterative methods for computing a fixed point have been successfully applied to contractive, generalized contractive, and nonexpansive mappings, a natural question arises that under which conditions, the mapping B_λ possesses certain contraction or generalized nonexpansiveness properties.

This is a survey paper, but it also contains a new result on a fixed point approach to equilibrium problem (MEP). Namely, first we outline from [4, 18] results on quasicontraction and contraction of the Moreau proximal mapping when the bifunction f is strongly monotone and satisfies a certain Lipschitz-type condition. Next, in the case f is not necessarily strongly monotone, but monotone, we present in Theorem 4 a new result on approximate nonexpansiveness of the proximal mapping for monotone equilibrium problems satisfying a certain strongly Lipschitz-type condition. Finally, we recall from [5] a result on quasinonexpansiveness of a composed proximal mapping defined by the equilibrium problem. This relationship allows the equilibrium problem can be solved by the existing methods in the fixed point theory (see e.g. [13, 17, 21, 31, 42, 43] and the references theirein).

The paper is organized as follows. The next section contains preliminaries on the equilibrium problem under consideration and on generalized contractions in real Hilbert spaces. In Section 3 we present some results on contraction, quasicontraction, nonexpansiveness, and approximate nonexpansiveness of the Moreau proximal mapping defined for the equilibrium problem. We close the paper by some conclusions in Section 4.

2. Preliminaries

The following definitions for a bifunction is commonly used in the literature, see e.g. [6].

Definition 1. *A bifunction $f : C \times C \to \mathbb{R}$ is said to be*

(i) strongly monotone with modulus $\gamma > 0$ (shortly γ-strongly monotone) on $S \subseteq C$ if

$$f(x, y) + f(y, x) \leq -\gamma \|x - y\|^2 \ \forall x, y \in S;$$

(ii) monotone on $S \subseteq C$ if

$$f(x, y) + f(y, x) \leq 0 \ \forall x, y \in S;$$

(iii) strongly pseudomonotone on $S \subseteq C$ with modulus $\gamma > 0$ (shortly γ-strongly pseudomonotone) if for all $x, y \in S$ we have

$$f(x, y) \geq 0 \Rightarrow f(y, x) \leq -\gamma \|x - y\|^2;$$

(iv) pseudomonotone on $S \subseteq C$ if for all $x, y \in S$ we have

$$f(x, y) \geq 0 \Rightarrow f(y, x) \leq 0.$$

The notions on monotonicity properties of a bifunction are generalized ones for operators. In fact, it is easy to see that when $f(x, y) := \langle F(x), y - x\rangle + \varphi(y) - \varphi(x)$, then f is γ-strongly monotone (resp. monotone, γ-strongly pseudomonotone, pseudomonotone) if and only if F is γ-strongly monotone (resp. monotone, γ-strongly pseudomonotone, pseudomonotone). The following Lipschitz-type conditions has been introduced in [29] and commonly used for Problem (EP).

Definition 2. *A bifunction $f : C \times C \to \mathbb{R}$ is said to be of Lipschitz-type on $S \subseteq C$ if there exists constants $L_1, L_2 > 0$ such that*

$$f(u, v) + f(v, w) \geq f(u, w) - L_1\|u - v\|^2 - L_2\|v - w\|^2 \ \forall u, v, w \in S.$$

For a bifunction f of Lipschitz-type on $S \subseteq C$, by taking $u = v = w$ in the above formula we see that $f(u, u) \geq 0$. In this case, if f is pseudomonotone in addition, then $f(u, u) = 0$.

The following concepts are well-known in the fixed point theory (see e.g. [1]).

Definition 3. *Let $T : \mathcal{H} \to C$.*

(i) T is said to be contractive on C if there exists $0 < \rho < 1$ such that

$$\|T(x) - T(y)\| \leq \rho\|x - y\| \ \forall x, y \in C.$$

If T satisfies this condition with $\rho = 1$, then it is said to be nonexpansive. It is said to be quasicontractive on C if

$$\|T(x) - T(y)\| \leq \rho\|x - y\| \ \forall x \in \mathrm{Fix}(T), y \in C,$$

where $\mathrm{Fix}(T)$ stands for the set of fixed points of T. If this condition holds for $\rho = 1$, then T is said to be quasinonexpansive.

(ii) T is said to be firmly nonexpansive on C if

$$\|T(x) - T(y)\|^2 \leq \|x - y\|^2 - \|(I - T)(x) - (I - T)(y)\|^2 \ \forall x, y \in C.$$

(iii) T is said to be ρ-strongly converse monotone or ρ-cocoercive on C with $\rho > 0$, if

$$\langle T(x) - T(y), x - y\rangle \geq \rho\|T(x) - T(y)\|^2 \ \forall x, y \in C.$$

3. Contraction and generalized nonexpansive properties of the proximal mapping

Let $g : C \to \mathbb{R}$ be a convex function and $\lambda > 0$. The proximal mapping P_λ with respect to C, g, λ (shortly proximal mapping) is defined as follows (see e.g. Definition 1.22 [38]):

$$P_\lambda(x) := \mathrm{argmin}\left\{\lambda g(y) + \frac{1}{2}\|y - x\|^2 : y \in C\right\}.$$

For the bifunction f where $f(x,\cdot)$ is convex and finite on C, the proximal mapping B_λ is defined by taking

$$B_\lambda(x) := \text{argmin}\left\{\lambda f(x,y) + \frac{1}{2}\|y-x\|^2 : y \in C\right\}$$

for each $x \in C$. Note that when either $f(x,\cdot)$ is continuous on C or C has an interior point, and $f(x,x)=0$, then it is well known from [6] that x^* is a fixed point of B_λ if and only if it is a solution to Problem (EP).

The following theorem, which is an extension of Theorem 2.1 in [34] to problem (MEP), says that when f is strongly monotone and satisfies the Lipschitz-type on C, then one can choose a regularization parameter such that the proximal mapping is quasicontractive on C. For applying optimality condition for the problem defining the proximal mapping, we always assume that either C has an interior point or, for any $x \in C$, $\phi(x,\cdot)$ is continuous with respect to C at a point of C.

Theorem 1. *Suppose that f is strongly monotone on C with modulus τ and satisfies the Lipschitz-type condition with constants L_1, L_2 satisfying $L_1 + L_2 > \tau$. Then, the proximal mapping B_λ is quasicontractive on C, namely*

$$\|B_\lambda(x) - x^*\| \le \sqrt{\alpha}\|x - x^*\| \;\; \forall x \in C, x^* \in \text{Fix}(B_\lambda),$$

whenever $\lambda \in (0, \frac{1}{2L_2})$, where $\alpha := 1 - 2\lambda(\tau - L_1) > 0$.

Proof. The following proof borrows some techniques from the one in [37]. For simplicity of notation we let

$$f_x(y) := \lambda f(x,y) + \frac{1}{2}\|y-x\|^2.$$

Since $\lambda > 0$ and $f(x,\cdot)$ is convex on C by assumption, f_x is strongly convex with modulus 1. As defined, $B_\lambda(x)$ is a minimizer of $f_x(\cdot)$ over the closed convex set C. Therefore, we have

$$f_x(B_\lambda(x)) + \frac{1}{2}\|B_\lambda(x) - x^*\|^2 \le f_x(x^*),$$

that is

$$\lambda f(x, B_\lambda(x)) + \frac{1}{2}\|B_\lambda(x) - x\|^2 + \frac{1}{2}\|B_\lambda(x) - x^*\|^2 \le \lambda f(x,x^*) + \frac{1}{2}\|x^* - x\|^2,$$

or equivalently

$$\|B_\lambda(x) - x^*\|^2 \le 2\lambda\left(f(x,x^*) - f(x, B_\lambda(x))\right) + \|x - x^*\|^2 - \|B_\lambda(x) - x\|^2. \quad (1)$$

Since f is strongly monotone on C with modulus τ, it follows from (1) that

$$\begin{aligned} &\|B_\lambda(x) - x^*\|^2 \\ &\le 2\lambda(-\tau\|x-x^*\|^2 - f(x^*,x) - f(x,B_\lambda(x))) + \|x-x^*\|^2 - \|B_\lambda(x) - x\|^2 \\ &\le (1-2\lambda\tau)\|x-x^*\|^2 - 2\lambda\left(f(x^*,x) + f(x,B_\lambda(x))\right) - \|B_\lambda(x)-x\|^2. \quad (2) \end{aligned}$$

Since f satisfies Lipschitz-type condition, we have

$$f(x^*,x) + f(x,B_\lambda(x)) \ge f(x^*,B_\lambda(x)) - L_1\|x^*-x\|^2 - L_2\|x - B_\lambda(x)\|^2.$$

Therefore, it follows from (2) that

$$\begin{aligned}&\|B_\lambda(x)-x^*\|^2\\ &\leq (1-2\lambda(\tau-L_1))\|x-x^*\|^2-(1-2\lambda L_2)\|x-B_\lambda(x)\|^2-2\lambda f(x^*,B_\lambda(x))\\ &\leq (1-2\lambda(\tau-L_1))\|x-x^*\|^2.\end{aligned}$$

The last inequality is due to the fact that $f(x^*,B_\lambda(x))\geq 0$ and the assumption that $0<\lambda<\frac{1}{2L_2}$. Since $\tau<L_1+L_2$, we see that if $0<\lambda<\frac{1}{2L_2}$, then $1-2\lambda(\tau-L_1)>0$ and hence B_λ is quasicontractive. □

By Theorem 1, the contraction iterative method can be used for solving equilibrium problem (EP). A question arises here: under which condition, the proximal mapping is contractive? The following result, which is stated in Theorem 1 [18], gives an answer for this question.

For the statement of the theorem, first we recall from [18] that a bifunction $f: C\times C\to\mathbb{R}$ is said to be *strongly Lipschitz-type* on C if there exist $\alpha_i: C\times C\to C$, $\beta_i: C\to C$, $K_i, L_i>0$ (with $i=1,\dots,p$) such that

$$f(x,y)+f(y,z)\geq f(x,z)+\sum_{j=1}^{p}\langle\alpha_i(x,y),\beta_i(y-z)\rangle\ \forall x,y,z\in C,$$

where

$$\begin{aligned}\|\beta_i(x)-\beta_i(y)\|&\leq K_i\|x-y\| && \forall x,y\in C, i=1,\dots,p,\\ \|\alpha_i(x,y)\|&\leq L_i\|x-y\| && \forall x,y\in C, i=1,\dots,p,\\ \alpha_i(x,y)+\alpha_i(y,x)&=0 && \forall x,y\in C, i=1,\dots,p.\end{aligned}$$

As also remarked in [18], the following facts are not hard to see.

(i) If f is strongly Lipschitz-type on C, then it is Lipschitz-type on C with both constants $\frac{1}{2}\sum_{i=1}^{p}K_iL_i$.

(ii) If $f(x,y)=\langle F(x),y-x\rangle+\varphi(y)-\varphi(x)$, then f is strongly Lipschitz-type on C if and only if F is Lipschitz on C.

Theorem 2. *Let C be a nonempty closed convex set, $f: C\times C\to\mathbb{R}$. Suppose that $f(x,\cdot)$ is lower semicontinuous and convex on C, f is γ-strongly monotone and strongly Lipschitz-type on C. Then the proximal mapping B_λ is contractive on C whenever $\lambda\in(0,\frac{2\gamma}{M})$ with $M=\sum_{i=1}^{p}K_iL_i$.*

In the case of mixed variational inequality, when $f(x,y):=\langle F(x),y-x\rangle+\varphi(y)-\varphi(x)$ with F being Lipschitz continuous and strongly monotone on C, the proximal mapping is contractive on C (see. e.g. [4]). This result also follows from the above theorem due to the fact that $f(x,y):=\langle F(x),y-x\rangle+\varphi(y)-\varphi(x)$ is strongly Lipschitz-type on C whenever F is Lipschitz on C. In the case that F is cocoercive (strongly inverse monotone) on C, the proximal mapping is nonexpansive on C as stated in the following theorem.

Theorem 3. *Suppose that $f(x,y):=\langle F(x),y-x\rangle+\varphi(y)-\varphi(x)$ with F being δ-cocoercive on C and φ being convex on C. Then, whenever $0<\lambda\leq 2\delta$, the proximal mapping B_λ is nonexpansive on C.*

Proof. By definition of B_λ we have

$$B_\lambda(x) = \operatorname{argmin}\left\{\lambda\langle F(x), z-x\rangle + \lambda\varphi(z) - \lambda\varphi(x) + \frac{1}{2}\|z-x\|^2 : z \in C\right\}.$$

Since φ is convex on C, this is a convex programming problem. The optimality condition for this convex program gives

$$\langle \lambda F(x) + \lambda v_x + B_\lambda(x) - x, z - B_\lambda(x)\rangle \geq 0 \quad \forall z \in C, \tag{3}$$

in which v_x is a subgradient of φ at $B_\lambda(x)$, i.e., $v_x \in \partial\varphi(B_\lambda(x))$. Similarly, for $B_\lambda(y)$ we have

$$\langle \lambda F(y) + \lambda v_y + B_\lambda(y) - y, z - B_\lambda(y)\rangle \geq 0 \quad \forall z \in C, \tag{4}$$

in which $v_y \in \partial\varphi(B_\lambda(y))$. Now, in (3) we replace $z = B_\lambda(y)$, in (4) we replace $z = B_\lambda(x)$, then adding side by side the obtained inequalities, we have

$$\langle \lambda F(x) + \lambda v_x + B_\lambda(x) - x - (\lambda F(y) + \lambda v_y + B_\lambda(y) - y), B_\lambda(y) - B_\lambda(x)\rangle \geq 0.$$

This is equivalent to

$$\langle \lambda(F(x) - F(y)) + \lambda(v_x - v_y) - (x-y), B_\lambda(y) - B_\lambda(x)\rangle \geq \|B_\lambda(y) - B_\lambda(x)\|^2.$$

Since $v_x \in \partial\varphi(B_\lambda(x))$ and $v_y \in \partial\varphi(B_\lambda(y))$, by monotonicity of subgradient of convex function φ, we have

$$\langle v_x - v_y, B_\lambda(x) - B_\lambda(y)\rangle \geq 0.$$

From the last two inequalities we obtain

$$\langle \lambda(F(x) - F(y)) - (x-y), B_\lambda(y) - B_\lambda(x)\rangle \geq \|B_\lambda(y) - B_\lambda(x)\|^2.$$

Since

$$\begin{aligned}&\langle \lambda(F(x) - F(y)) - (x-y), B_\lambda(y) - B_\lambda(x)\rangle\\ &\leq \|\lambda(F(x) - F(y)) - (x-y)\|\|B_\lambda(y) - B_\lambda(x)\|,\end{aligned}$$

we come up with

$$\|\lambda(F(x) - F(y)) - (x-y)\| \geq \|B_\lambda(y) - B_\lambda(x)\|. \tag{5}$$

Furthermore, for $x, y \in C$ we have

$$\begin{aligned}&\|\lambda(F(x) - F(y)) - (x-y)\|^2\\ &= \|x-y\|^2 + \lambda^2\|F(x) - F(y)\|^2 - 2\lambda\langle x-y, F(x) - F(y)\rangle\\ &\leq \|x-y\|^2 + \lambda^2\|F(x) - F(y)\|^2 - 2\lambda\delta\|F(x) - F(y)\|^2 \quad (6)\\ &= \|x-y\|^2 + \lambda(\lambda - 2\delta)\|F(x) - F(y)\|^2\\ &\leq \|x-y\|^2. \quad (7)\end{aligned}$$

Inequality (6) is due to the cocoerciveness of F on C, while inequality (7) is due to $0 < \lambda \leq 2\delta$ by the assumption. It follows from (5) and (7) that

$$\|B_\lambda(x) - B_\lambda(y)\| \leq \|x-y\|,$$

which means nonexpansiveness of B_λ on C. □

In relation to Theorem 3, one may think that the proximal mapping B_λ is nonexpansive when f is monotone. In the following we give a counter-example for this argument.

Let us consider the linear variational inequality problem

$$\text{Find } x \in \mathbb{R}^2 \text{ such that } f(x,y) := \langle Ax, y-x\rangle \geq 0 \text{ for all } y \in \mathbb{R}^2, \qquad \text{(VI)}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

For all $x, y \in \mathbb{R}^2$ we have

$$\begin{aligned} f(x,y) + f(y,x) &= \langle A(x-y), y-x\rangle \\ &= (x_2-y_2)(y_1-x_1) - (x_1-y_1)(y_2-x_2) \\ &= 0, \end{aligned}$$

therefore f is monotone on $\mathbb{R}^2$. It is easy to see that $x^* = (0,0)^t$ is a solution to the variational inequality (VI), since $f(x^*, y) = 0$ for all $y \in \mathbb{R}^2$. Furthermore, x^* is the unique solution to (VI). Indeed, if $\overline{x} = (\overline{x}_1, \overline{x}_2)^t$ is a solution to (VI), then

$$\langle A\overline{x}, y - \overline{x}\rangle \geq 0 \quad \forall y \in \mathbb{R}^2.$$

By taking $y = \overline{y} = (\overline{x}_1 - \overline{x}_2, \overline{x}_1 + \overline{x}_2)^t$, we have

$$0 \leq \langle A\overline{x}, \overline{y} - \overline{x}\rangle = \left\langle \begin{bmatrix} \overline{x}_2 \\ -\overline{x}_1 \end{bmatrix}, \begin{bmatrix} -\overline{x}_2 \\ \overline{x}_1 \end{bmatrix} \right\rangle = -(\overline{x}_1^2 + \overline{x}_2^2) \leq 0,$$

which implies $\overline{x} = (0,0)^t = x^*$. Now we see that

$$\begin{aligned} &\lambda\langle Ax, y-x\rangle + \frac{1}{2}\|y-x\|^2 \\ =&\lambda \left\langle \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}, \begin{bmatrix} y_1 - x_1 \\ y_2 - x_2 \end{bmatrix} \right\rangle + \frac{1}{2}\left((y_1-x_1)^2 + (y_2-x_2)^2\right) \\ =&\lambda(x_2y_1 - x_1y_2) + \frac{1}{2}\left((y_1-x_1)^2 + (y_2-x_2)^2\right) \\ =&\frac{1}{2}\left(y_1^2 - 2(x_1 - \lambda x_2)y_1 + y_2^2 - 2(x_2 + \lambda x_1)y_2 + x_1^2 + x_2^2\right) \\ =&\frac{1}{2}\left((y_1 - x_1 + \lambda x_2)^2 + (y_2 - x_2 - \lambda x_1)^2 - \lambda^2(x_1^2 + x_2^2)\right) \end{aligned}$$

which attains its minimum at $(y_1, y_2) = (x_1 - \lambda x_2, x_2 + \lambda x_1)$. We obtain the following explicit formula for the proximal mapping of (VI):

$$B_\lambda(x) = \operatorname{argmin}\{\lambda\langle Ax, y-x\rangle + \frac{1}{2}\|y-x\|^2 \mid y \in \mathbb{R}^2\} = \begin{bmatrix} x_1 - \lambda x_2 \\ x_2 + \lambda x_1 \end{bmatrix}.$$

Therefore, for any $\lambda > 0$ we have

$$\|B_\lambda(x) - B_\lambda(x^*)\| = \left\| \begin{bmatrix} x_1 - \lambda x_2 \\ x_2 + \lambda x_1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\| = \sqrt{1+\lambda^2}\sqrt{x_1^2 + x_2^2} > \|x - x^*\|,$$

which proves that B_λ is not nonexpansive for any $\lambda > 0$.

So in general the proximal mapping may not be nonexpansive even for the variational inequality when $f(x,y) = \langle F(x), y-x\rangle$ with F being Lipschitz and monotone. However, it is well known from [14, 42] that if f is monotone, $f(x,\cdot)$ is convex, lower semicontinuous, and $f(\cdot,y)$ is hemicontinuous, then the regularization proximal mapping R_λ is defined everywhere, single valued, and firmly nonexpansive for any $\lambda > 0$. Here, for each $x \in C$, $R_\lambda(x)$ is defined as the unique solution of the strongly monotone equilibrium problem

$$\text{Find } z \in C \text{ such that } f(z,y) + \frac{1}{2\lambda}\langle y-z, z-x\rangle \geq 0 \text{ for all } y \in C.$$

Moreover, the solution set of (EP) coincides with the fixed point set of the proximal mapping R_λ.

The main difference between the proximal mapping and the regularization proximal mapping is that the former is defined as the unique solution of a strongly convex program, while the latter is defined by the unique solution of a strongly monotone equilibrium problem.

We adopt the following definition.

Definition 4. *For given $\epsilon > 0$, the proximal mapping B_λ is said to be ϵ-nonexpansive on C if*

$$\|B_\lambda(x) - B_\lambda(y)\|^2 \leq (1+\epsilon)\|x-y\|^2 \ \forall x, y \in C.$$

The following theorem says that for monotone equilibrium problem, the proximal mapping is ϵ-nonexpansive.

Theorem 4. *Suppose that the bifunction ϕ is monotone and satisfies the strongly Lipschitz-type condition on C. Then for any $\epsilon > 0$, there exists $\lambda > 0$ such that the proximal mapping B_λ for Problem (MEP) is ϵ-nonexpansive.*

Proof. As before we see that if ϕ is monotone, strongly Lipschitz-type, then so is $f(x,y) := \phi(x,y) + \varphi(y) - \varphi(x)$ for any function $\varphi : C \to \mathbb{R}$. It is well known (see e.g. [30]) that

$$\langle B_\lambda(x) - x, B_\lambda(x) - z\rangle \leq \lambda\left(f(x,z) - f(x, B_\lambda(x))\right) \ \forall x, z \in C.$$

Applying this inequality with $z := B_\lambda(y)$ we obtain

$$\langle B_\lambda(x) - x, B_\lambda(x) - B_\lambda(y)\rangle \leq \lambda\left(f(x,B_\lambda(y)) - f(x, B_\lambda(x))\right) \ \forall x, y \in C.$$

Similarly with $B_\lambda(y)$, we have

$$\langle B_\lambda(y) - y, B_\lambda(y) - B_\lambda(x)\rangle \leq \lambda\left(f(y,B_\lambda(x)) - f(y, B_\lambda(y))\right) \ \forall x, y \in C.$$

Adding the two obtained inequalities we get

$$\begin{aligned} &\langle B_\lambda(x) - B_\lambda(y) + y - x, B_\lambda(x) - B_\lambda(y)\rangle \\ &\leq \lambda\left(f(x,B_\lambda(y)) - f(x,B_\lambda(x)) + f(y,B_\lambda(x)) - f(y,B_\lambda(y))\right). \end{aligned}$$

By simple arrangements we obtain

$$\begin{aligned} &\|B_\lambda(x) - B_\lambda(y)\|^2 \\ &+ 2\lambda\left(f(x,B_\lambda(y)) - f(x,B_\lambda(x)) + f(y,B_\lambda(x)) - f(y,B_\lambda(y))\right) \\ &\leq \|x-y\|^2. \end{aligned}$$

Now using the strongly Lipschitz-type condition, by the same argument as in the proof of Theorem 3.7 in [18] we arrive at the following inequality

$$\|B_\lambda(x) - B_\lambda(y)\|^2 \leq (1 + \lambda^2 M)\|x - y\|^2,$$

where $M = \sum_{j=1}^{p} K_i L_i$, with K_j and L_j being the Lipschitz constants defined in the strongly Litschitz-type. Hence, with $0 < \lambda^2 < \frac{\epsilon}{M}$, we obtain

$$\|B_\lambda(x) - B_\lambda(y)\|^2 \leq (1 + \epsilon)\|x - y\|^2 \ \forall x, y \in C.$$

□

Corollary 1. *Consider the mixed variational inequality*

$$\textit{Find } x^* \in C \textit{ such that } \langle F(x^*), y - x^*\rangle + \varphi(y) - \varphi(x^*) \geq 0 \textit{ for all } y \in C. \tag{MVI}$$

Suppose that F is monotone and Lipschitz on C. Then for any $\epsilon > 0$ there exists $\lambda > 0$ such that the proximal mapping B_λ defined by the bifunction $\langle F(x), y - x\rangle + \varphi(y) - \varphi(x)$ is ϵ-nonexpansive.

Proof. Since F is monotone and Lipschitz on C, the bifunction $f(x, y) := \langle F(x), y - x\rangle + \varphi(y) - \varphi(x)$ is strongly Lipschitz and monotone. Thus the corollary follows directly from Theorem 4. However, one can prove this result simply as follows.

From the definition of B_λ, by the same arguments as proof of (5) we have

$$\|B_\lambda(x) - B_\lambda(y)\| \leq \|x - y - \lambda(F(x) - F(y))\| \quad \forall x, y \in C. \tag{8}$$

We observe that

$$\begin{aligned} &\|x - y - \lambda(F(x) - F(y))\|^2 \\ &= \|x - y\|^2 - 2\lambda\langle x - y, F(x) - F(y)\rangle + \lambda^2\|F(x) - F(y)\|^2 \\ &\leq \|x - y\|^2 + \lambda^2\|F(x) - F(y)\|^2. \end{aligned}$$

The last inequality is due to $\langle x - y, F(x) - F(y)\rangle \geq 0$ by monotonicity of F on C. By (8) we can write

$$\|B_\lambda(x) - B_\lambda(y)\|^2 \leq \|x - y\|^2 + \lambda^2\|F(x) - F(y)\|^2,$$

from which, by Lipschitz continuity of F, it follows that

$$\|B_\lambda(x) - B_\lambda(y)\|^2 \leq (1 + L^2\lambda^2)\|x - y\|^2.$$

Hence the mapping B_λ is ϵ-nonexpansive on C whenever $\lambda^2 L^2 \leq \epsilon$. □

Now a natural question may arise: how to modify the proximal mapping for monotone equilibrium problems such that it has a generalized nonexpansiveness property? In order to answer this question, let us define the mapping T_λ from C to itself by taking, for every $x \in C$,

$$T_\lambda(x) := \operatorname{argmin}\Big\{\lambda f(B_\lambda(x), y) + \frac{1}{2}\|y - x\|^2 : y \in C\Big\}$$

where λ is a fixed positive number.

Theorem 5. ([5]). *Let $f : C \times C \to \mathbb{R}$ be a bifunction such that $f(x,\cdot)$ is subdifferentiable, pseudomonotone and Lipschitz-type on C. Suppose that the following conditions are satisfied:*

(A1) f is jointly weakly continuous on $C \times C$ in the sense that, if $x, y \in C$ and $\{x_n\}, \{y_n\} \subset C$ converge weakly to x and y, respectively, then $f(x_n, y_n) \to f(x,y)$ as $n \to \infty$.

(A2) The solution set of Problem (EP) *is nonempty.*

Then the mapping T_λ is quasi-nonexpansive on C if $0 < \lambda < \min\left\{\frac{1}{2L_1}, \frac{1}{2L_2}\right\}$. In addition, it is demiclosed at zero, in the sense that for every sequence $\{x_n\}$ contained in C weakly converging to x and $\|T(x_n) - x_n\| \to 0$, then $x \in Fix(T)$.

By this theorem, the algorithms for finding a fixed point of quasi-nonexpansive mappings (see e.g. [17, 21]) can be used for solving pseudomonotone equilibrium problems.

A disadvantage of the composite proximal mapping T_λ is that for evaluating it at a point, it requires solving two strongly convex programming problems. An open question is that how to define a nonexpansive or ϵ-nonexpansive mapping with any $\epsilon > 0$ for pseudomotone equilibrium problems, which requires solving only one strongly convex program.

As we have seen from the definition of the proximal mapping that when applying the iterative fixed point methods for solving mixed equilibrium problem (MEP), at an iterative point $x^k \in C$, we have to solve a strongly convex program of the form

$$\min\left\{ f(x^k, y) := \phi(x^k, y) + \varphi(y) - \varphi(x^k) + \frac{1}{2\lambda}\|y - x^k\|^2 : y \in C \right\}. \quad (P_k)$$

This problem can be solved by efficient algorithms of convex programming (see [12]).

4. Conclusions

The mixed equilibrium problem (MEP) and the regularized Moreau proximal mapping B_λ defined for it are equivalent in the following senses:

(i) The solution set of (MEP) coincides with the fixed point set of B_λ for any $\lambda > 0$.

In addition, we have shown in this paper the following results.

(ii) If (MEP) is strongly monotone and satisfies the Lipschitz-type condition, then one can choose λ such that B_λ is quasicontractive (Theorem 1).

(iii) If (MEP) is strongly monotone and satisfies the strongly Lipschitz-type condition, then one can choose λ such that B_λ is contractive (Theorem 2).

(iv) If (MEP) is strongly inverse monotone, then one can choose λ such that B_λ is nonexpansive (Theorem 3).

(v) If (MEP) is monotone and satisfies the strongly Lipschitz-type condition, then one can choose λ such that B_λ is ϵ-nonexpansive for any $\epsilon > 0$ (Theorem 4).

(vi) If (MEP) is pseudomonotone and satisfies the Lipschitz-type condition, then the composite proximal mapping is quasinonexpansive, and its fixed point-set coincides the solution-set of Problem (MEP) (Theorem 5).

Applications to mixed variational inequality problems with Lipschitz cost operator have been presented (Corollary 1).

The following questions seem to be interesting.

(1) How to extend these results for the equilibrium problem when the bifunction involved is quasiconvex with respect to its second variable?

(2) How to extend Theorem 4 to the case the bifunction is quasimonotone?

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