REGULARITY OF SYMBOLIC POWERS OF SQUARE-FREE MONOMIAL IDEALS

TRUONG THI HIEN AND TRAN NAM TRUNG

Abstract. We study the regularity of symbolic powers of square-free monomial ideals. We prove that if $I = I_\Delta$ is the Stanley-Reisner ideal of a simplicial complex Δ, then $\operatorname{reg}(I^{(n)}) \leqslant \delta(n-1) + b$ for all $n \geqslant 1$, where $\delta = \lim\limits_{n\to\infty} \operatorname{reg}(I^{(n)})/n$, and $b = \max\{\operatorname{reg}(I_\Gamma) \mid \Gamma \text{ is a subcomplex of } \Delta \text{ with } \mathcal{F}(\Gamma) \subseteq \mathcal{F}(\Delta)\}$. This bound is sharp for any n.

Introduction

Throughout the paper, let K be a field and $R = K[x_1, \ldots, x_r]$ the polynomial ring of r variables $x_1, \ldots, x_r$ with $r \geqslant 1$. Let I be a homogeneous ideal of R. Then the n-th symbolic power of I is defined by

$$I^{(n)} = \bigcap_{\mathfrak{p} \in \operatorname{Min}(I)} I^n R_{\mathfrak{p}} \cap R,$$

where $\operatorname{Min}(I)$ is as usual the set of minimal associated prime ideals of I.

Cutkosky, Herzog, Trung [4], and independently Kodiyalam [15], proved that the function $\operatorname{reg}(I^n)$ is a linear function in n for $n \gg 0$. The similar result for symbolic powers is not true even when I is a square-free monomial ideal (see e.g. [6, Theorem 5.15]) except for the case $\dim(R/I) \leqslant 2$ (see [13]).

If I is a square-free monomial ideal, Hoa and the second author (see [12, Theorem 4.9]) proved that the limit

$$\delta(I) = \lim_{n\to\infty} \frac{\operatorname{reg}(I^{(n)})}{n}, \tag{1}$$

does exist. Moreover, $\operatorname{reg}(I^{(n)}) < \delta(I)n + \dim(R/I) + 1$ for all $n \geqslant 1$. Obviously, this bound is not sharp for every integer n. In fact, the limit (1) exists for arbitrary monomial ideals (see [6]). Recently, many results established the sharp bounds for $\operatorname{reg}(I^{(n)})$ in the case I is the edge ideal of a simple graph (see e.g. [1, 8, 9, 14]).

In this paper we will focus on finding bounds of the regularity of symbolic powers of square-free monomial ideals in terms of combinatorial data from associated simplicial complexes and hypergraphs.

2020 *Mathematics Subject Classification.* 13D45, 05C90, 05E40, 05E45.

Key words and phrases. Castelnuovo-Mumford regularity, symbolic power, edge ideal, cover ideal.

For a square-free monomial ideal I, we denote $\delta(I)$ to be the limit defined by Formula (1). In order to compute this invariant, we can use the geometric interpretation of $\delta(I)$ by means of symbolic polyhedra defined in [3, 6] (see Formula (4)).

For a simplicial complex Δ on the set $V = \{1, \ldots, r\}$, the Stanley-Reisner ideal of Δ is defined by

$$I_\Delta = \left(\prod_{i \in \tau} x_i \mid \tau \subseteq V \text{ and } \tau \notin \Delta \right) \subseteq R.$$

Let $\mathcal{F}(\Delta)$ denote the set of facets of Δ.

The first main result of the paper is the following theorem.

Theorem 2.3. *Let Δ be a simplicial complex. Then,*

$$\operatorname{reg}(I_\Delta^{(n)}) \leqslant \delta(I_\Delta)(n-1) + b, \quad \textit{for all } n \geqslant 1,$$

where $b = \max\{\operatorname{reg}(I_\Gamma) \mid \Gamma$ *is a subcomplex of* Δ *with* $\mathcal{F}(\Gamma) \subseteq \mathcal{F}(\Delta)\}$.

This bound is sharp for every n (see Example 2.5).

Let $\mathcal{H} = (V, E)$ be a simple hypergraph with $V = \{1, \ldots, r\}$. The edge ideal of $\mathcal{H}$ is defined by

$$I(\mathcal{H}) = \left(\prod_{i \in e} x_i \mid e \in E \right) \subseteq R.$$

Let $\mathcal{H}^*$ be the simple hypergraph corresponding to the Alexander duality $I(\mathcal{H})^*$ of $I(\mathcal{H})$. Let $\epsilon(\mathcal{H}^*)$ be the minimum number of cardinality of edgewise dominant sets of $\mathcal{H}^*$, this concept was introduced by Dao and Schweig [5].

Then second main result of the paper is the following theorem.

Theorem 2.7. *Let $\mathcal{H}$ be a simple hypergraph. Then,*

$$\operatorname{reg}(I(\mathcal{H})^{(n)}) \leqslant \delta(I(\mathcal{H}))(n-1) + |V(\mathcal{H})| - \epsilon(\mathcal{H}^*), \quad \textit{for all } n \geqslant 1.$$

A hypergraph is a graph if every edge has exactly two vertices. For a graph G, a linear lower bound for $\operatorname{reg}(I(G)^{(n)})$ is given in [9]:

$$\operatorname{reg}(I(G)^{(n)}) \geqslant 2n + \nu(G) - 1,$$

where $\nu(G)$ is the induced matching of G. Note that this lower bound is also valid for ordinary powers (see [2, Theorem 4.5]).

On the upper bounds, Fakhari (see [8, Conjecture 1.3]) conjectured that

$$\operatorname{reg}(I(G)^{(n)}) \leqslant 2n + \operatorname{reg}(I(G)) - 2.$$

As a consequence of Theorem 2.3, we obtain a linear upper bound for $\operatorname{reg}(I(G)^{(n)})$, however it is weaker than the bound in this conjecture.

Corollary 2.8. *Let G be a graph. Then, for any $n \geqslant 1$, we have*

$$\operatorname{reg}(I(G)^{(n)}) \leqslant 2n + \dim(R/I(G)) - 1.$$

Let us explain the idea to prove Theorems 2.3 and 2.7 as follows. Let $i \geqslant 0$ such that $\operatorname{reg}(R/I^{(n)}) = a_i(R/I^{(n)}) + i$.

The first key point is to prove that $a_i(R/I^{(n)}) \leqslant \delta(I)(n-1)$. Assume that $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r) \in \mathbb{Z}^r$ such that

$$H^i_{\mathfrak{m}}(R/I^{(n)})_{\boldsymbol{\alpha}} \neq 0, \text{ and } a_i(R/I^{(n)}) = |\boldsymbol{\alpha}|,$$

where $\mathfrak{m} = (x_1, \ldots, x_r)$ and $|\boldsymbol{\alpha}| = \alpha_1 + \cdots + \alpha_r$. We reduce to the case $\boldsymbol{\alpha} \in \mathbb{N}^r$. In order to bound $|\boldsymbol{\alpha}|$, we use Takayama's formula (see Lemma 1.3) to compute $H^i_{\mathfrak{m}}(R/I^{(n)})_{\boldsymbol{\alpha}}$, which allows us to search for $\boldsymbol{\alpha}$ in a polytope in $\mathbb{R}^r$, so that we can get the desired bound of $|\boldsymbol{\alpha}|$ via theory of convex polytopes (see Theorem 2.2).

The second key point is to bound the index i by using the regularity of a Stanley-Reisner ideal in terms of the vanishing of reduced homology of simplicial complexes which derived from Hochster's formula about the Hilbert series of the local cohomology module of Stanley-Reisner ideals (see Lemma 1.1).

Our paper is structured as follows. In the next section, we collect notations and terminology used in the paper, and recall a few auxiliary results. In Section 2, we prove Theorems 2.3 and 2.7.

1. Preliminaries

We shall follow standard notations and terminology from usual texts in the research area (cf. [7, 11, 16]). For simplicity, we denote the set $\{1, \ldots, r\}$ by $[r]$.

1.1. **Regularity and projective dimension.** Through out this paper, let K be a field, and let $R = K[x_1, \ldots, x_r]$ be a standard graded polynomial ring of r variables over K. The object of our work is the Castelnuovo-Mumford regularity of graded modules and ideals over R. This invariant can be defined via either the minimal free resolutions or the local cohomology modules.

Let M be a nonzero finitely generated graded R-module and let

$$0 \to \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_{p,j}(M)} \to \cdots \to \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_{0,j}(M)} \to 0$$

be the minimal free resolution of M. The *Castelnuovo–Mumford regularity* (or regularity for short) of M is defined by

$$\operatorname{reg}(M) = \max\{j - i \mid \beta_{i,j}(M) \neq 0\},$$

and the *projective dimension* of M is the length of this resolution

$$\mathrm{pd}(M) = p.$$

Let us denote by $d(M)$ the maximal degree of a minimal homogeneous generator of M. The definition of the regularity implies

$$d(M) \leqslant \mathrm{reg}(M).$$

For any nonzero proper homogeneous ideal I of R, by looking at the minimal free resolution, it is easy to see that $\mathrm{reg}(I) = \mathrm{reg}(R/I) + 1$, so we shall work with $\mathrm{reg}(I)$ and $\mathrm{reg}(R/I)$ interchangeably.

The regularity of M can also be computed via the local cohomology modules of M. For $i = 0, \ldots, \dim(M)$, we define the a_i-invariant of M as follows

$$a_i(M) = \max\{t \mid H^i_{\mathfrak{m}}(M)_t \neq 0\}$$

where $H^i_{\mathfrak{m}}(M)$ is the i-th local cohomology module of M with the support $\mathfrak{m} = (x_1, \ldots, x_r)$ (with the convention $\max \emptyset = -\infty$). Then,

$$\mathrm{reg}(M) = \max\{a_i(M) + i \mid i = 0, \ldots, \dim(M)\},$$

and

$$\mathrm{pd}(M) = r - \min\{i \mid H^i_{\mathfrak{m}}(M) \neq 0\}.$$

As usual we shall make the convention that $\mathrm{reg}(M) = -\infty$ if $M = 0$.

1.2. **Simplicial complexes and Stanley-Reisner ideals.** A simplicial complex Δ over the vertex set $V = \{1, \ldots, r\}$ is a collection of subsets of V such that if $F \in \Delta$ and $G \subseteq F$ then $G \in \Delta$. Elements of Δ are called faces. Maximal faces (with respect to inclusion) are called facets. For $F \in \Delta$, the dimension of F is defined to be $\dim F = |F| - 1$. The empty set, $\emptyset$, is the unique face of dimension -1, as long as Δ is not the void complex $\{\}$ consisting of no subsets of V. The dimension of Δ is $\dim \Delta = \max\{\dim F \mid F \in \Delta\}$. The link of F inside Δ is its subcomplex:

$$\mathrm{lk}_\Delta(F) = \{H \in \Delta \mid H \cup F \in \Delta \ \text{ and } H \cap F = \emptyset\}.$$

If every facet of Δ has the same cardinality, then Δ is called a pure complex. If there is a vertex, say j, such that $\{j\} \cup F \in \Delta$ for every $F \in \Delta$, then Δ is called a cone over j. It is well-known that if Δ is a cone, then it is an acyclic complex.

For a subset $\tau = \{j_1, \ldots, j_i\}$ of V, denote $\mathbf{x}^\tau = x_{j_1} \cdots x_{j_i}$. Let Δ be a simplicial complex on V. The Stanley-Reisner ideal of Δ is defined to be the squarefree monomial ideal

$$I_\Delta = (\mathbf{x}^\tau \mid \tau \subseteq V \text{ and } \tau \notin \Delta) \ \text{ in } R = K[x_1, \ldots, x_r]$$

and the *Stanley-Reisner* ring of Δ to be the quotient ring $k[\Delta] = R/I_\Delta$. This provides a bridge between combinatorics and commutative algebra (see [16, 20]).

Note that if I is a square-free monomial ideal, then it is a Stanley–Reisner ideal of the simplicial complex $\Delta(I) = \{\tau \subseteq V \mid \mathbf{x}^\tau \notin I\}$. When I is a monomial ideal (maybe not square-free) we also use $\Delta(I)$ to denote the simplicial complex corresponding to the square-free monomial ideal $\sqrt{I}$.

The regularity of a square-free monomial ideal can compute via the vanishing of reduced homology of simplicial complexes. From Hochster's formula on the Hilbert series of the local cohomology module $H^i_{\mathfrak{m}}(I_\Delta)$ (see [16, Corollary 13.16]), one has

Lemma 1.1. *For a simplicial complex Δ, we have*

$$\operatorname{reg}(I_\Delta) = \max\{d \mid \widetilde{H}_{d-1}(\operatorname{lk}_\Delta(\sigma); K) \neq 0, \textit{ for some } \sigma \in \Delta\}.$$

The *Alexander dual* of Δ, denoted by Δ^*, is the simplicial complex over V with faces

$$\Delta^* = \{V \setminus \tau \mid \ \tau \notin \Delta\}.$$

Notice that $(\Delta^*)^* = \Delta$. If $I = I_\Delta$ then we shall denote the Stanley-Reisner ideal of the Alexander dual Δ^* by I^*. It is a well-known result of Terai [22] (or see [16, Theorem 5.59]) that the regularity of a squarefree monomial ideal can be related to the projective dimension of its Alexander dual.

Lemma 1.2. *Let $I \subseteq R$ be a square-free monomial ideal. Then,*

$$\operatorname{reg}(I) = \operatorname{pd}(R/I^*).$$

Let $\mathcal{F}(\Delta)$ denote the set of all facets of Δ. We say that Δ is generated by $\mathcal{F}(\Delta)$ and write $\Delta = \langle \mathcal{F}(\Delta)\rangle$. Note that I_Δ has the minimal primary decomposition (see [16, Theorem 1.7]):

$$I_\Delta = \bigcap_{F \in \mathcal{F}(\Delta)} (x_i \mid i \notin F),$$

and therefore the n-th symbolic power of I_Δ is

$$I_\Delta^{(n)} = \bigcap_{F \in \mathcal{F}(\Delta)} (x_i \mid i \notin F)^n.$$

We next describe a formula to compute the local cohomology modules of monomial ideals. Let I be a non-zero monomial ideal. Since R/I is an $\mathbb{N}^r$-graded algebra, $H^i_{\mathfrak{m}}(R/I)$ is an $\mathbb{Z}^r$-graded module over R/I for every i. For each degree $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r) \in \mathbb{Z}^r$, in order to compute $\dim_K H^i_{\mathfrak{m}}(R/I)_{\boldsymbol{\alpha}}$ we use a formula given by Takayama [21, Theorem 2.2] which is a generalization of Hochster's formula for the case I is square-free [20, Theorem 4.1].

Set $G_{\boldsymbol{\alpha}} = \{i \mid \alpha_i < 0\}$. For a subset $F \subseteq [r]$, we set $R_F = R[x_i^{-1} \mid i \in F \cup G_{\boldsymbol{\alpha}}]$. Define the simplicial complex $\Delta_{\boldsymbol{\alpha}}(I)$ by

$$\Delta_\alpha(I) = \{F \subseteq [r] \setminus G_{\boldsymbol{\alpha}} \mid x^\alpha \notin IR_F\}. \tag{2}$$

Lemma 1.3. [21, Theorem 2.2] $\dim_K H^i_{\mathfrak{m}}(R/I)_{\boldsymbol{\alpha}} = \dim_K \widetilde{H}_{i-|G_\alpha|-1}(\Delta_{\boldsymbol{\alpha}}(I); K)$.

The following result of Minh and Trung is very useful to compute $\Delta_{\boldsymbol{\alpha}}(I_\Delta^{(n)})$, which allows us to investigate $\operatorname{reg}(I_\Delta^{(n)})$ by using the theory of convex polyhedra.

Lemma 1.4. [17, Lemma 1.3] *Let Δ be a simplicial complex and $\boldsymbol{\alpha} \in \mathbb{N}^r$. Then,*

$$\mathcal{F}(\Delta_{\boldsymbol{\alpha}}(I_\Delta^{(n)})) = \left\{ F \in \mathcal{F}(\Delta) \mid \sum_{i \notin F} \alpha_i \leqslant n-1 \right\}.$$

This lemma can be generalized a bit as follows.

Lemma 1.5. [13, Lemma 1.3] *Let Δ be a simplicial complex and $\boldsymbol{\alpha} \in \mathbb{Z}^r$. Then,*

$$\mathcal{F}(\Delta_{\boldsymbol{\alpha}}(I_\Delta^{(n)})) = \left\{ F \in \mathcal{F}(\operatorname{lk}_\Delta(G_{\boldsymbol{\alpha}})) \mid \sum_{i \notin F \cup G_{\boldsymbol{\alpha}}} \alpha_i \leqslant n-1 \right\}.$$

1.3. **Hypergraphs.** Let V be a finite set. A simple hypergraph $\mathcal{H}$ with vertex set V consists of a set of subsets of V, called the edges of $\mathcal{H}$, with the property that no edge contains another. We use the symbols $V(\mathcal{H})$ and $E(\mathcal{H})$ to denote the vertex set and the edge set of $\mathcal{H}$, respectively.

In this paper we assume that all hypergraphs are simple unless otherwise specified.

In the hypergraph $\mathcal{H}$, an edge is *trivial* if it contains only one element, a vertex is *isolated* if it is not appearing in any edge, a vertex is a *neighbor* of another one if they are in some edge.

A hypergraph $\mathcal{H}'$ is a *subhypergraph* of $\mathcal{H}$ if $V(\mathcal{H}') \subseteq V(\mathcal{H})$ and $E(\mathcal{H}') \subseteq E(\mathcal{H})$. For an edge e of $\mathcal{H}$, we define $\mathcal{H} \setminus e$ to be the hypergraph obtained by deleting e from the edge set of $\mathcal{H}$. For a subset $S \subseteq V(\mathcal{H})$, we define $\mathcal{H} \setminus S$ to be the hypergraph obtained from $\mathcal{H}$ by deleting the vertices in S and all edges containing any of those vertices.

A set $S \subseteq E(\mathcal{H})$ is called an *edgewise dominant set* of $\mathcal{H}$ if every non-isolated vertex of $\mathcal{H}$ not contained in some edge of S or contained in a trivial edge has a neighbor contained in some edge of S. Define,

$$\epsilon(\mathcal{H}) = \min\{|S| \mid S \text{ is edgewise dominant}\}.$$

For a hypergraph $\mathcal{H}$ with $V(\mathcal{H}) \subseteq [r]$, we associate to the hypergraph $\mathcal{H}$ a square-free monomial ideal

$$I(\mathcal{H}) = (\mathbf{x}^e \mid e \in E(\mathcal{H})) \subseteq R,$$

which is called the *edge ideal* of $\mathcal{H}$.

Notice that if I is a square-free monomial ideal, then I is an edge ideal of a hypergraph with the edge set uniquely determined by the generators of I.

Let $\mathcal{H}^*$ be the simple hypergraph corresponding to the Alexander duality $I(\mathcal{H})^*$ of $I(\mathcal{H})$. We will determine the edge set of $\mathcal{H}^*$, it turns out that $E(\mathcal{H}^*)$ is the set of all minimal vertex covers of $\mathcal{H}$. A *vertex cover* in a hypergraph is a set of vertices, such that every edge of the hypergraph contains at least one vertex of that set. It is an extension of the notion of vertex cover in a graph. A vertex cover S is called minimal if no proper subset of S is a vertex cover. From the minimal primary decomposition (see [16, Definition 1.35 and Proposition 1.37]):

$$I(\mathcal{H}^*) = \bigcap_{e \in E(\mathcal{H})} (x_i \mid i \in e),$$

it follows that $E(\mathcal{H}^*)$ is just the set of minimal vertex covers of $\mathcal{H}$. Thus,

$$I(\mathcal{H}^*) = (\mathbf{x}^\tau \mid \tau \text{ is a minimal vertex cover of } \mathcal{H}),$$

so that it is also called the *cover ideal* of $\mathcal{H}$ and denoted by $J(\mathcal{H})$.

In the sequel, we need the following result of Dao and Schweig [5, Theorem 3.2].

Lemma 1.6. *Let $\mathcal{H}$ be a hypergraph. Then,* $\operatorname{pd}(R/I(\mathcal{H})) \leqslant |V(\mathcal{H})| - \epsilon(\mathcal{H})$.

1.4. **Convex polyhedra.** The theory of convex polyhedra plays a key role in our study.

For a vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r) \in \mathbb{R}^r$, we set $|\boldsymbol{\alpha}| := \alpha_1 + \cdots + \alpha_r$ and for a nonempty bounded closed subset S of $\mathbb{R}^r$ we set

$$\delta(S) := \max\{|\boldsymbol{\alpha}| \mid \boldsymbol{\alpha} \in S\}.$$

Let Δ be a simplicial conplex over $[r]$. In general, $\operatorname{reg}(I_\Delta^{(n)})$ is not a linear function in n for $n \gg 0$ (see e.g. [6, Theorem 5.15]), but a quasi-linear function as in the following result.

Lemma 1.7. [12, Theorem 4.9] *There exist positive integers N, n_0 and rational numbers $a, b_0, \ldots, b_{N-1} < \dim(R/I_\Delta) + 1$ such that*

$$\operatorname{reg}(I_\Delta^{(n)}) = an + b_k, \text{ for all } n \geqslant n_0 \text{ and } n \equiv k \mod N, \text{ where } 0 \leqslant k \leqslant N-1.$$

Moreover, $\operatorname{reg}(I_\Delta^{(n)}) < an + \dim(R/I_\Delta) + 1$ *for all* $n \geqslant 1$.

By virtue of this result, we define

$$\delta(I_\Delta) = a = \lim_{n \to \infty} \frac{\operatorname{reg}(I_\Delta^{(n)})}{n}.$$

In order to compute $\delta(I_\Delta)$, let $\mathcal{SP}(I_\Delta)$ be the convex polyhedron in $\mathbb{R}^r$ defined by the following system of linear inequalities:

$$(3) \qquad \begin{cases} \sum_{i \notin F} x_i \geqslant 1 & \text{for } F \in \mathcal{F}(\Delta), \\ x_1 \geqslant 0, \ldots, x_r \geqslant 0, \end{cases}$$

which is called the *symbolic polyhedron* of I_Δ.

Then, $\mathcal{SP}(I_\Delta)$ is a convex polyhedron in $\mathbb{R}^r$. By [6, Theorem 3.6] we have

$$\delta(I_\Delta) = \max\{\mathbf{v} \mid \mathbf{v} \text{ is a vertex of } \mathcal{SP}(I_\Delta)\}. \tag{4}$$

Now assume that

$$H^i_{\mathfrak{m}}(I_\Delta^{(n)})_{\boldsymbol{\alpha}} \neq 0$$

for some $0 \leqslant i \leqslant \dim(R/I_\Delta)$ and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r$.

By Lemma 1.3 we have

$$\dim_K \widetilde{H}_{i-1}(\Delta_{\boldsymbol{\alpha}}(I_\Delta^{(n)}); K) = \dim_K H^i_{\mathfrak{m}}(R/I_\Delta^{(n)})_{\boldsymbol{\alpha}} \neq 0. \tag{5}$$

In particular, $\Delta_{\boldsymbol{\alpha}}(I_\Delta^{(n)})$ is not acyclic.

Suppose that $\mathcal{F}(\Delta) = \{F_1, \ldots, F_t\}$ for $t \geqslant 1$. By Lemma 1.4 we may assume that

$$\mathcal{F}(\Delta_{\boldsymbol{\alpha}}(I_\Delta^{(n)})) = \{F_1, \ldots, F_s\}, \text{ where } 1 \leqslant s \leqslant t.$$

For each integer $m \geqslant 1$, let $\mathcal{P}_m$ be the convex polyhedron of $\mathbb{R}^r$ defined by:

$$\begin{cases} \sum_{i \notin F_j} x_i \leqslant m - 1 & \text{for } j = 1, \ldots, s, \\ \sum_{i \notin F_j} x_i \geqslant m & \text{for } j = s+1, \ldots, t, \\ x_1 \geqslant 0, \ldots, x_r \geqslant 0. \end{cases} \tag{6}$$

Then, $\boldsymbol{\alpha} \in \mathcal{P}_n$. Moreover, by Lemma 1.4 one has

$$\Delta_{\boldsymbol{\beta}}(I_\Delta^{(m)}) = \langle F_1, \ldots, F_s \rangle = \Delta_{\boldsymbol{\alpha}}(I_\Delta^{(n)}) \text{ whenever } \boldsymbol{\beta} \in \mathcal{P}_m \cap \mathbb{N}^r. \tag{7}$$

Note also that for such a vector $\boldsymbol{\beta}$, by Formula (7) we have

$$\dim_K \widetilde{H}_{i-1}(\Delta_{\boldsymbol{\beta}}(I_\Delta^{(m)}); K) = \dim_K \widetilde{H}_{i-1}(\Delta_{\boldsymbol{\alpha}}(I_\Delta^{(n)}); K) \neq 0.$$

Together with Lemma 1.3, this fact yields

$$H^i_{\mathfrak{m}}(R/I_\Delta^{(m)})_{\boldsymbol{\beta}} \neq 0. \tag{8}$$

In order to investigate the convex polyhedron $\mathcal{P}_m$ we also consider the convex polyhedron $\mathcal{C}_m$ in $\mathbb{R}^r$ defined by:

$$\begin{cases} \sum_{i \notin F_j} x_i \leqslant m & \text{for } j = 1, \ldots, s, \\ \sum_{i \notin F_j} x_i \geqslant m & \text{for } j = s+1, \ldots, t, \\ x_1 \geqslant 0, \ldots, x_r \geqslant 0. \end{cases} \tag{9}$$

Note that $\mathcal{C}_m = m\mathcal{C}_1$ for all $m \geqslant 1$, where $m\mathcal{C}_1 = \{m\mathbf{y} \mid \mathbf{y} \in \mathcal{C}_1\}$.

By the same way as in the proof of [10, Lemma 2.1] we obtain the following lemma.

Lemma 1.8. *$\mathcal{C}_1$ is a polytope with* $\dim \mathcal{C}_1 = r$.

The next lemma gives an upper bound for $\delta(\mathcal{C}_1)$.

Lemma 1.9. $\delta(C_1) \leqslant \delta(I_\Delta)$.

Proof. Since $\mathcal{C}_1$ is a polytope with $\dim \mathcal{C}_1 = r$ by Lemma 1.8, $\delta(\mathcal{C}_1) = |\boldsymbol{\gamma}|$ for some vertex $\boldsymbol{\gamma}$ of $\mathcal{C}_1$. By [19, Formula (23) in Page 104] we imply that $\boldsymbol{\gamma}$ is the unique solution of a system of linear equations of the form

$$\begin{cases} \sum_{i \notin F_j} x_i = 1 & \text{for } j \in S_1, \\ x_j = 0 & \text{for } j \in S_2, \end{cases} \tag{10}$$

where $S_1 \subseteq [t]$ and $S_2 \subseteq [r]$ such that $|S_1| + |S_2| = r$. By using Cramer's rule to get $\boldsymbol{\gamma}$, we conclude that $\boldsymbol{\gamma}$ is a rational vector. In particular, there is a positive integer, say p, such that $p\boldsymbol{\gamma} \in \mathbb{N}^r$. Note that $\mathcal{C}_p = p\mathcal{C}_1$, so $p\boldsymbol{\gamma} \in \mathcal{C}_p \cap \mathbb{N}^r$.

For every $j \geqslant 1$, let $\mathbf{y} = jp\boldsymbol{\gamma} + \boldsymbol{\alpha}$. Then, $\mathbf{y} \in \mathbb{N}^r$ and $|\mathbf{y}| = \delta(\mathcal{C}_1)jp + |\boldsymbol{\alpha}|$. On the other hand, by using the fact that $jp\boldsymbol{\gamma} \in \mathcal{C}_{jp}$, we can check that

$$\begin{cases} \sum_{i \notin F_j} y_i \leqslant jp + n - 1 & \text{for } j = 1, \ldots, s, \\ \sum_{i \notin F_j} y_i \geqslant jp + n & \text{for } j = s+1, \ldots, t, \end{cases}$$

and so $\mathbf{y} \in \mathcal{P}_{jp+n} \cap \mathbb{N}^r$.

Together with Equation (8), we deduce that $H^i_{\mathfrak{m}}(R/I_\Delta^{(jp+n)})_{\mathbf{y}} \neq 0$, and therefore

$$\operatorname{reg}(R/I_\Delta^{(jp+n)}) \geqslant |\mathbf{y}| + i = \delta(\mathcal{C}_1)jp + |\boldsymbol{\alpha}| + i.$$

Combining with Lemma 1.7, this inequality yields

$$\delta(\mathcal{C}_1)jp + |\boldsymbol{\alpha}| + i < \delta(I_\Delta)(jp + n) + \dim(R/I_\Delta).$$

Since this inequality valid for any positive integer j, it forces $\delta(\mathcal{C}_1) \leqslant \delta(I_\Delta)$. ☐

2. Regularity of symbolic powers of ideals

In this section we will prove the upper bound for $\operatorname{reg}(I_\Delta^{(n)})$. Firts we start with the following fact.

Lemma 2.1. *Let $\sigma \subseteq [r]$ with $\sigma \neq [r]$, $S = K[x_i \mid i \notin \sigma]$ and $J = IR_\sigma \cap S$. Then,*

$$\operatorname{reg}(J^{(n)}) \leqslant \operatorname{reg}(I^{(n)}) \quad \textit{for all } n \geqslant 1.$$

In particular, $\delta(J) \leqslant \delta(I)$.

Proof. We may assume that $S = K[x_1, \ldots, x_s]$ for some $1 \leqslant s \leqslant r$. Let i be an index and $\boldsymbol{\alpha}$ a vector in $\mathbb{Z}^s$ such that

$$H^i_{\mathfrak{n}}(S/J^{(n)})_{\boldsymbol{\alpha}} \neq 0 \text{ and } \operatorname{reg}(S/J^{(n)}) = |\boldsymbol{\alpha}| + i,$$

where $\mathfrak{n} = (x_1, \ldots, x_s)$ is the homogeneous maximal ideal of S.

Let $\boldsymbol{\beta} = (\alpha_1, \ldots, \alpha_s, -1, \ldots, -1) \in \mathbb{Z}^r$ so that $G_{\boldsymbol{\beta}} = G_{\boldsymbol{\alpha}} \cup \{s+1, \ldots, r\}$. By Formula (2) we deduce that

$$\Delta_{\boldsymbol{\alpha}}(J^{(n)}) = \Delta_{\boldsymbol{\beta}}(I^{(n)}). \tag{11}$$

By Lemma 1.3,

$$\dim_K H^i_{\mathfrak{n}}(S/J^{(n)})_{\boldsymbol{\alpha}} = \dim_K \widetilde{H}_{i-|G_{\boldsymbol{\alpha}}|-1}(\Delta_{\boldsymbol{\alpha}}(J^{(n)}); K),$$

and thus $\widetilde{H}_{i-|G_{\boldsymbol{\alpha}}|-1}(\Delta_{\boldsymbol{\alpha}}(J^{(n)}); K) \neq 0$. Together with Equation (11), it yields

$$\widetilde{H}_{i-|G_{\boldsymbol{\alpha}}|-1}(\Delta_{\boldsymbol{\beta}}(I^{(n)}); K) \neq 0.$$

By Lemma 1.3 again, it gives $H^{i+(r-s)}_{\mathfrak{m}}(R/I^{(n)})_{\boldsymbol{\beta}} \neq 0$ since $|G_{\boldsymbol{\beta}}| = |G_{\boldsymbol{\alpha}}| + (r-s)$. Therefore,

$$\operatorname{reg}(R/I^{(n)}) \geqslant |\boldsymbol{\beta}| + i + (r-s) = |\boldsymbol{\alpha}| + i = \operatorname{reg}(S/J^{(n)}),$$

it follows that $\operatorname{reg}(J^{(n)}) \leqslant \operatorname{reg}(I^{(n)})$.

Finally, together this inequality with Lemma 1.7 we have

$$\delta(J) = \lim_{n\to\infty} \frac{\operatorname{reg}(J^{(n)})}{n} \leqslant \lim_{n\to\infty} \frac{\operatorname{reg}(I^{(n)})}{n} = \delta(I),$$

and the lemma follows. □

Theorem 2.2. *Let I be a square-free monomial ideal. Then, for all $i \geqslant 0$ we have*

$$a_i(R/I^{(n)}) \leqslant \delta(I)(n-1).$$

Proof. If $n = 1$, the theorem follows from Hochster's formula on the Hilbert series of the local cohomology module $H^i_{\mathfrak{m}}(R/I_\Delta)$ (see [20, Theorem 4.1]).

We may assume that $n \geqslant 2$. If $a_i(R/I^{(n)}) = -\infty$, the theorem is obvious, so that we also assume that $a_i(R/I^{(n)}) \neq -\infty$.

Suppose $\boldsymbol{\alpha} \in \mathbb{Z}^r$ such that

$$H^i_{\mathfrak{m}}(R/I^{(n)})_{\boldsymbol{\alpha}} \neq 0 \text{ and } a_i(R/I^{(n)}) = |\boldsymbol{\alpha}|.$$

By Lemma 1.3 we have

$$\dim_K \widetilde{H}_{i-|G_{\boldsymbol{\alpha}}|-1}(\Delta_{\boldsymbol{\alpha}}(I^{(n)}); K) = \dim_K H^i_{\mathfrak{m}}(R/I^{(n)})_{\boldsymbol{\alpha}} \neq 0. \tag{12}$$

In particular, $\Delta_{\boldsymbol{\alpha}}(I^{(n)})$ is not acyclic.

If $G_{\boldsymbol{\alpha}} = [r]$, then $a_i(R/I^{(n)}) = |\boldsymbol{\alpha}| \leqslant 0$, and hence the theorem holds in this case.

We therefore assume that $G_{\boldsymbol{\alpha}} = \{m+1, \ldots, r\}$ for $1 \leqslant m \leqslant r$. Let $S = K[x_1, \ldots, x_m]$ and $J = IR_{G_{\boldsymbol{\alpha}}} \cap S$.

Let $\boldsymbol{\alpha}' = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$. By using Formula (2), we have

$$\Delta_{\boldsymbol{\alpha}'}(J^{(n)}) = \Delta_{\boldsymbol{\alpha}}(I^{(n)}). \tag{13}$$

Together with (12), it gives $\widetilde{H}_{i-|G_{\boldsymbol{\alpha}}|-1}(\Delta_{\boldsymbol{\alpha}'}(J^{(n)}); K) \neq 0$. By Lemma 1.3 we get

$$H_{\mathfrak{n}}^{i-|G_{\boldsymbol{\alpha}}|}(S/J^{(n)})_{\boldsymbol{\alpha}'} \neq 0,$$

where $\mathfrak{n} = (x_1, \ldots, x_m)$ is the homogeneous maximal ideal of S.

Let Δ be the simplicial complex over $[m]$ corresponding to the square-free monomial ideal J. Assume that $\mathcal{F}(\Delta) = \{F_1, \ldots, F_t\}$.

By Lemma 1.4 we may assume that $\mathcal{F}(\Delta_{\boldsymbol{\alpha}'}(J^{(n)})) = \{F_1, \ldots, F_s\}$ for $1 \leqslant s \leqslant t$. Let

$$\boldsymbol{\beta} = (\beta_1, \ldots, \beta_m) = \frac{1}{n-1}\boldsymbol{\alpha}' \in \mathbb{R}^m.$$

By Lemma 1.4 again, we deduce that

$$\begin{cases} \sum\limits_{i \notin F_j} \beta_i = \dfrac{1}{n-1} \sum\limits_{i \notin F_j} \alpha_i \leqslant 1 & \text{for } j = 1, \ldots, s, \\ \sum\limits_{i \notin F_j} \beta_i = \dfrac{1}{n-1} \sum\limits_{i \notin F_j} \alpha_i \geqslant \dfrac{n}{n-1} > 1 & \text{for } j = s+1, \ldots, t. \end{cases}$$

It follows that $\boldsymbol{\beta} \in C_1$, where C_1 is a polyhedron in $\mathbb{R}^m$ defined by

$$\begin{cases} \sum\limits_{i \notin F_j} x_i \leqslant 1 & \text{for } j = 1, \ldots, s, \\ \sum\limits_{i \notin F_j} x_i \geqslant 1 & \text{for } j = s+1, \ldots, t, \\ x_1 \geqslant 0, \ldots, x_m \geqslant 0. \end{cases}$$

By Lemma 1.8, C_1 is a polytope in $\mathbb{R}^m$.

Hence $|\boldsymbol{\beta}| \leqslant \delta(C_1)$, and hence $|\boldsymbol{\alpha}'| = (n-1)|\boldsymbol{\beta}| \leqslant \delta(C_1)(n-1)$. Observe that $\alpha_j < 0$ for all $j \in G_{\boldsymbol{\alpha}} = \{m+1, \ldots, r\}$, so

$$a_i(R/I^{(n)}) = |\boldsymbol{\alpha}| = |\boldsymbol{\alpha}'| + (\alpha_{m+1} + \cdots + \alpha_r) \leqslant |\boldsymbol{\alpha}'| \leqslant \delta(C_1)(n-1). \tag{14}$$

On the other hand, by Lemmas 1.9 and 2.1 we deduce that

$$\delta(C_1) \leqslant \delta(J) \leqslant \delta(I).$$

Together with Formula (14), it yields $a_i(R/I^{(n)}) \leqslant \delta(I)(n-1)$, and the proof of the theorem is complete. □

We are now in position to prove the main result of the paper.

Theorem 2.3. *Let Δ be a simplicial complex. Then,*

$$\operatorname{reg}(I_\Delta^{(n)}) \leqslant \delta(I_\Delta)(n-1) + b, \quad \textit{for all } n \geqslant 1,$$

where $b = \max\{\operatorname{reg}(I_\Gamma) \mid \Gamma$ is a subcomplex of Δ with $\mathcal{F}(\Gamma) \subseteq \mathcal{F}(\Delta)\}$.

Proof. For simplicity, we put $I = I_\Delta$. Let $i \in \{0, \dots, \dim(R/I)\}$ and $\boldsymbol{\alpha} \in \mathbb{Z}^r$ such that

$$H^i_{\mathfrak{m}}(R/I^{(n)})_{\boldsymbol{\alpha}} \neq 0, \text{ and } \operatorname{reg}(R/I^{(n)}) = a_i(R/I^{(n)}) + i = |\boldsymbol{\alpha}| + i.$$

By Lemma 1.3, we have

$$\dim_K \widetilde{H}_{i-|G_{\boldsymbol{\alpha}}|-1}(\Delta_{\boldsymbol{\alpha}}(I^{(n)}); K) = \dim_K H^i_{\mathfrak{m}}(R/I^{(n)})_{\boldsymbol{\alpha}} \neq 0. \tag{15}$$

In particular, $\Delta_{\boldsymbol{\alpha}}(I^{(n)})$ is not acyclic.

If $G_{\boldsymbol{\alpha}} = [r]$, then $\Delta_{\boldsymbol{\alpha}}(I^{(n)})$ is either $\{\emptyset\}$ or a void complex. Because it is not acyclic, $\Delta_{\boldsymbol{\alpha}}(I^{(n)}) = \{\emptyset\}$. By Formula (15) we deduce that $i = |G_{\boldsymbol{\alpha}}| = r$, and hence $\dim R/I = r$. It means that $I = 0$, so $I^{(n)} = 0$ as well. Therefore, $\operatorname{reg}(I^{(n)}) = -\infty$, and the theorem holds in this case.

We may assume that $G_{\boldsymbol{\alpha}} = \{m+1, \dots, r\}$ for some $1 \leqslant m \leqslant r$. Let $S = K[x_1, \dots, x_m]$ and $J = IR_{G_{\boldsymbol{\alpha}}} \cap S$.

Let $\boldsymbol{\alpha}' = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m$. By using Formula (2), we have

$$\Delta_{\boldsymbol{\alpha}'}(J^{(n)}) = \Delta_{\boldsymbol{\alpha}}(I^{(n)}). \tag{16}$$

Together with (15), it gives $\widetilde{H}_{i-|G_{\boldsymbol{\alpha}}|-1}(\Delta_{\boldsymbol{\alpha}'}(J^{(n)}); K) \neq 0$. By Lemma 1.3 we get

$$H^{i-|G_{\boldsymbol{\alpha}}|}_{\mathfrak{n}}(S/J^{(n)})_{\boldsymbol{\alpha}'} \neq 0,$$

where $\mathfrak{n} = (x_1, \dots, x_m)$ is the homogeneous maximal ideal of S. In particular,

$$|\boldsymbol{\alpha}'| \leqslant a_{i-|G_{\boldsymbol{\alpha}}|}(S/J^{(n)}).$$

Together with Lemma 2.1 and Theorem 2.2, it yields

$$|\boldsymbol{\alpha}'| \leqslant \delta(J)(n-1) \leqslant \delta(I)(n-1).$$

Therefore,

$$\operatorname{reg}(I^{(n)}) = |\boldsymbol{\alpha}| + i = |\boldsymbol{\alpha}'| + \sum_{j=m+1}^{r} \alpha_j + i \leqslant |\boldsymbol{\alpha}'| + i - |G_{\boldsymbol{\alpha}}| \leqslant \delta(I)(n-1) + i - |G_{\boldsymbol{\alpha}}|.$$

It remains to prove that $i - |G_{\boldsymbol{\alpha}}| \leqslant b$. By Lemma 1.5, we have

$$\Delta_{\boldsymbol{\alpha}'}(J^{(n)}) = \Delta_{\boldsymbol{\alpha}}(I^{(n)}) = \left\{ F \in \mathcal{F}(\operatorname{lk}_\Delta(G_{\boldsymbol{\alpha}})) \mid \sum_{j \notin F \cup G_{\boldsymbol{\alpha}}} \boldsymbol{\alpha}_j \leqslant n-1 \right\}.$$

It follows that there is a simplicial complex Γ with $\mathcal{F}(\Gamma) \subseteq \mathcal{F}(\Delta)$ such that

$$\Delta_{\boldsymbol{\alpha}'}(J^{(n)}) = \operatorname{lk}_\Gamma(G_{\boldsymbol{\alpha}}).$$

Since $\widetilde{H}_{i-|G_{\boldsymbol{\alpha}}|-1}(\operatorname{lk}_\Gamma(G_{\boldsymbol{\alpha}}); K) \neq 0$, by Lemma 1.1 we have $i - |G_{\boldsymbol{\alpha}}| \leqslant \operatorname{reg}(I_\Gamma) \leqslant b$, and then proof of the theorem is complete. □

As a direct consequence of Theorem 2.3, we have a simple bound:

Corollary 2.4. *Let I be a square-free monomial ideal. Then,*

$$\operatorname{reg}(I^{(n)}) \leqslant \delta(I)(n-1) + \dim(R/I) + 1, \quad \text{for all } n \geqslant 1.$$

Proof. Let Δ be the simplicial complex corresponding to the square-free ideal I. For every subcomplex Γ of Δ we have $\dim \Gamma \leqslant \dim \Delta$. It follows from Lemma 1.1 that

$$\operatorname{reg}(I_\Gamma) \leqslant \dim(R/I_\Gamma) + 1 \leqslant \dim(R/I_\Delta) + 1.$$

Therefore, $b = \max\{\operatorname{reg}(I_\Gamma) \mid \mathcal{F}(\Gamma) \subseteq \mathcal{F}(\Delta)\} \leqslant \dim(R/I_\Delta) + 1$. Now the corollary follows from Theorem 2.3. □

The following example shows that the bound in Corollary 2.4 is sharp and so is in Theorem 2.3. Recall that a simplicial complex Δ is called a *matroid complex* if for every subset σ of $V(\Delta)$, the simplicial complex $\Delta[\sigma]$ is pure (see e.g. [20, Chapter 3]). Here, $\Delta[\sigma]$ is the restriction of Δ to σ and defined by $\Delta[\sigma] = \{\tau \mid \tau \in \Delta \text{ and } \tau \subseteq \sigma\}$.

Example 2.5. Let Δ be a matroid complex. Assume that Δ is not a cone. Let $I = I_\Delta$ and $d = \dim(R/I)$. By [18, Theorem 4.5], for all $n \geqslant 1$ we deduce that:

(1) $\delta(I) = d(I)$.
(2) $a_d(R/I^{(n)}) = \delta(I)(n-1)$.
(3) $\operatorname{reg}(I^{(n)}) = \delta(I)(n-1) + d + 1$.

Theorem 2.3 states for a square-free monomial ideal arising from a hypergraph as follows.

Theorem 2.6. *Let $\mathcal{H}$ be a hypergraph. Then, for all $n \geqslant 1$, we have*

$$\operatorname{reg}(I(\mathcal{H})^{(n)}) \leqslant \delta(I(\mathcal{H}))(n-1) + b,$$

where $b = \max\{\operatorname{pd}(R/I(\mathcal{H}')) \mid \mathcal{H}'$ is a subhypergraph of $\mathcal{H}^*$ with $E(\mathcal{H}') \subseteq E(\mathcal{H}^*)\}$.

Proof. Let Δ be the corresponding simplicial complex of the square-free monomial ideal $I(\mathcal{H})$. Assume that $\mathcal{F}(\Delta) = \{F_1, \ldots, F_p\}$. Since

$$I(\mathcal{H}) = \bigcap_{j=1}^{p} (x_i \mid i \notin F_j),$$

so that $E(\mathcal{H}^*) = \{C_1, \ldots, C_p\}$, where $C_j = [r] \setminus F_j$ for all $j = 1, \ldots, p$.

Let Γ be a subcomplex of Δ with $\mathcal{F}(\Gamma) \subseteq \mathcal{F}(\Delta)$. We may assume that $\mathcal{F}(\Gamma) = \{F_1, \ldots, F_k\}$ for $1 \leqslant k \leqslant p$. Then, we have $I_\Gamma^* = I(\mathcal{H}')$ where $\mathcal{H}'$ is the subhypergraph of $\mathcal{H}^*$ with $E(\mathcal{H}') = \{C_1, \ldots, C_k\}$.

By Lemma 1.2 we have $\operatorname{reg}(I_\Gamma) = \operatorname{pd}(R/I_\Gamma^*) = \operatorname{pd}(R/I(\mathcal{H}'))$, and therefore the theorem follows from Theorem 2.3. □

The next theorem is the second main result of the paper. It bounds the regularity of symbolic powers of a square-free monomial ideal via the combinatorial properties of the associated hypergraph.

Theorem 2.7. *Let $\mathcal{H}$ be a simple hypergraph. Then,*

$$\operatorname{reg}(I(\mathcal{H})^{(n)}) \leqslant \delta(I(\mathcal{H}))(n-1) + |V(\mathcal{H})| - \epsilon(\mathcal{H}^*), \text{ for all } n \geqslant 1.$$

Proof. By Theorem 2.6, it suffices to show that

$$\operatorname{pd}(R/I(\mathcal{G})) \leqslant |V(\mathcal{H})| - \epsilon(\mathcal{H}^*)$$

for every hypergraph $\mathcal{G}$ with $E(\mathcal{G}) \subseteq E(\mathcal{H}^*)$. By Lemma 1.6, it suffices to prove that

$$|V(\mathcal{G})| - \epsilon(\mathcal{G}) \leqslant |V(\mathcal{H}^*)| - \epsilon(\mathcal{H}^*).$$

In order to prove this inequality, without loss of generality we may assume that $\mathcal{H}^*$ has no both trivial edges and isolated vertices.

Let S be an edgewise-dominant set of $\mathcal{G}$ such that $|S| = \epsilon(\mathcal{G})$. For each vertex $v \in V(\mathcal{H}^*) \setminus V(\mathcal{G})$, we take an edge of $\mathcal{H}^*$ containing v, and denote this edge by $F(v)$. Then,

$$S' = S \cup \{F(v) \mid v \in V(\mathcal{H}^*) \setminus V(\mathcal{G})\}$$

is an edgewise-dominant set of $\mathcal{H}^*$. It follows that

$$\epsilon(\mathcal{H}^*) \leqslant |S'| \leqslant |S| + |V(\mathcal{H}^*) \setminus V(\mathcal{G})| = |S| + |V(\mathcal{H}^*)| - |V(\mathcal{G})|,$$

and therefore $|V(\mathcal{G})| - \epsilon(\mathcal{G}) \leqslant |V(\mathcal{H}^*)| - \epsilon(\mathcal{H}^*)$, as required. □

The following result gives an upper bound for the regularity of symbolic powers of an edge ideal a graph.

Corollary 2.8. *Let G be a graph. Then,*

$$\operatorname{reg}(I(G)^{(n)}) \leqslant 2n + \dim(R/I(G)) - 1, \text{ for all } n \geqslant 1.$$

Proof. By [6, Example 4.4], we have $\delta(I(G)) = 2$, so the corollary follows from Corollary 2.4. □

Continuing with graphs, we will write down the bounds in Theorem 2.3 and 2.7 for a cover ideal of a graph.

Example 2.9. Let G be a graph. First we note that the formula for computing $\delta(J(G))$ is quite complicated, due to [6, Theorem 4.6], $\delta(J(G))$ is given by

$$\frac{r}{2} + \max\left\{\frac{|N(S)| - |S|}{2} \mid S \in \Delta(I(G)) \text{ and } G \setminus N[S] \text{ has no bipartite component}\right\},$$

where $N[S] = S \cup \{v \in V(G) \mid v$ is a neighbor of some vertex in $S\}$. The number $\delta(J(G))$ may be not an integer and even bigger than $\operatorname{reg}(J(G))$ (see [6, Lemma 5.14 and Theorem 5.15]).

Now since $J(G)^* = I(G)$, from Theorem 2.3, we obtain

$$\operatorname{reg}(J(G)^{(n)}) \leqslant \delta(J(G))(n-1) + b, \text{ for all } n \geqslant 1,$$

where $b = \max\{\operatorname{pd}(R/I(H)) \mid H \text{ is a subgraph of } G\}$.

The weaker bound obtained from Theorem 2.7 is

$$\operatorname{reg}(J(G)^{(n)}) \leqslant \delta(J(G))(n-1) + |V(G)| - \epsilon(G), \text{ for all } n \geqslant 1.$$

We conclude the paper with a remark on lower bounds.

Remark 2.10. Let I be a square-free monomial ideal. By [6, Lemma 4.2(ii)] we deduce that $d(I)n \leqslant d(I^{(n)})$, and therefore

$$\operatorname{reg}(I^{(n)}) \geqslant d(I)n, \quad \text{for all } n \geqslant 1.$$

In general, $d(I) < \delta(I)$ (see e.g. [6, Lemma 5.14]), so that the bound is not optimal.

On the other hand, by Lemma 1.7, there is a number b such that

$$\operatorname{reg}(I^{(n)}) \geqslant \delta(I)n + b, \text{ for all } n \geqslant 1.$$

The natural question is to find a good bound for b.

Acknowledgment. We are supported by Project ICRTM.02_2021.02 of the International Centre for Research and Postgraduate Training in Mathematics (ICRTM), Institute of Mathematics, VAST.

References

[1] A. Banerjee and E. Nevo, *Regularity of edge ideals via suspension*, arXiv:1908.03115.

[2] S. Beyarslan, H.T. Hà and T. N. Trung. *Regularity of powers of forests and cycles*, J.Algebraic Combin., **42** (2015), no. 4, 1077-1095.

[3] S. Cooper, R.J.D. Embree, H.T. Hà and A.H. Hoefel, *Symbolic powers of monomial ideals*, Proc.Edinb.Math. Soc. (2) **60** (2017), no. 1, 39-55.

[4] D. Cutkosky, J. Herzog and N.V. Trung, *Asymptotic behaviour of Castelnuovo-Mumford regularity*, Compositito Math. **118** (1999), 243-261.

[5] H. Dao and J. Schweig, *Bounding the projective dimension of a squarefree monomial ideal via domination in clutters*, Proc. Amer. Math. Soc. **143** (2015), no. 2, 555-565.

[6] L.X. Dung, T.T. Hien, N.D. Hop and T.N. Trung, *Regularity and Koszul property of symbolic powers of monomial ideals*, Math. Z. **298** (2021), no. 3-4, 1487-1522.

[7] D. Eisenbud, Commutative Algebra: with a View Toward Algebraic Geometry, Springer, New York (1995).

[8] S.A.S. Fakhari, *On the regularity of small symbolic powers of edge ideals of graphs*, arXiv:1908.10845.

[9] Y. Gu, H.T. Hà and J.K. O'Rourke and J.S. Skelton, *Symbolic powers of edge ideals of graphs*, Comm. Algebra **48** (2020), no. 9, 3743-3760.
[10] N.T. Hang and T.N. Trung, *The behavior of depth functions of powers of cover ideals of unimodular hypergraphs*, Ark. Mat. **55** (1) (2017), 89-104.
[11] J. Herzog and T. Hibi, Monomial Ideals, Graduate Texts in Mathematics, vol. **260**, Springer (2011).
[12] L.T. Hoa and T.N. Trung, *Partial Castelnuovo-Mumford regularities of sums and intersections of powers of monomial ideals*, Math. Proc. Cambridge Philos. Soc. **149** (2010), no. 2, 229-246.
[13] L.T. Hoa and T.N. Trung, *Castelnuovo - Mumford regularity of symbolic powers of two-dimensional square-free monomial ideals*, J. Commut. Algebra **8** (2016), no. 1, 77-88.
[14] A.V. Jayanthan, N. Narayanan and S. Selvaraja, *Regularity of powers of bipartite graphs*, J.Algebraic Combin. **47** (2018), no.1, 17-38.
[15] V. Kodiyalam, *Asymptotic behaviour of Castelnuovo-Mumford regularity*, Proc. Amer. Math. Soc. **128** (2000), 407-411.
[16] E. Miller and B. Sturmfels, Combinatorial commutative algebra, Springer, 2005.
[17] N.C. Minh and N.V. Trung, *Cohen - Macaulayness of powers of two-dimensional squarefree monomial ideals*, J. Algebra **322** (2009), 4219-4227.
[18] N.C. Minh and T.N. Trung, *Regularity of symbolic powers and arboricity of matroids*, Forum Math. **31** (2019), no. 2, 465-477.
[19] A. Schrijver, Theory of linear and integer programming, John Wiley & Sons, 1998.
[20] R. P. Stanley, Combinatorics and Commutative Algebra, second edition, Birkhauser, Boston, MA, 1996.
[21] Y. Takayama, *Combinatorial characterizations of generalized Cohen-Macaulay monomial ideals*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) **48** (2005), 327-344.
[22] N. Terai, *Alexander duality theorem and Stanley-Reisner rings. Free resolutions of coordinate rings of projective varieties and related topics* (Japanese) (Kyoto, 1998), Sūrikaisekikenkyūsho Kōkyūroku no. 1078 (1999), 174-184.

Faculty of Natural Sciences, Hong Duc University, No. 565 Quang Trung, Dong Ve, Thanh Hoa, Vietnam
Email address: hientruong86@gmail.com

Institute of Mathematics, VAST, 18 Hoang Quoc Viet, Hanoi, Viet Nam, and Institute of Mathematics and TIMAS, Thang Long University, Ha Noi, Vietnam.
Email address: tntrung@math.ac.vn