ON GORENSTEIN GRAPHS

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Abstract. We classify all Gorenstein claw-free graphs. Moreover we provide a new way to construct a Gorenstein graph from another one.

Introduction

Let $R = k[x_1, \dots, x_n]$ be a polynomial ring over a field k. Let G be a graph with vertex set $V = \{1, \dots, n\}$ and edge set E. We associate to the graph G a quadratic squarefree monomial ideal in R

$$I(G) = (x_i x_j \mid ij \in E),$$

which is called the *edge ideal* of G.

We say that G is *Cohen-Macaulay* (resp. *Gorenstein*) over k if so is $R/I(G)$. The classification of Cohen-Macaulay and Gorenstein graphs in terms of the underlying graphs is still widely open. In this paper we focus on Gorenstein graphs. In general we cannot read off the Gorenstein property of a graph just from its structure since this property as usual depends on the characteristic of k (see [6, Proposition 3.1]), so we are interested in some classes of graphs such as: bipartite graphs, chordal graphs, triangle-free graphs, locally triangle-free graphs, planar graphs, and so on (see [4, 5, 6, 7, 8, 13]).

In the paper we will add Gorenstein claw-free graphs into the list. A *claw* is the complete bipartite graph $K_{1,3}$, and a *claw-free* graph is a graph in which no induced subgraph is a claw. Let K_n be the complete graph of order n, C_n the cycle of length n and C_n^c be the complement of the cylce C_n (see Figure 1).

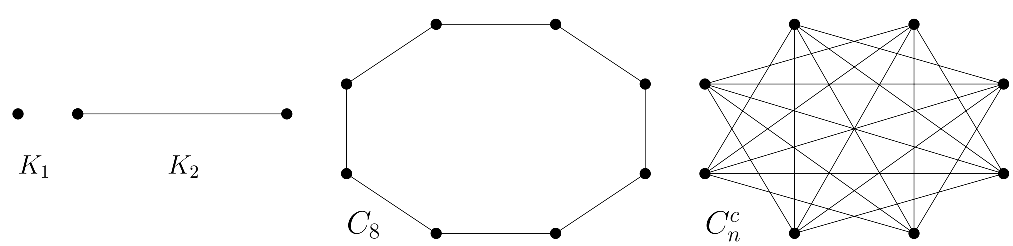

Figure 1. K_1, K_2, C_n and C_n^c with $n = 8$.

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The main result of the paper is the following theorem.

Theorem 3.4 *A claw-free graph G is Gorenstein if and only if every connected component of G is one of K_1, K_2 and C_n^c with $n \geqslant 5$.*

In the next result, we provide a way to construct a Gorenstein graph from another one. Let $\alpha(G)$ denote the independence number of the graph G. Then,

Theorem 4.4 *Let H be a Gorenstein graph and let x be a non isolated vertex of H. Let a, b and c be the three new points. Join a to b and every neighbor of x; join b to c; and join c to x. Let G be the graph obtained from this construction. Then, G is a Gorenstein graph with $\alpha(G) = \alpha(H) + 1$.*

The paper is organized as follows. In the section 1 we set up some basic notations, terminology for graph and the simplicial complex. In the section 2, we compute the Euler characteristics of the independence complexes of locally Gorenstein graphs. In the section 3, we characterize Gorenstein claw-free graphs. In the last section we give a construction of Gorenstein graphs from another Gorenstein graphs.

1. Preliminaries

Let Δ be a simplicial complex on the vertex set $V(\Delta) = \{1, \ldots, n\}$. We define the *Stanley-Reisner* ideal of the simplicial complex Δ to be the squarefree monomial ideal

$$I_\Delta = (x_{j_1} \cdots x_{j_i} \mid j_1 < \cdots < j_i \text{ and } \{j_1, \ldots, j_i\} \notin \Delta) \text{ in } R = k[x_1, \ldots, x_n]$$

and the *Stanley-Reisner* ring of Δ to be the quotient ring $k[\Delta] = R/I_\Delta$. Then, Δ is Cohen-Macaulay (resp. Gorenstein) if $k[\Delta]$ is Cohen-Macaulay (resp. Gorenstein).

If $F \in \Delta$, we define the dimension of $F \in \Delta$ to be $\dim F = |F| - 1$ and the dimension of Δ to be $\dim \Delta = \max\{\dim F \mid F \in \Delta\}$. The link of F inside Δ is its subcomplex:

$$\operatorname{lk}_\Delta F = \{H \in \Delta \mid H \cup F \in \Delta \text{ and } H \cap F = \emptyset\}.$$

For each i, let $\widetilde{\mathcal{C}}_i(\Delta; k)$ the vector space over k whose basis elements are the exterior products $e_F = e_{j_0} \wedge \cdots \wedge e_{j_i}$ that correspond to i-faces $F = \{j_0, \ldots, j_i\} \in \Delta$ with $j_0 < \cdots < j_i$. The *reduced chain complex* of Δ over k is the complex $\widetilde{\mathcal{C}}_\bullet(\Delta; k)$ whose differentials $\partial_i : \widetilde{\mathcal{C}}_i(\Delta; k) \longrightarrow \widetilde{\mathcal{C}}_{i-1}(\Delta; k)$ is given by

$$\partial_i(e_{j_0} \wedge \cdots \wedge e_{j_i}) = \sum_{s=0}^{i} (-1)^s e_{j_0} \wedge \cdots \wedge \widehat{e}_{j_s} \wedge \cdots \wedge e_{j_i},$$

and the i-th *homology group* of Δ is $\widetilde{H}_i(\Delta; k) = \ker(\partial_i)/\operatorname{im}(\partial_{i+1})$. For simplicity, if $\omega \in \widetilde{\mathcal{C}}_i(\Delta; k)$, we write $\partial\omega$ stands for $\partial_i\omega$. With this notation we have

(1) $\quad \partial(\omega \wedge \nu) = \partial\omega \wedge \nu + (-1)^{i+1}\omega \wedge \partial\nu$ for all $\omega \in \widetilde{\mathcal{C}}_i(\Delta; k)$ and $\nu \in \widetilde{\mathcal{C}}_j(\Delta; k)$.

The most widely used criterion for determining when a simplicial complex is Cohen-Macaulay is due to Reisner (see [10, Corollary 4.2]), which says that links have only top homology.

Lemma 1.1. *Δ is Cohen-Macaulay over k if and only if for all $F \in \Delta$ and all $i < \dim(\mathrm{lk}_\Delta F)$, we have $\widetilde{H}_i(\mathrm{lk}_\Delta F; k) = 0$.*

Let f_i be the number of i-dimensional faces of Δ. The *reduced Euler characteristic* $\widetilde{\chi}(\Delta)$ of Δ is defined by

$$\widetilde{\chi}(\Delta) := \sum_{i=-1}^{d} (-1)^i f_i = \sum_{F \in \Delta} (-1)^{|F|-1},$$

where $d := \dim(\Delta)$. This number can be represented via the reduced homology groups by (see e.g. [10]):

$$\widetilde{\chi}(\Delta) = \sum_{i=-1}^{d} (-1)^i \dim_k \widetilde{H}_i(\Delta; k).$$

The restriction of Δ to a subset S of $V(\Delta)$ is $\Delta|_S := \{F \in \Delta \mid F \subseteq S\}$. The star of a vertex v in Δ is $\mathrm{st}_\Delta(v) := \{F \in \Delta \mid F \cup \{v\} \in \Delta\}$. Let $\mathrm{core}(V(\Delta)) := \{x \in V(\Delta) \mid \mathrm{st}_\Delta(x) \neq V(\Delta)\}$, then the core of Δ is $\mathrm{core}(\Delta) := \Delta|_{\mathrm{core}(V(\Delta))}$. If $\Delta = \mathrm{st}_\Delta(v)$ for some vertex v, then Δ is a cone over v. Thus $\Delta = \mathrm{core}(\Delta)$ means Δ is not a cone.

Let Δ be a *pure* simplicial complex, i.e. every facet of Δ has the same cardinality. We say that Δ is an *Euler complex* if

$$\widetilde{\chi}(\mathrm{lk}_\Delta F) = (-1)^{\dim \mathrm{lk}_\Delta F} \text{ for all } F \in \Delta;$$

and Δ is a *semi-Euler complex* if $\mathrm{lk}_\Delta(x)$ is an Euler complex for all vertex x.

We then have a criterion for determining when Cohen-Macaulay complexes are Gorenstein due to Stanley (see [10, Theorem 5.1]).

Lemma 1.2. *Δ is Gorenstein if and only if and only if* $\mathrm{core}(\Delta)$ *is an Euler complex which is Cohen-Macaulay.*

Let S be a subset of the vertex set of Δ and let $\Delta \setminus S := \{F \in \Delta \mid F \cap S = \emptyset\}$, so that $\Delta \setminus S$ is a subcomplex of Δ. If $S = \{x\}$, then we write $\Delta \setminus x$ stands for $\Delta \setminus \{x\}$. Clearly, $\Delta \setminus x = \{F \in \Delta \mid x \notin F\}$.

The following lemma is the key to investigate Cohen-Macaulay simplicial complexes in the sequence of this paper (see [6, Lemma 1.4]).

Lemma 1.3. *Let Δ be a Gorenstein simplicial complex with $\Delta = \mathrm{core}(\Delta)$. If S is a subset of $V(\Delta)$ such that $\Delta|_S$ is a cone, then $\widetilde{H}_i(\Delta \setminus S, k) = 0$ for all i.*

We next recall some notations, terminology from Graph theory (see [1]). Let G be a graph. We use the symbols $V(G)$ and $E(G)$ to denote the vertex set and the edge set of G respectively. Let S be a subset of $V(G)$, we denote $G[S]$ to be the *induced subgraph* of G on S; and denote $G \setminus S$ to be the induced subgraph of G on $V(G) \setminus S$.

Two vertices in G which are incident with a common edge are *adjacent*, and two distinct adjacent vertices in G are *neighbors*. The set of neighbors of a vertex v in G is denoted by $N_G(v)$. For a subset S of vertices of G, we denote the *neighbors* of S by

$$N_G(S) := \{x \in V(G) \setminus S \mid N_G(x) \cap S \neq \emptyset\},$$

the *closed neighbors* of S by $N_G[S] := S \cup N_G(S)$, and the *localization* of G with respect to S by $G_S := G \setminus (S \cup N_G(S))$.

An *independent set* in G is a set of vertices no two of which are adjacent to each other. The *independence number* of G, denoted by $\alpha(G)$, is the cardinality of the largest independent set in G. The set of all independent sets of G is called the *independence complex* of G and denoted by $\Delta(G)$. It is well-known that $I_{\Delta(G)} = I(G)$ and $\dim(\Delta(G)) = \alpha(G) - 1$.

A graph G is called *well-covered* if very maximal independent set of G has the same size, that is $\alpha(G)$. A well-covered graph G is said to be a member of the class W_2 if $G \setminus v$ is well-covered with $\alpha(G \setminus v) = \alpha(G)$ for every vertex v (see [9, 12]). At first sight, if G is Gorenstein without isolated vertices, then G is in W_2 (see [6, Lemma 2.4]). Note that G is well-covered if and only if $\Delta(G)$ is pure; and $\Delta(G) = \mathrm{core}(\Delta(G))$ if and only if G has no isolated vertices.

The well-covered graphs behave well when taking localization.

Lemma 1.4. [3, Lemma 1] *If G be a well-covered graph and S is a an independent set of G then G_S is well-covered. Moreover, $\alpha(G_S) = \alpha(G) - |S|$.*

For any independent set S of G we have $\Delta(G_S) = \mathrm{lk}_{\Delta(G)}(S)$. Therefore, G_S is Cohen-Macaulay (resp. Gorenstein) if so is G by Lemma 1.1 (resp. Lemma 1.2). We say that G is locally Cohen-Macaulay (resp. Gorenstein) if G is well-covered and G_v is Cohen-Macaulay (resp. Gorenstein) for every vertex v.

Lemma 1.5. [6, Lemma 2.3] *Let G be a locally Gorenstein graph in W_2 and let S be a nonempty independent set of G. Then we have G_S is Gorenstein and $\Delta(G_S)$ is Eulerian with* $\dim(\Delta(G_S)) = \dim(\Delta(G)) - |S|$.

Remark 1.6. If $\Delta(G)$ is Eulerian, then G has no isolated vertices. Because if G has some isolated vertices, then $\Delta(G)$ is cone, and then $\widetilde{\chi}(\Delta(G)) = 0$, a contradiction.

2. Euler characteristics of semi-Eulerian independence complexes

In this section we will compute $\widetilde{\chi}(\Delta(G))$ when $\Delta(G)$ is semi-Eulerian. This formula plays a key tool for two remaining sections.

Lemma 2.1. *Let Δ be a simplicial complex. If Δ is not a void complex, then*

$$\sum_{F \in \Delta} \widetilde{\chi}(\mathrm{lk}_\Delta(F)) = -1.$$

Proof. Let $d := \dim(\Delta)$. We prove the lemma by induction on d. If $d = -1$, then $\Delta = \{\emptyset\}$, and the lemma holds for this case.

If $d = 0$, then Δ consists of isolated vertices, say $v_1, \ldots, v_n$, where $n = |V(G)|$. We have

$$\sum_{F \in \Delta} \widetilde{\chi}(\mathrm{lk}_\Delta(F)) = \widetilde{\chi}(\mathrm{lk}_\Delta(\{\emptyset\})) + \sum_{i=1}^{n} \widetilde{\chi}(\mathrm{lk}_\Delta(v_i)) = -1 + n - n = -1,$$

and hence the lemma holds.

Assume that $d \geqslant 1$. Let $F_1, \dots, F_t$ be the facets of Δ. We may assume that $\dim(F_i) = d$ for all $i = 1, \dots, s$; and $\dim(F_i) < d$ for all $i = s+1, \dots, t$; with $1 \leqslant s \leqslant t$. We now proceed to prove by induction on s. If $s = 1$, let $\Lambda = \Delta \setminus \{F_1\}$. Because $\dim \Lambda = \dim \Delta - 1 = d - 1$, by the induction hypothesis we have

$$\sum_{F \in \Lambda} \widetilde{\chi}(\mathrm{lk}_\Delta(F)) = -1,$$

so that

$$\sum_{F \in \Delta} \widetilde{\chi}(\mathrm{lk}_\Delta(F)) = \sum_{F \in \Lambda} \widetilde{\chi}(\mathrm{lk}_\Delta(F)) + \sum_{F \subseteq F_1} (-1)^{|F_1| - |F| - 1} = -1 + (-1)^{|F_1|} \sum_{F \subseteq F_1} (-1)^{|F|-1}$$
$$= -1 + (-1)^{|F_1|} \widetilde{\chi}(\langle F_1 \rangle),$$

where $\langle F_1 \rangle$ is the simplex on the vertex set F_1. Note that $F_1 \neq \emptyset$, so $\widetilde{\chi}(\langle F_1 \rangle) = 0$, and so $\sum_{F \in \Delta} \widetilde{\chi}(\mathrm{lk}_\Delta(F) = -1$.

Assume that $s \geqslant 2$. By the same argument as the previous case , let $\Lambda := \Delta \setminus \{F_s\}$. Since $\dim \Lambda = \dim \Delta$ and Λ has one facet less than Δ, by the induction hypothesis on s we have $\sum_{F \in \Delta} \widetilde{\chi}(\mathrm{lk}_\Lambda(F)) = -1$. Therefore,

$$\sum_{F \in \Delta} \widetilde{\chi}(\mathrm{lk}_\Delta(F)) = \sum_{F \in \Lambda} \widetilde{\chi}(\mathrm{lk}_\Delta(F)) + (-1)^{|F_s|} \widetilde{\chi}(\langle F_s \rangle) = -1.$$

□

Proposition 2.2. *Let G be a graph such that $\Delta(G)$ is semi-Eulerian. Let v be a vertex of G and $d := \dim(\Delta(G))$. Then,*

$$\widetilde{\chi}(\Delta(G)) = (-1)^d (1 + \widetilde{\chi}(\Delta(G[N_G(v)]))).$$

Proof. We prove by induction on d. If $d = 0$, then $\alpha(G) = 1$. In this case, G is a complete graph, so $G = K_n$ where $n = |V(G)|$. Let v be a vertex of G. Then, $G[N_G(v)] = K_{n-1}$. Hence,

$$\widetilde{\chi}(\Delta(G)) = -1 + n \text{ and } \widetilde{\chi}(G[N_G(v)]) = -1 + (n-1) = -2 + n,$$

and hence the proposition holds.

Assume that $d \geqslant 1$ so that $\alpha(G) \geqslant 2$. By [3, Lemma 2(i)] and Remark 1.6 we deduce that G is in W_2, in particular G is well-covered. Let $\Delta := \Delta(G)$ and $A := N_G(v)$. Let

$$\Gamma := \{F \in \Delta \mid F \cap A \neq \emptyset\},$$

so that Δ can be partitioned into $\Delta = \mathrm{st}_\Delta(v) \cup \Gamma$. Note that $\widetilde{\chi}(\mathrm{st}_\Delta(v)) = 0$ because $\mathrm{st}_\Delta(v)$ is a cone over v. Thus,

$$\widetilde{\chi}(\Delta) = \sum_{F \in \Delta} (-1)^{|F|-1} = \sum_{F \in \mathrm{st}_\Delta(v)} (-1)^{|F|-1} + \sum_{F \in \Gamma} (-1)^{|F|-1}$$
$$= \widetilde{\chi}(\mathrm{st}_\Delta(v)) + \sum_{F \in \Gamma} (-1)^{|F|-1} = \sum_{F \in \Gamma} (-1)^{|F|-1}.$$

Let $\Lambda := \Delta(G[N_G(v)])$ and let $\Omega := \Lambda \setminus \{\emptyset\}$. For each $S \in \Omega$, we define

$$g(S) := \sum_{F\in\Gamma, S\subseteq F} (-1)^{|F|-1}, \quad \text{and } \tau(S) := \sum_{F\in\Gamma, F\cap A=S} (-1)^{|F|-1}.$$

Then,

$$\widetilde{\chi}(\Delta) = \sum_{U\in\Omega} \tau(S), \quad \text{and} \quad g(S) = \sum_{F\in\Omega, S\subseteq F} \tau(F).$$

For every $S \in \Omega$, as S is a nonempty face of Δ and Δ is semi-Eulerian, we have $\Delta(G_S)$ is Eulerian. Since G is well-covered, $\alpha(G_S) = \alpha(G) - |S|$ by Lemma 1.4, and so $\dim(\Delta(G_S)) = \alpha(G_S) - 1 = d - |S|$. Hence, $\widetilde{\chi}(\Delta(G_S)) = (-1)^{d-|S|}$, and hence

$$\begin{aligned} g(S) &= \sum_{F\in\Gamma, S\subseteq F} (-1)^{|F|-1} = \sum_{F\in\Delta, S\subseteq F} (-1)^{|F|-1} = \sum_{F\in\Delta(G_S)} (-1)^{|F|+|S|-1} \\ &= (-1)^{|S|} \sum_{F\in\Delta(G_S)} (-1)^{|F|-1} = (-1)^{|S|}\widetilde{\chi}(G_S) = (-1)^{|S|}(-1)^{d-|S|} = (-1)^d. \end{aligned}$$

We now consider Ω as a poset with the partial order $\leqslant$ being inclusion. Then, $g(S)$ can be written as

$$g(S) = \sum_{F\in\Omega, F\geqslant S} \tau(F).$$

Let μ be the Mobius function of the poset Ω. Then by Mobius inversion formula (see [11, Proposition 3.7.2]) we have

$$\tau(S) = \sum_{F\in\Omega, F\geqslant S} \mu(S,F)g(F) = \sum_{F\in\Omega, F\geqslant S} \mu(S,F)(-1)^d = (-1)^d \sum_{F\in\Omega, F\geqslant S} \mu(S,F).$$

Observe that if $S \leqslant F$ in Ω, then every T such that $S \subseteq T \subseteq F$ we have $T \in \Omega$ and $S \leqslant T \leqslant F$. Hence, $\mu(S,F) = (-1)^{|F|-|S|}$, and hence

$$\begin{aligned} \tau(S) &= (-1)^d \sum_{F\in\Omega, F\geqslant S} (-1)^{|F|-|S|} = (-1)^d \sum_{F\in\Omega, F\geqslant S} (-1)^{|F|-|S|} \\ &= (-1)^d \sum_{F\in\Lambda, S\subseteq F} (-1)^{|F|-|S|} = -(-1)^d \sum_{F\in \mathrm{lk}_\Lambda(S)} (-1)^{|F|-1} = -(-1)^d\widetilde{\chi}(\mathrm{lk}_\Lambda(S)). \end{aligned}$$

Therefore,

$$\begin{aligned} \widetilde{\chi}(\Delta) &= \sum_{S\in\Omega} \tau(S) = \sum_{S\in\Lambda, S\neq\emptyset} \tau(S) = -(-1)^d \sum_{S\in\Lambda, S\neq\emptyset} \widetilde{\chi}(\mathrm{lk}_\Lambda(S)) \\ &= (-1)^d(\mathrm{lk}_\Lambda(\emptyset) - \sum_{S\in\Lambda} \widetilde{\chi}(\mathrm{lk}_\Lambda(S))) = (-1)^d(\widetilde{\chi}(\Lambda)) - \sum_{S\in\Lambda} \widetilde{\chi}(\mathrm{lk}_\Lambda(S))). \end{aligned}$$

On the other hand, by Proposition 2.1 we have $\sum_{S\in\Lambda} \widetilde{\chi}(\mathrm{lk}_\Lambda(S)) = -1$, and thus

$$\widetilde{\chi}(\Delta) = (-1)^d(1 + \widetilde{\chi}(\Delta(G[N_G(v)]))),$$

as required. $\square$

As a consequence we have.

Corollary 2.3. *Let G be a graph such that $\Delta(G)$ is semi-Eulerian. Then, the following conditions are equivalent:*

(1) $\Delta(G)$ is Eulerian;
(2) $\widetilde{\chi}(\Delta(G[N_G(v)])) = 0$ for every vertex v of G;
(3) $\widetilde{\chi}(\Delta(G[N_G(v)])) = 0$ for some vertex v of G.

Proof. Let $\Delta = \Delta(G)$ and $d := \dim(\Delta(G))$. (1) $\Longrightarrow$ (2): Since Δ is Eulerian, $\widetilde{\chi}(\Delta) = (-1)^d$. By Proposition 2.2 we have

$$\widetilde{\chi}(\Delta) = (-1)^d = (-1)^d(1 + \widetilde{\chi}(\Delta(G[N_G(v)])),$$

so that $\widetilde{\chi}(\Delta(G[N_G(v)])) = 0$.

(2) $\Longrightarrow$ (3): Obviously. (3) $\Longrightarrow$ (1): Since Δ is semi-Eulerian, by Proposition 2.2 we have

$$\widetilde{\chi}(\Delta) = (-1)^d(1 + \widetilde{\chi}(\Delta(G[N_G(v)])) = (-1)^d.$$

Together with the semi-Eulerian property of Δ, it follows that Δ is Eulerian. $\square$

3. Gorenstein Claw-free Graphs

This section devotes to classify Gorenstein claw-free graphs. In principle, we can use the classification of claw-free graphs announced in [2], but this classification is highly non-transparent and quite complicated, so we classify by using the result given in the previous section and the localization property of Gorenstein graphs. We start with graphs of small independence numbers.

Lemma 3.1. *Let G be a Gorenstein connected graph without isolated vertices. Then,*

(1) If $\alpha(G) = 1$, then G is K_2.
(2) If $\alpha(G) = 2$, then G is C_n^c with $n \geqslant 5$.

Proof. If $\alpha(G) = 1$, then $G = K_n$ is the complete graph where $n = |V(G)|$. Since G is a Gorenstein graph without isolated vertices, we have $\widetilde{\chi}(\Delta(G)) = -1 + n = 1$, and so $n = 2$.

If $\alpha(G) = 2$, for each $v \in V(G)$, G_v is Gorenstein. Since $\alpha(G_v) = 1$ by Lemma 1.4, G_v is just one edge by the previous case. It follows that $\deg_G(v) = n - 3$ for every $v \in V(G)$. Hence, $\deg_{G^c}(v) = 2$ for every $v \in V(G^c)$, so G^c is the cycle C_n where $n = |V(G)|$, and so $G = C_n^c$. Since $\alpha(G) = 2$ and G is connected, $n \geqslant 5$, as required. $\square$

Remark 3.2. A graph G is claw-free if and only if $\alpha(G[N_G(v)]) \leqslant 2$ for all $v \in V(G)$.

Lemma 3.3. *Let G be a connected Gorenstein claw-free graph without isolated vertices and ab an edge of G. If $N_G[a] \subseteq N_G[b]$, then G is just the edge ab.*

Proof. We prove by induction on $\alpha(G)$. If $\alpha(G) = 1$, then G is just ab by Lemma 3.1. In this case $N_G[a] = N_G[b]$.

Assume that $\alpha(G) \geqslant 2$ and we will derive a contradiction. Let $A := N_G(a) \setminus \{b\}$ and $B := N_G(b) \setminus \{a\}$. Then, $A \subseteq B$ because of $N_G[a] \subseteq N_G[b]$.

We first claim that every vertex of G_{ab} is adjacent with every vertex in B of G. Indeed, assume on the contrary that there is $v \in V(G_{ab})$ that is not adjacent with every vertex in B. Let H be a connected component of G_v that have the edge ab. Then, we have $N_H[a] \subseteq N_H[b]$ and $N_H(b) \neq \{a\}$. Since H is Gorenstein and $\alpha(H) \leqslant \alpha(G_v) = \alpha(G)-1$, H is just the edge ab by the induction hypothesis. This is impossible as $N_H(b) \neq \{a\}$, and the claim follows.

Since G is claw-free and $B \subset N_G(b)$, $\alpha(G[B]) \leqslant \alpha(G[N_G(b)]) \leqslant 2$. Let S be a maximal independent set of $G[B]$. Note that a and b both are adjacent with every vertex in S. Together with the claim above, we imply that S is a maximal independent set of G. Thus, $\alpha(G) = |S| \leqslant 2$. Together with the assumption $\alpha(G) \geqslant 2$, we get $\alpha(G) = 2$. Therefore, $G = C_n^c$ for $n = |V(G)| \geqslant 5$ by Lemma 3.1.

But in this case, we have $N_G[a] \nsubseteq N_G[b]$ by the structure of the graph C_n^c, a contradiction. Thus, the proof of the lemma is complete. $\square$

We are now in position to prove the main result of the paper.

Theorem 3.4. *A claw-free graph G is Gorenstein if and only if every connected component of G is one of K_1, K_2 and C_n^c with $n \geqslant 5$.*

Proof. Since G is Gorenstein if and only if every its connected component is Gorenstein, we may assume that G is connected.

If G is K_1, then G is Gorenstein. Hence, we may assume that G has no isolated vertices. If $\alpha(G) = 1$, then G is K_2 by Lemma 3.1.

Assume that $\alpha(G) \geqslant 2$. Let v be a vertex of G with maximal degree and let $H := G[N_G(v)]$. Since G is claw-free, H has at most 2 connected components and $\alpha(H) \leqslant 2$. We now consider two cases:

Case 1: H is connected. We first claim that H is well-covered with $\alpha(H) = 2$. Indeed, assume that the claim is not true so that either $\alpha(H) = 1$ or $\alpha(H) = 2$ and H is not well-covered. In both cases, there is a vertex u of H such that u is adjacent to every other vertices of H. It gives $N_G[v] \subseteq N_G[u]$. Then, G is just one edge by Lemma 3.3, so $\alpha(G) = 1$, a contradiction, and the claim follows.

By this claim we have $\Delta(H)$ is pure and $\dim(\Delta(H)) = 1$. Thus, we may regard $\Delta(H)$ as a graph on the vertex set $V(H)$, whose facets are couples $\{u, v\}$ where u, v are distinct vertices of H with $uv \notin E(G)$. In other words, $\Delta(H) = H^c$. Note that H^c has no isolated vertices since $\Delta(H)$ is pure and $\dim(\Delta(H)) = 1$.

Let m be the number of edges of the graph H^c. We then have $\widetilde{\chi}(\Delta(H)) = -1 + |V(H)| - m$. Together with Corollary 2.3, this equality gives $-1 + |V(H)| - m = 0$, or equivalently $m = |V(H)|-1$. Since H^c has no isolated vertices, from $m = |V(H^c)|-1$, we deduce that one of connected component of H^c, say T, is a tree.

Let u be a leaf of T and w the unique neighbor of u in T. Then, $\deg_H(u) = |V(H)| - 1 = \deg_G(v) - 1$. Now let U be set set of vertices of G_v that are adjacent with u and let W be the set of vertices of G_v that are adjacent with w. Then, both U and W are non-empty by Lemma 3.3.

Since $\deg_G(u) = \deg_H(u) + |U| = \deg_G(v) - 1 + |U| \leqslant \deg_G(v)$ and $U \neq \emptyset$, so $|U| = 1$. Hence, we may assume that $U = \{s\}$. Note that G is claw-free, therefore $G[W]$ is complete. Since $G_u[N_{G_u}]$ is an induced subgraph of $G[W]$, it is also a complete graph. This shows that w is a vertex of G_u which has $G_u[N_{G_u}]$ is a complete graph. By Lemma 3.3, the connected component of G_u containing the vertex w is just an edge, say wt.

Let G' be the induced subgraph of G on the vertex set $V(G_v)\setminus\{s,t\}$. We next claim that G' is the empty graph. Indeed, assume on the contrary that G' is not empty. Let b be a vertex of G' (see Figure 2). Observe that G_u is a disjoint of G' and the edge wt, therefore both w and t are not adjacent with b. Clearly, u is not adjacent with b neither.

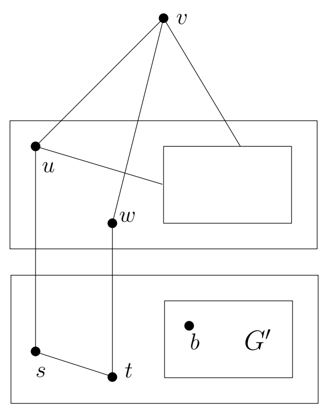

Figure 2. The configuration for the case 1.

We now show that b is not adjacent with s. Indeed, if s is adjacent with b. Then, $G_{\{v,b\}}$ would have an isolated vertex t. This is impossible by the virtue of Lemma 1.5, so b is not adjacent with s.

Therefore, b is not adjacent to any vertex of $\{u, w, s, t\}$. Since G is connected, b is adjacent with some vertex in $N_G(v) \setminus \{u, v\}$, say c.

On the other hand, when considering the graph G_t, we have $N_{G_t}[v] \subseteq N_{G_t}[u]$ since u is adjacent to every vertex of $N_G(v) \setminus \{w\}$ in G. By Lemma 3.3, G_t must be the edge vu. Thus, t is adjacent to s and every vertex of $N_G(v) \setminus \{w\}$. Especially, t is adjacent with c.

Consequently, we get $v, t, b \in N_G(c)$. But $\{v, t, b\}$ is an independent set of G, so $G[\{v, b, t, c\}]$ is a claw, a contradiction. It follows that G' is empty.

Finally, since G' is empty and t is adjacent with s and every vertex in $N_G(v)\setminus\{u\}$, we have G_t is just the edge vu. In particular, $\alpha(G_t) = 1$. Thus, $\alpha(G) = \alpha(G_t)+1 = 2$, and thus G is C_n^c with $n \geqslant 5$ by Lemma 3.3.

Case 2: H is disconnected. Since G is claw-free, H has exactly two connected components, say H_1 and H_2. Since $\alpha(H) \leqslant 2$, we have H_1 and H_2 are complete graphs. Let $p := |V(H_1)|$ and $q := |V(H_2)|$. Observe that $\dim \Delta(H) = 1$, and $\Delta(H)$

has $p + q$ vertices and pq faces of dimension 1. Hence,

$$\widetilde{\chi}(\Delta(H)) = -1 + (p + q) - pq = -(p - 1)(q - 1).$$

By Corollary 2.3 we have $\widetilde{\chi}(\Delta(H)) = 0$. Therefore, either $p = 1$ or $q = 1$. We may assume that $p = 1$.

If $q = 1$, then $\deg_G(v) = 2$. Thus, every vertex of G has degree at most 2, and thus G is either a path or a cycle. If G is a path then it must be K_2, and the $\alpha(G) = 1$. But $\alpha(G) \geqslant 2$, so G is a cycle, and so G is C_5. In this case $G = C_5 = C_5^c$.

Assume that $q \geqslant 2$. Since $H_1 = K_1$, it is just one vertex, say u. Let U be the set of all vertices of G_v which are adjacent with v. By Lemma 3.3, $U \neq \emptyset$ (see Figure 3). We first prove two following claims.

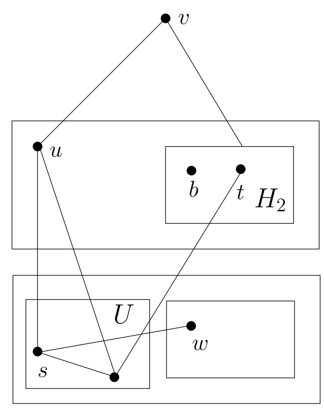

Figure 3. The configuration for the case 2.

Claim 1: $|U| \geqslant 2$. Indeed, assume on the contrary that, $|U| < 2$, i.e. $|U| = 1$. In this case $U = \{w\}$ for some vertex w of G_v. For any vertex b of H_2, since u is a leaf of G_b, we have the edge uw is just one connected component of G_b. In particular, w is not adjacent with any vertex of H_2. By applying Lemma 3.3 for the graph G_w, we have the connected component of G_g having a vertex v, must be an edge. It follows that H_2 is just one vertex, so $q = 1$, a contradiction. Thus, $|U| \geqslant 2$, as required

Claim 2: $V(G_v) = U$. Indeed, assume on the contrary that $U \neq V(G_v)$. Let w be a vertex in $V(G_b) \setminus U$. For any vertex b of H_2, since $G_b[N_{G_b}(u)]$ is a complete graph, by Lemma 3.3 we have G_b is just one edge. Together with Claim 1, it follows that b is adjacent to some vertex in U, say b'. Let B be the set of vertices of G_v which are adjacent with B. Then, $\deg_G(b) = q + |B| = \deg_G(v) - 1 + |B|$. Together with $\deg_G(b) \leqslant \deg_G(v)$, it yields $|B| = 1$. Therefore, b' is the unique vertex in G_b that adjacent with b in G. In particular, b is not adjacent with w. Hence, w is not adjacent to any vertex of H, and hence it must be adjacent with some vertex in U, say s. By Lemma 3.3, the connected component of G_s containing v is just one edge, say vt. In turn, it follows that G_t has a connected component containing the vertex u must be the edge us. But this graph also has the edge sw, a contradiction. Therefore, $V(G_v) = U$, as claimed.

We now turn to prove the theorem in this case. Since G is claw-free, and $\{v\} \cup U$ is the neighbors of u, we have $G[U]$ is a complete graph. By Claim 2, we have $G_v = G[U]$, hence $\alpha(G_v) = \alpha(G[U]) = 1$, and hence $\alpha(G) = \alpha(G_v) + 1 = 2$. By Lemma 3.1, $G = C_n^c$ for $n = |V(G)| \geqslant 5$, and the proof of the theorem is complete. $\square$

Remark 3.5. By looking into the structure of the graph C_n^c, we find out that the Gorenstein graph in the case 2 of the proof of Theorem 3.4 is exactly C_6^c as in the figure 4. It also has the name R_3 (see Plummer [9]).

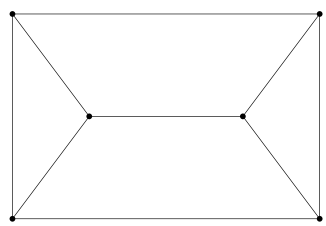

Figure 4. The graph C_6^c (or R_3).

4. A construction of Gorenstein graphs

In this section we give a way to construct a Gorenstein graph from another one. First we investigate the vanishing of homology groups of simplicial complexes

Lemma 4.1. *Let G be a locally Gorenstein graph in W_2, X an independent set of G and an integer $i < \dim(\Delta(G))$. Then every $\omega \in \mathcal{C}_i(\Delta(G); k)$ with $\partial\omega = 0$ can be written as $\omega = \partial\eta + \omega'$ where $\eta \in \mathcal{C}_{i+1}(\Delta(G); k)$ and $\omega' \in \mathcal{C}_i(\Delta(G) \setminus X; k)$.*

Proof. The lemma is trivial if $X = \emptyset$, so we assume that $X \neq \emptyset$. Let $\Delta := \Delta(G)$. We represent ω as:

$$\omega = \sum_{S \subseteq X} e_S \wedge \omega_S$$

where $\omega_S \in \mathcal{C}_{i-|S|}(\Delta(G_S \setminus X); k)$ for each S.

If there is $\emptyset \neq F \subseteq X$ such that $\omega_F \neq 0$, then we take such an F such that $|F|$ is maximal. By Equation (1) we get

$$\partial\omega = \sum_{S \subseteq X} \left(\partial e_S \wedge \omega_S + (-1)^{|S|} e_S \wedge \partial\omega_S\right) = 0.$$

Which implies $\partial\omega_F = 0$.

Next, we claim that

$$\widetilde{H}_{i-|F|}(\Delta(G_F \setminus X); k) = 0. \tag{2}$$

Indeed, we have G_F is Gorenstein because $F \neq \emptyset$. By Lemma 1.5 we imply that G_F has no isolated vertices, whence $\operatorname{core}(\Delta(G_F)) = \Delta(G_F)$. By Lemma 1.4 we have $\alpha(G_F) = \alpha(G) - |F|$, so $\dim(\Delta(G_F)) = \dim(\Delta) - |F|$. In particular, $i - |F| < \dim(\Delta(G_F))$. If $F = X$, then $G_F \setminus X = G_F$ is Cohen-Macaulay. By Lemma 1.1 we have $\widetilde{H}_{i-|F|}(\Delta(G_F \setminus X); k) = \widetilde{H}_{i-|F|}(\Delta(G_F); k) = 0$.

If F is a proper subset of X, then $G_F \setminus X = G_F \setminus (X \setminus F)$. Since $X \setminus F$ is an independent set of G_F, by Lemma 1.3 we have $\widetilde{H}_{i-|F|}(\Delta(G_F \setminus X); k) = 0$, as claimed.

We now prove the lemma. By Equation (2), we get $\omega_F = \partial(\eta_F)$ for some $\eta_F \in \mathcal{C}_{i-|F|+1}(\Delta(G_F \setminus X); k)$. Write $F = \{a_1, \dots, a_s\}$, where $a_1 < \cdots < a_s$, then

$$\partial e_F = \sum_{i=1}^{s} (-1)^{i-1} e_{a_1} \wedge \cdots \wedge \widehat{e}_{a_i} \wedge \cdots \wedge e_{a_s} = \sum_{i=1}^{s} (-1)^{i-1} e_{F \setminus \{a_i\}}.$$

By Equation (1) we have

$$\partial(e_F \wedge \eta_F) = \partial e_F \wedge \eta_F + (-1)^{|F|} e_F \wedge \partial \eta_F = \partial e_F \wedge \eta_F + (-1)^{|F|} e_F \wedge \omega_F,$$

so

$$\omega - \partial((-1)^{|F|} e_F \wedge \eta_F) = \sum_{i=1}^{s} e_{F \setminus \{a_i\}} \wedge ((-1)^{|F|+i} \eta_F) + \sum_{S \subseteq X, S \neq F} e_S \wedge \omega_S.$$

Note that $(-1)^{|F|} e_F \wedge \eta_F \in \mathcal{C}_{i+1}(\Delta; k)$. By repeating this process after finitely many steps, we can find an element $\eta \in \mathcal{C}_{i+1}(\Delta; k)$ such that $\omega - \partial\eta \in \mathcal{C}_i(\Delta(G) \setminus X; k)$, and hence $\omega = \partial\eta + \omega'$ for some $\omega' \in C_i(\Delta(G) \setminus X; k)$, as required. ☐

Lemma 4.2. *Let G be a locally Gorenstein graph in W_2. Let v be a vertex and let $H := G[N_G(v)]$. Assume that there is an independent set X of H such that for every nonempty independent sets S of $H \setminus X$ we have either H_S has an isolated vertex or H_S is empty. Then $\widetilde{H}_i(\Delta(G); k) = 0$ for all $i < \dim(\Delta(G))$.*

Proof. Let $\Delta := \Delta(G)$ and $d := \dim(\Delta(G))$. We first claim that

$$\widetilde{H}_{i-|S|}(\Delta(G_S) \setminus V(H); k) = 0 \quad \text{for any } \emptyset \neq S \in \Delta(H) \setminus X. \tag{3}$$

Indeed, observe that G_S is Gorenstein because G is locally Gorenstein. By Lemma 1.5 we have $\alpha(G_S) = d - |S|$. If H_S is empty. Since $i - |S| < d - |S| = \dim(\Delta(G_S))$, by Lemma 1.1 we get $\widetilde{H}_{i-|S|}(\Delta(G_S) \setminus V(H); k) = \widetilde{H}_{i-|S|}(\Delta(G_S); k) = 0$.

If H_S is not empty, then H_S has an isolated vertex, and so $\Delta(G_S)|_{V(H_S)}$ is a cone. By Lemma 1.3, $\widetilde{H}_{i-|F|}(\Delta(G_S) \setminus V(H); k) = \widetilde{H}_{i-|F|}(\Delta(G_F \setminus V(H_S)); k) = 0$, as claimed.

We now prove prove $\widetilde{H}_i(\Delta; k) = 0$. It suffices to show that if $\omega \in \mathcal{C}_i(\Delta; k)$ with $\partial\omega = 0$, then $\omega = \partial\zeta$ for some $\zeta \in \mathcal{C}_{i+1}(\Delta; k)$.

By Lemma 4.1, $\omega = \partial\eta + \omega'$ where $\eta \in C_{i+1}(\Delta; k)$ and $\omega' \in C_i(\Delta(G) \setminus X; k)$. Since $\omega' \in C_i(\Delta(G) \setminus X; k)$, we can write ω' as

$$\omega' = \sum_{S \in \Delta(H) \setminus X} e_S \wedge \omega_S$$

where $\omega_S \in \mathcal{C}_{i-|S|}(\Delta(G_S \setminus V(H)); k)$. By using the same argument as in the proof of Lemma 4.1 where the equation (2) is replaced by the equation (3), we can find an element $\eta' \in \mathcal{C}_{i+1}(\Delta; k)$ such that

$$\omega' - \partial\eta' \in \mathcal{C}_i(\Delta(G) \setminus V(H); k).$$

Note that $\Delta(G) \setminus V(H) = \mathrm{st}_\Delta(v)$ and $\widetilde{H}_i(\mathrm{st}_\Delta(v); k) = 0$. Since $\partial(\omega' - \partial\eta') = \partial\omega' - \partial^2\eta' = 0$, there is $\xi \in \mathcal{C}_{i+1}(\mathrm{st}_\Delta(v); k) \subseteq \mathcal{C}_{i+1}(\Delta; k)$ such that $\omega' - \partial\eta' = \partial\xi$, and hence $\omega' = \partial(\eta' + \xi)$.

Together with $\omega = \partial\eta + \omega'$, we obtain $\omega = \partial(\eta + \eta' + \xi)$, as required. □

Proposition 4.3. *Let G be a well-covered graph such that*

(1) G_x is a Gorenstein graph without isolated vertices, for every vertex x.

(2) $G[N_G(v)]$ has an isolated vertex for some vertex v of G.

Then, G is Gorenstein.

Proof. For every vertex v, by [6, Lemma 2.4] we have G_v is in W_2 because G_v is a Gorenstein graph without isolated vertices. Together with the fact that G is well-covered, we conclude that G is in W_2.

If $\alpha(G) = 1$, then G is the complete graph K_n where $n = |V(G)|$. Let v be a vertex of G such that $G[N_G(v)]$ has an isolated vertex. Note that $G[N_G(v)]$ is K_{n-1} so that $n = 1$. Hence G is an edge, and hence it is Gorenstein.

Assume that $\alpha(G) > 1$. For any vertex x of G we have G_x is a Gorenstein without isolated vertices, so $\Delta(G_x)$ is Eulerian. It yields $\Delta(G)$ is semi-Eulerian.

Let v be a vertex of G such that $G[N_G(v)]$ has an isolated vertex, say u. Then, $\widetilde{\chi}(\Delta(G[N_G(v)])) = 0$ since $\Delta(G[N_G(v)])$ is a cone over u. By Corollary 2.3 we have $\Delta(G)$ is Eulerian.

Since G is locally Gorenstein, by Lemmas 1.1 and 1.2, G is Gorenstein if

$$\widetilde{H}_i(\Delta(G); k) = 0, \text{ for } i < \dim(\Delta(G)).$$

In order to prove this, we let $H := G[N_G(v)]$ and $X = \{u\}$. Then, for any independent set S of $H \setminus X$ we have H_S has an isolated vertex u, so $\widetilde{H}_i(\Delta(G); k) = 0$, for $i < \dim(\Delta(G))$ by Lemma 4.2, as required. □

We now are in position to prove the main result of this section.

Theorem 4.4. *Let H be a Gorenstein graph and let x be a non isolated vertex of H. Let a, b and c be the three new points. Join a to b and every neighbor of x; join b to c; and join c to x. Let G be the graph obtained from this construction. Then, G is a Gorenstein graph with $\alpha(G) = \alpha(H) + 1$.*

Proof. We may assume that H has no isolated vertices so that H is in W_2. We prove the proposition by induction on $\alpha(H)$. If $\alpha(H) = 1$, then $H = K_2$. In this case, G is just a pentagon, so the proposition holds.

Assume that $\alpha(H) \geqslant 2$. We first prove that G is Gorenstein. Let v be an arbitrary vertex of G. We now claim that G_v is Gorenstein with $\alpha(G_v) = \alpha(H)$ and without isolated vertices. Indeed, observe that if $v \notin \{a, b, c, x\} \cup N_H(x)$ then $v \in V(H_x)$. We consider the four possible cases:

Case 1: $v \in \{b, c\}$. $G_v \cong H$, and the claim holds.

Case 2: $v \in \{a, x\}$. G_v is isomorphic to the disjoint union of H_x and K_2, and the claim holds

Case 3: $v \in N_H(x)$. G_v is the disjoint union of H_v and the edge bc, and the claim holds

Case 4: $v \in V(H_x)$. In this case, x is a vertex of H_v. Since H is in W_2, x is not an isolated vertex in H_v. Observe that G_v is exactly the graph obtained by the construction in the lemma when beginning with H_v. Since $\alpha(H_v) = \alpha(H) - 1$, by the induction hypothesis we have G_v is Gorenstein with $\alpha(G_v) = \alpha(H_v) + 1 = \alpha(H)$. Notice that G_v has no isolated vertices, and the claim follows.

Next we note that $G[N_G(b)]$ is just two isolated vertices a and c, so G is Gorenstein by Proposition 4.3.

Finally, clearly $\alpha(G) = \alpha(G_b) + 1 = \alpha(H) + 1$, and the proof is complete. □

Example 4.5. Start with the graph R_3, using our construction we get the following Gorenstein graph (see Figure 5).

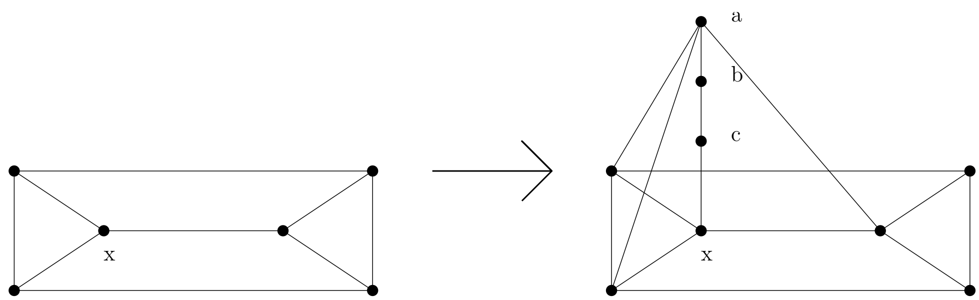

Figure 5. The Gorenstein graph obtained from R_3.

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