DECAY NEAR BOUNDARY OF VOLUME OF SUBLEVEL SETS OF $m-$SUBHARMONIC FUNCTIONS

NGUYEN QUANG DIEU [1,2] AND DO THAI DUONG [3]

ABSTRACT. We investigate decay near boundary of the volume of sublevel sets in Cegrell classes of $m-$ subharmonic function on bounded domains in $\mathbb{C}^n$. On the reverse direction, some sufficient conditions for membership in certain Cegrell's classes, in terms of the decay of the sublevel sets, are also discussed.

CONTENTS

1. INTRODUCTION

Let Ω be a domain in $\mathbb{C}^n$ and let u be a subharmonic function defined on Ω. Then, for an integer $m, 1 \leq m \leq n$, according to Li in [12], we say that u is $m-$subharmonic function if for every $\alpha_1, ..., \alpha_{m-1} \in \Gamma_m$, the inequality

$$dd^c u \wedge \alpha_1 \wedge ... \wedge \alpha_{m-1} \wedge \omega^{n-m} \geq 0,$$

holds in the sense of currents. Here we define

$$\Gamma_m := \{\alpha \in C_{(1,1)} : \alpha \wedge \omega^{n-1} \geq 0, ..., \alpha^m \wedge \omega^{n-m} \geq 0\},$$

where $\omega := dd^c|z|^2$ is the canonical Kähler form in $\mathbb{C}^n$ and $C_{(1,1)}$ is the set of $(1,1)-$forms with constant coefficients. Denote by $SH_m(\Omega)$ the set of all $m-$subharmonic functions in Ω, and $SH_m^-(\Omega)$ for the set of all non-positive $m-$subharmonic functions in Ω. A function $u \in SH_m(\Omega)$ is called *strictly* m-subharmonic if for every relatively compact domain $\Omega' \Subset \Omega$ there exists a constant $c > 0$ such that $u(z) - c|z|^2$ is $m-$subharmonic on Ω'.

We note the following chain of inclusions

$$PSH = SH_n \subset ... \subset SH_1 = SH.$$

The border cases, SH_1 and SH_n, of course, correspond to subharmonic function and plurisubharmonic functions which are of fundamental importance in potential theory and pluripotential theory respectively. Later on, using Bedford-Taylor's induction method in [3], Błocki extended the definition of the complex $m-$Hessian operator $(dd^c u)^m \wedge \omega^{n-m}$ to locally bounded m-subharmonic functions in [4]. In particular, if $u \in SH_m(\Omega) \cap L^\infty_{loc}(\Omega)$ then the Borel measure $(dd^c u)^m \wedge \omega^{n-m}$ is well-defined and is called the complex $m-$Hessian measure of u.

Date: August 31, 2021.
2000 *Mathematics Subject Classification.* Primary 32U15; Secondary 32B15.
Key words and phrases. $m-$subharmonic functions, $m-$complex Hessian measures, sublevel sets.

Recently, in [13], Lu following the framework of Cegrell (in [5] and [6]) studied the domain of existence for the complex $m-$Hessian operator. For this purpose, he introduced finite energy classes of m-subharmonic functions of Cegrell type on bounded $m-$ hyperconvex domains Ω, i.e., domains Ω that admit a negative $m-$subharmonic exhaustion function. More precisely, the following energy classes are defined in [13]

$$\mathscr{E}_m^0(\Omega) = \{u \in SH_m^-(\Omega) \cap L^\infty(\Omega) : \lim_{z\to\partial\Omega} u(z) = 0, \int_\Omega (dd^c u)^m \wedge \omega^{n-m} < \infty\},$$
$$\mathscr{F}_m(\Omega) = \{u \in SH_m^-(\Omega) : \exists \mathscr{E}_m^0(\Omega) \ni u_j \downarrow u, \sup_j \int_\Omega (dd^c u_j)^m \wedge \omega^{n-m} < \infty\},$$
$$\mathscr{E}_m(\Omega) = \{u \in SH_m^-(\Omega) : \forall G \Subset \Omega, \exists u_G \in \mathscr{F}_m(\Omega) \text{ such that } u = u_G \text{ on } G\}.$$

Then the complex $m-$Hessian operator can be defined on the class $\mathscr{E}_m(\Omega)$. Moreover, this is the largest subset of non-positive m-subharmonic functions defined on Ω for which the complex $m-$Hessian operator can be continuously extended.

Our paper studies behavior near boundary of *volume* of sublevel sets of the class $\mathscr{F}_m$. In fact, we are motivated by three sources: Firstly, we are inspired by the paper [15] where the author characterizes the classes $\mathscr{E}_m$, $\mathscr{F}_m$ in terms of the $m-$capacity of sublevel sets, secondly from the fact that, for a bounded $m-$hyperconvex domain Ω, although $\limsup_{z\to\partial\Omega} u(z) = 0$ for every $u \in \mathscr{F}_m(\Omega)$ (see [16]) but there exists $v \in \mathscr{F}_m(\Omega)$ such that $\liminf_{z\to\partial\Omega} v(z) = -\infty$ (see the end of the next section) and third (and perhaps most strongly) by the article [10] where some characterization of the Cegrell classes on a bounded hyperconvex domain are given in terms of the decay of the volume of the sublevel sets near the boundary.

Our first result gives some qualitative estimates on portion near the boundary of the sublevel sets of $u \in \mathscr{F}_m$.

Theorem A. *Let Ω be a bounded $m-$hyperconvex domain in $\mathbb{C}^n$, $\rho \in \mathscr{E}_m(\Omega)$ and $u \in \mathscr{F}_m(\Omega)$. For $\varepsilon, \delta > 0$ we set*

$$\Omega_{u,\varepsilon,\delta} := \{z \in \Omega : u(z) < -\varepsilon, \rho(z) > -\delta\}.$$

Then we have the following estimates:
(a) $\int\limits_{\Omega_{u,\varepsilon,\delta}} (dd^c\rho)^m \wedge \omega^{n-m} \le \left(\frac{\delta}{\varepsilon}\right)^m \int\limits_\Omega (dd^c u)^m \wedge \omega^{n-m}.$
(b) $\left(\frac{m}{m+1}\right)^{m+1} \int\limits_{\Omega_{u,\varepsilon,\delta}} d\rho \wedge d^c\rho \wedge (dd^c\rho)^{m-1} \wedge \omega^{n-m} \le \delta \left(\frac{\delta}{\varepsilon}\right)^m \int\limits_\Omega (dd^c u)^m \wedge \omega^{n-m}$*, if ρ is locally bounded.*

The proof of Theorem A uses a version of a classical comparison principle due to Bedford and Taylor in [3] but for $m-$subharmonic functions, and of course the structure of Cegrell's classes that involved. Under stronger convexity assumptions on Ω we are able to derive upper bounds for volume of $\Omega_{u,\varepsilon,\delta}$ that depend on ε, δ and the total $m-$ Hessian measure of u (cf. Corollary 3.2 and Corollary 3.3).

For $\eta \in \mathbb{C}^n$, we define $d(\eta) := \sup\{|z-\eta| : z \in \overline{\Omega}\}$. Using the same technique and a subextension result for $m-$ subharmonic functions coupled with a symmetrization trick, we prove the second main result which estimates the volumes of the sublevel sets near certain boundary points of Ω.

Theorem B. *Let Ω and u be as in Theorem A and $\xi \in \partial\Omega$. Let $\eta \in \mathbb{C}^n$ be a point such that $|\xi - \eta| = d(\eta)$. Then for all $\delta \in (0, d(\eta))$ and $t > 0$ we have*

$$vol_{2n}\{z \in \Omega : u(z) < -t, d(\eta) - \delta < |z - \eta| < d(\eta)\}$$
$$\leq a_n \Big(\int_\Omega (dd^c u)^m \wedge \omega^{n-m} \Big)^{1/m} \frac{\delta^2 d(\eta)^{2n-2}}{t},$$

where $a_n > 0$ is a constant depending only on n.

Remark 1.1. Note that not every $\xi \in \partial\Omega$ is an extremal for $d(\eta)$ for some $\eta \in \mathbb{C}^n$. Indeed, any point ξ in the inner sphere of the annulus $\{r < |z| < 1\}$ $(r \in (0,1))$ does not have this property.

In case Ω is the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$, by taking ξ to be an arbitrary point in $\partial\mathbb{B}^n$, η be the origin in Theorem B and considering $t := A\delta$ where $A > 0$ is a fixed constant, we easily obtain the following result.

Corollary C. *Let $u \in \mathscr{F}_m(\mathbb{B}^n)$. Then there exists $C > 0$ such that for $A > 0$ we have*

$$\limsup_{\delta \to 0^+} \frac{vol_{2n}\{z \in \mathbb{B}^n : u(z) < -A\delta,\ \|z\| > 1 - \delta\}}{\delta} < \frac{C}{A}.$$

Observe that Corollary C in the case $m = n$ was proved in Theorem 5 of [10]. Our next main result is a sufficient condition for membership of the class $\mathscr{F}_m$ in the case when Ω admits a nice defining $m-$subharmonic function.

Theorem D. *Let Ω be a bounded $m-$hyperconvex domain in $\mathbb{C}^n$ that admits a negative $m-$subharmonic exhaustion function ρ which is C^1-smooth on a neighbourhood of $\partial\Omega$ and satisfies $d\rho \neq 0$ on $\partial\Omega$. Let $u \in SH_m^-(\Omega)$ be such that there exist $A, C > 0$ and $\alpha > 2n$ satisfying*

$$vol_{2n}(\{z \in \Omega : d(z, \partial\Omega) < \delta,\ u(z) < -A\delta\}) \leq C\delta^\alpha,$$

for all $\delta > 0$ small enough. Then $u \in \mathscr{F}_m(\Omega)$.

The proof proceeds roughly as follows. First by averaging u over small balls, we may approximate u from above by a sequence u_ε of $m-$subharmonic functions defined on slightly smaller domains than Ω. Then, by the assumptions of the theorem we can glue each u_ε with a suitable defining function for Ω to obtain an element in $\mathscr{E}_m(\Omega)$ with uniform upper bound of the total complex $m-$Hessian measures.

Our last result focuses again on the special case when Ω is the unit ball in $\mathbb{C}^n$.

Theorem E. *Let $u \in SH_m^-(\mathbb{B}^n)$. Assume that there exists $A > 0$ such that*

$$\lim_{\delta \to 0^+} \frac{vol_{2n}(\{z \in \mathbb{B}^n : \|z\| > 1 - \delta, u(z) < -A\delta\})}{\delta} = 0. \tag{1.1}$$

Then $u \in \mathscr{F}_m(\mathbb{B}^n)$.

The proof is a small modification of Theorem 5 in [10] where the same statement is proved for plurisubharmonic functions (when $m = n$). The main step of the proof is to approximate from above u by a collection of $m-$subharmonic functions $u_{a,\varepsilon}$ which lives on slightly smaller balls. The function $u_{a,\varepsilon}$ is constructed by taking upper envelopes of a family generated by u and a sequence of rotations. Next, as in the proof of Theorem D, we will exploit the assumption on the volume decay of the set $\{u < -A\delta\}$ near the boundary to get a lower estimate of $u_{a,\varepsilon}$ in terms of some defining function for $\mathbb{B}^n$. Then we will glue these data together to obtain a sequence in $\mathscr{E}_m^0(\Omega)$ that approximate u from above.

We end this introductory section by comparing our work and its companion [10]. Of course, we have borrowed from [10], the patching methods of $m-$subharmonic functions and the striking symmetrization technique. These tools enable us to prove Theorem E that directly generalizes Theorem 5 in [10] and Theorem B that covers some intermediate results in the proof of

Theorem 5 in [10]. On the other hand, even in the case $m=n$, we have proved a technical result (Theorem A) yielding two precise estimates (Corollary 3.2 and Corollary 3.3) on decay near boundary of sublevel sets of plurisubharmonic functions. Moreover, in Theorem D we discuss the same problem as in Theorem E but for general $m-$ hyperconvex domains which might be more general than the balls, and in Theorem B we even treat volume estimate near certain boundary points of the domain. Finally, we note that in passing from plurisubharmonic case to the m-subharmonic one, there exists a technical difficulty, that is $m-$subharmonicity is in general not preserved under holomorphic changes of coordinates. Fortunately, in proving Theorem E, it suffices to use the much weaker fact that $m-$subharmonicity remains under unitary change of coordinates, see [2], [9].

Acknowledgments. We are indebted to the referees of this article for their constructive comments that help to improve significantly our exposition. This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.02-2019.304. The second named author would like to thank IMU, FIMU and TWAS for supporting his PhD studies through the IMU Breakout Graduate Fellowship.

2. PRELIMINARIES

In this short section, we will review some basic technical tools that will be used in our work and some properties of energy classes.

Let u be a locally bounded $m-$subharmonic function defined on a domain Ω in $\mathbb{C}^n$. Then, by following Bedford and Taylor's induction method in [3], we may define as in [4] the $m-$complex Hessian measure of u as follows

$$(dd^c u)^m \wedge \omega^{n-m} := dd^c(u(dd^c u)^{m-1} \wedge \omega^{n-m}).$$

A natural problem is to define the largest subset of $SH_m^-(\Omega)$ on which the above operator is well defined and enjoy the continuity property under monotone convergence. This problem was solved by introducing the classes $\mathscr{E}_m(\Omega)$ and $\mathscr{F}_m(\Omega)$ mentioned at the beginning of our article. A major tool in studying $m-$complex Hessian measures is the following comparison principle (see Theorem 2.13 in [13]).

Proposition 2.1. *Let u,v be locally bounded $m-$subharmonic functions on a bounded domain Ω in $\mathbb{C}^n$. Suppose that $\liminf\limits_{z\to\partial\Omega}(u(z)-v(z))\geq 0$. Then we have*

$$\int\limits_{\{u<v\}} (dd^c u)^m \wedge \omega^{n-m} \geq \int\limits_{\{u<v\}} (dd^c v)^m \wedge \omega^{n-m}.$$

The above result will be naturally referred to as Bedford-Taylor's comparison principle. A main consequence of this principle is the following useful fact that compares total complex $m-$Hessian masses of elements in $\mathscr{F}_m(\Omega)$.

Lemma 2.2. *Let Ω be a bounded $m-$hyperconvex domain in $\mathbb{C}^n$, and $u,v\in\mathscr{F}_m(\Omega)$. Suppose that $u\geq v$ on a small neighbourhood of $\partial\Omega$. Then*

$$\int\limits_{\Omega} (dd^c u)^m \wedge \omega^{n-m} \leq \int\limits_{\Omega} (dd^c v)^m \wedge \omega^{n-m}.$$

The proof is only a slight modification of the plurisubharmonic case. For details, we refer the reader to Theorem 3.22 in [13] or Proposition 2.6 (a) in [14]. An easy consequence of Lemma 2.2 is the following fact (see also Proposition 2.6 (b) in [14] for a related result).

Lemma 2.3. *Let Ω be a bounded $m-$hyperconvex domain in $\mathbb{C}^n$ and $u \in \mathscr{F}_m(\Omega)$. Then the following assertions hold true:*
(a) For every sequence $u_j \in \mathscr{E}_m^0(\Omega)$ such that $u_j \downarrow u$ on Ω we have

$$\lim_{j\to\infty} \int_\Omega (dd^c u_j)^m \wedge \omega^{n-m} = \int_\Omega (dd^c u)^m \wedge \omega^{n-m};$$

(b) If $v \in SH_m^-(\Omega)$ and $v \geq u$ then $v \in \mathscr{F}_m(\Omega)$.

Notice that property (a) was proved in [16], and property (b) was proved in [13]. More subtle aspect of $m-$subharmonic functions lies in their subextension property. Indeed, using the solvability of the complex $m-$Hessian equation, we have the following result about subextension of $m-$subharmonic. The proof follows closely the lines of [8] and [7] where similar results was proved for plurisubharmonic functions.

Theorem 2.4 ([11])**.** *Let $\Omega \subset \widetilde{\Omega} \subset \mathbb{C}^n$ be bounded $m-$hyperconvex domains and $u \in \mathscr{F}_m(\Omega)$. Then, there exists $v \in \mathscr{F}_m(\widetilde{\Omega})$ such that $v \leq u$ on Ω and*

$$(dd^c v)^m \wedge \omega^{n-m} = 1_\Omega (dd^c u)^m \wedge \omega^{n-m} \text{ on } \widetilde{\Omega}.$$

We recall the following result (Lemma 4.7 in [16]) that relaxes the pointwise convergence condition in the definition of $\mathscr{F}_m(\Omega)$ to almost everywhere (a.e.) convergence.

Lemma 2.5. *Let Ω be a $m-$hyperconvex domain in $\mathbb{C}^n$ and $u \in SH_m^-(\Omega)$. Assume that there exists a sequence $\{u_j\} \in \mathscr{F}_m(\Omega)$ such that u_j converges a.e. to u and*

$$\sup_{j>0} \int_\Omega (dd^c u_j)^m \wedge \omega^{n-m} < \infty.$$

Then $u \in \mathscr{F}_m(\Omega)$.

Our next ingredient is a device that creates elements in Cegrell's classes by integrating with parameters a family of $m-$subharmonic functions.

Lemma 2.6. *Let $\Omega \subset \mathbb{C}^n$ be a bounded $m-$hyperconvex domain and X be a compact metric space equipped with a probability measure μ. Let $u : \Omega \times X \to [-\infty, 0)$ such that*

(i) *For every $a \in X$, $u(.,a) \in \mathscr{F}_m(\Omega)$ and*

$$\int_\Omega (dd^c u(z,a))^m \wedge \omega^{n-m} \leq M,$$

where $M > 0$ is a constant.

(ii) *For every $z \in \Omega$, the function $u(z,.)$ is upper semicontinuous on X.*

Then the following assertions hold true:
(a) $\tilde{u}(z) := \int_X u(z,a) d\mu(a) \in \mathscr{F}_m(\Omega)$.
(b) $\int_\Omega (dd^c \tilde{u})^m \wedge \omega^{n-m} \leq M$.

The proof is just a small modification of the averaging lemma in [9]. The point is to replace the Hölder inequality for plurisubharmonic functions in Cegrell classes in [4] by its $m-$subharmonic analogue (see Proposition 3.3 in [19]).

The next auxiliary result, which was proved in [2] and [9], reflects an invariant property of elements in $\mathscr{F}_m(\Omega)$.

Lemma 2.7. *Let $\mathbb{B}$ be the unit ball in $\mathbb{C}^n$ and $u \in \mathscr{F}_m(\mathbb{B})$. Then for every unitary transformation $\lambda : \mathbb{C}^n \to \mathbb{C}^n$ we have $u \circ \lambda \in \mathscr{F}_m(\mathbb{B})$.*

We end up this preliminary section with a construction of a function in $\mathscr{F}_m$ whose singular locus contains a given $m-$polar set.

Lemma 2.8. *Let Ω be a bounded $m-$hyperconvex domain and $E \subset \Omega$ be a $m-$polar set, i.e. for any $z \in E$ there exists a neighborhood V of z and $v \in SH_m(\Omega)$ such that $E \cap V \subset \{v = -\infty\}$. Then there exists $u \in \mathscr{F}_m(\Omega)$ such that $u = -\infty$ on E.*

Proof. The proof is a simple adaptation of the proof of Theorem 5.8 in [6]. For the reader convenience we provide the sketch of the proof. By a result in [17] (see also [13]) we have $C_m^*(E,\Omega) = 0$, where

$$C_m^*(E,\Omega) := \inf\{C_m(U,\Omega) : \ E \subset U, \ U \text{ is open}\},$$

and

$$C_m(U,\Omega) := \sup\{\int_U (dd^c u)^m \wedge \omega^{n-m} : \ u \in SH_m(\Omega), u < 0, u|_U \leq -1\}.$$

Thus we can find a sequence of relatively compact open subsets U_j of Ω such that every point of E is contained in all but finite open sets U_j and that

$$\int\limits_{U_j} (dd^c u_j)^m \wedge \omega^{n-m} < \frac{1}{j},$$

where

$$u_j := (h_{m,U_j,\Omega})^*, h_{m,U_j,\Omega}(z) := \sup\{u(z) : \ u < 0, \ u|_{U_j} \leq -1\}.$$

It is clear that $u_j \in \mathscr{E}_{0,m}(\Omega)$. Now by a Cegrell-Hölder's type inequality for the $m-$Hessian operator (Proposition 3.3 in [19]), we obtain a sequence k_j that increases to ∞ so fast such that the sequence $\{v_j\}$ defined by

$$v_j := u_{k_1} + \cdots + u_{k_j}$$

satisfies

$$\sup_j \int\limits_{\Omega} (dd^c v_j)^m \wedge \omega^{n-m} < \infty.$$

It follows that the series $\sum\limits_{j \geq 1} u_{k_j}$ indeed creates an element u of $\mathscr{F}_m(\Omega)$, and by the choice of U_j, we have $u = -\infty$ on E. □

Note that, for a bounded $m-$hyperconvex domain Ω, if we choose E to be a countable set such that every point on $\partial\Omega$ is a limit point to E, then by Lemma 2.8, there exists $u \in \mathscr{F}_m(\Omega)$ such that $u = -\infty$ on E, and therefore

$$\liminf_{z \to \partial\Omega} u(z) = -\infty.$$

3. Proofs of the results

In this section we will provide detailed proofs of the results that are announced at the beginning of the article. We first deal with Theorem A. The main technique is the comparison theorem for class $\mathscr{F}$ and the structure of Cegrell classes that involved.

Proof of Theorem A. (a) By Lemma 2.3 we may find a sequence $u_j \in \mathscr{E}_m^0(\Omega)$ such that $u_j \downarrow u$ and

$$\lim_{j \to \infty} \int\limits_{\Omega} (dd^c u_j)^m \wedge \omega^{n-m} = \int\limits_{\Omega} (dd^c u)^m \wedge \omega^{n-m}.$$

Fix an open subset $\Omega' \Subset \Omega$, we can find $\rho' \in \mathscr{F}_m(\Omega)$ with $\rho'|_{\Omega'} = \rho$. Then we note the inclusion

$$\Omega(u_j, \varepsilon, \delta) := \{z \in \Omega : \ u_j(z) < -\varepsilon, \ \rho'(z) > -\delta\} \subset \{\rho' > \frac{\delta}{\varepsilon} u_j\}.$$

Thus, by using Theorem 3.22 in [13], we get the following chain of estimates

$$
\begin{aligned}
\left(\frac{\delta}{\varepsilon}\right)^m \int_\Omega (dd^c u_j)^m \wedge \omega^{n-m} &\geq \int_{\left\{\rho' > \frac{\delta}{\varepsilon} u_j\right\}} \left(\frac{\delta}{\varepsilon}\right)^m (dd^c u_j)^m \wedge \omega^{n-m} \\
&\geq \int_{\left\{\rho' > \frac{\delta}{\varepsilon} u_j\right\}} (dd^c \rho')^m \wedge \omega^{n-m} \\
&\geq \int_{\Omega(u_j,\varepsilon,\delta)} (dd^c \rho')^m \wedge \omega^{n-m} \\
&\geq \int_{\Omega(u_j,\varepsilon,\delta)\cap\Omega'} (dd^c \rho)^m \wedge \omega^{n-m}
\end{aligned}
$$

Since $\Omega(u_j,\varepsilon,\delta)\cap\Omega' \uparrow \Omega_{u,\varepsilon,\delta}\cap\Omega'$, by letting $j\to\infty$ and then if $\Omega'\uparrow\Omega$ we obtain the desired estimate.
(b) For each $a\in(0,1)$ we set

$$\rho_a := -(-\rho)^a.$$

Then, by a direct computation, we obtain the following identity in the sense of currents

$$dd^c\rho_a = a(1-a)(-\rho)^{a-2} d\rho\wedge d^c\rho + a(-\rho)^{a-1} dd^c\rho.$$

Then ρ_a is a negative locally bounded $m-$plurisubharmonic function on Ω. Moreover,

$$(dd^c\rho_a)^m\wedge\omega^{n-m} \geq ma^m(1-a)(-\rho)^{m(a-1)-1} d\rho\wedge d^c\rho\wedge(dd^c\rho)^{m-1}\wedge\omega^{n-m}.$$

Since $0 < -\rho < \delta$ on $\Omega_{u,\varepsilon,\delta}$, we may combine the above inequality and the estimate in (a) to obtain

$$
\begin{aligned}
& ma^m(1-a)\delta^{m(a-1)-1} \int_{\Omega_{u,\varepsilon,\delta}} d\rho\wedge d^c\rho\wedge(dd^c\rho)^{m-1}\wedge\omega^{n-m} \\
&\leq \int_{\Omega_{u,\varepsilon,\delta}} (dd^c\rho_a)^m\wedge\omega^{n-m} \\
&= \int_{\{u<-\varepsilon,\rho_a>-\delta^a\}} (dd^c\rho_a)^m\wedge\omega^{n-m} \\
&\leq \left(\frac{\delta^a}{\varepsilon}\right)^m \int_\Omega (dd^c u)^m\wedge\omega^{n-m}.
\end{aligned}
$$

Now our inequality follows by rearranging these estimates and taking $a=\frac{m}{m+1}$. □

It is natural to ask if the following converse to Theorem A is true.

Question 3.1. Let u be a negative $m-$subharmonic function on a bounded hyperconvex domain Ω. Suppose that there exists $A>0$ such that for all $\varepsilon>0,\delta>0$ and for all $\rho\in\mathscr{E}_m(\Omega)$ we have

$$\int_{\Omega_{u,\varepsilon,\delta}} (dd^c\rho)^m\wedge\omega^{n-m} \leq A\left(\frac{\delta}{\varepsilon}\right)^m.$$

Does u belong to $\mathscr{F}_m(\Omega)$?

Theorem E is, thus, an attempt, to answer this question in the affirmative when Ω is the unit ball in $\mathbb{C}^n$. The following result follows directly from Theorem A (a).

Corollary 3.2. *Let Ω be a bounded $B-$regular domain, i.e., there exists a negative plurisubharmonic exhaustion function ρ on Ω satisfying $dd^c\rho \geq \omega$. Then for all $u \in \mathscr{F}_m(\Omega)$ we have*

$$vol_{2n}(\Omega_{u,\varepsilon,\delta}) \leq \frac{\delta^m}{\varepsilon^m} \int_\Omega (dd^c u)^m \wedge \omega^{n-m}.$$

Notice that we are using here the notion of $B-$regular domains taken from the seminal work [18]. Under a stronger assumption on convexity and smoothness of Ω we may refine the above estimate as follows.

Corollary 3.3. *Let Ω be a bounded strictly $m-$pseudoconvex domain with C^2-smooth boundary, i.e., there exists a C^2- smooth defining function ρ for Ω which is strictly m-subharmonic on a neighbourhood of $\overline{\Omega}$. For $\delta > 0$ and $u \in \mathscr{F}_m(\Omega)$ we set*

$$\Omega_u(\varepsilon, \delta) := \{z \in \Omega : \ u(z) < -\varepsilon, \ d(z, \partial\Omega) < \delta\},$$

where d is the distance function. Then there exist $\delta_0 = \delta_0(\Omega) > 0$ and $C = C(\Omega, \delta_0, n) > 0$ such that for all $u \in \mathscr{F}_m(\Omega)$, $\delta \in (0, \delta_0)$ and $\varepsilon > 0$ we have

$$vol_{2n}(\Omega_u(\varepsilon, \delta)) \leq C\frac{\delta^{m+1}}{\varepsilon^m} \int_\Omega (dd^c u)^m \wedge \omega^{n-m}.$$

Proof. Let ρ be a C^2 smooth strictly $m-$subharmonic function on a neighbourhood of $\overline{\Omega}$ that defines Ω. Then,

$$dd^c\rho \geq A dd^c|z|^2 = A\omega,$$

for some constant $A > 0$. Therefore,

$$d\rho \wedge d^c\rho \wedge (dd^c\rho)^{m-1} \wedge \omega^{n-m} \geq B d\rho \wedge d^c\rho \wedge \omega^{n-1} = B\|\text{grad } \rho\|^2 \omega^n,$$

for some constant $B > 0$. Moreover, by strictly $m-$pseudoconvexity of ρ, we can find a positive constant δ_0 depending on Ω such that

$$d\rho \neq 0 \text{ on } \{z \in \Omega : \ d(z, \partial\Omega) \leq \delta_0\}.$$

Therefore, since $\|\text{grad } \rho\|$ is bounded from below by a positive constant, we have, on $\{z \in \Omega : d(z, \partial\Omega) \leq \delta_0\}$,

$$d\rho \wedge d^c\rho \wedge (dd^c\rho)^{m-1} \wedge \omega^{n-m} \geq C'\omega^n,$$

for some constant C'. It follows that

$$\int_{\Omega_u(\varepsilon,\delta)} d\rho \wedge d^c\rho \wedge (dd^c\rho)^{m-1} \wedge \omega^{n-m} \geq C' \int_{\Omega_u(\varepsilon,\delta)} \omega^n,$$

for all $\varepsilon > 0$ and $\delta \in (0, \delta_0)$. The desired estimate follows by combining this with Theorem A(b). □

Regarding boundary behavior of $\mathscr{F}_m(\Omega)$, we have the following result which will also be used in the proof of Proposition 3.6.

Proposition 3.4. *Let $u, \rho \in \mathscr{F}_m(\Omega)$. Then we have*

$$\liminf_{z \to \partial\Omega} \frac{u(z)}{\rho(z)} \leq M,$$

where

$$M := \Big(\frac{\int_\Omega (dd^c u)^m \wedge \omega^{n-m}}{\int_\Omega (dd^c \rho)^m \wedge \omega^{n-m}} \Big)^{1/m} \in (0, \infty).$$

Proof. Fix $j \geq 1$. We claim that

$$M + \frac{1}{j} \geq \liminf_{z \to \partial\Omega} \frac{u(z)}{\rho(z)}.$$

Assume the contrary holds, then we have $u \leq (M + \frac{1}{2j})\rho$ on a small neighbourhood of $\partial\Omega$. Thus

$$u \leq v_j := \max\{u, (M + \frac{1}{2j})\rho\} \in \mathscr{F}(\Omega)$$

and $v_j = (M + \frac{1}{2j})\rho$ near $\partial\Omega$. Then by the comparison principle we obtain

$$\begin{aligned} M^m \int_\Omega (dd^c\rho)^m \wedge \omega^{n-m} &= \int_\Omega (dd^c u)^m \wedge \omega^{n-m} \\ &\geq \int_\Omega (dd^c v_j)^m \wedge \omega^{n-m} \\ &= (M + \frac{1}{2j})^m \int_\Omega (dd^c\rho)^m \wedge \omega^{n-m}. \end{aligned}$$

Here we used Stokes' theorem for the last equality. So we obtain a contradiction and thus the claim follows. By letting $j \to \infty$, we obtain the desired conclusion. □

The above result can be used to characterized radial elements in $\mathscr{F}_m(\Omega)$ when Ω is a *ball* in $\mathbb{C}^n$, a problem of independent interest.
A word of caution: From now on we always use a_n (which may change from line to line) to mean an absolute constant that depends only on n.

Proposition 3.5. *Let $u \in SH_m^-(\mathbb{B}^n(0,r))$ be a radial function on the ball $\mathbb{B}^n(0,r)$. Then the following conditions are equivalent.*
(a) $u \in \mathscr{F}_m(\mathbb{B}^n(0,r))$;
(b) There exists $a_n > 0$ such that

$$\sup_{0 \leq t < r} \frac{u(t)}{t - r} \leq a_n M(r),$$

where

$$M(r) := \frac{1}{r}\Big(\int_{\mathbb{B}^n(0,r)} (dd^c u)^m \wedge \omega^{n-m}\Big)^{1/m}.$$

Proof. We first treat the case $1 \leq m < n$. For simplicity of notation we set

$$\alpha := 2\Big(\frac{n}{m} - 1\Big),\ \rho(z) := 1 - \frac{r^\alpha}{|z|^\alpha},\ z \in \mathbb{B}^n(0,r).$$

According to [4], ρ is $m-$subharmonic on $\mathbb{C}^n$, moreover it is also a fundamental solution to the complex $m-$Hessian operator. Thus we get $\rho \in \mathscr{F}_m(\mathbb{B}^n(0,r))$.
$(b) \Rightarrow (a)$. The mean value theorem implies that for each $t \in (0,r)$ we have

$$\frac{r^\alpha - t^\alpha}{t^\alpha} \geq \frac{\alpha(r-t)t^{\alpha-1}}{t^\alpha} \geq \frac{\alpha}{r}(r-t),$$

so

$$t - r \geq \frac{r}{\alpha}\Big(1 - \frac{r^\alpha}{t^\alpha}\Big).$$

Thus for $0 < |z| = t < r$, by (b) and the above estimate we get the following bound for $u(z)$

$$\begin{aligned} u(z) = u(t) &\geq a_n M(r)(t-r) \\ &\geq a_n \frac{r}{\alpha} M(r)(1 - \frac{r^\alpha}{|z|^\alpha}) \\ &= a_n \frac{r}{\alpha} M(r) \rho(z). \end{aligned}$$

Since $\rho \in \mathscr{F}_m(\mathbb{B}^n(0,r))$, we may use Lemma 2.3 to see that $u \in \mathscr{F}_m(\mathbb{B}^n(0,r))$ as well.
$(a) \Rightarrow (b)$. Since $\rho(t) \approx t - r$ as $t \to r$, we may apply Proposition 3.4 to ρ and obtain some constant $a_n > 0$ such that

$$\liminf_{t \to r} \frac{u(t)}{t-r} \leq a_n M(r). \tag{3.1}$$

Now suppose (b) is false then there exists $t_0 \in (0,r)$ and $\lambda > a_n M(r)$ such that

$$u(t_0) < \lambda(t_0 - r).$$

Since u is subharmonic and radial on the ball $\mathbb{B}(0,r)$ we infer that the function $u : t \mapsto u(t)$ is increasing on $(0,r)$. Notice that $\lim_{t \uparrow r} u(t) = 0$, so for $t \in (t_0, r)$ we have

$$u(t) < \lambda(t - r).$$

This is a contradiction to (3.1). We are done.
For the case $m = n$, it suffices to repeat the same reasoning with $\rho(z) := \max\{\log(|z|/r), -1\}$. □

We now proceed to the proof of Theorem B. The proof requires the following auxiliary result.

Lemma 3.6. *Let $u \in \mathscr{F}_m(\mathbb{B}^n(0,r))$. Then there exists $a_n > 0$ such that for all $\delta \in (0,r)$ we have*

$$\int_{|z|=\delta} u(z) d\sigma(z) \geq \frac{a_n(\delta - r)\delta^{2n-1}}{r} \Big(\int_{\mathbb{B}^n(0,r)} (dd^c u)^m \wedge \omega^{n-m} \Big)^{1/m},$$

where σ denotes the surface measure on the sphere $|z| = \delta$.

Proof. We use a symmetrization trick as in [10]. More precisely, let μ be the unique invariant probability measure on the unitary group $U(n)$. Set

$$\begin{aligned} \tilde{u}(z) &:= \int_{U(n)} u(\lambda(z)) d\mu(\lambda) \\ &= \frac{1}{c_{2n-1}|z|^{2n-1}} \int_{|w|=|z|} u(w) d\sigma(w), \end{aligned}$$

where c_{2n-1} is the area of the unit sphere. Then $\tilde{u}$ is radial. Moreover, by Lemma 2.7 each function $u \circ \lambda \in \mathscr{F}_m(\mathbb{B}^n(0,r))$, so using the averaging Lemma 2.6 we obtain $\tilde{u} \in \mathscr{F}_m(\Omega)$. It then follows from Proposition 3.5 that

$$\tilde{u}(z) \geq (|z| - r) a_n M(r) \ \forall z \in \mathbb{B}^n(0,r).$$

By putting $z = \delta$ in the above estimate we obtain

$$\frac{1}{\delta^{2n-1}} \int_{|z|=\delta} u(z) d\sigma(z) \geq (\delta - r) a_n M(r).$$

After rearranging we get our desired inequality. □

Proof of Theorem B. The proof is spitted into two steps.
Step 1. We will show that for $r \in (0, d(\eta))$ we have

$$\int\limits_{\{\{|z-\eta|=r\}\cap\Omega\}} u(z)d\sigma(z) \geq \frac{a_n(r-d(\eta))r^{2n-1}}{d(\eta)}\Big(\int\limits_{\Omega}(dd^cu)^m \wedge \omega^{n-m}\Big)^{1/m}.$$

Consider the open ball $\Omega' := \mathbb{B}(\eta, d(\eta))$. Then $\Omega \subset \Omega'$ and $\xi \in \partial\Omega' \cap \partial\Omega$. By Theorem 2.4, we can find $u' \in \mathscr{F}_m(\Omega')$ such that $u' \leq u$ on Ω but

$$(dd^cu')^m \wedge \omega^{n-m} = \chi_\Omega(dd^cu)^m \wedge \omega^{n-m}.$$

Thus, by applying Lemma 3.6 to u' and Ω', we obtain

$$\begin{aligned}\int\limits_{\{\{|z-\eta|=r\}\cap\Omega\}} u(z)d\sigma(z) &\geq \int\limits_{\{\{|z-\eta|=r\}\}} u'(z)d\sigma(z)\\ &\geq \frac{a_n(r-d(\eta))r^{2n-1}}{d(\eta)}\Big(\int\limits_{\Omega'}(dd^cu')^m \wedge \omega^{n-m}\Big)^{1/m}.\end{aligned}$$

Therefore, we obtain the required estimate.
Step 2. Completion of the proof. For $t > 0$ and $r \in (0, d(\eta))$ we have

$$\int\limits_{\{\{|z-\eta|=r\}\cap\Omega\}} u(z)d\sigma(z) \leq -t\sigma\{z \in \Omega : \ u(z) < -t, |z-\eta| = r\}.$$

Thus, by the estimate obtained in the first step we get

$$\sigma\{z \in \Omega : \ u(z) < -t, |z-\eta| = r\} \leq \frac{a_n(r-d(\eta))r^{2n-1}}{td(\eta)}\Big(\int\limits_{\Omega}(dd^cu)^m \wedge \omega^{n-m}\Big)^{1/m}.$$

Thus, for $\delta \in (0, d(\eta))$, we obtain

$$\begin{aligned}&\mathrm{vol}_{2n}\{z \in \Omega : \ u(z) < -t, d(\eta) - \delta < |z-\eta| < d(\eta)\}\\ &= \int\limits_{d(\eta)-\delta}^{d(\eta)} \sigma\{z \in \Omega : \ u(z) < -t, |z-\eta| = r\}dr\\ &\leq \frac{a_n}{td(\eta)}\Big(\int\limits_{\Omega}(dd^cu)^m \wedge \omega^{n-m}\Big)^{1/m} \int\limits_{d(\eta)-\delta}^{d(\eta)} (d(\eta)-r)r^{2n-1}dr\\ &\leq \frac{a_n}{td(\eta)}\Big(\int\limits_{\Omega}(dd^cu)^m \wedge \omega^{n-m}\Big)^{1/m} \int\limits_{d(\eta)-\delta}^{d(\eta)} \delta d(\eta)^{2n-1}dr\\ &\leq a_n\Big(\int\limits_{\Omega}(dd^cu)^m \wedge \omega^{n-m}\Big)^{1/m}\frac{\delta^2 d(\eta)^{2n-2}}{t}.\end{aligned}$$

Thus we have arrive at the desired estimate. □

Concerning the geometry of the domain Ω in Theorem D, it is proved in [1] that if Ω is a bounded $m-$hyperconvex domain with C^2-smooth boundary then Ω admit a C^2-smooth negative exhaustion function which is $m-$subharmonic on Ω.
Next we proceed to the proof of Theorem D.

Proof of Theorem D. By multiplying ρ with a small positive constant we can assume $\rho > -1$ on Ω. Since the gradient of ρ is nowhere zero on $\partial\Omega$, using the implicit function theorem, we can find positive constants C_1, C_2 such that

$$C_1 d(z,\partial\Omega) \leq -\rho(z) \leq C_2 d(z,\partial\Omega)\ \forall z \in \Omega. \tag{3.2}$$

We consider two cases.

Case 1. $u \geq a\rho$ in Ω for some $a > 0$. For $\varepsilon > 0$, we let

$$\Omega_\varepsilon := \{z \in \Omega :\ \mathrm{d}(z,\partial\Omega) > \varepsilon\}.$$

We then define on Ω_ε the function

$$u_\varepsilon(z) := \frac{1}{c_n \varepsilon^{2n}} \int\limits_{\mathbb{B}(z,\varepsilon)} u(\xi) dV(\xi) = \frac{1}{c_n \varepsilon^{2n}} \int\limits_{\mathbb{B}(0,\varepsilon)} u(z+\xi) dV(\xi),$$

where dV denote the Lebesgue measure on $\mathbb{C}^n$ and c_n is the volume of the unit ball in $\mathbb{C}^n$. We have $u_\varepsilon \in SH_m^-(\Omega_\varepsilon)$ and $u_\varepsilon \downarrow u$ when $\varepsilon \downarrow 0$. Our key step is to estimate u_ε from below by a *fixed* multiple of ρ for ε small enough. For $\delta > 1$ and $0 < \varepsilon_0 < 1$, we consider the following annulus

$$\Omega^{\delta,\varepsilon_0} := \{z \in \Omega :\ \varepsilon_0 < \mathrm{d}(z,\partial\Omega) < 2\delta^2\varepsilon_0\}. \tag{3.3}$$

So for $z \in \Omega^{\delta,\varepsilon_0}$, let $\varepsilon = \mathrm{d}(z,\partial\Omega)$, we have

$$u_{\varepsilon_0}(z) = \frac{1}{c_n \varepsilon_0^{2n}} \Big(\int\limits_{B_1} u(\xi) dV(\xi) + \int\limits_{B_2} u(\xi) dV(\xi) \Big),$$

where

$$B_1 := \{\xi \in \mathbb{B}(z,\varepsilon_0) :\ u(\xi) < -A(\varepsilon+\varepsilon_0)\},\ B_2 := \mathbb{B}(z,\varepsilon_0) \setminus B_1.$$

Since $u \geq a\rho$ in Ω, using (3.2) we obtain

$$u_{\varepsilon_0}(z) \geq \frac{1}{c_n \varepsilon_0^{2n}} \Big(\int\limits_{B_1} -aC_2 \mathrm{d}(\xi,\partial\Omega) dV(\xi) + \int\limits_{B_2} -A(\varepsilon+\varepsilon_0) dV(\xi) \Big).$$

Observe that

$$B_1 \subset \{\xi \in \Omega :\ \mathrm{d}(\xi,\partial\Omega) < \varepsilon+\varepsilon_0,\ u(\xi) < -A(\varepsilon+\varepsilon_0)\}.$$

So by the assumption of the theorem we obtain

$$\mathrm{vol}_{2n}(B_1) \leq C(\varepsilon+\varepsilon_0)^\alpha.$$

Combining this with (3.3), we obtain for $z \in \Omega^{\delta,\varepsilon_0}$ the lower estimate for u_{ε_0}

$$\begin{aligned} u_{\varepsilon_0}(z) &\geq \frac{-aCC_2}{c_n \varepsilon_0^{2n}} (\varepsilon+\varepsilon_0)^{\alpha+1} - A(\varepsilon+\varepsilon_0) \\ &\geq \frac{-2aCC_2}{c_n \varepsilon_0^{2n}} (\varepsilon+\varepsilon_0)^{\alpha} \varepsilon - 2A\varepsilon \\ &\geq \frac{-2aCC_2}{c_n} (2\delta^2+1)^\alpha \varepsilon_0^{\alpha-2n} \varepsilon - 2A\varepsilon. \end{aligned}$$

Thus, by applying again (3.2) we get

$$u_{\varepsilon_0}(z) \geq \Big[\frac{2aCC_2}{c_n C_1} (2\delta^2+1)^\alpha \varepsilon_0^{\alpha-2n} + \frac{2A}{C_1} \Big] \rho(z).$$

Since $\alpha - 2n > 0$, the first term inside the bracket tends to 0 when ε_0 tends to 0. Hence, there exists $\varepsilon_0^* > 0$ depending only on a such that

$$u_{\varepsilon_0} \geq C_3\rho \text{ in } \Omega_{\varepsilon_0},\ \text{for all } \varepsilon_0 < \varepsilon_0^* \tag{3.4}$$

where $C_3 := \frac{2A}{C_1} + 1$. Set

$$\delta := 2\frac{C_2}{C_1} \text{ and } \lambda := \frac{C_3}{\frac{1}{\delta C_2} - \frac{1}{\delta^2 C_1}}.$$

For $\varepsilon_0 < \varepsilon_0^*$, we will estimate $u_{\varepsilon_0}(z) - \lambda\varepsilon_0$ from above and from below on $\partial\Omega_{\delta\varepsilon_0}$ and $\partial\Omega_{\delta^2\varepsilon_0}$ respectively. We first use (3.2) to obtain

$$u_{\varepsilon_0}(z) - \lambda\varepsilon_0 = u_{\varepsilon_0}(z) - \frac{\lambda}{\delta}\mathrm{d}(z,\partial\Omega) \leq \frac{\lambda}{\delta C_2}\rho(z) \text{ for } z \in \partial\Omega_{\delta\varepsilon_0}. \tag{3.5}$$

By (3.4) and (3.2), we have

$$u_{\varepsilon_0}(z) - \lambda\varepsilon_0 = u_{\varepsilon_0}(z) - \frac{\lambda}{\delta^2}\mathrm{d}(z,\partial\Omega) \geq \Big(C_3 + \frac{\lambda}{\delta^2 C_1}\Big)\rho(z) \text{ for } z \in \partial\Omega_{\delta^2\varepsilon_0}. \tag{3.6}$$

Combining (3.5), (3.6) and noting that

$$\frac{\lambda}{\delta C_2} = C_3 + \frac{\lambda}{\delta^2 C_1},$$

we derive for $\varepsilon_0 < \varepsilon_0^*$ the following estimates

$$\begin{cases} u_{\varepsilon_0}(z) - \lambda\varepsilon_0 \leq \beta\rho(z) \text{ for } z \in \partial\Omega_{\delta\varepsilon_0} \\ u_{\varepsilon_0}(z) - \lambda\varepsilon_0 \geq \beta\rho(z) \text{ for } z \in \partial\Omega_{\delta^2\varepsilon_0} \end{cases},$$

where $\beta = \frac{\lambda}{\delta C_2}$. Now, for $\varepsilon_0 < \varepsilon_0^*$, we consider

$$\widetilde{u}_{\varepsilon_0}(z) = \begin{cases} \beta\rho, & \text{in } \Omega\backslash\Omega_{\delta\varepsilon_0} \\ \max(\beta\rho, u_{\varepsilon_0} - \lambda\varepsilon_0), & \text{in } \Omega_{\delta\varepsilon_0}\backslash\overline{\Omega_{\delta^2\varepsilon_0}} \\ u_{\varepsilon_0} - \lambda\varepsilon_0, & \text{in } \overline{\Omega_{\delta^2\varepsilon_0}} \end{cases}.$$

We have $\widetilde{u}_{\varepsilon_0} \in \mathscr{E}_m^0(\Omega)$, $\widetilde{u}_{\varepsilon_0} \downarrow u$ when $\varepsilon_0 \downarrow 0$ and by the comparison principle Lemma 2.2, we have

$$\int_\Omega (dd^c\widetilde{u}_{\varepsilon_0})^m \wedge \omega^{n-m} \leq \beta^m \int_\Omega (dd^c\rho)^m \wedge \omega^{n-m},$$

for ε_0 small enough. Therefore $u \in \mathscr{F}_m(\Omega)$ as we want.
Case 2. Now we treat the general case. For $N \geq 1$, we set $u_N := \max\{u, N\rho\}$. Then $u_N \in \mathscr{F}_m(\Omega)$ and $u_N \downarrow u$. By the result obtained in Case 1, we have

$$\sup_{N\geq 1} \int_\Omega (dd^c u_N)^m \wedge \omega^{n-m} \leq \beta^m \int_\Omega (dd^c\rho)^m \wedge \omega^{n-m}.$$

Therefore $u \in \mathscr{F}_m(\Omega)$. The proof is thereby completed. □

Proof of Theorem E. Denote by $U(n)$ the set of unitary transformations from $\mathbb{C}^n$ to $\mathbb{C}^n$. For $0 < a < 1$, $\varepsilon > 0$ and $z \in \mathbb{B}^n_{1-\varepsilon} := \{w \in \mathbb{C}^n : \|w\| < 1 - \varepsilon\}$, we define

$$u_{a,\varepsilon}(z) := (\sup\{u((1+r)\phi(z)) : \phi \in S_a, 0 \leq r \leq \varepsilon\})^*,$$

where $S_a := \{\phi \in U(n) : \|\phi - Id\| < a\}$. Since $m-$subharmonicity is preserved under unitary transformations, we infer that $u_{a,\varepsilon}$ is $m-$subharmonic on $\mathbb{B}^n_{1-\varepsilon}$. Now the rest of our proof is almost identical to Theorem 5 in [10]. However, for the convenience of the reader we repeat some details. By upper-semicontinuity of u we obtain

$$\lim_{\max(a,\varepsilon)\to 0^+} u_{a,\varepsilon}(z) = u(z),\ \forall z \in \Omega. \tag{3.7}$$

We also note that if $z \neq 0$ then

$$u_{a,\varepsilon}(z) := (\sup\{u(\xi) : \ \xi \in B_{a,\varepsilon,z}\})^*, \tag{3.8}$$

where

$$B_{a,\varepsilon,z} := \Big\{\xi \in \mathbb{C}^n : \ \|\frac{z}{\|z\|} - \frac{\xi}{\|\xi\|}\| < a, \|z\| \leq \|\xi\| \leq (1+\varepsilon)\|z\|\Big\}.$$

Next we observe that there exist positive constants C_1, C_2 which do not depend on $a \in (0, 1/2), \varepsilon > 0$ such that

$$C_1 a^{2n-1}\varepsilon < \mathrm{vol}_{2n}(B_{a,\varepsilon,z}) < C_2 a^{2n-1}\varepsilon. \tag{3.9}$$

On the other hand, by the assumption (1.1) we deduce that for $0 < a < 1/2$, there exists $\varepsilon_a \in (0, a)$ such that

$$\mathrm{vol}_{2n}\{\xi \in \mathbb{B}^{2n} : \ \|\xi\| > 1 - 3\varepsilon, u(\xi) < -3A\varepsilon\} < C_1 a^{2n-1}\varepsilon, \ \forall \varepsilon \in (0, \frac{\varepsilon_a}{3}).$$

Hence, by (3.9), we have, for every $3\varepsilon \geq 1 - \|z\| \geq \varepsilon$,

$$B_{a,\varepsilon,z} \not\subseteq \{\xi \in \mathbb{B}^n : \ \|\xi\| > 1 - 3\varepsilon, u(\xi) < -3A\varepsilon\}.$$

Combining this fact with (3.8) we conclude that for $a \in (0, 1/2)$, there exists $\varepsilon_a > 0$ such that, for every $\varepsilon_a > 3\varepsilon \geq 1 - \|z\| \geq \varepsilon > 0$, we have the following crucial estimate

$$u_{a,\varepsilon}(z) \geq -3A\varepsilon. \tag{3.10}$$

Now for $a \in (0, 1/2)$ and $0 < \varepsilon < \varepsilon_a/3$, consider the following function

$$\tilde{u}_{a,\varepsilon}(z) := \begin{cases} 3A(-1+|z|^2) & 1-\varepsilon \leq \|z\| < 1, \\ \max\{3A(-1+|z|^2), u_{a,\varepsilon}(z) - 6A\varepsilon\} & 1-3\varepsilon \leq \|z\| \leq 1-\varepsilon, \\ u_{a,\varepsilon}(z) - 6A\varepsilon & \|z\| \leq 1-3\varepsilon. \end{cases}$$

Then $\lim\limits_{z\to\partial\mathbb{B}^n} \tilde{u}_{a,\varepsilon}(z) = 0$, and by (3.10) $\tilde{u}_{a,\varepsilon} \in SH_m^-(\mathbb{B}^n)$. Furthermore, from Lemma 2.2, we infer $\tilde{u}_{a,\varepsilon} \in \mathscr{E}_m^0(\mathbb{B}^n)$. Finally, for $j \geq 1$, we consider $u_j := \widetilde{u}_{2^{-j}, \frac{\varepsilon_{2^{-j}}}{3}}$. By (3.7), we have $u_j \to u$ pointwise on Ω. Moreover $\sup\limits_j \int\limits_{\mathbb{B}^n} (dd^c \tilde{u}_j)^m \wedge \omega^{n-m} < \infty$, then by Lemma 2.5, we conclude $u \in \mathscr{F}_m(\Omega)$ as desired. □

REFERENCES

[1] P. Åhag, R. Czyż and L. Hed, *The geometry of m−hyperconvex domains*, J. Geom. Anal., **28** (2018), no. 4, 3196-3222.

[2] P. Åhag, R. Czyż and L. Hed, *The classification of holomorphic (m,n)−subharmonic morphisms*, Complex Variables and Elliptic Equations, **65** (2020), 152-177.

[3] E. Bedford and B.A. Taylor, *A new capacity for plurisubharmonic functions,* Acta Mathematica, **149** (1982), 1-40.

[4] Z. Błocki, *Weak solutions to the complex Hessian equation,* Annales de l'Institute Fourier (Grenoble), **55** (2005) no. 5, 1735-1756.

[5] U. Cegrell, *Pluricomplex energy,* Acta Mathematica, **180** (1998), no. 2, 187-217.

[6] U. Cegrell, *The general definition of the complex Monge-Ampère operator,* (English, French summary) Annales de l'Institute Fourier (Grenoble), **54** (2004), no. 1, 159-179.

[7] U. Cegrell, S. Kołodziej and A. Zeriahi, *Subextension of plurisubharmonic functions with weak singularities,* Mathematische Zeitschrift, **250** (2005), 7-22.

[8] U. Cegrell, A. Zeriahi, *Subextension of plurisubharmonic functions with bounded Monge–Ampère operator mass,* Comptes Rendus Mathematique, **336** (2003), 305-308.

[9] S. Dinew and S. Kołodziej, *Non standard properties of m−subharmonic functions*, Dolomites Research Notes on Approximation, **11** (2018), 35-50.

[10] H.S. Do and T.D. Do, *Some remarks on Cegrell's class $\mathscr{F}$*, Annales Polonici Mathematici **125** (2020), no. 1, 13-24.
[11] M.H. Le and V.D. Trieu, *Subextension of $m-$Subharmonic Functions,* Vietnam Journal of Mathematics, **48** (2020), 47-57.
[12] S.Y. Li, *On the Dirichlet problems for symmetric function equations of the eigenvalues of the complex Hessian,* Asian Journal of Mathematics, **8** (2004), Issue 1, 87-106.
[13] H.C. Lu, *A variational approach to complex Hessian equations in $\mathbb{C}^n$,* Journal of Mathematical Analysis and Applications, **431** (2015), Issue 1, 228-259.
[14] Q.D. Nguyen, H.B. Pham and X.H. Nguyen, *A uniqueness properties of m-subharmonic functions in Cegrell classes,* Journal of Mathematical Analysis and Applications, **420** (2014), Issue 1, 669-683.
[15] V.T. Nguyen, *A characterization of the Cegrell classes and generalized m-capacities,* Annales Polonici Mathematici, **121** (2018), Issue 1, 33-43.
[16] V.T. Nguyen, *Maximal $m-$subharmonic functions and the Cegrell class $\mathscr{N}_m$*, Indagationes Mathematicae, **30** (2019), Issue 4, 717-739.
[17] A.S. Sadullaev and B.I. Abdullaev, *Capacities and Hessians in the class of m-subharmonic functions,* Doklady Mathematics, **87** (2013), 88-90.
[18] N. Sibony, *Une classe de domaines pseudoconvexes,* Duke Mathematical Journal, **55** (1987), 299-319.
[19] V.H. Vu and V.P. Nguyen, *Hessian measures on m-polar sets and applications to the complex Hessian equations,* Complex Variables and Elliptic Equations, **62** (2017). no.8, 1135-1164.

[1] DEPARTMENT OF MATHEMATICS, HANOI NATIONAL UNIVERSITY OF EDUCATION, 136 XUAN THUY, CAU GIAY, HANOI, VIETNAM

[2] THANG LONG INSTITUTE OF MATHEMATICS AND APPLIED SCIENCES, NGHIEM XUAN YEM, HOANG MAI, HANOI, VIETNAM
Email address: dieu_vn@yahoo.com, ngquang.dieu@hnue.edu.vn

[3] INSTITUTE OF MATHEMATICS, VIETNAM ACADEMY OF SCIENCE AND TECHNOLOGY, 18 HOANG QUOC VIET, CAU GIAY, 100000 HANOI, VIETNAM
Email address: dtduong@math.ac.vn