Attractors of Caputo fractional differential equations with triangular vector fields

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Abstract

It is shown that the attractor of an autonomous Caputo fractional differential equation of order $\alpha \in (0,1)$ in $\mathbb{R}^d$ whose vector field has a certain triangular structure and satisfies a smooth condition and dissipativity condition is essentially the same as that of the ordinary differential equation with the same vector field. As an application, we establish several one-parameter bifurcations for scalar fractional differential equations including the saddle-node and the pichfork bifurcations. The proof uses a result of Cong & Tuan [2] which shows that no two solutions of such a Caputo FDE can intersect in finite time.

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1 Introduction

The asymptotic behaviour of Caputo fractional differential equations (Caputo FDE) in $\mathbb{R}^d$ has attracted much attention in the literature in recent years. It has often been asked if such equations on $\mathbb{R}^d$, generate an autonomous dynamical system, since that would allow the theory of attractors to applied to them.

Consider an autonomous Caputo FDE of order $\alpha \in (0,1)$ in $\mathbb{R}^d$ of the following form

$$^{C}D^{\alpha}_{0+}x(t) = g(x(t)) \tag{1.1}$$

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where $g : \mathbb{R}^d \to \mathbb{R}^d$ is Lipschitz continuous and satisfies a growth bound. The Caputo FDE (1.1) with the initial condition $x(0) = x_0$ is the integral equation

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(x(s))ds, \tag{1.2}$$

where $\Gamma(\alpha) := \int_0^\infty t^{\alpha-1} e^{-t} dt$ is the Gamma function.

It is easy to show that the ordinary differential equation (ODE) with the same vector field, i.e.,

$$\frac{d}{dt} x(t) = g(x(t)), \tag{1.3}$$

has an attractor when the vector field satisfies a dissipativity condition such as for $a, b > 0$

$$\langle x, g(x) \rangle \leq a - b\|x\|^2. \tag{1.4}$$

Specifically, by the chain rule along a solution of (1.3),

$$\frac{d}{dt}\|x(t)\|^2 = 2\,\langle x(t), g(x(t)) \rangle \leq 2a - 2b\|x(t)\|^2,$$

which integrates to give

$$\|x(t)\|^2 \leq \|x_0\|^2 e^{-2bt} + \frac{a}{b}\left(1 - e^{-2bt}\right).$$

Hence the set

$$\mathcal{B} := \left\{ x \in \mathbb{R}^d \,:\, \|x\|^2 \leq 1 + \frac{a}{b} \right\}$$

is an absorbing set for the autonomous semi-dynamical system generated by the solution mapping of the ODE (1.3), which is positive invariant. In particular, this means that this system has a global attractor,

$$\mathcal{A} = \bigcap_{t \geq 0} x(t, \mathcal{B}) = \Omega_{\mathcal{B}},$$

where $x(t, \mathcal{B}) = \cup_{\eta \in B} x(t, \eta)$ and $\Omega_{\mathcal{B}}$ is the omega limit set defined by

$$\Omega_{\mathcal{B}} := \{ y \in \mathcal{B} \,:\, \exists\, x_{0,n} \in \mathcal{B}, t_n \to \infty \text{ such that } x(t_n, x_{0,n}) \to y \}.$$

By a recent result of Aguila-Camacho *et al.* [1, Lemma 1] it is known that a solution of the Caputo FDE (1.1) satisfies

$${}^C D^\alpha_{0+} \|x(t)\|^2 \leq 2 \left\langle x(t), {}^C D^\alpha_{0+} x(t) \right\rangle.$$

Hence, if the vector field g of (1.1) satisfies the dissipativity condition (1.4), then along the solutions of (1.1)

$$^{C}D^{\alpha}_{0+}\|x(t)\|^2 \leq 2\left\langle x(t), g(x(t))\right\rangle \leq 2a - 2b\|x(t)\|^2$$

as in the ODE case. Then, by Wang & Xiao [11, Theorem 1], the corresponding set $\mathcal{B}$ defined in terms of the solutions of the Caputo FDE is an absorbing set for the solutions of the Caputo FDE (1.1). Since this set is compact in $\mathbb{R}^d$, the corresponding omega limit set $\Omega_{\mathcal{B}}$ exists and is a nonempty compact subset of $\mathcal{B}$, which attracts all of the future dynamics of the Caputo FDE. It is clear that $\Omega_{\mathcal{B}}$ contains all of the steady state solutions of (1.1).

In general, $\Omega_{\mathcal{B}}$ cannot be called the attractor of the autonomous Caputo FDE (1.1). Recently Cong & Tuan [2] confirmed the conjecture in [3, 5] by showing that the solution mapping of a general autonomous Caputo FDE (1.1) on $\mathbb{R}^d$ does not generate a semi-group on $\mathbb{R}^d$ and, hence, there is no autonomous semi-dynamical system on $\mathbb{R}^d$ corresponding to (1.1). Consequently, since solutions cannot be in general concatenated, there may also be omega limit points of solutions starting outside $\mathcal{B}$ that are not in $\Omega_{\mathcal{B}}$. Also, strictly speaking, mathematically, a general Caputo FDE (1.1) has no attractor on $\mathbb{R}^d$ since this concept is defined in terms of an autonomous semi-dynamical system.

However, Cong & Tuan [2] showed that a Caputo FDE (1.1) with a triangular vector field does generate a semi-dynamical system on $\mathbb{R}^d$. Recall that a vector field $g : \mathbb{R}^d \to \mathbb{R}^d$ is called triangular if it has components with the structure $g_1(x_1), g_2(x_1, x_2), \cdots, g_d(x_1, x_2, \cdots, x_d)$ which covers scalar vector fields as a special case. Our aim in this paper is to use the result in [2] to investigate the attractor of Caputo fractional differential equations with a triangular vector field. The result for an attractor of scalar Caputo fractional differential equations is presented in Section 2. A generalization to Caputo fractional differential equations with a certain triangular vector field is presented in Section 3. Several examples of bifurcations of scalar Caputo FDEs are presented in Section 4. Section 5 is devoted to discussing a potential approach to attractors of general Caputo FDEs by using the existing theory of attractors for semi-dynamical systems on function spaces.

2 Attractor of scalar Caputo fractional differential equations

In this subsection, we consider following scalar fractional differential equation

$$ {}^{C}D^{\alpha}_{0+}x(t) = g(x(t)), \qquad t \geq 0, \tag{2.1} $$

where $g : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function and satisfies that

(H1) (Dissipative condition): There exist $a, b > 0$ such that

$$ g(x)x \leq a - bx^2 \quad \text{for all } x \in \mathbb{R}. $$

(H2) (Non-degenerate condition): $g'(x) \neq 0$ for all $x \in \mathcal{N}(g) := \{x \in \mathbb{R} : g(x) = 0\}$.

Remark 2.1. By (H1), we have $\mathcal{N}(g) \subseteq [-\sqrt{\frac{a}{b}}, \sqrt{\frac{a}{b}}]$. By (H2), $\mathcal{N}(g)$ has no accumulation point and therefore $\mathcal{N}(g)$ has a finite elements and the number of elements is odd. Furthermore, let $\mathcal{N}(g) = \{x_1, \ldots, x_{2k+1}\}$. Then, for all $i = 0, 1, \ldots, k$

$$ g(x) > 0 \quad \text{for all } x \in (x_{2i}, x_{2i+1}), \tag{2.2} $$

and

$$ g(x) < 0 \quad \text{for all } x \in (x_{2i+1}, x_{2i+2}), \tag{2.3} $$

where we use the conventions that $x_0 := -\infty$ and $x_{2k+2} = \infty$.

The main result of this section is the following theorem about attractors for a scalar Caputo fractional differential equation (2.1).

Theorem 2.2 (Attractor for scalar Caputo fractional differential equations)**.** *Consider system* (2.1)*. Suppose that the assumptions (H1) and (H2) hold. Then, the following statements hold:*

(i) The global attractor attracting all solutions starting from a bounded set is

$$ \mathcal{A} = [\min \mathcal{N}(g), \max \mathcal{N}(g)]. $$

(ii) Each solution of (2.1) *converges to an element of* $\mathcal{N}(g)$ *and the rate of convergence is* $t^{-\alpha}$*.*

(iii) Each pair of successive values of $\mathcal{N}(g)$ *is a heteroclinic solution of* (2.1)

To prove the above theorem, we need several preparatory results. The following proposition indicates that the set $A = [x_1, x_{2k+1}]$ attracts all solutions of (2.1). Note that this set A includes all steady states $x_1, \ldots, x_{2k+1}$.

For $\alpha, \beta \in (0,1)$ the Mittag-Leffler function $E_{\alpha,\beta} : \mathbb{R} \to \mathbb{R}$ is defined as

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \qquad E_\alpha(z) := E_{\alpha,1}(z).$$

Let $d(A, B)$ denote the distance between two subsets A and B of $\mathbb{R}$.

Proposition 2.3. Let B be a bounded set of $\mathbb{R}$. For any $t \geq 0$, let $x(t, B) := \{x(t, \eta) : \eta \in B\}$. Then, there exists $\lambda > 0$ such that

$$d(x(t,B), [x_1, x_{2k+1}]) \leq E_\alpha(-\lambda t^\alpha) d(B, [x_1, x_{2k+1}]) \quad \text{for all } t \geq 0.$$

Consequently,

$$\lim_{t\to\infty} d(x(t,B), [x_1, x_{2k+1}]) = 0.$$

Proof. Due to the non-intersection of two trajectories of (2.1), we have

$$x(t,\eta) \in [x(t, \inf B), x(t, \sup B)] \qquad \text{for all } \eta \in B.$$

Then, to conclude the proof it is sufficient to show that for all $\eta \in \mathbb{R}$ there exists $\gamma > 0$ such that

$$d(x(t,\eta), [x_1, x_{2k+1}]) \leq E_\alpha(-\gamma t^\alpha) d(\eta, [x_1, x_{2k+1}]). \tag{2.4}$$

By using the non-intersection of two trajectories of (2.1) and the fact that x_1, x_{2k+1} are steady state solutions, for all $\eta \in [x_1, x_{2k+1}]$ we have

$$d(x(t,\eta), [x_1, x_{2k+1}]) = d(\eta, [x_1, x_{2k+1}]) = 0,$$

which implies that the preceding conclusion obviously holds. Then, it is enough to deal with the case that $\eta < x_1$ and use analogous arguments for the case $\eta > x_{2k+1}$. Choose and fix $\eta < x_1$ and to conclude the proof we will show that

$$|x(t,\eta) - x_1| \leq E_\alpha(-\gamma t^\alpha)|x_1 - \eta| \quad \text{for } t \geq 0. \tag{2.5}$$

for some $\gamma > 0$. The proof of the preceding fact is divided into two steps:

Step 1: Consider a new fractional differential equation

$$^{C}D^{\alpha}_{0+}y(t) = f(y(t)), \tag{2.6}$$

where $f : \mathbb{R} \to \mathbb{R}$ is defined as

$$f(x) := g(x + x_1) \qquad \text{for all } x \in \mathbb{R}. \tag{2.7}$$

Then, we show that $x(t, \eta) = x_1 + y(t, \eta - x_1)$, where $y(\cdot, \zeta)$ denotes the solution of (2.6) satisfying $y(0) = \zeta$. To see that, the integral form of (2.1) yields that

$$x(t, \eta) = \eta + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} g(x(s, \eta))\, ds.$$

Thus,

$$\begin{aligned} x(t, \eta) - x_1 &= \eta - x_1 + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} g(x(s, \eta) - x_1 + x_1)\, ds \\ &= \eta - x_1 + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(x(s, \eta) - x_1)\, ds, \end{aligned}$$

which implies that $y(t, x_1 - \eta) = x(t, \eta) - x_1$. So, to prove (2.5) it is sufficient to show that

$$|y(t, \zeta)| \le E_\alpha(-\gamma t^\alpha)|\zeta| \qquad \text{for } \zeta := x - x_1 < 0. \tag{2.8}$$

Step 2: For this purpose, we first show that there exists $\gamma > 0$ such that

$$f(x) \ge \gamma |x| \qquad \text{for all } x \le 0. \tag{2.9}$$

Indeed, by (H1) we have $g(x) > 0$ for $x < x_1$ and therefore by (H2), $g'(x_1) < 0$. Equivalently, by (2.7) we have $f(x) > 0$ for all $x < 0$, $f(0) = 0$ and $f'(0) < 0$. Hence, by mean value theorem, there exists $\varepsilon > 0$ such that

$$|f(x)| \ge \frac{|f'(0)|}{2}|x| \qquad \text{for all } x \in [-\epsilon, \epsilon].$$

Hence, if $x \ge -\epsilon$ then (2.9) holds for $\gamma := \frac{|f'(0)|}{2}$. In the other case, i.e. $x < -\epsilon$, let

$$\gamma := \min\left\{ \frac{|f'(0)|}{2}, \min_{w \in [\zeta, -\epsilon]} \frac{f(w)}{|w|} \right\}.$$

Then, by strictly positivity of f on $[\zeta, -\epsilon]$ we have $\gamma > 0$ and obviously (2.9) also holds for this choice of γ. So, in both cases there exists $\gamma > 0$ satisfying (2.9). We now rewrite (2.6) in the following form

$$^{C}D^{\alpha}_{0+}y(t) = -\gamma y(t) + h(y(t)),$$

where $h : \mathbb{R} \to \mathbb{R}$ is defined by

$$h(y) := f(y) + \gamma y.$$

On the one hand, by (2.9) we have

$$h(y) \geq 0 \qquad \text{for all } y \leq 0. \tag{2.10}$$

Thanks to the variation of constants formula (see e.g. [2, Lemma 3.1]) we arrive at the following representation of the solution $y(t, \zeta)$ as

$$y(t,\zeta) = E_\alpha(-\gamma t^\alpha)\zeta + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\gamma(t-s)^\alpha) h(y(s,\zeta))\, ds. \tag{2.11}$$

By the non-intersection of two solutions of (2.6) we have $y(t,\zeta) \in [\zeta, 0]$ for all $t \geq 0$. This together with (2.10) gives that

$$h(y(s,\zeta)) \geq 0 \qquad \text{for all } s \geq 0.$$

Consequently, by (2.11) and positivity of the function $E_{\alpha,\alpha}$ we arrive at

$$y(t,\zeta) \in [E_\alpha(-\gamma t^\alpha)\zeta, 0] \qquad \text{for all } t \geq 0,$$

which shows (2.8). Furthermore, since $\lim_{t\to\infty} E_\alpha(-\gamma t^\alpha) = 0$ it follows that $\lim_{t\to\infty} y(t,\zeta) = 0$. The proof is complete. □

In the following result, we establish the asymptotic behavior of solutions starting inside the attractor. The idea of the proof of this proposition is quite similar to Proposition 2.3 and we only sketch the main points of the proof.

Proposition 2.4. The following statements hold:

(i) For $i = 0, \ldots, k$ and $\eta \in (x_{2i}, x_{2i+1})$ there exists $\gamma > 0$ such that $|x(t,\eta) - x_{2i+1}| \leq E_\alpha(-\gamma t^\alpha)|\eta - x_{2i+1}|$. Consequently, $\lim_{t\to\infty} x(t,\eta) = x_{2i+1}$.

(ii) For $i = 0, \ldots, k$ and $\eta \in (x_{2i+1}, x_{2i+2})$ there exists $\gamma > 0$ such that $|x(t,\eta) - x_{2i+1}| \leq E_\alpha(-\gamma t^\alpha)|\eta - x_{2i+1}|$. Consequently, $\lim_{t\to\infty} x(t,\eta) = x_{2i+1}$.

Proof. We only give a proof of (i) and by using analogous arguments we also obtain (ii). Let $i \in \{0, 1, \ldots, k-1\}$ be arbitrary but fixed. From (2.6) and (2.3), we have

$$g(x) > 0 \quad \text{for all } x \in (x_{2i}, x_{2i+1}) \quad \text{and } g'(x_{2i+1}) < 0. \tag{2.12}$$

Now, choose and fix an arbitrary $\eta \in (x_{2i}, x_{2i+1})$. Consider a new fractional differential equation

$$^{C}D^{\alpha}_{0+}y(t) = f(y(t)), \tag{2.13}$$

where $f : \mathbb{R} \to \mathbb{R}$ is defined as

$$f(y) := g(y + x_{2i+1}) \qquad \text{for all } x \in \mathbb{R}.$$

Then, for $y(t, \zeta)$ denoting the solution of (2.13) we have $x(t, \eta) = y(t, \eta - x_{2i+1}) + x_{2i+1}$ for all $t \geq 0$. Then it is sufficient to show that for all $\zeta \in (x_{2i} - x_{2i+1}, 0)$

$$|y(t, \zeta)| \leq E_\alpha(-\gamma t^\alpha)|\zeta| \qquad \text{for some } \gamma > 0. \tag{2.14}$$

Now, the property (2.12) is translated into the function f as

$$f(y) > 0 \quad \text{for all } y \in (x_{2i} - x_{2i+1}, 0) \quad \text{and } f'(0) < 0,$$

which gives that there exists $\gamma > 0$ (depending on $\zeta \in (x_{2i} - x_{2i+1}, 0)$) such that $f(y) \geq \gamma|y|$ for all $y \in [\zeta, 0]$. Thus, by variation of constants formula we have

$$y(t, \zeta) \geq E_\alpha(-\gamma t^\alpha)\zeta \qquad \text{for all } t \geq 0,$$

proving (2.14). The proof is complete. □

Next, we discuss the existence of heteroclinic orbits joining the steady state solutions. Here, we need to discuss how to define the value of solution in the negative time axis. Roughly speaking, we can extend the solution $x(\cdot, \eta)$ in the negative time axis as follows: for any $t \leq 0$ then $x(t, \eta)$ is the unique value $\zeta \in \mathbb{R}$ satisfying that $x(-t, \zeta) = \eta$, it means that

$$x(-t, x(t, \eta)) = \eta.$$

The well-defined property of this way of extension is confirmed by the result in [2, Theorem 4.8].

Proposition 2.5 (Heteroclinic trajectory joining the steady states)**.** The following statements hold:

(i) For $i = 0, \dots, k-1$ and $\eta \in (x_{2i}, x_{2i+1})$ the solution $x(t, \eta)$ is a heteroclinic trajectory joining the steady states x_{2i} and x_{2i+1}. More precisely, $\lim_{t\to-\infty} x(t, \eta) = x_{2i}$ and $\lim_{t\to\infty} x(t, \eta) = x_{2i+1}$.

(ii) For $i = 0, \dots, k$ and $\eta \in (x_{2i+1}, x_{2i+2})$ the solution $x(t, \eta)$ is a heteroclinic trajectory joining the steady states x_{2i+1} and x_{2i+2}. More precisely, $\lim_{t\to-\infty} x(t, \eta) = x_{2i+2}$ and $\lim_{t\to\infty} x(t, \eta) = x_{2i+1}$.

Proof. We only give a proof for the part (i) and refer an analogous argument for the proof of part (ii). In fact, by Proposition 2.4 it is only required to prove that

$$\lim_{t\to-\infty} x(t,\eta) = x_{2i} \qquad \text{for all } \eta \in (x_{2i}, x_{2i+1}). \tag{2.15}$$

Analog to the proof of Proposition 2.3 and Proposition 2.4(i), we can introduce the new system to have the property that $x_{2i+1} = 0$. So, in what follows we can assume additionally that $x_{2i+1} = 0$. Choose and fix $\eta \in (x_{2i}, 0)$. We divide the remaining proof into several steps:

Step 1: We show that there exists $\gamma > 0$ such that

$$g(\zeta) \geq \gamma(\zeta - x_{2i})|\zeta| \qquad \text{for all } \zeta \in [x_{2i}, 0]. \tag{2.16}$$

To prove this, since $g'(x_{2i}) > 0 > g'(0)$ it follows that there exists $\varepsilon \in (0, -\frac{x_{2i}}{3})$ such that

$$g(\zeta) \geq \frac{g'(x_{2i})}{2}(\zeta - x_{2i}) \qquad \text{for all } \zeta \in (x_{2i}, x_{2i} + \epsilon).$$

and

$$g(\zeta) \geq \frac{|g'(0)|}{2}|\zeta| \qquad \text{for all } \zeta \in (-\epsilon, 0).$$

Then, (2.16) holds for

$$\gamma := \min\left\{\frac{g'(x_{2i})}{2|x_{2i}|}, \frac{|g'(0)|}{2|\epsilon + x_{2i}|}, \min_{\zeta\in[x_{2i}+\epsilon,-\epsilon]} \frac{g(\zeta)}{(\zeta - x_{2i})|\zeta|}\right\}.$$

The positivity of γ follows from the fact that $g(\zeta) > 0$ for all $\zeta \in [x_{2i}+\epsilon, -\epsilon]$.

Step 2: For any $\zeta \in (x_{2i}, 0)$, we show that

$$x(t,\zeta) \geq E_\alpha(-\gamma(\zeta - x_{2i})t^\alpha)\zeta \qquad \text{for all } t \geq 0. \tag{2.17}$$

To show this inequality, choose and fix $\zeta \in (x_{2i}, 0)$ and let $\widehat{\gamma} := \gamma(\zeta - x_{2i})$. Then, we can write the Caputo fractional differential equation (2.1) as

$${}^C D^\alpha_{0+} x(t) = -\widehat{\gamma} x(t) + (g(x(t)) + \widehat{\gamma} x(t)).$$

By the variation of constants formula, we have

$$\begin{aligned} x(t,\zeta) &= E_\alpha(-\widehat{\gamma} t^\alpha)\zeta \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\widehat{\gamma}(t-s)^\alpha)(g(x(s,\zeta)) + \widehat{\gamma} x(s,\zeta))\, ds. \end{aligned}$$

Since $x(s,\zeta) \geq \zeta$ for all $s \geq 0$ it follows with (2.17) that

$$g(x(s,\zeta)) + \widehat{\gamma}x(s,\zeta) \geq \gamma(\zeta - x_{2i})|x(s,\zeta)| + \widehat{\gamma}x(s,\zeta) \geq 0.$$

Thus, $x(t,\zeta) \geq E_\alpha(-\widehat{\gamma}t^\alpha)\zeta$ and (2.17) is proved.

Step 3: Let $\eta \in (x_{2i}, 0)$ be arbitrary. Then,

$$x(-t, x(t,\eta)) = \eta \qquad \text{for all } t < 0,$$

which together with (2.17) implies that

$$\begin{aligned} \eta &\geq E_\alpha(-\gamma(x(t,\eta) - x_{2i})(-t)^\alpha)x(t,\eta) \quad \text{for all } t < 0 \\ &\geq E_\alpha(-\gamma(x(t,\eta) - x_{2i})(-t)^\alpha)x_{2i} \quad \text{for all } t < 0 \end{aligned}$$

Since $E_\alpha(\cdot)$ is a montononically increasing function and

$$\lim_{t\to-\infty} E_\alpha(-\rho(-t)^\alpha) = 0 \quad \text{for all } \rho > 0$$

it follows that $\lim_{t\to-\infty} x(t,\eta) = x_{2i}$. The proof is complete. □

We are now in a position to prove the main result of this section.

Proof of Theorem 2.2. The proof of (i) and (iii) are given in Proposition 2.3 and Proposition 2.5, respectively. The first statement in (ii) that each solution of (2.1) converges to an element of $\mathcal{N}(g)$ is given in Proposition 2.4. It remains to show the rate of convergence. In fact, by using Proposition 2.4 for any η there exists $\gamma > 0$ such that

$$d(x(t,\eta), \mathcal{N}(g)) \leq E_\alpha(-\gamma t^\alpha)d(\eta, \mathcal{N}(g)) \qquad \text{for all } t \geq 0. \tag{2.18}$$

On the other hand, by [2, Theorem 4.1] there exists $L > 0$ such that

$$d(x(t,\eta), \mathcal{N}(g)) \geq E_\alpha(-Lt^\alpha)d(\eta, \mathcal{N}(g)) \qquad \text{for all } t \geq 0. \tag{2.19}$$

Furthermore, for any $\lambda > 0$ we have $\lim_{t\to\infty} t^\alpha E_\alpha(-\lambda t^\alpha)$ is finite. Then, by using (2.18) and (2.19) the rate of convergence of any solution of (2.1) to the steady states of (2.1) is $t^{-\alpha}$. □

Remark 2.6 (Comparison to the proof in the scalar ordinary differential equations)**.** Consider a scalar ordinary differential equation

$$\dot{x}(t) = g(x(t)) \tag{2.20}$$

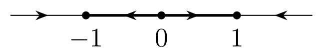

Figure 1: Attractor $\mathcal{A} = [-1, 1]$ for $g(x) = x - x^3$.

Let a, b be two successive zeros of g, i.e. $g(a) = g(b) = 0$ and $g(x) \neq 0$ for $x \in (a, b)$. Then, by continuity of g either $g(x) > 0$ for all $x \in (a, b)$ or either $g(x) < 0$ for all $x \in (a, b)$. Then, any solution starting from a value in (a, b) will be either strictly monotonically increasing or strictly monotonically decreasing. Consequently, any solution of (2.20) will converges to one of two steady states a, b.
The above monotonicity argument of ODEs cannot extend to FDEs with the same vector field. The main reason is the appearance of the singular kernel in the integral form

$$x(t, \eta) = \eta + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} g(x(s, \eta))\, ds.$$

Example 2.7. Two specific scalar Caputo FDE , i.e., with a vector field $g : \mathbb{R}^1 \to \mathbb{R}^1$, namely will be investigated.

$$g(x) = -x, \quad g(x) = x - x^3.$$

These satisfy a dissipativity condition and have steady state solutions 0 and 0, ± 1, respectively. The ODEs

$$\frac{d}{dt} x(t) = g(x(t))$$

with these vector fields have global attractors $\mathcal{A} = \{0\}$ and $\mathcal{A} = [-1, 1]$, respectively. Then, the corresponding Caputo FDEs

$${}^{C}D^{\alpha}_{0+} x(t) = g(x(t))$$

have the same steady state solutions and attractors, see Figure 1. A major difference is that attraction or repulsion of the steady state solutions is not at an exponential rate in the Caputo case. Also, in the second example, the heteroclinic trajectories joining the steady state solutions have the same geometric image in $\mathbb{R}^1$, but different functional representations in the ODE and Caputo systems.

3 Attractor of Caputo fractional differential equations with triangular vector fields

In this section, we first generalize the result in previous section to a special class of Caputo fractional differential equations with triangular vector fields of the following form

$$^{C}D^{\alpha}_{0+}x(t) = g(x(t)) = (g_1(x(t)), \dots, g_d(x(t)))^{\mathrm{T}}, \tag{3.1}$$

where for $i = 1, \dots, d$ we assume that the function $g_i : \mathbb{R}^d \to \mathbb{R}$ is of the following form

$$g_i(x) = h_i(x_1, \dots, x_{i-1}) f_i(x_i) \qquad \text{for } i = 1, \dots, d.$$

The function g is assumed to be continuously differentiable and to satisfy the following hypothesises

(H1) (Dissipative condition): There exist $a, b > 0$ such that

$$\langle x, g(x) \rangle \le a - b\|x\|^2 \quad \text{ for all } x \in \mathbb{R}^d.$$

(H2) (Non-degenerate condition): $g'_{ii}(u) \neq 0$ for all $u \in \mathcal{N}(g_{ii}) := \{u \in \mathbb{R} : g_{ii}(u) = 0\}$.

Since the structure of vector field in (3.1) is of product form it follows with the assume (H1) that for all $i = 1, \dots, d$ the function $h_i : \mathbb{R}^{i-1} \to \mathbb{R}$ does not vanishing. Thus, the sign of g_i depends only the scalar function f_i. So, applying Theorem 2.2 to every components leads to the following result.

Theorem 3.1 (Attractor for Caputo fractional differential equations of triangular vector fields)**.** *Consider system* (3.1)*. Suppose that the assumptions (H1) and (H2) hold. Then, the following statements hold:*

(i) The global attractor attracting all solutions starting from a bounded set is

$$\mathcal{A} = [\min \mathcal{N}(g_{11}), \max \mathcal{N}(g_{11})] \times \dots \times [\min \mathcal{N}(g_{dd}), \max \mathcal{N}(g_{dd})].$$

(ii) Each solution of (2.1) *converges to an element of the following set*

$$\mathcal{N}(g) := \mathcal{N}(g_{11}) \times \dots \times \mathcal{N}(g_{dd})$$

and the rate of convergence is $t^{-\alpha}$.

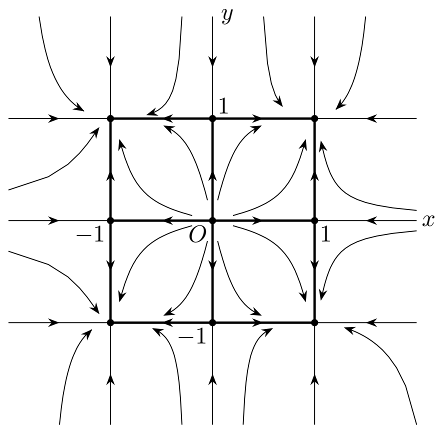

Figure 2: Attractor $\mathcal{A} = [-1,1] \times [-1,1]$ with steady state and heteroclinic trajectories for the vector field $g(x) = x(1-x^2), f(x,y) = y(1-y^2)(1+x^2)$.

Example 3.2. Consider the following FDEs with the following vector fields

$$\begin{aligned} {}^{C}D^{\alpha}_{0+}x(t) &= x(t)(1-x(t)), \\ {}^{C}D^{\alpha}_{0+}y(t) &= y(t)(1-y(t)^2)(1+x(t)^2). \end{aligned}$$

So, the attractor for the above equation is $\mathcal{A} = [-1,1]\times[-1,1]$. The asymptotically behavior of solutions are depicted in the Figure 2.

Remark 3.3. It is interesting to know whether Theorem 3.1 remains true when the vector field is of a more general form of triangular vector fields, e.g.

$$\begin{aligned} {}^{C}D^{\alpha}_{0+}x(t) &= f(x(t)), \\ {}^{C}D^{\alpha}_{0+}y(t) &= g(x(t),y(t)). \end{aligned}$$

Note the result of non-intersection of two solutions is still true for this equation, see [2]. However, the sign of the vector field $g(x(t),y(t))$ depends on both $x(t)$ and $y(t)$ and we can not use the approach in Section 2 to this problem.

4 One-parameter bifurcations for scalar Caputo fractional differential equations

Consider a family of scalar Caputo FDEs (2.1)

$$^{C}D^{\alpha}_{0+}x(t) = g(\gamma, x), \tag{4.1}$$

where γ is a parameter. Let $x_\gamma(t,\eta)$ denote the solution of (4.1) satisfying $x(0) = \eta$. It follows from Theorem 2.2 that (4.1) has the same bifurcations as an ODE with the same vector field. This means, in particular, that the simpler steady states and sign of vector fields for the ODE can be used to determine the bifurcations of the corresponding Caputo FDE. In what follows, we study the saddle-node and pitchfork bifurcations for fractional differential equations. We refer the readers to [7, Section 2.1] for a corresponding bifurcation analysis of ODE.

Example 4.1 (Saddle-node bifurcation)**.** Consider the following family of scalar Caputo FDEs

$$^{C}D^{\alpha}_{0+}x(t) = \gamma - x(t)^2. \tag{4.2}$$

Then, the following statements hold:

(i) For $\gamma < 0$, then all solutions of (4.2) tends to $-\infty$.

(ii) For $\gamma \geq 0$, then (4.2) has two steady states $x = -\sqrt{\gamma}$ and $x = \sqrt{\gamma}$ and

$$\lim_{t\to\infty} x_\gamma(t,\eta) = \begin{cases} -\infty, & \text{if } \eta < -\sqrt{\gamma}; \\ \sqrt{\gamma}, & \text{if } \eta > -\sqrt{\gamma}. \end{cases}$$

An analytical proof for (i) comes from the fact that

$$x_\gamma(t,\eta) = \eta + \frac{1}{\Gamma(\alpha)}\int_0^t (\gamma - x_\gamma(s,\eta)^2)\, ds \leq \eta + \frac{\gamma t}{\Gamma(\alpha)}.$$

For the case $\gamma \geq 0$, an analogous argument as in (i) implies that $\lim_{t\to\infty} x_\gamma(t,\eta) = -\infty$ for $\eta < -\sqrt{\gamma}$. Meanwhile, using Theorem 2.2 for the restriction of (4.2) on $(-\sqrt{\gamma}, \infty)$ leads to $\lim_{t\to\infty} x_\gamma(t,\eta) = \sqrt{\gamma}$ if $\eta > -\sqrt{\gamma}$.

Example 4.2 (Pitchfork bifurcation)**.** Consider the following family of scalar Caputo FDEs

$$^{C}D^{\alpha}_{0+}x(t) = \gamma x(t) - x(t)^3. \tag{4.3}$$

Then, by Theorem 2.2 we obtain the following description of the bifurcation of the asymptotical behavior of solutions of (4.3) on the parameter γ:

(i) For $\gamma < 0$, then (4.2) has a steady state $x = 0$ attracting all solutions of (4.2).

(ii) For $\gamma \geq 0$, then (4.2) has three steady states $x = -\sqrt{\gamma}, x = 0, x = \sqrt{\gamma}$ and

$$\lim_{t\to\infty} x_\gamma(t,\eta) = \begin{cases} -\sqrt{\gamma}, & \text{if } \eta < 0; \\ \sqrt{\gamma}, & \text{if } \eta > 0. \end{cases}$$

5 Attractors of Caputo semi-dynamical systems: the general case

Doan & Kloeden [6] showed recently that a general autonomous Caputo fractional differential equation

$$^{C}D^{\alpha}_{0+}x(t) = g(x(t)), \qquad \text{where } g : \mathbb{R}^d \to \mathbb{R}^d, \tag{5.1}$$

generates a semi-dynamical system on the function space $\mathfrak{C}$ of continuous functions $f : \mathbb{R}^+ \to \mathbb{R}^d$ with the topology uniform convergence on compact subsets. This topology is induced by the metric

$$\rho(f,h) := \sum_{n=1}^{\infty} \frac{1}{2^n}\rho_n(f,h), \quad \text{where } \rho_n(f,h) := \frac{\sup_{t\in[0,n]} \|f(t)-h(t)\|}{1+\sup_{t\in[0,n]} \|f(t)-h(t)\|}.$$

Define the operators $T_\tau : \mathfrak{C} \to \mathfrak{C}$, $\tau \in \mathbb{R}^+$, by

$$(T_\tau f)(\theta) = f(\tau+\theta) + \frac{1}{\Gamma(\alpha)}\int_0^\tau (\tau+\theta-s)^{\alpha-1} g(x_f(s))\, ds, \qquad \theta \in \mathbb{R}^+, \tag{5.2}$$

where x_f is a solution of the singular Volterra integral equation for this f, i.e.,

$$x_f(t) = f(t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(x_f(s))\, ds. \tag{5.3}$$

It was shown by Doan & Kloeden [6] that the operators T_τ, $\tau \in \mathbb{R}^+$, form a semi-group on the space $\mathfrak{C}$. (The proof in [6] follows Chapter XI, pages 178-179, in Sell [10] closely). This semi-group represents the Caputo FDE (5.1) as an autonomous semi-dynamical system on the space $\mathfrak{C}$.

The theory of autonomous semi-dynamical systems (see e.g., [8]) can be applied to the Caputo semi-group defined above.

Theorem 5.1. *Suppose that the semi-dynamical system $\{T_\tau, \tau \in \mathbb{R}^+\}$ on the space $\mathfrak{C}$ has a closed and bounded positively invariant absorbing set $\mathfrak{B}$ in*

$\mathfrak{C}$ *and is asymptotically compact. Then the semi-dynamical system* $\{T_\tau, \tau \in \mathbb{R}^+\}$ *has a global attractor given by*

$$\mathfrak{A} = \bigcap_{t \geq 0} T_t(\mathfrak{B}).$$

The solution $x(t, x_0)$ of the autonomous Caputo FDE (5.1) on $\mathbb{R}^d$ corresponds to a constant function $f_0(t) \equiv x_0$ and

$$x(t, x_0) \equiv (T_t f_0)(0).$$

Thus, when the semi-group $\{T_\tau, \tau \in \mathbb{R}^+\}$ has an attractor $\mathfrak{A} \subset \mathfrak{C}$, then an omega limit point $x \in \mathbb{R}^d$ of trajectories of the Caputo FDE satisfies $x = f(0)$ for some function $f \in \mathfrak{A}$. In particular, if $g(x^*) = 0$, then $f^* \in \mathfrak{A}$ for the constant function $f^*(t) \equiv x^*$, i.e., x^* is a steady state solution of the system. But there may be functions $f^* \in \mathfrak{A}$ that are not constant functions, so the strict inclusion, $\Omega_{\mathcal{B}} \subsetneq \mathfrak{A}(0)$ usually holds, where $\Omega_{\mathcal{B}}$ the omega limit point set discussed in the introduction section and $\mathfrak{A}(0)$ is the set of values in $\mathbb{R}^d$ of the functions in $\mathfrak{A}$ when evaluated at $t = 0$.

The application of Theorem 5.1 requires determining an absorbing set $\mathfrak{B}$ in $\mathfrak{C}$ and showing that the semi-dynamical system $\{T_\tau, \tau \in \mathbb{R}^+\}$ is asymptotically compact in some sense. This will be investigated in another paper.

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