LEFT ORDERABLE SURGERIES OF DOUBLE TWIST KNOTS II

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Abstract. A slope r is called a left orderable slope of a knot $K \subset S^3$ if the 3-manifold obtained by r-surgery along K has left orderable fundamental group. Consider two-bridge knots $C(2m, \pm 2n)$ and $C(2m+1, -2n)$ in the Conway notation, where $m \geq 1$ and $n \geq 2$ are integers. By using *continuous* families of hyperbolic $\mathrm{SL}_2(\mathbb{R})$-representations of knot groups, it was shown in [HTe1, Tr2] that any slope in $(-4n, 4m)$ (resp. $[0, \max\{4m, 4n\}))$ is a left orderable slope of $C(2m, 2n)$ (resp. $C(2m, -2n)$) and in [Ga2] that any slope in $(-4n, 0]$ is a left orderable slope of $C(2m+1, -2n)$. However, the proofs of these results have gaps since the *continuity* of the families of representations was not proved. In this paper, we fix these gaps and moreover we show that any slope in $(-4n, 4m)$ is a left orderable slope of $C(2m+1, -2n)$ detected by hyperbolic $\mathrm{SL}_2(\mathbb{R})$-representations of the knot group.

1. Introduction

The study of left orderability of fundamental groups of 3-manifolds obtained by Dehn surgeries along knots is motivated by the L-space conjecture of Boyer, Gordon and Watson [BGW] which states that an irreducible rational homology 3-sphere is an L-space if and only if its fundamental group is not left orderable. Here a rational homology 3-sphere Y is an L-space if its Heegaard Floer homology $\widehat{\mathrm{HF}}(Y)$ has rank equal to the order of $H_1(Y; \mathbb{Z})$, and a non-trivial group G is left orderable if it admits a total ordering $<$ such that $g < h$ implies $fg < fh$ for all elements f, g, h in G.

Many hyperbolic 3-manifolds are obtained by Dehn surgeries along knots. A slope r is called a left orderable slope of a knot $K \subset S^3$ if the 3-manifold obtained by r-surgery along K has left orderable fundamental group. Consider two-bridge knots $C(2m, \pm 2n)$ and $C(2m+1, -2n)$ in the Conway notation, where $m \geq 1$ and $n \geq 2$ are integers. By using *continuous* families of hyperbolic $\mathrm{SL}_2(\mathbb{R})$-representations of knot groups, it was shown in [HTe1, Tr2] that any slope in $(-4n, 4m)$ (resp. $[0, \max\{4m, 4n\}))$ is a left orderable slope of $C(2m, 2n)$ (resp. $C(2m, -2n)$) and in [Ga2] that any slope in $(-4n, 0]$ is a left orderable slope of $C(2m+1, -2n)$. However, the proofs of these results have gaps since the *continuity* of the families of representations was not proved. More precisely, [HTe1, Proposition 4.2], [Tr2, Lemma 2.1] and [Ga2, Proposition 4.2] proved the existence of families of $\mathrm{SL}_2(\mathbb{R})$-representations of the knot groups but did not prove the continuity of

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these families. In this paper, we fix these gaps in Proposition 4.1, Proposition 4.2 and Proposition 5.1 respectively. Moreover, we extend the range of left orderable slopes of $C(2m+1,-2n)$ detected by hyperbolic $\mathrm{SL}_2(\mathbb{R})$-representations of their knot groups.

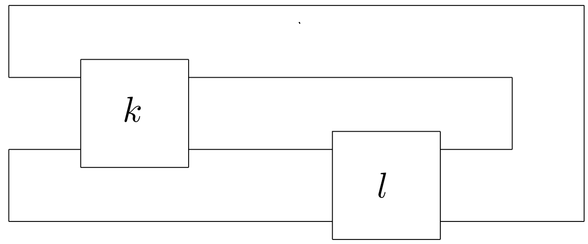

FIGURE 1. The two-bridge knot/link $C(k,l)$ in the Conway notation.

Theorem 1. *Let $m \geq 1$ and $n \geq 2$ be integers. Then any slope in $(-4n, 4m)$ is a left orderable slope of $C(2m+1,-2n)$ detected by hyperbolic $\mathrm{SL}_2(\mathbb{R})$-representations of the knot group.*

Remark 1.1. In [Tr1], by following the method of Culler-Dunfield's paper [CD] the third author used *continuous* families of elliptic $\mathrm{SL}_2(\mathbb{R})$-representations of knot groups to show that if K is a two-bridge knot of the form $C(2m,-2n)$, $C(2m+1,2n)$ or $C(2m+1,-2n)$, where $m \geq 1$ and $n \geq 1$ are integers, and

$$\mathrm{LO}_K = \begin{cases} (-\infty, 1) & \text{if } K = C(2m,-2n), \\ (-\infty, 2n-1) & \text{if } K = C(2m+1,2n), \\ (3-2n, \infty) & \text{if } K = C(2m+1,-2n) \text{ and } n \geq 2, \end{cases}$$

then any slope in LO_K is a left orderable slope of K.

Gao [Ga2] independently showed that if K is a two-bridge knot of the form $C(2m+1,2n)$ or $C(2m+1,-2n)$, where $m \geq 1$ and $n \geq 1$ are integers, and

$$\mathrm{LO}'_K = \begin{cases} (-\infty, 1) & \text{if } K = C(2m+1,2n), \\ (-1, \infty) & \text{if } K = C(2m+1,-2n) \text{ and } n \geq 2, \end{cases}$$

then any slope in LO'_K is a left orderable slope of K. Her proof also used families of elliptic $\mathrm{SL}_2(\mathbb{R})$-representations of knot groups. Note that LO'_K is a subset of LO_K.

For more on the study of left orderable slopes of knots using hyperbolic $\mathrm{SL}_2(\mathbb{R})$-representations of knot groups, see [BGW, HTe2, Ga1].

This paper is organized as follows. In Section 2, we recall some facts about the universal covering group $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ of $\mathrm{SL}(2,\mathbb{R})$ and we study the lifting problem of a connected curve of hyperbolic $\mathrm{SL}(2,\mathbb{R})$-representations of knot groups. In Section 3, we review the Riley polynomial of two-bridge knots $C(k,-2p)$, whose zero locus describes all non-abelian representations of the knot group into $\mathrm{SL}_2(\mathbb{C})$. In Section 3 we prove the existence of continuous families of hyperbolic $\mathrm{SL}_2(\mathbb{R})$-representations of the knot groups of $C(2m,\pm 2n)$

and therefore fix the gaps in [HTe1, Tr2]. In Section 4 we prove the existence of continuous families of hyperbolic $\mathrm{SL}_2(\mathbb{R})$-representations of the knot groups of $C(2m+1,-2n)$ and use it to give a proof of Theorem 1.

2. Lifting of a curve of hyperbolic representations

2.1. **The group $\widetilde{\mathrm{SL}(2,\mathbb{R})}$.** We recall some facts about the universal covering group $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ (see [Kh] pages 763-764). Let $\psi : \widetilde{\mathrm{SL}(2,\mathbb{R})} \to \mathrm{SL}(2,\mathbb{R})$ be the covering map. We can parameterize the universal covering group as

$$\widetilde{\mathrm{SL}(2,\mathbb{R})} = \{(\gamma,\omega)| \quad |\gamma|<1, -\infty < w < \infty\}.$$

For an element $g=(\gamma,\omega)\in\widetilde{\mathrm{SL}(2,\mathbb{R})}$, we will write $g[1]=\gamma$ and $g[2]=\omega$.

An element of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ is called *elliptic/parabolic/hyperbolic* if it covers a matrix in $\mathrm{SL}(2,\mathbb{R})$ of the corresponding type.

The multiplication rule in the group $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ is given by $(\gamma,\omega)(\gamma',\omega')=(\gamma'',\omega'')$ where

$$\begin{aligned}\gamma'' &= \frac{\gamma+\gamma' e^{-2i\omega}}{1+\bar{\gamma}\gamma' e^{-2i\omega}}\\ \omega'' &= \omega+\omega'+\arg(1+\bar{\gamma}\gamma' e^{-2i\omega}).\end{aligned}$$

Let $A=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$ be a matrix in $\mathrm{SL}(2,\mathbb{R})$ then

$$\psi^{-1}(A)=\left\{\left(\frac{a-d+(b+c)i}{a+d+(b-c)i}, \arg(a+d+(b-c)i)+2n\pi\right)|n\in\mathbb{Z}\right\}.$$

Here, the function argument takes value in the interval $(-\pi,\pi]$. We note that if $\mathrm{Tr}(A)=a+d>0$ then in the above formula we have $\arg(a+d+(b-c)i)=\arctan(\frac{b-c}{a+d})\in(-\frac{\pi}{2},\frac{\pi}{2})$.

2.2. **Lifting of a curve of hyperbolic representations.** For a knot K in S^3, let X be an open tubular neighborhood of K and let $G(K)=\pi_1(X)$ be the knot group of K which is the fundamental group of X. Let μ be a meridian and λ the canonical longitude. Recall that any representation $\rho : G(K)\to\mathrm{SL}(2,\mathbb{R})$ can be lifted to a representation $\widetilde{\rho}$ into $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ because $H^2(K,\mathbb{Z})=0$. We know from [Kh] that the lifts of ρ come in a family. If we fix a lift $\widetilde{\rho}_0$, then we have a $\mathbb{Z}$-family of lifts $\widetilde{\rho}_n$ given by $\widetilde{\rho}_n(g)=\widetilde{\rho}_0(g)h_n(g)$, where h_n is the representation

$$\begin{aligned} G(K) &\longrightarrow H_1(X)\equiv\mathbb{Z} &&\longrightarrow \widetilde{\mathrm{SL}(2,\mathbb{R})}\\ \mu &\longmapsto 1 &&\longmapsto (0,2n\pi).\end{aligned}$$

From this, we see that $\widetilde{\rho}_n(\lambda)$ does not depend on n.

An $\mathrm{SL}(2,\mathbb{R})$ representation ρ of a knot group $G(K)$ is called *hyperbolic* if $\rho(\mu)$ and $\rho(\lambda)$ are hyperbolic elements.

Recall that a hyperbolic element $g \in \widetilde{\mathrm{SL}(2,\mathbb{R})}$ can be conjugated to a unique normal form $(\tanh(a), k\pi)$ if and only if $k\pi - \frac{\pi}{2} < g[2] < k\pi + \frac{\pi}{2}$. Let us fix an arbitrary lift $\widetilde{\rho}_0 : G(K) \to \widetilde{\mathrm{SL}(2,\mathbb{R})}$ and suppose that $\widetilde{\rho}_0(\lambda)$ is conjugate to $(\tanh(b), k\pi)$. As noted above, the number k does not depend on the chosen lift $\widetilde{\rho}_0$. We call k the *index* of the representation ρ.

Lemma 2.1. *Let C be a connected curve of hyperbolic representations of a knot group $G(K)$ into $\mathrm{SL}(2,\mathbb{R})$. Then the indexes of all representations of C are the same.*

Proof. Note that as ρ varies in C, the image $\widetilde{\rho}_0(\lambda)$ varies in a connected component of hyperbolic elements. Since each connected component corresponds to a single value of k (see Figure 1 of [Kh]), the number k is the same for all representations in C. □

We will also say that the curve C in the above lemma has *index k*.

Corollary 2.2. *Let C be a connected curve of hyperbolic representations of a knot group $G(K)$ into $\mathrm{SL}(2,\mathbb{R})$. If C contains a reducible representation then it has index 0.*

Proof. Suppose that $\rho_1 \in C$ is a reducible hyperbolic representation given by

$$\rho_1(\mu_i) = \begin{pmatrix} s & a_i \\ 0 & s^{-1} \end{pmatrix},$$

where μ_i are generators of $G(K)$ which are conjugate to the standard meridian. Then we can connect ρ_1 to an abelian representation by using the curve

$$\rho_t(\mu_i) = \begin{pmatrix} s & ta_i \\ 0 & s^{-1} \end{pmatrix}, t \in \mathbb{R}.$$

It is easy to verify that if ρ_1 is a representation then so is ρ_t for all t. As the abelian representation ρ_0 has index 0, the corollary follows from the previous lemma. □

The next proposition tells us how to find left orderable slopes of a knot, given a connected curve of hyperbolic representations of the knot group into $\mathrm{SL}(2,\mathbb{R})$.

Proposition 2.3. *(1) Let $\{\rho_y : G(K) \to \mathrm{SL}(2,\mathbb{R})\}_y$ be a connected curve of hyperbolic representations of index 0, and choose a lift $\widetilde{\rho}_y : G(K) \to \widetilde{\mathrm{SL}(2,\mathbb{R})}$ such that $\widetilde{\rho}_y(\mu) = (\tanh a(y), 0)$ and $\widetilde{\rho}_y(\lambda) = (\tanh b(y), 0)$. If $\frac{p}{q}$ is a slope such that $\frac{p}{q} = -\frac{b(y)}{a(y)}$ for some y, then $\frac{p}{q}$ is a left orderable slope of K.*

(2) Let $\{\rho_y : G(K) \to \mathrm{SL}(2,\mathbb{R})\}_y$ be a connected curve of hyperbolic representations of index $k \neq 0$, and choose a lift $\widetilde{\rho}_y : G(K) \to \widetilde{\mathrm{SL}(2,\mathbb{R})}$ such that $\widetilde{\rho}_y(\mu) = (\tanh a(y), 0)$ and $\widetilde{\rho}_y(\lambda) = (\tanh b(y), k\pi)$. If $\frac{p}{q}$ is a slope such that $\frac{p}{q} = -\frac{b(y)}{a(y)}$ for some y and $p|k$, then $\frac{p}{q}$ is a left orderable slope of K.

Proof. Let $X_{p/q}$ denote by 3-manifold obtained from S^3 by $\frac{p}{q}$-surgery along K.

(i) Since $\widetilde{\rho_y}(\mu^p\lambda^q) = (\tanh(pa(y)+qb(y)), 0) = (0,0)$, $\widetilde{\rho_y}$ gives a representation from $\pi_1(X_{p/q})$ to $\widetilde{\mathrm{SL}(2,\mathbb{R})}$. Note that $X_{p/q}$ is an irreducible 3-manifold (by [HTh]) and $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ is a left orderable group (by [Be]). Hence, by Theorem 1.1 of [BRW], $\pi_1(X_{p/q})$ is a left orderable group. This means that $\frac{p}{q}$ is a left orderable slope of K.

(ii) We choose another lift $\widetilde{\rho_y}' : G(K) \to \widetilde{\mathrm{SL}(2,\mathbb{R})}$ such that $\widetilde{\rho_y}'(\mu) = (\tanh a(y), -\frac{kq}{p}\pi)$ and $\widetilde{\rho_y}'(\lambda) = (\tanh b(y), k\pi)$. We then have

$$\widetilde{\rho_y}'(\mu^p\lambda^q) = (\tanh(pa(y)+qb(y)), -kq\pi + kq\pi) = (0,0).$$

Therefore $\widetilde{\rho_y}'$ gives a representation from $\pi_1(X_{p/q})$ to $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ and, as in (i), the assertion follows. □

3. Representations of double twist knots

Consider the two-bridge knot/link $C(k,l)$ in the Conway notation, where k,l are integers such that $|kl| \geq 3$. Note that $C(k,l)$ is the rational knot/link corresponding to continued fraction $k+1/l$. It is easy to see that $C(k,l)$ is the mirror image of $C(l,k) = C(-k,-l)$. Moreover, $C(k,l)$ is a knot if kl is even and is a two-component link if kl is odd. In this paper, we only consider knots and so we can assume that $k>0$ and $l=-2p$ is even.

Note that $C(k,-2p)$ is the mirror image of the double twist knot $J(k,2p)$ in [HS]. Then, by [HS], the knot group of $C(k,-2p)$ has a presentation

$$G(C(k,-2p)) = \langle a, b \mid aw^p = w^p b \rangle$$

where a,b are meridians and

$$w = \begin{cases} (ab^{-1})^m(a^{-1}b)^m & \text{if } k=2m, \\ (ab^{-1})^m ab(a^{-1}b)^m & \text{if } k=2m+1. \end{cases}$$

Moreover, the canonical longitude of $C(k,-2p)$ corresponding to the meridian $\mu = a$ is $\lambda = (w^p(w^p)^* a^{-2\varepsilon})^{-1}$, where $\varepsilon = 0$ if $k=2m$ and $\varepsilon = 2p$ if $k=2m+1$. Here, for a word u in the letters a,b we let u^* be the word obtained by reading u backwards.

Suppose $\rho\colon G(C(k,-2p)) \to \mathrm{SL}_2(\mathbb{C})$ is a nonabelian representation. Up to conjugation, we may assume that

$$\rho(a) = \begin{bmatrix} M & 1 \\ 0 & M^{-1} \end{bmatrix} \quad \text{and} \quad \rho(b) = \begin{bmatrix} M & 0 \\ 2-y & M^{-1} \end{bmatrix} \tag{3.1}$$

where $(M,y) \in \mathbb{C}^2$ satisfies the matrix equation $\rho(aw^p) = \rho(w^p b)$. It is known that this matrix equation is equivalent to a single polynomial equation $R_{C(k,-2p)}(x,y) = 0$, where $x = (\mathrm{tr}\rho(a))^2$ and $R_K(x,y)$ is the Riley polynomial of a two-bridge knot K, see [Ri]. This polynomial can be described via the Chebychev polynomials as follows.

Let $\{S_j(v)\}_{j\in\mathbb{Z}}$ be the Chebychev polynomials in the variable v defined by $S_0(v)=1$, $S_1(v)=v$ and $S_j(v) = vS_{j-1}(v) - S_{j-2}(v)$ for all integers j. Note that $S_j(v) = -S_{-j-2}(v)$ and $S_j(\pm 2) = (\pm 1)^j (j+1)$. Moreover, we see that $S_j(v) = (s^{j+1} - s^{-(j+1)})/(s - s^{-1})$ for

$v = s + s^{-1} \neq \pm 2$ from the recurrence relation. Using these identities one can prove the following.

Lemma 3.1. *We have*

(1) $S_j^2(v) - vS_j(v)S_{j-1}(v) + S_{j-1}^2(v) = 1$ *for any integer* j,
(2) $S_n(v) = \prod_{j=1}^n (v - 2\cos\frac{j\pi}{n+1})$ *for any positive integer* n,
(3) $S_n(v) - S_{n-1}(v) = \prod_{j=1}^n (v - 2\cos\frac{(2j-1)\pi}{2n+1})$ *for any positive integer* n.

Proof. (1) By the recurrence relation,

$$\begin{aligned} S_j^2(v) - vS_j(v)S_{j-1} + S_{j-1}^2(v) &= S_j(v)(S_j(v) - vS_{j-1}(v)) + S_{j-1}^2(v) \\ &= (vS_{j-1}(v) - S_{j-2}(v))(-S_{j-2}(v)) + S_{j-1}^2(v) \\ &= S_{j-1}^2(v) - vS_{j-1}(v)S_{j-2}(v) + S_{j-2}^2(v). \end{aligned}$$

Since $S_1(v)^2 - vS_1(v)S_0(v) + S_0^2(v) = 1$, we have the conclusion.

(2) For any positive integer n, $S_n(v)$ is a polynomial of degree n. Since ± 2 are not roots of $S_n(v)$, all roots come from solving $s^{n+1} - s^{-(n+1)} = 0$, where $v = s + s^{-1}$. Hence, the conclusion follows from the observation that $2\cos j\pi/(n+1)$ $(j = 1, 2, \dots, n)$ give all roots of $S_n(v)$.

(3) For any positive integer n, $S_n(v) - S_{n-1}(v)$ is a polynomial of degree n, and

$$\begin{aligned} S_n(v) - S_{n-1}(v) &= \frac{s^{n+1} - s^{-(n+1)}}{s - s^{-1}} - \frac{s^n - s^{-n}}{s - s^{-1}} \\ &= \frac{s^{n+1} - s^{-(n+1)} - s^n + s^{-n}}{s - s^{-1}} \\ &= \frac{s^{-(n+1)}}{s - s^{-1}} \cdot (s^{2n+2} - 1 - s^{2n+1} + s) \\ &= \frac{s^{-(n+1)}}{s - s^{-1}}(s - 1)(s^{2n+1} + 1). \end{aligned}$$

Hence all roots of $S_n(v) - S_{n-1}(v)$ come from solving $s^{2n+1} + 1 = 0$. That is, $2\cos(2j - 1)\pi/(2n + 1)$ $(j = 1, 2, \dots, n)$ give all the roots. □

The Riley polynomial of $C(k, -2p)$, whose zero locus describes all non-abelian representations of the knot group of $C(k, -2p)$ into $\mathrm{SL}_2(\mathbb{C})$, is

$$R_{C(k,-2p)}(x, y) = S_p(t) - zS_{p-1}(t)$$

where

$$t = \mathrm{tr}\rho(w) = \begin{cases} 2 + (y + 2 - x)(y - 2)S_{m-1}^2(y) & \text{if } k = 2m, \\ 2 - (y + 2 - x)(S_m(y) - S_{m-1}(y))^2 & \text{if } k = 2m + 1, \end{cases}$$

and

$$z = \begin{cases} 1 + (y + 2 - x)S_{m-1}(y)(S_m(y) - S_{m-1}(y)) & \text{if } k = 2m, \\ 1 - (y + 2 - x)S_m(y)(S_m(y) - S_{m-1}(y)) & \text{if } k = 2m + 1. \end{cases}$$

Moreover, for the representation $\rho\colon G(C(k,-2p)) \to \mathrm{SL}_2(\mathbb{C})$ of the form (3.1) the image of the canonical longitude $\lambda = (w^p(w^p)^* a^{-2\varepsilon})^{-1}$ has the form $\rho(\lambda) = \begin{bmatrix} L & * \\ 0 & L^{-1} \end{bmatrix}$, where

$$L = -\frac{M^{-1}(S_m(y) - S_{m-1}(y)) - M(S_{m-1}(y) - S_{m-2}(y))}{M(S_m(y) - S_{m-1}(y)) - M^{-1}(S_{m-1}(y) - S_{m-2}(y))} \quad \text{if } k = 2m$$

and

$$L = -M^{4p}\frac{M^{-1}S_m(y) - MS_{m-1}(y)}{MS_m(y) - M^{-1}S_{m-1}(y)} \quad \text{if } k = 2m+1.$$

See e.g. [Tr2, Pe].

4. The case of $C(2m, \pm 2n)$

In this section we prove the existence of continuous families of hyperbolic $\mathrm{SL}_2(\mathbb{R})$-representations of knot groups of $C(2m, \pm 2n)$ and hence fix the gaps in [HTe1, Tr2].

Proposition 4.1. *There exist $n-1$ continuous real functions $x_j\colon (2,\infty) \to (0,\infty)$, where $1 \le j \le n-1$, in the variable y such that $R_{C(2m,2n)}(x_j(y), y) = 0$ and*

$$y + 2 + \frac{4\sin^2\frac{(2j-1)\pi}{4n+2}}{(y-2)S^2_{m-1}(y)} < x_j(y) < y + 2 + \frac{4\sin^2\frac{(2j+1)\pi}{4n+2}}{(y-2)S^2_{m-1}(y)}$$

for all $y > 2$.

Proof. Let $K = C(2m, 2n)$. We have $R_K(x,y) = S_{-n}(t) - zS_{-n-1}(t)$ where

$$\begin{aligned} t &= 2 + (y+2-x)(y-2)S^2_{m-1}(y), \\ z &= 1 + (y+2-x)S_{m-1}(y)(S_m(y) - S_{m-1}(y)). \end{aligned}$$

Note that $R_K(x,y) = (t-z)S_{-n-1}(t) - S_{-n-2}(t) = S_n(t) - (t-z)S_{n-1}(t)$.

Let $t_j = 2\cos\frac{(2j-1)\pi}{2n+1}$ for $j = 1, \dots, n$. Then, Lemma 3.1(3) gives $S_n(t) - S_{n-1}(t) = \prod_{j=1}^n (t - t_j)$, and the signs of $S_n(t_j)$ change alternately as $S_n(t_1) > 0, S_n(t_2) < 0, \dots$, because of the inequality

$$\frac{j-1}{n+1} < \frac{2j-1}{2n+1} < \frac{j}{n+1} \ (j = 1, 2, \dots, n)$$

and Lemma 3.1(2).

Fix a real number $y > 2$. Let $s_j(y) = y + 2 + \frac{2-t_j}{(y-2)S^2_{m-1}(y)}$ for $j = 1, \dots, n$. Since $-2 < t_n < \cdots < t_1 < 2$, we have $s_n(y) > \cdots > s_1(y) > y+2$. At $x = s_j(y)$ we have $t = t_j$ and so $S_n(t) = S_{n-1}(t)$. This implies that

$$\begin{aligned} R_K(s_j(y), y) &= (1 - (t-z))S_n(t_j) \\ &= (y + 2 - s_j(y))S_{m-1}(y)(S_{m-1}(y) - S_{m-2}(y))S_n(t_j) \\ &= -\frac{2-t_j}{(y-2)S_{m-1}(y)}(S_{m-1}(y) - S_{m-2}(y))S_n(t_j). \end{aligned}$$

Since $y > 2$, we have $S_{m-1}(y) - S_{m-2}(y) > 0$ and $S_{m-1}(y) > 0$ by Lemma 3.1. Hence $R_K(s_j(y), y)$ and $S_n(t_j)$ have opposite signs, so the sign of $R_K(s_j(y), y)$ changes alternately as $R_K(s_1(y), y) < 0, R_K(s_2(y), y) > 0, \dots$.

For each $1 \leq j \leq n-1$, since $R_K(s_j(y), y)R_K(s_{j+1}(y), y) < 0$, there exists $x_j(y) \in (s_j(y), s_{j+1}(y))$ such that $R_K(x_j(y), y) = 0$. Also, since $R_K(y+2, y) = 1$ and $R_K(s_1(y), y) < 0$, there exists $x_0(y) \in (y+2, s_1(y))$ such that $R_K(x_0(y), y) = 0$.

Since $R_K(x, y) = S_n(t) - (t-z)S_{n-1}(t) = zS_{n-1}(t) - S_{n-2}(t)$, we see that $R_K(x, y)$ is a polynomial of degree n in x for each fixed real number $y > 2$. This polyomial has exactly n simple real roots $x_0(y), \dots, x_{n-1}(y)$ satisfying $x_{n-1}(y) > \cdots > x_0(y) > y+2$, hence the implicit function theorem implies that each $x_j(y)$ is a continuous function in $y > 2$. The continuous functions $x_1(y), \dots, x_{n-1}(y)$ satisfy the conditions of Proposition 4.1. $\square$

Proposition 4.2. *There exist $n-1$ continuous real functions $x_j \colon (2, \infty) \to (0, \infty)$, where $1 \leq j \leq n-1$, in the variable y such that $R_{C(2m,-2n)}(x_j(y), y) = 0$ and*

$$y + 2 + \frac{4\sin^2 \frac{(2j-1)\pi}{4n+2}}{(y-2)S^2_{m-1}(y)} < x_j(y) < y + 2 + \frac{4\sin^2 \frac{(2j+1)\pi}{4n+2}}{(y-2)S^2_{m-1}(y)}$$

for all $y > 2$.

Proof. Let $K = C(2m, -2n)$. We have $R_K(x, y) = S_n(t) - zS_{n-1}(t)$ where

$$\begin{aligned} t &= 2 + (y+2-x)(y-2)S^2_{m-1}(y), \\ z &= 1 + (y+2-x)S_{m-1}(y)(S_m(y) - S_{m-1}(y)). \end{aligned}$$

Fix a real number $y \geq 2$. Choose t_j and $s_j(y)$ for $1 \leq j \leq n$ as in the proof of Proposition 4.1. Recall $S_n(t) = S_{n-1}(t)$ at $t = t_j$. Since

$$\begin{aligned} R_K(s_j(y), y) &= (1-z)S_n(t_j) \\ &= -(y+2-s_j(y))S_{m-1}(y)(S_m(y) - S_{m-1}(y))S_n(t_j) \\ &= \frac{2-t_j}{(y-2)S_{m-1}(y)}(S_m(y) - S_{m-1}(y))S_n(t_j), \end{aligned}$$

$R_K(s_j(y), y)$ and $S_n(t_j)$ have the same sign. So, the sign of $R_K(s_j(y), y)$ changes alternately. Since $R_K(s_j(y), y)R_K(s_{j+1}(y), y) < 0$, there exists $x_j(y) \in (s_j(y), s_{j+1}(y))$ such that $R_K(x_j(y), y) = 0$ for each $1 \leq j \leq n-1$.

We now claim that there exists $x_0(y) \in (0, y+2)$ such that $R_K(x_0(y), y) = 0$. Indeed, at $x = 0$ we have $t = 2 + (y^2-4)S^2_{m-1}(y)$ and $z = 1 + (y+2)S_{m-1}(y)(S_m(y) - S_{m-1}(y))$. Write $y = \xi + \xi^{-1}$ for some $\xi > 1$. Then $S_k(y) = \frac{\xi^{k+1} - \xi^{-(k+1)}}{\xi - \xi^{-1}}$ for all integers k.

Claim 4.3. (1) $t = \xi^{2m} + \xi^{-2m}$.
(2) $z = \xi^{-2m}\frac{\xi^{4m+1}-1}{\xi - 1}$.

Proof of Claim 4.3. (1) follows from

$$t = 2 + (y^2-4)S^2_{m-1}(y)$$

$$= 2 + (\xi - \xi^{-1})^2 \left(\frac{\xi^m - \xi^{-m}}{\xi - \xi^{-1}}\right)^2$$
$$= 2 + (\xi^m - \xi^{-m})^2$$
$$= \xi^{2m} + \xi^{-2m}.$$

(2) Similarly,

$$\begin{aligned}
z &= 1 + (\xi + \xi^{-1} + 2)\frac{\xi^m - \xi^{-m}}{\xi - \xi^{-1}} \left(\frac{\xi^{m+1} - \xi^{-(m+1)}}{\xi - \xi^{-1}} - \frac{\xi^m - \xi^{-m}}{\xi - \xi^{-1}}\right)\\
&= \frac{1}{(\xi - \xi^{-1})^2}\big(1 + (\xi + \xi^{-1} + 2)(\xi^m - \xi^{-m})(\xi^{m+1} - \xi^{-(m+1)} - \xi^m + \xi^{-m})\big)\\
&= \frac{1}{(\xi - \xi^{-1})^2}\left(\xi^{2m+2} + \xi^{-(2m+2)} + \xi^{2m+1} + \xi^{-(2m+1)} - \xi^{2m} - \xi^{-2m} - \xi^{2m-1} - \xi^{-(2m-1)}\right)\\
&= \frac{\xi^2 - 1}{(\xi - \xi^{-1})^2}(\xi^{2m} + \xi^{2m-1} - \xi^{-(2m+1)} - \xi^{-(2m+2)})\\
&= \frac{\xi^2}{\xi^2 - 1}\xi^{-(2m+2)}(\xi^{4m+2} + \xi^{4m+1} - \xi - 1)\\
&= \xi^{-2m}\frac{\xi^{4m+1} - 1}{\xi - 1}.
\end{aligned}$$

□

By Claim 4.3(1), $S_k(t) = \frac{\xi^{2m(k+1)} - \xi^{-2m(k+1)}}{\xi^{2m} - \xi^{-2m}}$ for all integers k. Hence we have

$$\begin{aligned}
R_K(0, y) &= S_n(t) - zS_{n-1}(t)\\
&= \frac{\xi^{2m(n+1)} - \xi^{-2m(n+1)}}{\xi^{2m} - \xi^{-2m}} - \xi^{-2m}\frac{\xi^{4m+1} - 1}{\xi - 1}\frac{\xi^{2mn} - \xi^{-2mn}}{\xi^{2m} - \xi^{-2m}}\\
&= \frac{1}{\xi^{2m} - \xi^{-2m}}\left((\xi^{2m(n+1)} - \xi^{-2m(n+1)}) - \xi^{-2m}\frac{\xi^{4m+1} - 1}{\xi - 1}(\xi^{2mn} - \xi^{-2mn})\right)\\
&= \frac{\xi^{-2mn}}{\xi^{2m} - \xi^{-2m}}\left((\xi^{2m(2n+1)} - \xi^{-2m}) - \xi^{-2m}\frac{\xi^{4m+1} - 1}{\xi - 1}(\xi^{4mn} - 1)\right)\\
&= \frac{\xi^{-2mn}}{\xi^{2m} - \xi^{-2m}}\frac{1}{\xi - 1}(\xi^{4mn+2m} - \xi^{4mn-2m} - \xi^{2m+1} + \xi^{-(2m-1)})\\
&= -\xi^{-2mn}\,\frac{\xi^{4mn} - \xi}{\xi - 1} < 0.
\end{aligned}$$

Since $R_K(y + 2, y) = 1$, there exists $x_0(y) \in (0, y + 2)$ such that $R_K(x_0(y), y) = 0$.

By writing $R_K(x, y) = S_n(t) - zS_{n-1}(t) = (t - z)S_{n-1}(t) - S_{n-2}(t)$ and noting that

$$t - z = 1 + (y + 2 - x)S_{m-1}(y)(-S_{m-1}(y) + S_{m-2}(y)),$$

we see that $R_K(x, y)$ is a polynomial of degree n in x for a fixed real number $y > 2$. This polynomial has exactly n simple real roots $x_0(y), \dots, x_{n-1}(y)$ satisfying $x_{n-1}(y) > \cdots > x_1(y) > y + 2 > x_0(y) > 0$, hence the implicit function theorem implies that each $x_j(y)$

is a continuous function in $y > 2$. The continuous functions $x_1(y), \dots, x_{n-1}(y)$ satisfy the conditions of Proposition 4.2. $\square$

5. The case of $C(2m+1, -2n)$

In this section we prove the existence of continuous families of hyperbolic $\mathrm{SL}_2(\mathbb{R})$-representations of the knot groups of $C(2m+1, -2n)$ and use it to give a proof of Theorem 1.

5.1. **Real roots of the Riley polynomial.**

Proposition 5.1. *There exists a unique continuous real function* $x\colon (2\cos\frac{\pi}{2m+1}, \infty) \to (0, \infty)$ *in the variable* y *such that* $R_{C(2m+1,-2n)}(x(y), y) = 0$ *and* $x(y) > y + 2$ *for all* $y > 2\cos\frac{\pi}{2m+1}$.

Proof. Let $K = C(2m+1, -2n)$. We have $R_K(x, y) = S_n(t) - zS_{n-1}(t)$ where

$$\begin{aligned} t &= 2 - (y + 2 - x)(S_m(y) - S_{m-1}(y))^2, \\ z &= 1 - (y + 2 - x)S_m(y)(S_m(y) - S_{m-1}(y)). \end{aligned}$$

Choose t_j for $1 \le j \le n$ as in the proof of Proposition 4.1. Fix a real number $y > 2\cos\frac{\pi}{2m+1}$. Then $S_m(y) > S_{m-1}(y) > 0$ by Lemma 3.1(3). Let $s_j(y) = y + 2 - \frac{2-t_j}{(S_m(y)-S_{m-1}(y))^2}$ for $j = 1, \dots, n$. Then $s_n(y) < \cdots < s_1(y) < y + 2$. Since

$$R_K(s_j(y), y) = (1 - z)S_n(t_j) = (y + 2 - s_j(y))S_m(y)(S_m(y) - S_{m-1}(y))S_n(t_j),$$

$R_K(s_j(y), y)$ and $S_n(t_j)$ have the same sign. Thus the sign of $R_K(s_j(y), y)$ changes alternately. Hence, there exists $x_j(y) \in (s_{j+1}(y), s_j(y))$ such that $R_K(x_j(y), y) = 0$ for each $1 \le j \le n - 1$.

By writing $R_K(x, y) = (t - z)S_{n-1}(t) - S_{n-2}(t)$ and noting that

$$t - z = 1 + (y + 2 - x)(S_m(y) - S_{m-1}(y))S_{m-1}(y),$$

we see that $R_K(x, y)$ is a polynomial of degree n in x with negative highest coefficient for each fixed real number $y > 2\cos\frac{\pi}{2m+1}$. For, t (resp. $t - z$) has degree one in the variable x with positive (resp. negative) coefficient, and the Chebyshev polynomials $S_{n-1}(t)$ and $S_{n-2}(t)$ are polynomials of degree $n-1$ and $n-2$, respectively, in t with positive highest coefficient. Since $\lim_{x\to\infty} R_K(x, y) = -\infty$ and $R_K(y + 2, y) = 1$, there exists $x_0(y) \in (y + 2, \infty)$ such that $R_K(x_0(y), y) = 0$. For a fixed real number $y > 2\cos\frac{\pi}{2m+1}$, the polynomial $R_K(x, y)$ of degree n in x has exactly n simple real roots $x_0(y), \dots, x_{n-1}(y)$ satisfying $x_{n-1}(y) < \cdots < x_1(y) < y + 2 < x_0(y)$, hence the implicit function theorem implies that each $x_j(y)$ is a continuous function in $y > 2\cos\frac{\pi}{2m+1}$.

Letting $x(y) = x_0(y)$ for $y > 2\cos\frac{\pi}{2m+1}$, we have $x(y) > y + 2$ and $R_K(x(y), y) = 0$. $\square$

Proposition 5.2. *The continuous real function* $x(y)$ *in Proposition 5.1 satisfies the following properties:*

(1) $x(y) > 2 + \frac{S_m(y)}{S_{m-1}(y)} + \frac{S_{m-1}(y)}{S_m(y)} > 4$ *for all* $y > 2\cos\frac{\pi}{2m+1}$,
(2) $x(y) \to \infty$ *as* $y \to (2\cos\frac{\pi}{2m+1})^+$,
(3) $y^{2m+2n-2}\big(x(y) - 2 - \frac{S_m(y)}{S_{m-1}(y)} - \frac{S_{m-1}(y)}{S_m(y)}\big) \to 1$ *as* $y \to \infty$.

Proof. (1) Since $R_K(x(y), y) = 0$ we have $S_n(t) = zS_{n-1}(t)$. By Lemma 3.1(1), $S_n^2(t) - tS_n(t)S_{n-1}(t) + S_{n-1}^2(t) = 1$. Thus we have $(z^2 - tz + 1)S_{n-1}^2(t) = 1$. Let $G = S_m(y)$ and $H = S_{m-1}(y)$. Then $G > H > 0$ for $y > 2\cos\frac{\pi}{2m+1}$ by Lemma 3.1(3). By using $G^2 - yGH + H^2 = 1$ and $t - 2 = (x - y - 2)(G - H)^2$, we have

$$\begin{aligned} z^2 - tz + 1 &= (z-1)^2 - (t-2)z \\ &= (x-y-2)^2G^2(G-H)^2 - (x-y-2)(G-H)^2\,(1 + (x-y-2)G(G-H)) \\ &= (x-y-2)(G-H)^2((x-y-2)GH - 1) \\ &= (x-y-2)(G-H)^2((x-2)GH - G^2 - H^2) \\ &= (t-2)((x-2)GH - G^2 - H^2). \end{aligned}$$

Hence $(t-2)((x-2)GH - G^2 - H^2)S_{n-1}^2(t) = 1$. Since $t - 2 = (x - y - 2)(G - H)^2 > 0$, we get $(x-2)GH - G^2 - H^2 > 0$. This implies that

$$x > 2 + \frac{G^2 + H^2}{GH} = 4 + \frac{(G-H)^2}{GH} > 4.$$

(2) As $y \to (2\cos\frac{\pi}{2m+1})^+$ we have $G - H \to 0$ by Lemma 3.1(3). If x is bounded, then $t - 2 = (x - 2 - y)(G - H)^2 \to 0$ and $1 = (t-2)((x-2)GH - G^2 - H^2)S_{n-1}^2(t) \to 0$, a contradiction. Hence $x(y) \to \infty$ as $y \to (2\cos\frac{\pi}{2m+1})^+$.

(3) We now consider the case $y > 2$. First,

$$\begin{aligned} t + 2 - x &= (x-y-2)(G-H)^2 + 4 - x \\ &= (x-y-2)(1 + (y-2)GH) + 4 - x \\ &= (x-y-2)(y-2)GH - y + 2 \\ &= (y-2)\left((x-y-2)GH - 1\right) \\ &= (y-2)((x-2)GH - G^2 - H^2) > 0. \end{aligned}$$

Hence $t > x - 2 > y$. Noting that $(t-2)((x-2)GH - G^2 - H^2)S_{n-1}^2(t) = 1$, we have $(x-2)GH - G^2 - H^2 \to 0$ as $y \to \infty$. This is equivalent to $GH(x - 2 - y) - 1 = GH\frac{t-2}{(G-H)^2} - 1 \to 0$ as $y \to \infty$. Since

$$GH\frac{t-2}{(G-H)^2} = \frac{\frac{t}{y} - \frac{2}{y}}{1 - \frac{2}{y} + \frac{1}{yGH}},$$

$\frac{t}{y} \to 1$ as $y \to \infty$. The equation $(t-2)((x-2)GH - G^2 - H^2)S_{n-1}^2(t) = 1$ gives

$$(t-2)GH\left(x - 2 - \frac{G}{H} - \frac{H}{G}\right)S_{n-1}^2(t) = 1.$$

Since G and H have degree m and $m-1$ respectively in y with positive highest coefficient and $S_{n-1}(t)^2$ has degree $2n-2$ in t, $y^{2m+2n-2}(x-2-\frac{G}{H}-\frac{H}{G}) \to 1$ as $y \to \infty$. This completes the proof of Proposition 5.2. ☐

5.2. **Proof of Theorem 1.** Let X be the complement of an open tubular neighborhood of $K = C(2m+1, -2n)$ in S^3, and X_r the 3-manifold obtained from S^3 by r-surgery along K. Consider the function $x(y)$ in Proposition 5.1. For each $y > 2\cos\frac{\pi}{2m+1}$, we have $x(y) > 2 + \frac{S_m(y)}{S_{m-1}(y)} + \frac{S_{m-1}(y)}{S_m(y)} > 4$ by Proposition 5.2(1). Let $M(y) = \frac{1}{2}(\sqrt{x(y)} + \sqrt{x(y)-4}) > 1$, then $\sqrt{x(y)} = M(y) + M(y)^{-1}$. Since $R_K(x(y), y) = 0$, there exists a non-abelian representation $\rho_y \colon \pi_1(X) \to \mathrm{SL}_2(\mathbb{R})$ such that

$$\rho_y(a) = \begin{bmatrix} M(y) & 1 \\ 0 & M(y)^{-1} \end{bmatrix} \quad \text{and} \quad \rho_y(b) = \begin{bmatrix} M(y) & 0 \\ 2-y(x) & M(y)^{-1} \end{bmatrix}.$$

Moreover, the image of the canonical longitude λ corresponding to the meridian $\mu = a$ has the form $\rho_y(\lambda) = \begin{bmatrix} L(y) & * \\ 0 & L(y)^{-1} \end{bmatrix}$, where

$$L(y) = -M(y)^{4n} \frac{M(y)^{-1}S_m(y) - M(y)S_{m-1}(y)}{M(y)S_m(y) - M(y)^{-1}S_{m-1}(y)}.$$

As in the proof of Proposition 5.2, we let $G = S_m(y)$ and $H = S_{m-1}(y)$. Then $G > H > 0$ for $y > 2\cos\frac{\pi}{2m+1}$ and $L(y) = M(y)^{4n}\frac{M(y)^2 - \frac{G}{H}}{M(y)^2\frac{G}{H} - 1}$. Since $M(y)^2 + M(y)^{-2} = x(y) - 2 > \frac{G}{H} + \frac{H}{G}$, we have $M(y)^2 > \frac{G}{H} > 1$. This implies that

$$L(y) = M(y)^{4n}\frac{M(y)^2 - \frac{G}{H}}{M(y)^2\frac{G}{H} - 1} > 0.$$

As $y \to (2\cos\frac{\pi}{2m+1})^+$ we have $\frac{G}{H} = \frac{S_m(y)}{S_{m-1}(y)} \to 1$ and $M(y) \to \infty$ (by Proposition 5.2(2)), so $\frac{M(y)^2 - \frac{G}{H}}{M(y)^2\frac{G}{H} - 1} \to 1$. Hence

$$\frac{\log L(y)}{\log M(y)} = 4n + \frac{\log \frac{M(y)^2 - \frac{G}{H}}{M(y)^2\frac{G}{H} - 1}}{\log M(y)} \to 4n.$$

As $y \to \infty$ we have $y^{2m+2n-2}\big(x(y) - 2 - \frac{G}{H} - \frac{H}{G}\big) \to 1$ by Proposition 5.2(3). This is equivalent to $y^{2m+2n-2}(M(y)^2 - \frac{G}{H})(1 - \frac{1}{M(y)^2\frac{G}{H}}) \to 1$, which implies that $y^{2m+2n-2}(M(y)^2 - \frac{G}{H}) \to 1$. Thus $M(y)^2 - \frac{G}{H} \to 0$. Since G and H have degree m and $m-1$ respectively in y with positive highest coefficient, $\frac{G}{H} - y \to 0$. Then $M(y)^2 - y \to 0$. Asymptotically, $M(y)^2 - \frac{G}{H} \sim y^{2-2m-2n}$ and $M(y)^2\frac{G}{H} - 1 \sim y^2$. Hence

$$\frac{\log L(y)}{\log M(y)} = 4n + \frac{\log \frac{M(y)^2 - \frac{G}{H}}{M(y)^2\frac{G}{H} - 1}}{\log M(y)} \to 4n - (4m+4n) = -4m.$$

Consider the continuous function $f(y) := -\frac{\log L(y)}{\log M(y)}$ for $y > 2\cos\frac{\pi}{2m+1}$. Then from the above arguments we conclude that the image of f contains the interval $(-4n, 4m)$.

This implies that for any slope $r \in (-4n, 4m)$ there exists $y > 2\cos\frac{\pi}{2m+1}$ such that $r = f(y) = -\frac{\log L(y)}{\log M(y)}$.

The continuous family C of nonabelian representations $\{\rho_y\}$, $y > 2\cos\frac{\pi}{2m+1}$, contains a special one which is the reducible nonabelian representation ρ_2 (i.e. ρ_y at $y = 2$). This representation has index 0 by Corollary 2.2 and so by Lemma 2.1 the continuous family C has index 0. Applying Proposition 2.3(i) to C, with $a(y) = \log M(y)$ and $b(y) = \log L(y)$, we conclude that any slope $r \in (-4n, 4m)$ is a left orderable slope of $K = C(2m+1, -2n)$ detected by hyperbolic $\mathrm{SL}_2(\mathbb{R})$-representations of the knot group.

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