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Solution Existence Theorems for Finite Horizon Optimal Economic Growth Problems

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Abstract The solution existence of finite horizon optimal economic growth problems is studied by invoking Filippov's Existence Theorem for optimal control problems from the monograph of L. Cesari [*Optimization Theory and Applications*, Springer-Verlag, New York, 1983]. Our results are obtained not only for general problems but also for typical ones, where the production function is given by either the AK function or the Cobb–Douglas one, while the utility function can be in a linear or power form. Some open questions and conjectures about the regularity of global solutions of finite horizon optimal economic growth problems are formulated in this paper.

Keywords Optimal economic growth · optimal control · finite horizon · solution existence · regularity

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1 Introduction

Models of economic growth have played an essential role in economics and mathematical studies since the 30s of the twentieth century. Based on different consumption behavior hypotheses, they allow ones to analyze, plan, and predict relations between global factors, which include *capital*, *labor force*, *production technology*, and *national product*, of a particular economy in a given planning interval of time. Principal models and their basic properties have been investigated by Ramsey [1], Harrod [2], Domar [3], Solow [4], Swan [5], Cass [6], Koopmans [7], Romer [8], Lucas [9] and others. Surveys and details about the origins and developments of economic growth theories can be found in the recent papers [10–13] of Spear and Young and in the books [14] of Barro and Sala-i-Martin and [15] of Acemoglu.

Along with the analysis of the global economic factors, another major issue regarding an economy is the so-called *optimal economic growth problem*, which

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can be roughly stated as follows: *Define the amount of consumption (and therefore, saving) at each time moment of a given planning interval to maximize a certain target of consumption satisfaction while fulfilling given relations in the growth model of that economy.* This optimal consumption/saving problem was first formulated and solved to a certain extent by Ramsey [1]. Later, significant extensions of the model in [1] were suggested by Cass [6] and Koopmans [7]. Note that the planning interval of optimal economic growth problems can be finite or infinite (see, e.g., [14, Section 3.6], [16, p. 407], and esp., [17, pp. 445–446, 450 − 459] for detail discussions). When it is finite (resp., infinite), one has *optimal economic growth problems with finite horizon* (resp., *infinite horizon*). Thus, these two classes of optimal economic growth problems can be studied independently. However, an optimal economic growth problem with finite horizon might be considered as a special case of the infinite horizon counterpart (see, e.g., [15, p. 260]) and an optimal economic growth problem with infinite horizon can be approximated by the corresponding problems with finite horizon, provided that the planning interval is sufficiently long (see, e.g., [20, p. 144]).

A major part in the literature on optimal economic growth problems is devoted to the characterization of the solutions. Necessary optimality conditions and sufficient optimality conditions have been discussed in the books [15, Chapters 7 and 8], [16, Chapter16], [17, Chapter 5], [18, Chapters 5, 7, 10, and 11], [19, Chapter 20] and papers cited therein while computational aspects can be found in [20, Chapters 4 and 6]. It is worthy to note that necessary optimality conditions only allow us to sort out "*candidates*" for solutions. To conclude that those candidates are solutions, we need to verify sufficient optimality conditions, which normally require that the problem under consideration possesses enough concavity (see, e.g., [15, Theorems 7.11 and 7.14]). If the problem does not satisfy such requirements, the justification for using the necessary optimality conditions is much weaker and sufficient conditions to assure the solution existence are really needed. However, the results on solution existence of optimal economic growth problems seem much fewer than the results on solution characterizations (see, [15, p. 259]). For infinite horizon models, some existence results were given in [15, Example 7.4] and [21, Subsection 4.1]. For finite horizon models, our careful searching in the literature leads just to [22, Theorem 1]. Though one might consider a finite horizon problem as a special case of its infinite horizon counterpart (see the comments in [15, p.260] and [21, p. 328]), there are examples (see, e.g., [23] and [17, Subsection b, pp. 450 − 459]) showing that while solutions of finite horizon problems exist, the corresponding one with infinite horizon does not have a solution. These observations motivate the present investigation on solution existence of optimal economic growth problems with finite horizon, which are treated as an independent class of problems.

This paper considers the solution existence of finite horizon optimal economic growth problems of an aggregative economy; see, e.g., [17, Sections C and D in Chapter 5]. Our main tool is Filippov's Existence Theorem for optimal control problems with state constraints of the Bolza type from the monograph of Cesari [24]. Our new results on the solution existence are obtained under some mild conditions on the *per capita production function* and the *utility function*, which are two major inputs of the model in question. The results for general problems are also specified for typical ones, where the *production function* is given by either the *AK function* or the *Cobb-Douglas one* while the utility function can be in a linear or power form. Some interesting open questions and conjectures about the *regularity of global solutions* of finite horizon optimal economic growth problems are formulated in the final part of the paper. Note that, since the saving policy on a compact segment of time would not be practicable if it has an infinite number of discontinuities, our concept of regularity of solutions of the optimal economic growth problem has a clear practical meaning.

The solution existence theorems in this paper for finite horizon optimal economic growth problems can neither be obtained from [22, Theorem 1] for problems in a similar finite horizon setting nor be derived from [15, Example 7.4] and [21, Subsection 4.1] for problems with infinite horizon, even if one might consider finite horizon problems as special cases of the counterpart with infinite horizon. Among other things, the assumptions therein on the strict concavity or the differentiability (such as continuous differentiability, vanishing at infinity of the derivative, and Inada conditions) on the utility function and the per capita production function are not required herein. In Remark 3.1, we will see a per capita production that is not concave and differentiable on $\mathbb{R}_+$ but satisfies our assumptions in Theorem 3.1 on the solution existence of general problems. Also, when the per capita production function and utility function are both linear, which means they are concave and continuously differentiable everywhere but does not meet the criteria about strict concavity, vanishing at infinity of the derivative, and Inada conditions, we can still guarantee the solution existence of the problem by Theorem 4.1.

The rest of this paper is organized as follows. Section 2 presents the modeling of optimal economic growth problems with finite horizon and some background materials including the above-mentioned Filippov's theorem. Results on the solution existence for general and typical problems are addressed, respectively, in Sections 3 and 4. Further discussions about the assumptions on the per capita production function and about the regularity of global solutions are given in Section 5.

2 Preliminaries

This section collects some notations, definitions, and results that will be used in the sequel. The emphasis will be made on optimal economic growth problems.

By $\mathbb{R}$ (resp., $\mathbb{R}_+$ and $\mathbb{N}$) we denote the set of real numbers (resp., the set of nonnegative real numbers and the set of positive integers). The Euclidean norm in the n-dimensional space $\mathbb{R}^n$ is denoted by $\|.\|$. The *Sobolev space* $W^{1,1}([t_0,T],\mathbb{R}^n)$ (see, e.g., [25, p. 21]) is the linear space of the *absolutely continuous functions* $x : [t_0,T] \to \mathbb{R}^n$ equipped with the norm

$$\|x\|_{W^{1,1}} = \|x(t_0)\| + \int_{t_0}^{T} \|\dot{x}(t)\| dt.$$

It is well-known (see, e.g., [26]) that any absolutely continuous function $x : [t_0,T] \to \mathbb{R}^n$ is Fréchet differentiable almost everywhere on $[t_0,T]$. Moreover, the function $\dot{x}(\cdot)$ is integrable on $[t_0,T]$ with the integral being understood in the Lebesgue sense. The space $W^{1,1}([t_0,T],\mathbb{R})$ is vital for us, because the capital-to-labor ratio function $k(\cdot)$ (see Subsection 2.1 below) in the economic growth models is sought in that space.

2.1 Optimal Economic Growth Models

Following Takayama [17, Sections C and D in Chapter 5], we consider the problem of *optimal growth of an aggregative economy.* Suppose that the economy can be characterized by one sector, which produces the *national product*

$Y(t)$ at time t. Suppose that $Y(t)$ depends on two factors, the *labor* $L(t)$ and the *capital* $K(t)$, and the dependence is described by a *production function* F. Namely, one has

$$Y(t) = F(K(t), L(t)), \quad t \geq 0.$$

It is assumed that $F : \mathbb{R}^2_+ \to \mathbb{R}_+$ is a function defined on the nonnegative orthant $\mathbb{R}^2_+$ of $\mathbb{R}^2$ having nonnegative real values, and that it *exhibits constant returns to scale*, i.e.,

$$F(\alpha K, \alpha L) = \alpha F(K, L), \quad \forall \alpha > 0, \forall (K, L) \in \mathbb{R}^2_+. \tag{1}$$

For every $t \geq 0$, by $C(t)$ and $I(t)$, respectively, we denote the *consumption amount* and the *investment amount* of the economy. The *equilibrium relation* in the output market is depicted by

$$Y(t) = C(t) + I(t), \quad \forall t \geq 0. \tag{2}$$

Since the consumption amount can be neither negative nor exceed the total outcome, one has

$$0 \leq C(t) \leq Y(t), \quad \forall t \geq 0. \tag{3}$$

The relationship between the capital $K(t)$ and the investment amount $I(t)$ is given by the differential equation

$$\dot{K}(t) = I(t), \quad \forall t \geq 0, \tag{4}$$

where $\dot{K}(t) = \dfrac{dK(t)}{dt}$ denotes the Fréchet derivative of $K(\cdot)$ at time instance t (see, e.g., [19, pp. 465–466]). If the investment function $I(\cdot)$ is continuous, then one can compute the capital stock $K(t)$ at time t by the formula

$$K(t) = K(0) + \int_0^t I(\tau) d\tau,$$

where the integral is Riemannian and $K(0)$ signifies the initial capital stock. In particular, the rate of increase of the capital stock $\dot{K}(t)$ at every time moment t exists and it is finite.

If the initial labor amount is $L_0 > 0$ and the *growth rate of labor force* is a constant $\sigma > 0$ (i.e., $\dot{L}(t) = \sigma L(t)$ for all $t \geq 0$), then the labor amount at any time t is

$$L(t) = L_0 e^{\sigma t}, \quad \forall t \geq 0. \tag{5}$$

As $L(t) > 0$ for any $t \geq 0$, it follows from (1) that $\frac{Y(t)}{L(t)} = F\left(\frac{K(t)}{L(t)}, 1\right)$ for all $t \geq 0$. By introducing the *capital-to-labor ratio* $k(t) := \frac{K(t)}{L(t)}$ for $t \geq 0$ and the function $\phi(k) := F(k, 1)$ for $k \geq 0$, from the last equality we have

$$\phi(k(t)) = \frac{Y(t)}{L(t)}, \quad \forall t \geq 0. \tag{6}$$

Due to (6), one calls $\phi(k(t))$ the *output per capita* at time t and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ the *per capita production function*. This function $\phi(\cdot)$ is well-defined on $\mathbb{R}_+$ and has nonnegative values because the production function $F(\cdot, \cdot)$ is defined on $\mathbb{R}^2_+$ and has nonnegative values.

Combining the continuous differentiability of $K(\cdot)$ and $L(\cdot)$, which is guaranteed by (4) and (5), with the equality defining the capital-to-labor ratio, one can asserts that $k(\cdot)$ is continuously differentiable. Thus, from the relation

$K(t) = k(t)L(t)$ one obtains $\dot{K}(t) = \dot{k}(t)L(t) + k(t)\dot{L}(t)$ for all $t \geq 0$. Dividing both sides of the last equality by $L(t)$ and recalling that $\dot{L}(t) = \sigma L(t)$, we get

$$\frac{\dot{K}(t)}{L(t)} = \dot{k}(t) + \sigma k(t), \quad \forall t \geq 0. \tag{7}$$

Similarly, dividing both sides of the equality in (4) by $L(t)$ and using (2), we have $\frac{\dot{K}(t)}{L(t)} = \frac{Y(t)}{L(t)} - \frac{C(t)}{L(t)}$ for all $t \geq 0$. So, by considering the *per capita consumption* $c(t) := \frac{C(t)}{L(t)}$ of the economy at time t and invoking (6), one obtains $\frac{\dot{K}(t)}{L(t)} = \phi(k(t)) - c(t)$ for all $t \geq 0$. Combining this with (7) yields

$$\dot{k}(t) = \phi(k(t)) - \sigma k(t) - c(t), \quad \forall t \geq 0. \tag{8}$$

The dynamic constraint (8) is called the *fundamental differential equation of neoclassical economic growth* [16, p. 402]. By (6), the constraint on the consumption amount (3) becomes

$$0 \leq c(t) \leq \phi(k(t)), \quad \forall t \geq 0. \tag{9}$$

Thus, the growth problem of the aggregative economy under consideration is: *Find pairs of functions* (k, c) *that together satisfy the dynamic constraint* (8)*, the inequality constraint* (9)*, and an initial condition* $k(t_0) = k_0$ *with* $k_0 > 0$ *being a given value for the capital-to-labor ratio at initial time* t_0*.*

Note that the just-mentioned growth problem is not easy to study. Firstly, the differential equation(8) is always a nonlinear, unless the function $\phi(\cdot)$ is linear or constant. Secondly, though the per capita consumption $c(t)$ appears linearly in the dynamic constraint (8), which is a quite nice property, the second inequality in (9) means that the per capita consumption $c(t)$ depends on the capital-to-labor ratio $k(t)$ at each reference time t via the per capita production function $\phi(\cdot)$; hence, $c(t)$ might be unbounded. The latter properties usually cause difficulties in the study of optimal economic growth problems. To avoid this, one can use an alternative way to formulate the growth problem as follows.

Introduce the *propensity to save* $s(t)$ at time t with

$$0 \leq s(t) \leq 1, \quad \forall t \geq 0 \tag{10}$$

and present the consumption amount by

$$C(t) = (1 - s(t))Y(t), \quad \forall t \geq 0. \tag{11}$$

Then, by dividing both sides of (11) by $L(t)$ and referring to (6), one gets

$$c(t) = (1 - s(t))\phi(k(t)), \quad \forall t \geq 0. \tag{12}$$

Thanks to (12), one can rewrite (8) equivalently as

$$\dot{k}(t) = s(t)\phi(k(t)) - \sigma k(t), \quad \forall t \geq 0. \tag{13}$$

The economic growth problem now becomes: *Find pairs of functions* (k, s) *that fulfill the dynamic constraint* (13)*, the inequality constraint* (10)*, and the initial condition* $k(t_0) = k_0$*.*

As we can see, these two formulations for the economic growth problem are linked by the relation (12) between the per capita consumption $c(\cdot)$ and the propensity to save $s(\cdot)$. The constraint on the consumption amount (3) is represented by (9) in the first formulation and by (10) in the second one. Obviously, the constraint in (10) is simpler than (9). For this technical reason, we will consider in this paper the economic growth problem which is formulated via the propensity to save $s(\cdot)$, instead of the per capita consumption $c(\cdot)$. It is worthy to note that one might consider the economic growth problem in the second formulation with some special saving behaviors, such as the *constancy of the saving rate*, i.e., (13) is satisfied with $s(t) = s \in [0,1]$ for all $t \geq 0$, as in growth models of Solow [4] and Swan [5] or the *classical saving behavior* as in [17, p. 439].

One major concern of the planners is to *choose a pair of functions* (k, c) *(resp.,* (k, s)*) defined on a planning interval* $[t_0, T]$*, that satisfies* (8), (9) *(resp.,* (10), (13)*) and the initial condition* $k(t_0) = k_0$*, to maximize a certain target of consumption.* A target function one may choose is $\int_{t_0}^{T} c(t)dt$, which is the total amount of per capita consumption on the time period $[t_0, T]$. A more general kind of the target function is

$$\int_{t_0}^{T} \omega(c(t))e^{-\lambda t}dt, \tag{14}$$

where $\omega : \mathbb{R}_+ \to \mathbb{R}$ is a *utility function* associated with the representative individual consumption $c(t)$ in the society, $e^{-\lambda t}$ is the *time discount factor*, and $\lambda \geq 0$ is the *real interest rate*. Clearly, the former target function is a particular case of (14) with $\omega(c) = c$ being a linear utility function and the real interest rate $\lambda = 0$. For more discussions about the choice the utility function $\omega(\cdot)$ (*it must be linear, or it can be nonlinear?*) as well as the choice of the real interest rate (*one must have* $\lambda = 0$*, or one can have* $\lambda > 0$*?*), we refer the reader to [17, pp. 445–447]. Note that the length of the planning interval of optimal economic growth problems can be finite or infinite (see, e.g., [16, p. 407], [14, Section 3.6] and esp., [17, pp. 445–446, 450 − 459] for detail discussions). When it is finite (resp., infinite), one has *optimal economic growth problems with finite horizon* (resp., *infinite horizon*). Thus, these two classes of optimal economic growth problems can be studied and treated independently. In this paper, we will work with optimal economic growth problems with finite horizon where the target function is in the general setting (14) and the real interest rate λ can be either zero or a positive number. Note also that, because of the relationship between $c(t)$ and $s(t)$ in (12), the target function in (14) can be expressed via $k(t)$ and $s(t)$ by

$$\int_{t_0}^{T} \omega[(1 - s(t))\phi(k(t))]e^{-\lambda t}dt.$$

In summary, the optimal economic growth problem that we are going to consider is formulated as follows. Let there be given a production function $F : \mathbb{R}^2_+ \to \mathbb{R}_+$ satisfying (1) and a utility function $\omega : \mathbb{R}_+ \to \mathbb{R}$. Define the per capita production function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ by setting $\phi(k) = F(k, 1)$. Assume that a finite planning interval $[t_0, T]$ with $T > t_0 \geq 0$, a growth rate of labor force $\sigma > 0$, a real interest rate $\lambda \geq 0$, and an initial capital-to-labor ratio $k_0 > 0$ are given. The problem of finding an optimal growth process for an aggregative economy is:

$$\text{Maximize } \ I(k, s) := \int_{t_0}^{T} \omega[(1 - s(t))\phi(k(t))]e^{-\lambda t}dt \tag{15}$$

over $k \in W^{1,1}([t_0, T], \mathbb{R})$ and measurable functions $s : [t_0, T] \to \mathbb{R}$ satisfying

$$\begin{cases} \dot{k}(t) = s(t)\phi(k(t)) - \sigma k(t), & \text{a.e. } t \in [t_0, T] \\ 0 \leq s(t) \leq 1, & \text{a.e. } t \in [t_0, T] \\ k(t) \geq 0, & \forall\, t \in [t_0, T] \\ k(t_0) = k_0. \end{cases} \tag{16}$$

Denote the problem in (15)–(16) by (GP). This an *optimal control problem* with $k(\cdot)$ playing the role of the *state variable* and $s(\cdot)$ playing the role of the *control variable*. By the integral form in the objective function (15) and the state constraint $k(t) \geq 0$ in (16), (GP) is an optimal control problem of the *Lagrange type* with *state constraints*.

To make (GP) competent with the given modeling presentation, one has to explain why the state trajectory can be sought in $W^{1,1}([t_0,T],\mathbb{R})$ and the control function is just required to be measurable. If one assumes that the investment function $I(\cdot)$ is continuous on $[t_0,T]$, then (4) implies that $K(\cdot)$ is continuously differentiable; hence so is $k(\cdot)$. However, in practice, the investment function $I(\cdot)$ can be discontinuous at some points $t \in [t_0,T]$ (say, the policy has a great change, and the government decides to allocate a large amount of money into the production field, or to cancel a large amount of money from it). Thus, the requirement that $k(\cdot)$ is differentiable at these points may not be fulfilled. To deal with this situation, it is reasonable to assume that the state trajectory $k(\cdot)$ belongs to the space of continuous, piecewise continuously differentiable functions on $[t_0,T]$, which is endowed with the norm $\|k\| = \max\limits_{t\in[t_0,T]} |k(t)|$. Since the latter space is incomplete one embeds it into the space $W^{1,1}([t_0,T],\mathbb{R})$, which possesses many good properties (see [26]). In that way, tools from the Lebesgue integration theory and results from the conventional optimal control theory can be used for (GP). Now, concerning the control function $s(\cdot)$, one has the following observation. Since the derivative $\dot{k}(t)$ exists almost everywhere on $[t_0,T]$ and $\dot{k}(\cdot)$ is a measurable function, for the fulfillment of the relation $\dot{k}(t) = s(t)\phi(k(t)) - \sigma k(t)$ almost everywhere on $[t_0,T]$, it suffices to assume that $s(\cdot)$ is a measurable function. Recall that a function $\varphi : [t_0,T] \to \mathbb{R}$ is said to be *measurable* if for any $\alpha \in \mathbb{R}$ the set $\{t \in [t_0,T] : \varphi \in (-\infty,\alpha)\}$ is Lebesgue measurable.

2.2 Filippov's Existence Theorem for Bolza Problems

To recall a solution existence theorem for finite horizon optimal control problems with state constraints of the Bolza type, we will use the notations and concepts given in the monograph of Cesari [24, Sections 9.2, 9.3, and 9.5]. For solution existence theorems in optimal control theory, apart from [24], the reader is referred to [27,28], and the references therein. Let $A \subset \mathbb{R} \times \mathbb{R}^n$ and $U : A \rightrightarrows \mathbb{R}^m$ be a set-valued map defined on A. Let

$$M := \{(t,x,u) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \ : \ (t,x) \in A,\ u \in U(t,x)\},$$

$f_0(t,x,u)$ and $f(t,x,u) = (f_1, f_2, \ldots, f_n)$ be functions defined on M. Let B be a given subset of $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ and $g(t_1,x_1,t_2,x_2)$ be a real valued function defined on B. Let there be given an interval $[t_0,T] \subset \mathbb{R}$. Consider the problem of minimizing the function

$$I(x,u) := g(t_0,x(t_0),T,x(T)) + \int_{t_0}^{T} f_0(t,x(t),u(t))dt \qquad (17)$$

over pairs of functions (x,u) such that $x : [t_0,T] \to \mathbb{R}^n$ is absolutely continuous, $u : [t_0,T] \to \mathbb{R}^m$ is measurable, $f_0(.,x(\cdot),u(\cdot)) : [t_0,T] \to \mathbb{R}$ is Lebesgue integrable, and

$$\begin{cases} \dot{x}(t) = f(t,x(t),u(t)), & \text{a.e. } t \in [t_0,T] \\ u(t) \in U(t,x(t)), & \text{a.e. } t \in [t_0,T] \\ (t,x(t)) \in A, & \forall\, t \in [t_0,T] \\ (t_0,x(t_0),T,x(T)) \in B. & \end{cases} \qquad (18)$$

Such a pair (x, u) is called a *feasible process.* The problem (17)–(18), which is an optimal control of the Bolza type with state constraints, is denoted by $\mathcal{B}$.

If (x, u) is a feasible process for $\mathcal{B}$, then x is said to be a *feasible state*, and u a *feasible control.* The set of all the feasible processes for $\mathcal{B}$ is denoted by Ω. A feasible process $(\bar{x}, \bar{u})$ is said to be a *global minimizer* for $\mathcal{B}$ if one has $I(\bar{x}, \bar{u}) \le I(x, u)$ for any feasible process (x, u).

Let $A_0 := \{t : \exists x \in \mathbb{R}^n \text{ s.t. } (t, x) \in A\}$. Set $A(t) = \{x \in \mathbb{R}^n : (t, x) \in A\}$ for each $t \in A_0$ and

$$\widetilde{Q}(t, x) = \{(z^0, z) \in \mathbb{R}^{n+1} : z^0 \ge f_0(t, x, u),\ z = f(t, x, u) \text{ for some } u \in U(t, x)\}$$

for every $(t, x) \in A$.

The forthcoming statement is known as *Filippov's Existence Theorem for Bolza problems.*

Theorem 2.1 (see [24, Theorem 9.3.i, p. 317, and Section 9.5]) *Suppose that Ω is nonempty, B is closed, g is lower semicontinuous on B, f_0 and f is continuous on M and, for almost every $t \in [t_0, T]$, the sets $\widetilde{Q}(t, x)$, $x \in A(t)$, are convex. Moreover, assume either that A and M are compact or that A is not compact but closed and contained in a slab $[t_1, t_2] \times \mathbb{R}^n$ with t_1 and t_2 being finite, and the following conditions are fulfilled:*

(a) *For any $\varepsilon \ge 0$, the set $M_\varepsilon := \{(t, x, u) \in M : \|x\| \le \varepsilon\}$ is compact;*
(b) *There is a compact subset P of A such that every feasible trajectory x of $\mathcal{B}$ passes through at least one point of P;*
(c) *There exists $c \ge 0$ such that $x_1 f_1(t, x, u) + \cdots + x_n f_n(t, x, u) \le c(\|x\|^2 + 1)$ for all $(t, x, u) \in M$.*

Then, $\mathcal{B}$ has a global minimizer.

Clearly, condition (b) is satisfied if the initial point $(t_0, x(t_0))$ or the end point $(T, x(T))$ is fixed. As shown in [24, p. 317], the following condition implies (c):

(c_0) *There exists $c \ge 0$ such that $\|f(t, x, u)\| \le c(\|x\| + 1)$ for all $(t, x, u) \in M$.*

In the next two sections, several results on the solution existence of optimal economic growth problems will be derived from Theorem 2.1.

3 General Optimal Economic Growth Problems

Our first result on the solution existence of the finite horizon optimal economic growth problem (GP) in (15)–(16) is stated as follows.

Theorem 3.1 *For the problem (GP), suppose that $\omega(\cdot)$ and $\phi(\cdot)$ are continuous on $\mathbb{R}_+$. If, in addition, $\omega(\cdot)$ is concave on $\mathbb{R}_+$ and the function $\phi(\cdot)$ satisfies the condition*

(c_1) *There exists $c \ge 0$ such that $\phi(k) \le (c - \sigma)k + c$ for all $k \in \mathbb{R}_+$,*

then (GP) has a global solution.

Proof To apply Theorem 2.1, we have to interpret (GP) in the form of $\mathcal{B}$. For doing so, we let the variable k (resp., the variable s) play the role of the state variable x in $\mathcal{B}$ (resp., the control variable u in $\mathcal{B}$). Then, (GP) has the form of $\mathcal{B}$ with $n = m = 1$, $A = [t_0, T] \times \mathbb{R}_+$, $U(t, k) = [0, 1]$ for all $(t, k) \in A$, $B = \{t_0\} \times \{k_0\} \times \{T\} \times \mathbb{R}$, $M = [t_0, T] \times \mathbb{R}_+ \times [0, 1]$, $g \equiv 0$ on B, $f_0(t, k, s) = -\omega((1 - s)\phi(k))e^{-\lambda t}$, and $f(t, k, s) = s\phi(k) - \sigma k$ for all $(t, k, s) \in M$.

Setting $s(t) = 0$ and $k(t) = k_0 e^{-\sigma(t-t_0)}$ for all $t \in [t_0, T]$, one can easily verify that the pair (k, s) is a feasible process for (GP). Thus, the set Ω of the feasible processes is nonempty. It is clear that B is closed, g is continuous on B and, by the assumed continuity of $\omega(\cdot)$ and $\phi(\cdot)$, f_0 and f are continuous on M. Besides, the formula for A implies that $A_0 = [t_0, T]$ and $A(t) = \mathbb{R}_+$ for all $t \in A_0$. In addition, by the formulas for f_0, f and U, one has for any $(t, k) \in A$ the following:

$$\begin{aligned}
&\widetilde{Q}(t,k)\\
&= \big\{(z^0, z) \in \mathbb{R}^2 : z^0 \geq f_0(t,k,s),\ z = f(t,k,s) \ \text{for some } s \in U(t,k)\big\}\\
&= \big\{(z^0, z) \in \mathbb{R}^2 : \exists s \in [0,1] \ \text{s.t. } z^0 \geq -\omega((1-s)\phi(k))e^{-\lambda t},\ z = s\phi(k) - \sigma k\big\}.
\end{aligned}$$

Let us show that, for any $t \in [t_0, T]$ and $k \in A(t) = \mathbb{R}_+$, the set $\widetilde{Q}(t,k)$ is convex. Indeed, given any (z_1^0, z_1), $(z_2^0, z_2) \in \widetilde{Q}(t,k)$ and $\mu \in [0,1]$, one can find $s_1, s_2 \in [0,1]$ such that

$$\begin{aligned}
z_1^0 &\geq -\omega((1-s_1)\phi(k))e^{-\lambda t},\ \ z_1 = s_1\phi(k) - \sigma k,\\
z_2^0 &\geq -\omega((1-s_2)\phi(k))e^{-\lambda t},\ \ z_2 = s_2\phi(k) - \sigma k.
\end{aligned}$$

Therefore, it holds that

$$\mu z_1^0 + (1-\mu)z_2^0 \geq -\mu\omega((1-s_1)\phi(k))e^{-\lambda t} - (1-\mu)\omega((1-s_2)\phi(k))e^{-\lambda t} \tag{19}$$

and

$$\mu z_1 + (1-\mu)z_2 = \mu[s_1\phi(k) - \sigma k] + (1-\mu)[s_2\phi(k) - \sigma k]. \tag{20}$$

Setting $s_\mu = \mu s_1 + (1-\mu)s_2$, one has $s_\mu \in [0,1]$ and it follows from (20) that

$$\mu z_1 + (1-\mu)z_2 = s_\mu\phi(k) - \sigma k. \tag{21}$$

Clearly, the concavity of $\omega(\cdot)$ on $\mathbb{R}_+$ yields

$$\begin{aligned}
&-\mu\omega((1-s_1)\phi(k)) - (1-\mu)\omega((1-s_2)\phi(k))\\
&\geq -\omega[\mu(1-s_1)\phi(k) + (1-\mu)(1-s_2)\phi(k)] = -\omega((1-s_\mu)\phi(k)).
\end{aligned}$$

Hence, by (19) we obtain $\mu z_1^0 + (1-\mu)z_2^0 \geq -\omega[(1-s_\mu)\phi(k)]e^{-\lambda t}$, which together with (21) implies that $\mu(z_1^0, z_1) + (1-\mu)(z_2^0, z_2) \in \widetilde{Q}(t,k)$.

Now, although $A = [t_0, T] \times \mathbb{R}_+$ is noncompact, the fact that A is closed and contained in a slab $[t_1, t_2] \times \mathbb{R}$ with t_1 and t_2 being finite is clear. It remains to check the conditions (a)–(c) in Theorem 2.1.

For any $\varepsilon \geq 0$, the set M_ε is compact because

$$\begin{aligned}
M_\varepsilon &= \{(t,k,s) \in [t_0,T] \times \mathbb{R}_+ \times [0,1] \ :\ |k| \leq \varepsilon\}\\
&= [t_0,T] \times [0,\varepsilon] \times [0,1].
\end{aligned}$$

So, condition (a) is satisfied. As $P := \{(t_0, k_0)\}$ is a compact subset of A, and every feasible trajectory of (GP) passes through (t_0, k_0), condition (b) is fulfilled. Applied to the case of (GP), where $f(t,k,s) = s\phi(k) - \sigma k$ and $M = [t_0,T] \times \mathbb{R}_+ \times [0,1]$ as explained above, condition (c) in Theorem 2.1 can be rewritten as

(c′) *There exists $c \geq 0$ such that $sk\phi(k) \leq (c+\sigma)k^2 + c$ for all (k,s) in $\mathbb{R}_+ \times [0,1]$.*

By the comment given after Theorem 2.1, condition (c) is valid if condition (c_0) holds. As $f(t,k,s) = s\phi(k) - \sigma k$ and $M = [t_0,T] \times \mathbb{R} \times [0,1]$, the latter can be stated as

(c_0') *There exists* $c \geq 0$ *such that* $|s\phi(k) - \sigma k| \leq c(k+1)$ *for all* (k,s) *in* $\mathbb{R}_+ \times [0,1]$.

To prove (c_0'), observe that the estimates

$$|s\phi(k) - \sigma k| \leq s\phi(k) + \sigma k \leq \phi(k) + \sigma k \tag{22}$$

hold for any $(k,s) \in \mathbb{R}_+ \times [0,1]$. Furthermore, thanks to the assumption (c_1), we can find a constant $c \geq 0$ such that $\phi(k) \leq (c-\sigma)k + c$ for all $k \in \mathbb{R}_+$. Since the last inequality can be rewritten as $\phi(k) + \sigma k \leq c(k+1)$, from (22) we get ($c_0'$).

Since our problem (GP) in the interpretation given above satisfies all the assumptions of Theorem 2.1, we conclude that it has a global solution. □

In Theorem 3.1, it is not required that $\phi(\cdot)$ is concave on $\mathbb{R}_+$. It turns out that if the concavity of $\phi(\cdot)$ is available, then there is no need to check (c_1). Since the assumption saying that the per capita production function $\phi(k) := F(k,1)$ is concave on $\mathbb{R}_+$ is reasonable in practice, next theorem seems to be interesting.

Theorem 3.2 *If both functions* $\omega(\cdot)$ *and* $\phi(\cdot)$ *are continuous and concave on* $\mathbb{R}_+$*, then* (GP) *has a global solution.*

Proof Set $\psi = -\phi$ and put $\psi(k) = +\infty$ for every $k \in (-\infty, 0)$. Then, the function $\psi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is a proper convex function and the effective domain $\mathrm{dom}\,\psi$ of ψ is $\mathbb{R}_+$. Select any $\bar{k} > 0$. Since $\bar{k}$ belongs to the interior of $\mathrm{dom}\,\psi$, by [29, Theorem 23.4] we know that the *subdifferential* (see, e.g., [29, p. 215]) $\partial\psi(\bar{k})$ of ψ at $\bar{k}$ is nonempty. Thus, taking an element $a \in \partial\psi(\bar{k})$, one has

$$\psi(k) - \psi(\bar{k}) \geq a(k - \bar{k}), \quad \forall k \geq 0,$$

or, equivalently,

$$\phi(k) \leq -ak + a\bar{k} + \phi(\bar{k}), \quad \forall k \geq 0. \tag{23}$$

For $c := \max\{0, \sigma - a, \phi(\bar{k}) + a\bar{k}\}$, one has $c \geq 0$ and

$$-ak + a\bar{k} + \phi(\bar{k}) \leq (c - \sigma)k + c, \quad \forall k \geq 0. \tag{24}$$

Combining (23) and (24), one can assert that condition (c_1) in Theorem 3.1 is fulfilled. Thus, the assumed continuity of $\omega(\cdot)$ and $\phi(\cdot)$ together with the concavity of $\omega(\cdot)$ allows us to apply Theorem 3.1 to conclude that (GP) has a global solution. □

The next proposition reveals the nature of condition (c_1), which is essential for the validity of Theorem 3.1.

Proposition 3.1 *Condition* (c_1) *and the conditions* (c') *and* (c_0')*, which were formulated in the proof of Theorem 3.1, are equivalent. Moreover, each of these conditions is equivalent to the condition*

$$\limsup_{k \to +\infty} \frac{\phi(k)}{k} < +\infty \tag{25}$$

on the asymptotic behavior of ϕ.

Proof The implications (c_1) $\Rightarrow$ (c'_0) and (c'_0) $\Rightarrow$ (c') were obtained in the proof of Theorem 3.1. So, the proposition will be proved if we can show that (c') implies (25) and (25) implies (c_1).

To get the implication (c') $\Rightarrow$ (25), suppose that (c') holds. Then, there exists $c \geq 0$ satisfying $sk\phi(k) \leq (c+\sigma)k^2 + c$ for all $(k,s) \in \mathbb{R}_+ \times [0,1]$. Thus, choosing $s = 1$, one has

$$\frac{\phi(k)}{k} \leq c + \sigma + \frac{c}{k^2}, \quad \forall k > 0.$$

By taking the limsup on both sides of the last inequality when $k \to +\infty$, one gets (25).

Now, to obtain the implication (25) $\Rightarrow$ (c'), suppose that (25) holds. Then, there exist $\gamma_1 > 0$ and $\mu > 0$ such that $\frac{\phi(k)}{k} \leq \gamma_1$ for every $k > \mu$. Thanks to the continuity of ϕ at $k = 0$, one can find $\gamma_2 > 0$ and $\varepsilon \in (0,\mu)$ such that $\phi(k) \leq \gamma_2$ for all $k \in [0,\varepsilon)$. Moreover, by the continuity of the function $k \mapsto \phi(k)/k$ on the compact interval $[\varepsilon,\mu]$, the number $\gamma_3 := \max\left\{\frac{\phi(k)}{k} \ : \ k \in [\varepsilon,\mu]\right\}$ is well defined. Thus, for any $c \geq \max\{\gamma_1+\sigma,\ \gamma_2,\ \gamma_3+\sigma\}$, it holds that

$$\phi(k) \leq \gamma_2 \leq c \leq (c-\sigma)k + c \quad \forall k \in [0,\varepsilon),$$

$$\phi(k) \leq \gamma_3 k \leq (c-\sigma)k + c \quad \forall k \in [\varepsilon,\mu]$$

and

$$\phi(k) \leq \gamma_1 k \leq (c-\sigma)k + c \quad \forall k \in (\mu,+\infty).$$

Therefore, one has $\phi(k) \leq (c-\sigma)k + c$ for every $k \geq 0$, which justifies (c_1). □

Remark 3.1 There are many continuous functions $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ that are not concave on $\mathbb{R}_+$ but satisfy condition (c_1) in Theorem 3.1. Indeed, suppose that the values $\bar{k} > 0$, $\phi_0 \geq 0$, and $a > 0$ are given arbitrarily. Setting

$$\phi(k) = \begin{cases} \phi_0, & \text{if } \ k \in [0,\bar{k}] \\ a(k-\bar{k}) + \phi_0, & \text{if } \ k \in (\bar{k},-\infty), \end{cases}$$

one has a function ϕ, that is continuous and not concave on $\mathbb{R}_+$. But, since the coercivity condition (25) is fulfilled, this ϕ satisfies (c_1). More generally, the continuous function

$$\phi(k) = \begin{cases} \phi_1(k), & \text{if } \ k \in [0,\bar{k}] \\ a(k-\bar{k})^\alpha + \phi_1(\bar{k}), & \text{if } \ k \in (\bar{k},-\infty), \end{cases}$$

where $\alpha \in (0,1]$ is a constant and $\phi : [0,\bar{k}] \to \mathbb{R}_+$ is a continuous function, also satisfies (c_1) because (25) is fulfilled. Clearly, there are many ways to choose $\phi_1(k)$ such that this function ϕ in nonconcave on $\mathbb{R}_+$.

Economic growth problems with utility functions $\omega(\cdot)$ and production functions $F(\cdot)$ of two typical types will be the subject of our consideration in next section.

4 Typical Optimal Economic Growth Problems

As observed by Takayama [17, p. 450], the production function given by

$$F(K,L) = \frac{1}{a}K, \quad \forall (K,L) \in \mathbb{R}^2_+, \tag{26}$$

where $a > 0$ is a constant representing the *capital-to-output ratio*, is of a great importance. This function is in the form of the *AK function* (see, e.g., [14, Subsection 1.3.2]) with *the diminishing returns to capital being absent*, which is a key property of endogenous growth models. The function in (26) is also referred to in connection with the Harrod-Domar model of which a main assumption is that the labor factor is not explicitly involved in the production function (see, e.g., [17, Footnote 5, p. 464]). In the notations of Subsection 2.1, by (26) one has

$$\phi(k) = \frac{1}{a}k, \quad \forall k \geq 0.$$

So, the differential equation in (16) becomes

$$\dot{k}(t) = \frac{1}{a}s(t)k(t) - \sigma k(t), \quad \text{a.e. } t \in [t_0, T].$$

Another important type of the production function F is the *Cobb-Douglas function* (see, e.g., [14, p. 29]), which is given by

$$F(K,L) = AK^\alpha L^{1-\alpha}, \quad \forall (K,L) \in \mathbb{R}^2_+, \tag{27}$$

with $A > 0$ and $\alpha \in (0,1)$ being constants. The exponent α (resp., $1-\alpha$) refers to the *output elasticity of capital* (resp., the *output elasticity of labor*), which represents the share of the contribution of the capital (resp., of the labor) to the total product $F(K,L)$. Meanwhile, A expresses the *total factor productivity* (TFP; see, e.g., https://en.wikipedia.org/wiki/Total_factor_productivity). This measure of economic efficiency is the ratio of output over the weighted average of labor and capital input. TFP represents the increase in total production which is in excess of the increase that results from increase in inputs and depends on some intangible factors such as technological change, education, research and development, etc. As $\alpha \in (0,1)$, *F exhibits diminishing returns to capital and labor* (see, e.g., [17, p. 433]). The latter means that the *marginal products* of both capital and labor are diminishing (see, e.g.,[15, p. 29]). The presence of diminishing returns to capital, which plays a very important role in many results of the basic growth model (see, e.g., [15, p. 29]), distinguishes the production given by (27) with the one in (26). The per capita production function corresponding to (27) is

$$\phi(k) = Ak^\alpha, \quad \forall k \geq 0. \tag{28}$$

Therefore, (16) collapses to

$$\dot{k}(t) = As(t)k^\alpha(t) - \sigma k(t), \quad \text{a.e. } t \in [t_0, T]. \tag{29}$$

Since (26) can be written in the form of (27) with $\alpha := 1$ and $A := 1/a$, one can combine the above two types of production functions in a general one by considering (27) with $A > 0$ and $\alpha \in (0,1]$. This means that one has deal with the model (28)–(29), where $A > 0$ and $\alpha \in (0,1]$ are given constants. In the same manner, concerning the utility function $\omega(\cdot)$, the formula

$$\omega(c) = c^\beta, \quad \forall c \geq 0 \tag{30}$$

with $\beta \in (0,1]$ can be considered. The function $\omega(\cdot)$ is linear when $\beta = 1$ and nonlinear when $\beta \in (0,1)$.

In the rest of this section, for the problem (GP), we assume that $\phi(\cdot)$ and $\omega(\cdot)$ are given respectively by (28) and (30). Then, the target function of (GP) is

$$I(k,s) = \int_{t_0}^{T} [1 - s(t)]^{\beta} \phi^{\beta}(k(t)) e^{-\lambda t} dt = A^{\beta} \int_{t_0}^{T} [1 - s(t)]^{\beta} k^{\alpha\beta}(t) e^{-\lambda t} dt.$$

Thus, we have to solve the following equivalent problem:

$$\text{Maximize} \quad \int_{t_0}^{T} [1 - s(t)]^{\beta} k^{\alpha\beta}(t) e^{-\lambda t} dt \tag{31}$$

over $k \in W^{1,1}([t_0,T],\mathbb{R})$ and measurable functions $s : [t_0,T] \to \mathbb{R}$ satisfying

$$\begin{cases} \dot{k}(t) = Ak^{\alpha}(t)s(t) - \sigma k(t), & \text{a.e. } t \in [t_0,T] \\ 0 \le s(t) \le 1, & \text{a.e. } t \in [t_0,T] \\ k(t) \ge 0, & \forall\, t \in [t_0,T] \\ k(t_0) = k_0. \end{cases} \tag{32}$$

with $A > 0$, $\alpha \in (0,1]$, $\beta \in (0,1]$, $T > t_0 \ge 0$, $\sigma > 0$, $\lambda \ge 0$, and $k_0 \ge 0$ being eight given parameters.

The forthcoming result is a consequence of Theorem 3.2.

Theorem 4.1 *For any $A > 0$, $\alpha \in (0,1]$ and $\beta \in (0,1]$, the optimal economic growth problem in* (31)–(32) *possesses a global solution.*

Proof By the assumptions $A > 0$, $\alpha \in (0,1]$, and $\beta \in (0,1]$, the functions $\phi(k) = Ak^{\alpha}$ and $\omega(c) = c^{\beta}$ are continuous on $\mathbb{R}_+$. The concavity of $\phi(\cdot)$ on $(0,+\infty)$ follows from the fact that $\phi''(k) = A\alpha(\alpha-1)k^{\alpha-2} < 0$ for every $k \in (0,+\infty)$ (see, e.g., [29, Theorem 4.4]). As $\phi(\cdot)$ is continuous at 0, we can assert that $\phi(\cdot)$ is concave on $\mathbb{R}_+$. The concavity of $\phi(\cdot)$ on $\mathbb{R}_+$ is verified similarly. Since both functions $\omega(\cdot)$ and $\phi(\cdot)$ are continuous and concave on $\mathbb{R}_+$, Theorem 3.2 assures the solution existence for the problem (31)–(32). □

Depending on the displacement of α and β on $(0,1]$, we have four types of the model (31)–(32):

- "*Linear-linear*": $\phi(k) = Ak$ and $\omega(c) = c$ (both the per capita production function and the utility function are linear);
- "*Linear-nonlinear*": $\phi(k) = Ak$ and $\omega(c) = c^{\beta}$ with $\beta \in (0,1)$ (the per capita production function is linear, but the utility function is nonlinear);
- "*Nonlinear-linear*": $\phi(k) = Ak^{\alpha}$ and $\omega(c) = c$ with $\alpha \in (0,1)$ (the per capita production function is nonlinear, but the utility function is linear);
- "*Nonlinear-nonlinear*": $\phi(k) = Ak^{\alpha}$ and $\omega(c) = c^{\beta}$ with $\alpha \in (0,1)$ and $\beta \in (0,1)$ (both the per capita production function and the utility function are nonlinear).

Although the problem in question of each type has a global solution by Theorem 4.1, the above classification arranges the difficulties of solving (31)–(32), say, by the Maximum Principle given in [30, Theorem 9.3.1]. Obviously, problems of the first type are the easiest ones, while those of the fourth type are the most difficult ones.

5 Further Discussions

In this section, first we discuss some assumptions used for getting Theorems 3.1 and 3.2. Then we will look deeper into these theorems and the typical optimal economic growth problems in Section 4 by raising some open questions and conjectures about the *uniqueness* and the *regularity* of the global solutions of (GP).

5.1 The asymptotic behavior of ϕ and its concavity

The results in Section 3 were obtained under certain assumptions on the per capita production function ϕ, which is defined via the production function $F(K, L)$ by the formula

$$\phi(k) = F(k, 1) = \frac{F(K, L)}{L} \tag{33}$$

with $k := \dfrac{K}{L}$ signifying the capital-to-labor ratio. We want to know: *How the assumptions made on ϕ can be traced back to F?*

Proposition 5.1 *The per capita production function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies condition (c_1) if and only if the production function $F : \mathbb{R}^2_+ \to \mathbb{R}_+$ has the following property:*

(c_1') *There exists $c \geq 0$ such that $F(K, L) \leq (c - \sigma)K + cL$ for all $K \geq 0$ and $L > 0$.*

Proof Suppose that (c_1) is satisfied, i.e., there exists $c \geq 0$ such that $\phi(k) \leq (c-\sigma)k + c$ for all $k \in \mathbb{R}_+$. Then, given any $K \geq 0$ and $L > 0$, by substituting $k = \dfrac{K}{L}$ into the last inequality and using (33), one gets

$$\frac{F(K, L)}{L} \leq (c - \sigma)\frac{K}{L} + c.$$

This justifies (c_1'). Conversely, suppose that $F(K, L) \leq (c-\sigma)K + cL$ holds for all $K \geq 0$ and $L > 0$, where $c \geq 0$ is a constant. Then, letting $L = 1$ and $K = k$, where $k \geq 0$ is given arbitrarily, one gets the inequality $\phi(k) \leq (c - \sigma)k + c$. Thus, (c_1) is fulfilled. $\square$

Proposition 5.2 *The function ϕ satisfies* (25) *if and only if F fulfills the following inequality:*

$$\limsup_{\frac{K}{L} \to +\infty} \frac{F(K, L)}{K} < +\infty. \tag{34}$$

Proof By (33), for any $K > 0$ and $L > 0$, one has

$$\frac{F(K, L)}{K} = \frac{L^{-1}F(K, L)}{L^{-1}K} = \frac{\phi(k)}{k}$$

with $k := \dfrac{K}{L}$. So, the equivalence between (25) and (34) is straightforward. $\square$

Propositions 5.1 and 5.2 show that the assumption made on $\phi(\cdot)$ in Theorem 3.1 and its equivalent representations given in Proposition 3.1 can be checked directly on the original function $F(\cdot, \cdot)$.

Proposition 5.3 *The per capita production function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is concave on $\mathbb{R}_+$ if and only if the production function $F : \mathbb{R}^2_+ \to \mathbb{R}_+$ is concave on $\mathbb{R}_+ \times (0, +\infty)$.*

Proof Firstly, suppose that F is concave on $\mathbb{R}_+ \times (0, +\infty)$. Let $k_1, k_2 \in \mathbb{R}_+$ and $\lambda \in [0, 1]$ be given arbitrarily. The concavity of F and (33) yield

$$F(\lambda(k_1, 1)+(1-\lambda)(k_2, 1)) \geq \lambda F(k_1, 1)+(1-\lambda)F(k_2, 1) = \lambda\phi(k_1)+(1-\lambda)\phi(k_2).$$

Since $F(\lambda(k_1, 1) + (1-\lambda)(k_2, 1)) = F(\lambda k_1 + (1-\lambda)k_2, 1)$, combining this with (33), one obtains $\phi(\lambda k_1 + (1-\lambda)k_2) \geq \lambda\phi(k_1) + (1-\lambda)\phi(k_2)$. This justifies the concavity of ϕ.

Now, suppose that ϕ is concave on $\mathbb{R}_+$. If F is not concave on $\mathbb{R}_+ \times (0, +\infty)$, then there exist $(K_1, L_1), (K_2, L_2)$ in $\mathbb{R}_+ \times (0, +\infty)$ and $\lambda \in (0, 1)$ such that

$$F(\lambda K_1 + (1-\lambda)K_2, \lambda L_1 + (1-\lambda)L_2) < \lambda F(K_1, L_1) + (1-\lambda)F(K_2, L_2).$$

By (33), it holds that $F(K, L) = L\phi\Big(\dfrac{K}{L}\Big)$ for any $(K, L) \in \mathbb{R}_+ \times (0, +\infty)$. Therefore, we have

$$[\lambda L_1 + (1-\lambda)L_2]\phi\Big(\frac{\lambda K_1 + (1-\lambda)K_2}{\lambda L_1 + (1-\lambda)L_2}\Big) < \lambda L_1\phi\Big(\frac{K_1}{L_1}\Big) + (1-\lambda)L_2\phi\Big(\frac{K_2}{L_2}\Big).$$

Dividing both sides of this inequality by $\lambda L_1 + (1-\lambda)L_2$ gives

$$\phi\Big(\frac{\lambda K_1 + (1-\lambda)K_2}{\lambda L_1 + (1-\lambda)L_2}\Big) < \frac{\lambda L_1}{\lambda L_1 + (1-\lambda)L_2}\phi\Big(\frac{K_1}{L_1}\Big) + \frac{(1-\lambda)L_2}{\lambda L_1 + (1-\lambda)L_2}\phi\Big(\frac{K_2}{L_2}\Big). \tag{35}$$

Setting $\mu = \dfrac{\lambda L_1}{\lambda L_1 + (1-\lambda)L_2}$, one has $1-\mu = \dfrac{(1-\lambda)L_2}{\lambda L_1 + (1-\lambda)L_2}$, $\mu \in (0, 1)$, and

$$\mu\frac{K_1}{L_1} + (1-\mu)\frac{K_2}{L_2} = \frac{\lambda K_1 + (1-\lambda)K_2}{\lambda L_1 + (1-\lambda)L_2}.$$

Thus, (35) means that

$$\phi\Big(\mu\frac{K_1}{L_1} + (1-\mu)\frac{K_2}{L_2}\Big) < \mu\phi\Big(\frac{K_1}{L_1}\Big) + (1-\mu)\phi\Big(\frac{K_2}{L_2}\Big).$$

This contradicts to the assumed concavity of ϕ on $\mathbb{R}_+$ and completes the proof. □

We have seen that the assumption on the concavity of ϕ used in Theorem 3.2 can be verified directly on F.

5.2 Regularity of the optimal economic growth processes

Solution regularity is an important concept which helps one to look deeper into the structure of the problem in question. One may have deal with Lipschitz continuity, Hölder continuity, and degree of differentiability of the obtained solutions. We refer to [30, Chapter 11] for a solution regularity theory in optimal control and to [31, Theorem 9.2, p. 140] for a result on the solution regularity for variational inequalities.

The results of Sections 3 and 4 assure that, if some mild assumptions on the per capital function and the utility function are satisfied, then (GP) has a global solution $(\bar{k}, \bar{s})$ with $\bar{k}(\cdot)$ being absolutely continuous on $[t_0, T]$ and $\bar{s}(\cdot)$ being measurable. Since the saving policy $\bar{s}(\cdot)$ on the time segment $[t_0, T]$ cannot be implemented if it has an infinite number of discontinuities, the following concept of regularity of the solutions of the optimal economic growth problem (GP) appears in a natural way.

Definition 5.1 A global solution $(\bar{k}, \bar{s})$ of (GP) is said to be *regular* if the propensity to save function $\bar{s}(\cdot)$ only has finitely many discontinuities of first type on $[t_0, T]$. This means that there is a positive integer m such that the segment $[t_0, T]$ can be divided into m subsegments $[\tau_i, \tau_{i+1}]$, $i = 0, \ldots, m-1$, with $\tau_0 = t_0$, $\tau_m = T$, $\tau_i < \tau_{i+1}$ for all i, $\bar{s}(\cdot)$ is continuous on each open interval (τ_i, τ_{i+1}), and the one-sided limit $\lim_{t\to\tau_i{}^+} \bar{s}(t)$ (resp., $\lim_{t\to\tau_i{}^-} \bar{s}(t)$) exists for each $i \in \{0, 1, \ldots m-1\}$ (resp., for each $i \in \{1, \ldots m\}$).

In Definition 5.1, as $\bar{s}(t) \in [0, 1]$ for every $t \in [t_0, T]$, the one-sided limit $\lim_{t\to\tau_i{}^+} \bar{s}(t)$ (resp., $\lim_{t\to\tau_i{}^-} \bar{s}(t)$) must be finite for each $i \in \{0, 1, \ldots m-1\}$ (resp., for each $i \in \{1, \ldots m\}$).

Proposition 5.4 *Suppose that the function ϕ is continuous on $[t_0, T]$. If $(\bar{k}, \bar{s})$ is a regular global solution of (GP), then the capital-to-labor ratio $\bar{k}(t)$ is a continuous, piecewise continuously differentiable function on the segment $[t_0, T]$. In particular, the function $\bar{k}(\cdot)$ is Lipschitz on $[t_0, T]$.*

Proof Since $(\bar{k}, \bar{s})$ is a regular global solution of (GP), there is a positive integer m such that the segment $[t_0, T]$ can be divided into m subsegments $[\tau_i, \tau_{i+1}]$, $i = 0, \ldots, m-1$, and all the requirements stated in Definition 5.1 are fulfilled. Then, for each $i \in \{0, \ldots, m-1\}$, from the first relation in (16) we have

$$\dot{\bar{k}}(t) = \bar{s}(t)\phi(\bar{k}(t)) - \sigma\bar{k}(t), \quad \text{a.e. } t \in (\tau_i, \tau_{i+1}). \tag{36}$$

Hence, by the continuity of ϕ on $[t_0, T]$ and the continuity of $\bar{s}(\cdot)$ on (τ_i, τ_{i+1}), we can assert that the derivative $\dot{\bar{k}}(t)$ exists for every $t \in (\tau_i, \tau_{i+1})$. Indeed, fixing any point $\bar{t} \in (\tau_i, \tau_{i+1})$ and using the Lebesgue Theorem [26, Theorem 6, p. 340] for the absolutely continuous function $\bar{k}(\cdot)$, we have

$$\bar{k}(t) = \int_{\bar{t}}^{t} \dot{\bar{k}}(\tau)d\tau, \quad \forall t \in (\tau_i, \tau_{i+1}), \tag{37}$$

where integral on the right-hand-side of the equality is understood in the the Lebesgue sense. Since the Lebesgue integral does not change if one modifies the integrand on a set of zero measure, thanks to (36) we have

$$\bar{k}(t) = \int_{\bar{t}}^{t} [\bar{s}(\tau)\phi(\bar{k}(\tau)) - \sigma\bar{k}(\tau)]d\tau. \tag{38}$$

As the integrand of the last integral is a continuous function on (τ_i, τ_{i+1}), the integration in the Lebesgue sense coincides with that in the Riemanian sense, (38) proves our claim that the derivative $\dot{\bar{k}}(t)$ exists for every $t \in (\tau_i, \tau_{i+1})$. Moreover, taking derivative of both sides of the equality (37) yields

$$\dot{\bar{k}}(t) = \bar{s}(t)\phi(\bar{k}(t)) - \sigma\bar{k}(t), \quad \forall t \in (\tau_i, \tau_{i+1}). \tag{39}$$

So, the function $\bar{k}(\cdot)$ is continuously differentiable of (τ_i, τ_{i+1}). In addition, the relation (39) and the existence of the finite one-sided limit $\lim_{t\to\tau_i{}^+} \bar{s}(t)$ (resp., $\lim_{t\to\tau_i{}^-} \bar{s}(t)$) for each i in $\{0, 1, \ldots m-1\}$ (resp., for each i in $\{1, \ldots m\}$) implies that the one-sided limit $\lim_{t\to\tau_i{}^+} \dot{\bar{k}}(t)$ (resp., $\lim_{t\to\tau_i{}^-} \dot{\bar{k}}(t)$) is finite for each i in $\{0, 1, \ldots m-1\}$ (resp., for each i in $\{1, \ldots m\}$. Thus, the restriction of $\bar{k}(\cdot)$ on each segment $[\tau_i, \tau_{i+1}]$, $i = 0, \ldots, m-1$, is a continuously differentiable function. We have shown that the capital-to-labor ratio $\bar{k}(t)$ is a continuous, piecewise continuously differentiable function on the segment $[t_0, T]$.

We omit the proof of the Lipschitz property of on $[t_0, T]$ of $\bar{k}(\cdot)$, which follows easily from the continuity and piecewise continuously differentiablity of the function by using the classical mean value theorem. □

We conclude this subsection by two open questions and three independent conjectures, whose solutions or partial solutions will reveal more the beauty of the optimal economic growth model (GP).

Open question 1: *The assumptions of Theorem 3.1 are not enough to guarantee that* (GP) *has a regular global solution?*

Open question 2: *The assumptions of Theorem 3.2 are enough to guarantee that every global solution of* (GP) *is a regular one?*

Conjectures: *The assumptions of Theorem 4.1 guarantee that*

(a) (GP) *has a unique global solution*;

(b) *Any global solution of* (GP) *is a regular one*;

(c) *If* $(\bar{k}, \bar{s})$ *is a regular global solution of* (GP)*, then the optimal propensity to save function* $\bar{s}(\cdot)$ *can have at most one discontinuity on the time segment* $[t_0, T]$.

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