ON THE VISCOSITY APPROACH TO A CLASS OF FULLY NONLINEAR ELLIPTIC EQUATIONS

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ABSTRACT. In this paper, we study some properties of viscosity sub/super-solutions of a class of fully nonlinear elliptic equations relative to the eigenvalues of the complex Hessian. We show that every viscosity subsolution is approximated by a decreasing sequence of smooth subsolutions. When the equations satisfy some conditions on the limit at infinity, we verify that the comparison principle holds, and as a sequence, we obtain a result about the existence of solution of the Dirichlet problem. Using the comparison principle, we show that, under suitable conditions, a Perron-Bremermann envelope can be approximated by a decreasing sequence of viscosity solutions.

CONTENTS

1. INTRODUCTION

Let $\Gamma \subsetneq \mathbb{R}^n$ be an open, convex, symmetric cone with vertex at the origin such that $\Gamma_n \subseteq \Gamma \subseteq \Gamma_1$, where

$$\Gamma_k = \{x \in \mathbb{R}^n : \sigma_1(x) > 0, ..., \sigma_k(x) > 0\},$$

for every $1 \leq k \leq n$. Here $\sigma_k(x)$ is the k-th elementary symmetric sum of the coefficients of x, i.e.,

$$\sigma_k(x) = \sum_{1\leq j_1<...<j_k\leq n} x_{j_1}...x_{j_k}.$$

Let $f : \overline{\Gamma} \to (0, \infty)$ be a continuous function such that

- f is symmetric, strictly increasing in each variable and concave.
- $f|_\Gamma > 0$, $f|_{\partial\Gamma} = 0$.

We define $F : \mathcal{H}^n \to [-\infty, \infty)$ by

$$F(H) = \begin{cases} f(\lambda(H)) & \text{if} \quad H \in \overline{M(\Gamma, n)}, \\ -\infty & \text{if} \quad H \in \mathcal{H}^n \setminus \overline{M(\Gamma, n)}, \end{cases} \tag{1}$$

where $\mathcal{H}^n$ is the set of all $n \times n$ Hermitian matrices and $M(\Gamma, n)$ is the subset of $\mathcal{H}^n$ containing matrices H with the eigenvalues $\lambda(H) = (\lambda_1, ..., \lambda_n) \in \Gamma$. The conditions on f

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imply that F is concave on $\overline{M(\Gamma, n)}$ (see [3]) and

$$F(M+N) > F(M), \tag{2}$$

for every $M \in M(\Gamma, n)$ and for each positive semidefinite matrix $N \neq 0$.

We consider the fully nonlinear elliptic equation:

$$F(Hu) = \psi(z, u), \tag{3}$$

in a bounded domain $\Omega \subset \mathbb{C}^n$, where Hu is the complex Hessian of u and $\psi \geq 0$ is a continuous function in $\overline{\Omega} \times \mathbb{R}$ which is non-decreasing in the last variable.

In the smooth setting, the existence and uniqueness of the classical solution of the Dirichlet problem of (3) has been studied in [16] (see also [20] for corresponding problems on compact Hermitian manifolds). The real version of (3) has been studied earlier by Caffarelli-Nirenberg-Spruck [3], Guan [6], Trudinger [21]... (see also [13], [7], [8], [9], [14] for some recent developments). Some important equations of the form (3) are the complex Monge-Ampère equations, the complex Hessian equations and the complex Hessian quotient equations, where we take, respectively, $f(x) = (\sigma_n(x))^{1/n}$, $f(x) = (\sigma_k(x))^{1/k}(1 \leq k \leq n)$ and $f(x) = (\sigma_k(x)/\sigma_l(x))^{k-l}(1 \leq l < k \leq n)$.

The viscosity method introduced in [1, 17] (see [2] for a survey) is useful for studying partial differential equations in the non-smooth setting. A viscosity approach to the equation (3) has been studied in [4]. The goal of this paper is to expand this research direction. Following [2], a function $u \in USC(\Omega)$ is a viscosity subsolution (resp., supersolution) of (3) iff for every $z_0 \in \Omega$ and for any C^2-smooth function q in a neighbourhood U of z_0 such that $(u-q)(z_0) = 0 = \max_U(u-q)$ (resp., $(u-q)(z_0) = 0 = \min_U(u-q)$), we have $F(Hq(z_0)) \geq \psi(z_0, u(z_0))$ (resp., $F(Hq(z_0)) \geq \psi(z_0, u(z_0))$). The reader can find more details about the defintions and properties of viscosity sub/super-solutions in [4] (see also the Preliminaries). By [11, Lemma 4.6, Remark 4.9 and Theorem B.8], for every $u \in USC(\Omega)$, the following conditions are equivalent:

(i) $F(Hu) \geq 0$ in the viscosity sense in Ω.
(ii) For every open set $U \Subset \Omega$ there exists a decreasing sequence $\{u_j\}$ of smooth Γ-subharmonic functions on U such that $u_j \to u$ as $j \to \infty$. Here a smooth function w is Γ-subharmonic if $Hw(z) \in M(\Gamma, n)$ for every z.

We say that a function $u \in USC(\Omega)$ is Γ-subharmonic if it satisfies the above equivalent conditions. Since $\Gamma \subset \Gamma_1$, every Γ-subharmonic function is subharmonic. An alternative proof for the equivalence of (i) and (ii) is provided in this paper (see Corollary 3.3). Futhermore, we generalize this fact for viscosity subsolutions of (3) in the case where $\psi(z, r)$ is independent of r. Our first main result is as follows:

Theorem 1.1. *Assume that $\psi(z, r)$ does not depend on the last variable $r \in \mathbb{R}$. Then a function $u \in USC(\Omega)$ is a viscosity subsolution of (3) iff u is Γ-subharmonic and $F(H(u * \chi_\epsilon)) \geq \psi * \chi_\epsilon$ in Ω_ϵ (in the classical sense). Here χ_ϵ is the standard modifier, $*$ is the convolution operator and $\Omega_\epsilon = \{z \in \Omega : d(z, \partial\Omega) > \epsilon\}$.*

When f satisfies some conditions on the limit at infinity, we use Theorem 1.1 to show that every viscosity subsolution of (3) can be approximated by a decreasing sequence of classical subsolution of (3). In the cases of Monge-Ampère equations and Hessian equations, this fact has been proved in [5] and [18].

Corollary 1.2. *Assume that $\psi(z, r)$ does not depend on r and*

$$\lim_{R \to \infty} f(R, R, ..., R) > \sup_{\Omega} \psi.$$

Then a function $u \in USC(\Omega)$ is a viscosity subsolution of (3) iff for every open set $U \Subset \Omega$, there exists a decreasing sequence $\{u_j\}$ of smooth Γ-subharmonic functions on U such that $u_j \to u$ as $j \to \infty$ and $F(Hu_j(z)) \geq \psi(z)$ in U for every j.

Our second purpose is to study the comparison principle for (3). It follows from [12] that the comparison principle holds if $\psi(z, r) - \epsilon r$ is non-decreasing in r for some $\epsilon > 0$. Under appropriate growth restrictions on the behavior of F, one can permit $\epsilon = 0$ (see [4]). In this paper, we establish a version of the comparison principle with weaker conditions for F:

Theorem 1.3. *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain. Let $u \in USC \cap L^\infty(\overline{\Omega})$ and $v \in LSC \cap L^\infty(\overline{\Omega})$, respectively, be a bounded subsolution and a bounded supersolution of the equation*

$$F(Hw) = \psi(z, w), \tag{4}$$

in Ω. Assume that $u \leq v$ in $\partial\Omega$ and

$$\lim_{R\to\infty} f(R, R, ..., R) > \sup_K \psi(z, v(z)),$$

for every $K \Subset \Omega$. Then $u \leq v$ in Ω.

By Theorem 1.3 and the Perron method, if v is a viscosity supersolution of (3) satisfying some suitable conditions then the function

$$\Phi_v = \sup\{w : w \text{ is a subsolution of (3)}, \ w \leq v\},$$

is a discontinuous viscosity solution of (3) (see Proposition 4.2). In the case where $\psi(z, r)$ is independent of r, we show that Φ_v is further locally approximated by a decreasing sequence of viscosity solutions.

Theorem 1.4. *Assume that ψ does not depend on the last variable and*

$$\lim_{R\to\infty} f(R, R, ..., R) > \psi(z),$$

for every $z \in \Omega$. Suppose that $v \in L^\infty \cap LSC(\overline{\Omega})$ is a bounded viscosity supersolution of (3) in Ω which is continuous at every point $z \in \partial\Omega$. Then, for every relatively compact open subset U of Ω, there exists a decreasing sequence u_j of viscosity solutions of (3) in U such that $\lim_{j\to\infty} u_j = \Phi_v$ in U.

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2. Preliminaries

In this section, we recall the definitions and some properties of viscosity sub/super-solutions.

Definition 2.1. *(Test functions) Let $w : \Omega \longrightarrow \mathbb{R}$ be any function defined in Ω and $z_0 \in \Omega$ a given point. An upper test function (resp., a lower test function) for w at the point z_0 is a C^2-smooth function q in a neighbourhood of z_0 such that $w(z_0) = q(z_0)$ and $w \leq q$ (resp., $w \geq q$) in a neighbourhood of z_0.*

Definition 2.2. *1. A function $u \in USC(\Omega)$ is said to be a (viscosity) subsolution of*

$$F(Hu) = \psi(z, u), \tag{5}$$

in Ω if for any point $z_0 \in \Omega$ and any upper test function q for u at z_0, we have $F(Hq(z_0)) \geq \psi(z_0, u(z_0))$ (and then $Hq(z_0) \in \overline{M(\Gamma, n)}$). In this case, we also say that $F(Hu) \geq \psi(z, u)$ in the viscosity sense in Ω.

2. A function $v \in LSC(\Omega)$ is said to be a (viscosity) supersolution of (5) in Ω if for any point $z_0 \in \Omega$ and any lower test function q for v at z_0, we have $F(Hq(z_0)) \leq \psi(z_0, u(z_0))$. In this case, we also say that $F(Hv) \leq \psi(z, v)$ in the viscosity sense in Ω.
3. A function $u \in C(\Omega)$ is said to be a (viscosity) solution of (5) in Ω if it is a subsolution and a supersolution of (5) in Ω.
4. A function $u \in L^\infty(\Omega)$ is said to be a discontinuous viscosity solution of (5) in Ω if u^* is a subsolution and u_* is a supersolution of (5) in Ω.

It follows from the definition directly that if u, v are viscosity subsolutions of (5) then $\max\{u, v\}$ is a viscosity subsolution of (5). Furthermore, we also have:

Proposition 2.3. *Assume that $G \subsetneq \Omega$ is an open set. Suppose that v is a viscosity subsolution of (5) in G and u is a viscosity subsolution of (5) in Ω such that*

$$\limsup_{G \ni z \to z_0} v(z) \leq u(z_0),$$

for every $z_0 \in \partial G \cap \Omega$. Then, the function

$$\tilde{u} = \begin{cases} u \quad in \quad \Omega \setminus G, \\ \max\{u, v\} \quad in \quad G, \end{cases}$$

is a viscosity subsolution of (5) in Ω.

In the case of Hessian equations (i.e., $f(x) = (\sigma_k(x))^{1/k}$ and $\Gamma = \Gamma_k$), we use the notation F_k instead of F. The following result has been proved in [18]:

Proposition 2.4. *Let $u \in USC(\Omega)$. Then the following conditions are equivalent:*

a) *u is a viscosity subsolution of the equation $F_k(Hw) = \psi(z, w)$ in the sense of Definition 2.2;*
b) *for every $z_0 \in \Omega$, for every upper test function q for u at z_0, we have $\sigma_k(\lambda(Hq(z_0)) \geq \psi^k(z_0, u(z_0))$ (it does not require that $Hq(z_0) \in \overline{M(\Gamma_k, n)}$).*

Actually, in [18], a function $u \in USC(\Omega)$ is called a viscosity subsolution of the equation $\sigma_k(\lambda(Hw)) = \psi^k(z, w)$ if it satisfies the condition b) in the above proposition. By the proof of [18, Lemma 3.7], if b) is satisfied then, for every $z_0 \in \Omega$ and for every upper test function q for u at z_0, the Hessian matrix $Hq(z_0) = (\frac{\partial^2 q}{\partial z_\alpha \partial \bar{z}_\beta}(z_0))$ is k-positive, i.e., $Hq(z_0) \in \overline{M(\Gamma_k, n)}$. Then $b) \Rightarrow a)$. The fact $a) \Rightarrow b)$ is obvious.

If $u \in USC(\Omega)$ (resp., $u \in LSC(\Omega)$) then for every $x \in \Omega$, the set

$$\mathcal{J}^{\pm} u(z) = \{(Dw(z), Hw(z)) \in \mathbb{R}^{2n} \times \mathcal{H}^n : w \text{ is an upper test function (resp. a lower test function) for } u \text{ at } z\}$$

is called the super-(resp., sub-)differential of u at z. The set

$$\overline{\mathcal{J}}^{\pm} u(z) = \{(p, Z) \in \mathbb{R}^{2n} \times \mathcal{H}^n : \exists z_m \to z \text{ and } (p_m, Z_m) \in \mathcal{J}^{\pm} u(z_m) \text{ such that } (p_m, Z_m) \to (p, Z) \text{ and } u(z_m) \to u(z)\}$$

is called the limiting super-(resp. sub-)differential of u at z. By the continuity of F and ψ, the limiting super/sub-differentials can be used to identify viscosity sub/super-solutions as follows:

Proposition 2.5. *a) Let $u \in USC(\Omega)$. Then u is a viscosity subsolution of the equation (5) iff for any point $x \in \Omega$, for every $(p, X) \in \overline{\mathcal{J}}^{+} u(z)$, we have*

$$F(X) \geq \psi(z, u).$$

b) Let $u \in LSC(\Omega)$. Then u is a viscosity supersolution of the equation (5) iff for any point $x \in \Omega$, for every $(p, X) \in \overline{\mathcal{J}}^{-} u(z)$, we have

$$F(X) \leq \psi(z, u).$$

The following proposition is deduced by combining Proposition 2.5 and [2, Proposition 4.3]:

Proposition 2.6. *a) Assume that $\{u_\alpha\}$ is a family of viscosity subsolutions of the equation*

$$F(Hw) = \psi(z, w), \tag{6}$$

in Ω. If $u = \sup_\alpha u_\alpha$ is locally bounded from above then its usc regularization u^* is a viscosity subsolution of (6) in Ω.
b) Assume that $\{u_\alpha\}$ is a family of viscosity supersolutions of (6) in Ω. If $u = \inf_\alpha u_\alpha$ is locally bounded from below then its lsc regularization u_* is a viscosity supersolution of (6) in Ω.
b) Assume that u_j is a decreasing (resp., increasing) sequence of viscosity subsolutions (resp., supersolutions) to (6). Then $u = \lim_{j\to\infty} u_j$ is either a viscosity subsolutions (resp., supersolutions) to (6) or identically $-\infty$ (resp., ∞).

3. Approximation of subsolutions

In this section, we will prove the Theorem 1.1 and Corollary 1.2. First, we have the following lemma:

Lemma 3.1. *There exists a mapping*

$$\Phi : M(\Gamma, n) \to M(\Gamma_n, n)$$
$$H \mapsto \widetilde{H} = \Phi(H)$$

depending on F such that

a) *For all $B \in \overline{M(\Gamma, n)}$,*

$$F(B) = \inf\{\Delta_{\widetilde{H}} B + F(H) - \Delta_{\widetilde{H}} H : H \in M(\Gamma, n)\}, \tag{7}$$

where $\Delta_{\widetilde{H}} B = trace(\widetilde{H}B) = \sum_{j,k=1}^{n} \widetilde{h}_{jk} b_{kj}$.

b) *For all $B \in \mathcal{H}^n$, if*

$$\inf\{\Delta_{\widetilde{H}} B + F(H) - \Delta_{\widetilde{H}} H : H \in M(\Gamma, n)\} \geq 0, \tag{8}$$

then $B \in \overline{M(\Gamma, n)}$.

Proof. a) By the concavity of F in $M(\Gamma, n)$, for every $H \in M(\Gamma, n)$, the subdifferential $\partial(-F(H))$ is nonempty, i.e., there exists $\tilde{H} \in \mathcal{H}^n \setminus \{0\}$ such that

$$F(B) - F(H) \leq \Delta_{\tilde{H}}(B - H) = \Delta_{\tilde{H}}(B) - \Delta_{\tilde{H}}(H), \tag{9}$$

for all $B \in \overline{M(\Gamma, n)}$. Moreover,

$$F(B) = \lim_{H\to B}(F(H) + \Delta_{\tilde{H}}(B - H)). \tag{10}$$

Combining (9) and (10), we have

$$F(B) = \inf\{\Delta_{\tilde{H}}(B) + F(H) - \Delta_{\tilde{H}}(H) : H \in M(\Gamma, n)\}. \tag{11}$$

By (2) and (9), for every $N \in \overline{M(\Gamma_n, n)} \setminus \{0\}$ and for each $H \in M(\Gamma, n)$, we have

$$\Delta_{\tilde{H}}(N) = \Delta_{\tilde{H}}(H + N - H) \geq F(H + N) - F(H) > 0. \tag{12}$$

Hence, $\widetilde{H} \in M(\Gamma_n, n)$ for every $H \in M(\Gamma, n)$.
b) Assume that $B \notin \overline{M(\Gamma, n)}$ and the condition (8) is satisfied. Let $t_0 > 0$ such that $B + t_0 I \in \partial M(\Gamma, n)$. Then, for every $t > t_0$, we have $B + tI \in M(\Gamma, n)$. By the assumption, we get

$$\Delta_{\widetilde{B+tI}} B + F(B + tI) - \Delta_{\widetilde{B+tI}}(B + tI) \geq \inf_{H\in M(\Gamma,n)}\{\Delta_{\widetilde{H}} B + F(H) - \Delta_{\widetilde{H}} H\} \geq 0,$$

for every $t > t_0$. Then

$$F(B+tI) - t\Delta_{\widetilde{B+tI}} I \geq 0,$$

for every $t > t_0$. Letting $t \searrow t_0$, we get

$$\limsup_{t\to t_0^+} \Delta_{\widetilde{B+tI}} I \leq \lim_{t\to t_0^+} \frac{F(B+tI)}{t} = \frac{F(B+t_0 I)}{t_0} = 0. \tag{13}$$

Moreover, it follows from (9) that

$$F(B+(1+t_0)I) - F(B+tI) \leq (1+t_0-t)\Delta_{\widetilde{B+tI}} I,$$

for every $t_0 < t < t_0+1$. Letting $t \searrow t_0$, we get

$$\liminf_{t\to t_0^+} \Delta_{\widetilde{B+tI}} I \geq \lim_{t\to t_0^+} \frac{F(B+(1+t_0)I) - F(B+tI)}{1+t_0-t} = F(B+(1+t_0)I) > 0. \tag{14}$$

By (13) and (14), we get a contradiction.

Thus, the condition (8) implies that $B \in \overline{M(\Gamma,n)}$. □

Corollary 3.2. *Let $t \in [0,1]$ and $0 \leq \psi_1, \psi_2 \in C(\Omega)$. Assume that u_j is a viscosity subsolution of the equation $F(Hw) = \psi_j(z)$ in Ω for $j = 1,2$. Then, the function $tu_1 + (1-t)u_2$ is a subsolution of the equation $F(Hw) = t\psi_1(z) + (1-t)\psi_2(z)$.*

Proof. By Lemma 3.1, we have, for $j = 1,2$,

$$\Delta_{\widetilde{H}} u_j + F(H) - \Delta_{\widetilde{H}} H \geq \psi_j, \tag{15}$$

in the viscosity sense in Ω for every $H \in M(\Gamma,n)$. Then, it follows from [10, Proposition 3.2.10', page 147] that (15) holds in the distribution sense. Therefore, we have

$$\Delta_{\widetilde{H}}(tu_1 + (1-t)u_2) + F(H) - \Delta_{\widetilde{H}} H \geq t\psi_1(z) + (1-t)\psi_2(z), \tag{16}$$

in the distribution sense. Using again [10, Proposition 3.2.10', page 147], we get (16) holds in the viscosity sense. Thus, by Lemma 3.1, we obtain

$$F(H(tu_1 + (1-t)u_2)) \geq t\psi_1(z) + (1-t)\psi_2(z),$$

in the viscosity sense. □

Corollary 3.3. *Let $u \in USC(\Omega)$. Then the following conditions are equivalent*

a) *u is subharmonic and for every $\epsilon > 0$, $u * \chi_\epsilon$ is Γ-subharmonic in Ω_ϵ. Here χ_ϵ is the standard modifier, $*$ is the convolution operator and $\Omega_\epsilon = \{z \in \Omega : d(z, \partial\Omega) > \epsilon\}$.*
b) *u is Γ-subharmonic.*
c) *$F(Hu) \geq 0$ in the viscosity sense, i.e., for any point $z_0 \in \Omega$ and any upper test function q for u at z_0, we have $Hq(z_0) \in \overline{M(\Gamma,n)}$.*

Proof. $(a) \Rightarrow (b)$ and $(b) \Rightarrow (c)$ are clear. It remains to show $(c) \Rightarrow (a)$.

Assume that $F(Hu) \geq 0$ in the viscosity sense. Then, by the definition and by the condition $\Gamma \subset \Gamma_1$, we have $\Delta u \geq 0$ in the viscosity sense. Hence, by [10, Proposition 3.2.10', page 147], we get $u \in SH(\Omega)$.

Moreover, it follows from Lemma 3.1 that

$$\Delta_{\widetilde{H}} u + F(H) - \Delta_{\widetilde{H}} H \geq 0, \tag{17}$$

in the viscosity sense for every $H \in M(\Gamma,n)$. Then, it follows from Lemma 3.4 that (17) holds in the distribution sense. Hence

$$\Delta_{\widetilde{H}}(u * \chi_\epsilon) + F(H) - \Delta_{\widetilde{H}} H \geq 0,$$

in the classical sense in Ω_ϵ for every $H \in M(\Gamma, n)$. Using again Lemma 3.1, we have $H(u*\chi_\epsilon)(z) \in \overline{M(\Gamma,n)}$ for every $\epsilon > 0$ and $z \in \Omega_\epsilon$. Thus $u * \chi_\epsilon$ is Γ-subharmonic in Ω_ϵ.

The proof is completed. □

Lemma 3.4. *Let $U \subset \mathbb{R}^N (N \geq 2)$ be a bounded domain. Assume that $g \in C(U)$ and $u \in USC(U)$. Then, $\Delta u \geq g$ in the viscosity sense iff $\Delta u \geq g$ in the distribution sense.*

Proof. If $g \in C^2_c(\mathbb{R}^N)$ then $v = E*g$ is a classical solution to the Poisson equation $\Delta w = g$, where

$$E(x) = \begin{cases} \frac{1}{2\pi}\log|x| & (N=2), \\ \frac{-1}{N(N-2)c_N}\frac{1}{|x|^{N-2}} & (N \geq 3), \end{cases}$$

and c_N is the volume of $\mathbb{B}^N$. It follows from [10, Proposition 3.2.10', page 147] that $\Delta(u-v) \geq 0$ in the viscosity sense iff $\Delta(u-v) \geq 0$ in the distribution sense. Hence, $\Delta u \geq g (= \Delta v)$ in the viscosity sense iff $\Delta u \geq g$ in the distribution sense.

In the general case, since the problem is local, we can assume that $g \in C_c(\mathbb{R}^N)$. Then, we can choose a sequence $g_j \in C^2_c(\mathbb{R}^N)$ such that $g_j \nearrow g$ as $j \to \infty$. Hence, by the above argument, we have

$$\begin{aligned} (\Delta u \geq g \text{ in the viscosity sense}) &\Leftrightarrow (\Delta u \geq g_j \text{ in the viscosity sense for every j}) \\ &\Leftrightarrow (\Delta u \geq g_j \text{ in the distribution sense for every j}) \\ &\Leftrightarrow (\Delta u \geq g \text{ in the distribution sense}). \end{aligned}$$

□

Proof of Theorem 1.1. If u is Γ-subharmonic and $F(H(u*\chi_\epsilon)) \geq \psi * \chi_\epsilon$ in Ω_ϵ for every $\epsilon > 0$ then, by the definition, we have $F(H(u*\chi_\epsilon)) \geq \psi*\chi_\epsilon$ in the viscosity sense in Ω_ϵ for every $\epsilon > 0$. Hence, $F(H(u*\chi_\epsilon)) \geq \psi_r$ in the viscosity sense in Ω_r for every $0 < \epsilon < r$, where $\psi_r(z) = \inf_{|z-w|<r}\psi(w)$. Since u is subharmonic, we have $u * \chi_\epsilon \searrow u$ as $\epsilon \searrow 0$. Using Proposition 2.6, we get $F(Hu) \geq \psi_r$ in the viscosity sense in Ω_r for every $r > 0$. Letting $r \searrow 0$, we obtain $F(Hu) \geq \psi$ in the viscosity sense in Ω.

Conversely, assume that u is a viscosity subsolution of the equation $F(Hw) = \psi(z)$ in Ω. By Corollary 3.3, we have u and $u*\chi_\epsilon$ are Γ-subharmonic ($\epsilon > 0$). Moreover, by the same argument as in the proof of Corollary 3.3, we also have

$$\Delta_{\widetilde{H}}(u*\chi_\epsilon) + F(H) - \Delta_{\widetilde{H}}H \geq \psi * \chi_\epsilon,$$

in the classical sense in Ω_ϵ for every $H \in M(\Gamma, n)$. Hence, it follows from Lemma 3.1 that $F(H(u*\chi_\epsilon)) \geq \psi*\chi_\epsilon$ in Ω_ϵ in the classical sense. □

Proof of Corollary 1.2. If there exists a decreasing sequence $\{u_j\}$ of smooth Γ-subharmonic functions on U such that $u_j \to u$ as $j \to \infty$ and $F(Hu_j(x)) \geq \psi(z)$ in Ω for every j then, by Proposition 2.6, u is a viscosity subsolution of (3).

For the converse, assume that u is a viscosity subsolution of (3) and U is a relatively compact open subset of Ω. By Theorem 1.1, we have $u*\chi_\epsilon \searrow u$ as $\epsilon \searrow 0$ and $F(Hu*\chi_\epsilon) \geq \psi*\chi_\epsilon$ in U for every $0 < \epsilon < \epsilon_0$, where $\epsilon_0 > 0$ is small enough such that $U \Subset \Omega_{\epsilon_0}$. Since $\psi * \chi_\epsilon$ converges uniformly to ψ in U, there exists $0 < ... < \epsilon_{j+1} < \epsilon_j < ... < \epsilon_1 < \epsilon_0$ such that $\lim_{j\to\infty}\epsilon_j = 0$ and

$$|\psi * \chi_\epsilon - \psi| < \frac{1}{2^j},$$

in U. By the assumption, there exists $R \gg 1$ such that $f(R, R, ..., R) > \sup_U \psi + r$ for some $0 < r < 1$. For every $j > -\log_2 r$, we denote:

$$u_j(z) = u * \chi_{\epsilon_j} + \frac{R|z|^2}{2^{j+1}r}.$$

Then, u_j is a decreasing sequence of smooth Γ-subharmonic functions in U satisfying $\lim_{j\to\infty} u_j = u$. Moreover, for every $j > -\log_2 r$ and for each $z \in U$, we have

$$\begin{aligned} F(Hu_j) &\geq (1 - \frac{1}{2^j r})F(Hu * \chi_{\epsilon_j}) + \frac{1}{2^j r}F(H(u * \chi_{\epsilon_j} + \frac{R|z|^2}{2})) \\ &\geq (1 - \frac{1}{2^j r})\psi * \chi_\epsilon + \frac{1}{2^j r}F(RI) \\ &\geq (1 - \frac{1}{2^j r})(\psi(z) - \frac{1}{2^j}) + \frac{1}{2^j r}(\psi(z) + r) \\ &> \psi(z), \end{aligned}$$

in U. □

4. Comparison principle and applications

Now we prove the second main theorem of this paper:

Theorem 4.1. *Let $u \in USC \cap L^\infty(\overline{\Omega})$ and $v \in LSC \cap L^\infty(\overline{\Omega})$, respectively, be a bounded subsolution and a bounded supersolution of the equation*

$$F(Hw) = \psi(z, w), \tag{18}$$

in Ω. Assume that $u \leq v$ in $\partial\Omega$ and

$$\lim_{R\to\infty} f(R, R, ..., R) > \sup_{\Omega} \psi(z, v(z)). \tag{19}$$

Then $u \leq v$ in Ω.

Proof. First, we consider the case where $u - \delta|z|^2$ is a subsolution of (18) for some $\delta > 0$. Assume that there exists $z_0 \in \Omega$ such that

$$(u - v)(z_0) = \max_{\overline{\Omega}}(u - v) > 0. \tag{20}$$

For each $N > 0$, we denote

$$\phi_N(z, w) = u(z) - v(w) - N|z - w|^2,$$

for all $(z, w) \in \overline{\Omega}^2$. Since $\overline{\Omega}^2$ is compact and ϕ_N is upper semicontinuous, there exists $(z_N, w_N) \in \overline{\Omega}^2$ such that

$$\phi_N(z_N, w_N) = \max_{\overline{\Omega}^2} \phi_N.$$

Moreover, by [2, Lemma 3.1], we can assume that z_N and w_N converge to z_0 as $N \to \infty$. In particular, there exists $N_0 > 0$ such that $z_N, w_N \in B(z_0, R)$ for every $N > N_0$, where $0 < R < d(z_0, \partial\Omega)$. By the maximum principle [2, Theorem 3.2], there exist $Z_N, W_N \in \mathcal{H}^n$ such that $(2N(z_N - w_N), Z_N) \in \overline{\mathcal{J}}^+ u(z_N)$, $(2N(z_N - w_N), W_N) \in \overline{\mathcal{J}}^- v(w_N)$ and $Z_N \leq W_N$ for all $N > N_0$. Hence, we have

$$F(Z_N - 2\delta I) \geq \psi(z_N, u(z_N)), \tag{21}$$

and

$$F(W_N) \leq \psi(w_N, v(w_N)), \tag{22}$$

and

$$F(Z_N) \leq F(W_N). \tag{23}$$

Combining (22) and (23), we get

$$F(Z_N) \leq M, \tag{24}$$

for all $N > N_0$, where $M := \sup\{\psi(z, v(z)) : x \in B(z_0, R)\}$. Since $\lim_{R\to\infty} f(R, ..., R) > M$, there exist $R_0 \gg 1$ and $0 < r \ll 1$ such that

$$F(RI) = f(R, ..., R) > M + r.$$

for all $R > R_0$. Then, by (24) and by the concavity of F, we have, for every $0 < \epsilon < 1$,

$$\begin{aligned}\frac{F(Z_N) - F(Z_N - 2\delta I)}{2\delta} &\geq \frac{F(Z_N + R_0 I) - F(Z_N)}{R_0} \\ &\geq \frac{\epsilon F(Z_N/\epsilon) + (1-\epsilon)F(R_0 I/(1-\epsilon)) - M}{R_0} \\ &\geq \frac{(1-\epsilon)(M+r) - M}{R_0},\end{aligned}$$

for all $N > N_0$. Letting $\epsilon = \dfrac{r}{2M + r}$, we get

$$F(Z_N) \geq F(Z_N - \delta I) + \frac{2M\delta^2}{(2M + r)R_0}, \tag{25}$$

for every $N > N_0$. Combining (21), (22), (23) and (25), we obtain

$$\psi(z_N, u(z_N)) + \frac{2M\delta^2}{(2M + r)R_0} \leq \psi(w_N, v(w_N)), \tag{26}$$

for every $N > N_0$. Since $z_N, w_N \to z_0$ and ψ is uniformly continuous, we also have

$$\lim_{N\to\infty} (\psi(z_N, u(z_N)) - \psi(w_N, u(z_N))) = 0. \tag{27}$$

Moreover, it follows from [2, Lemma 3.1] that $\lim_{N\to\infty}(u(z_N) - v(w_N)) = \max_{\overline{\Omega}}(u - v) > 0$. Hence, since ψ is non decreasing in the last variable, we get

$$\liminf_{N\to\infty}(\psi(w_N, u(z_N)) - \psi(w_N, v(w_N))) \geq 0. \tag{28}$$

Combining (26), (27) and (28), we get

$$0 \leq \liminf_{N\to\infty}(\psi(z_N, u(z_N)) - \psi(w_N, v(w_N))) \leq -\frac{2M\delta^2}{(2M + r)R_0},$$

and this is a contradiction. Thus, (20) is not true.

In the general case, for each $\delta > 0$, we denote $u_\delta(z) = u(z) + \delta(|z|^2 - A)$, where $A = \max\{|w|^2 : w \in \overline{\Omega}\}$. By the above argument, we have $u_\delta \leq v$ in Ω for all $\delta > 0$. Letting $\delta \searrow 0$, we get $u \leq v$ in Ω.

The proof is completed. □

By using the Perron method [2] and Theorem 1.3, we obtain the following result:

Proposition 4.2. *Let $v \in LSC \cap L^\infty(\overline{\Omega})$ be a bounded supersolution of the equation*

$$F(Hw) = \psi(z, w), \tag{29}$$

in Ω such that

$$\lim_{\Omega \ni z \to \hat{z}} v(z) = v(\hat{z}),$$

for all $\hat{z} \in \partial\Omega$ and

$$\lim_{R\to\infty} f(R, R, ..., R) > \sup_{\Omega} \psi(z, v(z)).$$

Denote by S the set of all viscosity subsolutions w to (29) satisfying $w \leq v$. Then the function

$$u(z) = \sup\{w(z) : w \in S\},$$

is a discontinuous viscosity solution of (29) with $u = u^* \in S$.

Proof. By Proposition 2.6, we have u^* is a viscosity subsolution of the equation $F(D^2 w) = \psi(x, w)$ in Ω. Moreover, since $u \leq v$ in Ω, we have

$$\limsup_{\Omega\ni z\to\hat{z}} u^*(z) = \limsup_{\Omega\ni z\to\hat{z}} u(z) \leq \lim_{\Omega\ni z\to\hat{z}} v(z) = v(\hat{z}),$$

for all $\hat{z} \in \partial\Omega$. Then, it follows from Theorem 1.4 that $u^* \leq v$. Hence, $u^* \in S$ and $u = u^*$. It remains to show that u_* is a viscosity supersolution.

Assume that there exist a point $z_0 \in \Omega$, an open neighbourhood $U \subset \Omega$ of z_0 and a function $\eta \in C^2(U)$ such that $\eta(z_0) = u_*(z_0)$, $\eta \leq u_*|_U$, $H\eta(z_0) \in M(\Gamma, n)$ and

$$F(H\eta(z_0)) > \psi(z_0, \eta(z_0)).$$

By the continuity of F, ψ, η and $H\eta$, there exist $r, s > 0$ such that $\overline{B(z_0, r)} \subset U$, $H\eta(x) - 2sI \in M(\Gamma, n)$ for all $z \in B(z_0, r)$ and

$$F(H\eta(z) - 2sI) > \psi(z, \eta(z) + s),$$

for every $z \in B(z_0, r)$. Denote

$$\tilde{\eta}(z) = \eta(z) - s|z - z_0|^2 + \min\{s, \frac{sr^2}{4}\}.$$

We have

$$F(H\tilde{\eta}(z)) \geq \psi(z, \tilde{\eta}(z)), \quad \forall |z - z_0| \leq r, \tag{30}$$

and

$$\tilde{\eta}(z) \leq u(z), \quad \forall r/2 \leq |z - z_0| \leq r. \tag{31}$$

Denote

$$\tilde{u}(z) = \begin{cases} u(z) & \text{if} \quad z \in \Omega \setminus B(z_0, r), \\ \max\{u(z), \tilde{\eta}(z)\} & \text{if} \quad z \in B(z_0, r). \end{cases}$$

Then $\tilde{u} \in S$ and $\tilde{u} \geq u$. Since $u = \sup\{w : w \in S\}$, we have $\tilde{u} = u$. Moreover,

$$\tilde{u}_*(z_0) \geq \tilde{\eta}(z_0) \geq u_*(z_0) + \min\{s, \frac{sr^2}{4}\} > u_*(z_0).$$

and it implies that $\tilde{u}$ is not identical to u. We get a contradiction. Thus, u_* is a supersolution of (29). □

Note that every harmonic function is a supersolution of (29). By using Theorem 1.3 and Proposition 4.2, we obtain the following result which will be used in the next section:

Proposition 4.3. *Assume that Ω is a bounded smooth domain and φ is a continuous function on $\partial\Omega$ satisfying*

$$\lim_{R\to\infty} f(R, R, ..., R) > \psi(z, \sup_{\partial\Omega} \varphi),$$

for every $z \in \Omega$. Suppose that there exists $\underline{u} \in USC(\overline{\Omega})$ such that $\underline{u}|_{\partial\Omega} = \varphi$ and $F(H\underline{u}) \geq \psi(z, \underline{u})$ in the viscosity sense in Ω. Then, there exists a unique $u \in C(\overline{\Omega})$ such that $u|_{\partial\Omega} = \varphi$ and $F(Hu) = \psi(z, u)$ in the viscosity sense in Ω.

5. Maximal viscosity subsolutions

In this section, we study some properties of maximal viscosity subsolutions (see below for the defintion). Theorem 1.4 is deduced by combining Proposition 5.1 and Theorem 5.2.

Similar to the concept of maximal plurisubharmonic functions [19] (see also [15]), we say that a viscosity subsolution u for (3) is maximal if u satisfies the following condition: For every open set $U \Subset \Omega$ and for each $v \in USC(\overline{U})$ such that v is a subsolution for (3) in U and $v \leq u$ in ∂U, we have $v \leq u$ in U.

Proposition 5.1. *Under the assumption of Theorem 1.4, the function Φ_v is a maximal viscosity subsolution for* (3).

Proof. By Proposition 4.2, we have Φ_v is a viscosity subsolution of (3). We will show that Φ_v is maximal.

Let U be a relatively open subset of Ω. Let $w \in USC(\overline{U})$ such that w is a subsolution for (3) in U and $w \leq \Phi_v$ in ∂U. By Proposition 2.3, the function

$$u(z) = \begin{cases} \Phi_v(z) & \text{if} \quad z \in \Omega \setminus U, \\ \max\{w(z), \Phi_v(z)\} & \text{if} \quad z \in U, \end{cases}$$

is a subsolution of (3) in Ω. Since $u = \Phi_v \leq v$ in $\Omega \setminus U$, it follows from Theorem 1.3 that $u \leq v$ in Ω. Then, by the definition of Φ_v, we get $u \leq \Phi_v$ in Ω.

Thus Φ_v is a maximal viscosity subsolution of (3). □

Theorem 5.2. *Assume that ψ does not depend on the last variable and*

$$\lim_{R\to\infty} f(R, R, ..., R) > \psi(z), \tag{32}$$

for every $z \in \Omega$. Suppose that u is a maximal viscosity subsolution for (3) *in Ω. Then, for every relatively compact open subset U of Ω, there exists a decreasing sequence u_j of viscosity solutions of* (3) *in U such that $\lim_{j\to\infty} u_j = u$ in U.*

In order to prove Theorem 5.2, we need the following lemma:

Lemma 5.3. *For every $\epsilon > 0$, there exists an open set U with smooth boundary such that $\Omega_\epsilon \Subset U \Subset \Omega$, where*

$$\Omega_\epsilon = \{z \in \Omega : d(z, \partial\Omega) > \epsilon\}.$$

Proof. Consider the function $g(z) = (d * \chi_{\epsilon/4})(z)$, where $\chi_{\epsilon/4}$ is the standard modifier, $*$ is the convolution operator and $d(z) = -d(z, \partial\Omega)$. We have g is well-defined and smooth in $\Omega_{\epsilon/4}$. Moreover, for every $\epsilon/2 < t < 3\epsilon/4$,

$$\Omega_\epsilon \Subset U_t \Subset \Omega_{\epsilon/4},$$

where $U_t = \{z \in \Omega_{\epsilon/4} : g(z) < -t\}$. In particular, we have $\partial U_t \subseteq g^{-1}(t) \Subset \Omega_{\epsilon/4}$ for every $\epsilon/2 < t < 3\epsilon/4$. By Sard's Theorem, there exists $t_0 \in (\epsilon/2, 3\epsilon/4)$ such that $Dg(z) \neq 0$ for every $z \in g^{-1}(t_0)$. Then, $\partial U_{t_0} = g^{-1}(t_0)$ and $U := U_{t_0}$ is a smooth open set satisfying $\Omega_\epsilon \Subset U \Subset \Omega$.

The proof is completed. □

Proof of Theorem 5.2. By Lemma 5.3, there exists a smooth open set V such that $U \Subset V \Subset \Omega$. By the compactness of $\overline{U}$, we can assume that V has finite (open) connected components. Then the problem is reduced to the case where U is a smooth domain.

By Corollary 1.2, for every open neighbourhood $W \Subset \Omega$ of $\overline{U}$, there exists a decreasing sequence $\{v_j\}$ of smooth Γ-subharmonic functions on W such that $v_j \to u$ as $j \to \infty$ and $F(Hv_j(z)) \geq \psi(z)$ in W for every j. By Proposition 4.3, for each $j \in \mathbb{Z}^+$, there exists a unique $u_j \in C(\overline{U})$ such that $u_j|_{\partial U} = v_j|_{\partial U}$ and $F(Hu_j) = \psi(z)$ in the viscosity sense in U. We will show that u_j decreases to u as $j \to \infty$.

It follows from Theorem 1.3 that $u_j \geq u_{j+1} \geq u$, and then

$$\tilde{u} := \lim_{j\to\infty} u_j \geq u, \tag{33}$$

in U. It follows from Proposition 2.6 that $\tilde{u}$ is a viscosity subsolution of the equation $F(Hw) = \psi(z)$. Moreover, we have

$$\tilde{u}|_{\partial U} = \lim_{j\to\infty} u_j|_{\partial U} = \lim_{j\to\infty} v_j|_{\partial U} = u|_{\partial U}.$$

Since u is a maximal viscosity subsolution of the equation $F(Hw) = \psi(z,w)$, we get

$$u \geq \tilde{u}, \tag{34}$$

in U. Combining (33) and (34), we obtain

$$u = \tilde{u} = \lim_{j\to\infty} u_j,$$

in U.

The proof is completed. □

Corollary 5.4. *Under the assumption of Theorem 5.2, if u is bounded then u is also a discontinuous viscosity solution of (3).*

Proof. By Theorem 5.2, for every relatively compact open subset U of Ω, there exists a decreasing sequence u_j of viscosity solutions of (3) in U such that $\lim_{j\to\infty} u_j = u$ in U. Then, by Proposition 2.6, $u_* = (\inf_j u_j)_*$ is a viscosity supersolution of (3) in U. Since U is arbitrary, we get u_* is a viscosity supersolution of (3) in Ω. Hence, u is a discontinuous viscosity solution of (3). □

Remark 5.5. *By the same arguments as in the proof of Theorem 5.2, if we assume further that Γ, f and ψ satisfy the hypothesis of* [16, Theorem 1.1] *then each maximal viscosity subsolution for (3) is approximated on every relatively compact subset of Ω by a decreasing sequence u_j of classical solutions of (3).*

Remark 5.6. *In general, if $u \in USC \cap L^\infty(\Omega)$ is a discontinuous viscosity solution of (3) then u may not be a maximal visocity subsolution. For example, let $\{a_j\}_{j=1}^\infty$ be a dense subset of the unit ball $\mathbb{B}^{2n}$ in $\mathbb{C}^n$ and let*

$$v(z) = |z|^2 + \sum_{j=1}^{\infty} \frac{\log|z-a_j|}{2^j}.$$

We have $u = e^v$ is a bounded plurisubharmonic function and $u_* = 0$ in $\mathbb{B}^{2n}$. Therefore, u is a discontinuous viscosity solution of the equation $F_n(Hw) = 0$ in $\mathbb{B}^{2n}$. However, u is not a maximal plurisubharmonic function in $\mathbb{B}^{2n}$, since its Monge-Ampère measure $(dd^c u)^n \geq e^{nv}(dd^c v)^n \geq e^{nv}(dd^c|z|^2)^n$ is not identically 0. Then u is not a maximal viscosity solution of the equation $F_n(Hw) = 0$ in $\mathbb{B}^{2n}$.

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