NICHTNEGATIVSTELLENSÄTZE FOR DEFINABLE FUNCTIONS IN O-MINIMAL STRUCTURES

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ABSTRACT. This paper addresses to Nichtnegativstellensätze for definable functions in o-minimal structures on $(\mathbb{R}, +, \cdot)$. Namely, let $f, g_1, \ldots, g_l \colon \mathbb{R}^n \to \mathbb{R}$ be definable C^p-functions ($p \geq 2$) and assume that f is non-negative on $S := \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \ldots, g_l(x) \geq 0\}$. Under some natural hypotheses on zeros of f in S, we show that f is expressible in the form $f = \phi_0 + \sum_{i=1}^{l} \phi_i g_i$, where each ϕ_i is a sum of squares of definable C^{p-2}-functions. As a consequence, we derive global optimality conditions which generalize the Karush–Kuhn–Tucker optimality conditions for nonlinear optimization.

1. INTRODUCTION

A classical Positivstellensätz proved by Krivine [13], and independently by Stengle [22], states that a polynomial f is non-negative over a basic closed semi-algebraic set

$$S := \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \ldots, g_l(x) \geq 0\}$$

if and only if there exist an integer number $d \geq 0$ and polynomials ψ, ϕ in the preordering generated by $g_1, \ldots, g_l$ over the sums of squares (of polynomials) such that

$$\psi f = \phi + f^{2d}.$$

Note that, the denominator ψ cannot be omitted (see [5, 16]).

Schmüdgen [21] showed that if S is compact and f is strictly positive on S, then no denominators are needed; that is, ψ can be chosen as 1 in the above expression. Moreover, under some more restrictive hypotheses on the g_i, Putinar [19] proved that the polynomial f can be represented as

$$f = \phi_0 + \sum_{i=1}^{l} \phi_i g_i,$$

where each ϕ_i is a sum of squares; that is, f belongs to the quadratic module generated by the g_i's over the sums of squares, rather than the preordering generated by them. With additional conditions on zeros of f in S, Scheiderer [20] and Marshall [15] showed that the

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above equation remains true if we replace the assumption "f is strictly positive on S" by "f is non-negative on S".

In [2] (see also [1, 3, 8, 9]), Acquistapace, Andradas and Broglia established Positivstellensätze for differentiable functions in o-minimal structures. In particular, they proved that a function f that is non-negative on a closed set (not necessarily compact)

$$S = \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \ldots, g_l(x) \geq 0\}$$

admits a representation of the form

$$\psi^2 f = \phi_0^2 + \sum_{i=1}^{l} \phi_i^2 g_i.$$

Again, denominators are necessary. Indeed, for any S with non-empty interior, there are definable functions that are non-negative over S and do not belong to the preordering generated by $g_1, \ldots, g_l$ over the sums of squares; that is, the denominator ψ in the above equation cannot be omitted; for more details, see [2, Remark 3.9].

This paper deals with Nichtnegativstellensätze (local and global) for definable functions of class C^p $(p \geq 2)$ in o-minimal structures on $(\mathbb{R}, +, \cdot)$ without any compactness assumption. Indeed, let f be a definable function, which is non-negative on a basic definable set

$$S = \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \ldots, g_l(x) \geq 0\}.$$

We give natural sufficient conditions in terms of the first and second derivatives of f at its zeros in S, so that f can be represented as

$$f = \phi_0 + \sum_{i=1}^{l} \phi_i g_i,$$

where each ϕ_i is a sum of squares. Our proof is elementary, using Morse's lemma and partitions of unity.

As a consequence, we obtain global optimality conditions which generalize the Karush–Kuhn–Tucker optimality conditions for nonlinear optimization.

We finish this section by noting that all the statements and proofs in this paper remain true if we remove in them the term "definable". Also, all the statements still hold if we replace $\mathbb{R}^n$ by any real (definable) manifold; however, to lighten the exposition, we do not pursue this extension here.

The rest of this paper is organized as follows. Section 2 contains some properties of definable sets and functions in o-minimal structures. For the convenience of the reader, local optimality conditions in nonlinear programming theory are also recalled here. Nichtnegativstellensätze for definable functions (Theorems 3.1 and 3.2) are established in Section 3. Finally, global optimality conditions (Theorem 4.1) are presented in Section 4.

2. Preliminaries

In this paper, we deal with the Euclidean space $\mathbb{R}^n$ equipped with the usual scalar product $\langle \cdot, \cdot \rangle$ and the corresponding Euclidean norm $\| \cdot \|$. Let $\mathbb{R}_+$ denote the set of positive real numbers. If f is a function in x, $\nabla f(x)$ (resp., $\nabla^2 f(x)$) denotes the gradient vector (resp., Hessian matrix) of f at x.

2.1. **O-minimal structures.** The notion of o-minimality was developed in the late 1980s after it was noticed that many proofs of analytic and geometric properties of semi-algebraic sets and mappings can be carried over verbatim for sub-analytic sets and mappings. The reader is referred to [6, 23, 24] for more details.

Definition 2.1. An *o-minimal structure* on $(\mathbb{R}, +, \cdot)$ is a sequence $\mathcal{D} := (\mathcal{D}_n)_{n\in\mathbb{N}}$ such that for each $n \in \mathbb{N}$:

(a) $\mathcal{D}_n$ is a Boolean algebra of subsets of $\mathbb{R}^n$.
(b) If $X \in \mathcal{D}_m$ and $Y \in \mathcal{D}_n$, then $X \times Y \in \mathcal{D}_{m+n}$.
(c) If $X \in \mathcal{D}_{n+1}$, then $\pi(X) \in \mathcal{D}_n$, where $\pi\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection on the first n coordinates.
(d) $\mathcal{D}_n$ contains all algebraic subsets of $\mathbb{R}^n$.
(e) Each set belonging to $\mathcal{D}_1$ is a finite union of points and intervals.

A set belonging to $\mathcal{D}$ is said to *definable* (in that structure). *Definable mappings* in structure $\mathcal{D}$ are mappings whose graphs are definable sets in $\mathcal{D}$.

Examples of o-minimal structures are

- the semi-algebraic sets (by the Tarski–Seidenberg theorem),
- the globally sub-analytic sets, i.e., the sub-analytic sets of $\mathbb{R}^n$ whose (compact) closures in the real projective space $\mathbb{RP}^n$ are sub-analytic (using Gabrielov's complement theorem).

In this paper, we fix an o-minimal structure on $(\mathbb{R}, +, \cdot)$. The term "definable" means definable in this structure. We recall some useful facts which we shall need later.

Lemma 2.1. *Every definable set has a finite number of connected components.*

Proof. See [24, Properties 4.3]. □

Lemma 2.2. *Let U be a definable open subset of $\mathbb{R}^n$ containing 0 and $f\colon U \to \mathbb{R}$ be a definable C^p-function ($p \geq 1$) with $f(0) = 0$. Then there are definable C^{p-1}-functions $f_i\colon U \to \mathbb{R}$ such that on U we have*

$$f = x_1 f_1 + \cdots + x_n f_n.$$

Proof. See [18, Lemma A.6]. □

To simplify notation in what follows, the notation "$F\colon (\mathbb{R}^n, x^*) \to (\mathbb{R}^m, y^*)$" means that F is a mapping from a definable open neighbourhood of $x^* \in \mathbb{R}^n$ into $\mathbb{R}^m$ with $F(x^*) = y^*$.

Lemma 2.3 (Morse's lemma)**.** *Let U be a definable open subset of $\mathbb{R}^n$ containing 0 and $f\colon U \to \mathbb{R}$ be a definable C^p-function ($p \geq 2$). If 0 is a non-degenerate critical point of f (i.e., $\nabla f(0) = 0$ and the Hessian matrix $\nabla^2 f(0)$ of f at 0 is non-singular), then there is a definable C^{p-2}-diffeomorphism $\Phi\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that*

$$f \circ \Phi(y) = f(0) - y_1^2 - \cdots - y_\ell^2 + y_{\ell+1}^2 + \cdots + y_n^2,$$

where ℓ is the number of negative eigenvalues (multiplicity taken into account) of $\nabla^2 f(0)$.

Proof. Confer [18, Lemma A.7] (see also [11, Section 6.1] and [12, Theorem 2.8.2]). $\square$

Lemma 2.4 (Definable partition of unity)**.** *Let $\{U_k\}_{k=1,\ldots,K}$ be a finite definable open covering of $\mathbb{R}^n$. Then for any $p \geq 0$, there exist definable C^p-functions $\theta_k\colon \mathbb{R}^n \to [0,1], k = 1, \ldots, K$, such that the following statements hold:*

(i) $\operatorname{supp}\theta_k \subset U_k$, *where* $\operatorname{supp}\theta_k$ *denotes the closure of the set* $\{x \in \mathbb{R}^n \mid \theta_k(x) \neq 0\}$;
(ii) $\sum_{k=1}^K [\theta_k(x)]^2 = 1$ *for all* $x \in \mathbb{R}^n$.

Proof. It is well-known that (see, for example, [7, Theorem 3.4.2], [11, Theorem 2.1], [23, Lemma 3.7]), there exists a definable partition of unity $\{\phi_k\}_{k=1,\ldots,K}$ subordinated to the covering $\{U_k\}_{k=1,\ldots,K}$. Clearly, the functions $\theta_k := \frac{\phi_k}{\sqrt{\sum_{k=1}^K \phi_k^2}}$ have the desired properties. $\square$

2.2. **Optimality conditions.** We give here a short review of optimality conditions in nonlinear programming theory (confer [4, Section 4.3]).

Let $f, g_1, \ldots, g_l, h_1, \ldots, h_m\colon \mathbb{R}^n \to \mathbb{R}$ be C^p-functions ($p \geq 2$) and assume that

$$S := \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \ldots, g_l(x) \geq 0, h_1(x) = 0, \ldots, h_m(x) = 0\} \neq \emptyset.$$

Definition 2.2. The constraint set S is said to be *regular at $x \in S$* if the gradient vectors $\nabla g_i(x),\ i \in I(x)$ and $\nabla h_j(x),\ j = 1, \ldots, m$, are linearly independent, where

$$I(x) := \{i \in \{1, \ldots, l\} \mid g_i(x) = 0\}$$

is called the *set of active constraint indices at x*. The set S is called *regular* if it is regular at every point $x \in S$.

Let x^* be a local minimizer of the restriction of f on S and assume that S is regular at x^*. It is well-known that there exist (unique) Lagrange multipliers $\lambda_i, i = 1, \ldots, l$, and $\nu_j, j = 1, \ldots, m$, satisfying the *Karush–Kuhn–Tucker optimality conditions* (KKT optimality conditions for short)

$$\begin{aligned} &\nabla f(x^*) - \sum_{i=1}^l \lambda_i \nabla g_i(x^*) - \sum_{j=1}^m \nu_j \nabla h_j(x^*) = 0, \\ &\lambda_i g_i(x^*) = 0,\ \lambda_i \geq 0, \quad \text{for} \quad i = 1, \ldots, l. \end{aligned}$$

Recall that the *strict complementarity condition* holds at x^* if it holds that

$$\lambda_1 + g_1(x^*) > 0, \ldots, \lambda_l + g_l(x^*) > 0.$$

Note that strict complementarity is equivalent to $\lambda_i > 0$ for every $i \in I(x^*)$.

Let $L(x)$ be the associated Lagrangian function

$$L(x) := f(x) - \sum_{i \in I(x^*)} \lambda_i g_i(x) - \sum_{j=1}^{m} \nu_j h_j(x),$$

where $I(x^*)$ is the set of active constraint indices at x^*. Then the *second-order necessity condition* holds at x^*, that is

$$v^T \nabla^2 L(x^*) v \geq 0 \quad \text{for all} \quad v \in T_{x^*} S.$$

Here $\nabla^2 L(x^*)$ is the Hessian matrix of L at x^* and $T_{x^*}S$ stands for the *(generalized) tangent space* of S at x^*:

$$T_{x^*} S := \left\{ \begin{array}{ccl} v \in \mathbb{R}^n & : & \langle v, \nabla g_i(x^*) \rangle = 0, \ i \in I(x^*) \ \text{ and} \\ & & \langle v, \nabla h_j(x^*) \rangle = 0, \ j = 1, \ldots, m \end{array} \right\}.$$

If it holds that

$$v^T \nabla^2 L(x^*) v > 0 \quad \text{for all} \quad v \in T_{x^*} S \setminus \{0\},$$

we say the *second-order sufficiency condition* holds at x^*.

Remark 2.1. (i) Let $x^* \in S$ be a local minimizer of f on S and assume that S is regular at x^*. According to [12, Lemma 3.2.16], the point x^* is a *(generalized) non-degenerate critical point* of the restriction of f to S if and only if the strict complementarity and second-order sufficiency conditions hold at x^*.

(ii) Using transversality arguments, one can show that the regularity, strict complementarity and second order sufficiency conditions hold generically. For related works, see [10,17].

3. Nichtnegativstellensätze for definable functions

In this section we prove two Nichtnegativstellensätze for definable functions. So let $f, g_1, \ldots, g_l, h_1, \ldots, h_m \colon \mathbb{R}^n \to \mathbb{R}$ be definable C^p-functions ($p \geq 2$) and assume that

$$S := \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \ldots, g_l(x) \geq 0, h_1(x) = 0, \ldots, h_m(x) = 0\} \neq \emptyset.$$

The first main result of the paper reads as follows.

Theorem 3.1 (Local Nichtnegativstellensätz)**.** *Assume that f is non-negative on S and let $x^* \in S$ be a zero of f in S. If S is regular at x^* and the strict complementarity and second-order sufficiency conditions hold at x^*, then there are a definable open neighbourhood U of*

x^* and definable C^{p-2}-functions $\phi_i, \psi_j : U \to \mathbb{R}$ for $i = 0, \ldots, l$ and $j = 1, \ldots, m$, where each ϕ_i is a sum of squares of definable C^{p-2}-functions, such that on U we have

$$f = \phi_0 + \sum_{i=1}^{l} \phi_i g_i + \sum_{j=1}^{m} \psi_j h_j.$$

Proof. (cf. [12, 17]). For convenience, we can generally assume $x^* = 0$, up to a shifting.

Recall that $I(x^*) := \{i \in \{1, \ldots, l\} \mid g_i(x^*) = 0\}$ is the index set of inequality constraints that are active at x^*. By renumbering, we may assume that $I(x^*) := \{1, \ldots, k\}$. Since S is regular at 0, the gradient vectors

$$\nabla g_1(0), \ldots, \nabla g_k(0), \nabla h_1(0), \ldots, \nabla h_m(0)$$

are linearly independent. Let $d := n - m - k$. Up to a linear coordinate transformation, we can further assume that

$$\begin{array}{rclcrcl} \nabla g_1(0) & = & e^{d+1}, & \ldots, & \nabla g_k(0) & = & e^{d+k}, \\ \nabla h_1(0) & = & e^{d+k+1}, & \ldots, & \nabla h_m(0) & = & e^{n}, \end{array}$$

where $e^1, \ldots, e^n$ are the canonical basis vectors in $\mathbb{R}^n$. Note that the space $T_{x^*}S$ is determined by the vectors $e^1, \ldots, e^d$.

Define the definable C^p-mapping $\Phi : \mathbb{R}^n \longrightarrow \mathbb{R}^n, x \mapsto \Phi(x)$, by

$$\Phi(x) := (x_1, \ldots, x_d, g_1(x), \ldots, g_k(x), h_1(x), \ldots, h_m(x)).$$

Clearly, $\Phi(0) = 0$ and the Jacobian matrix $D\Phi(0)$ of Φ at 0 is the identity matrix I_n. Thus, by the inverse function theorem, Φ is a local C^p-diffeomorphism in some neighbourhood of 0 with the inverse

$$\Phi^{-1} : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0), \quad t \mapsto x := \Phi^{-1}(t).$$

So, $t := (t_1, \ldots, t_n)$ can serve as a coordinate system for $\mathbb{R}^n$ around 0. In the t-coordinate system and in a neighborhood of 0, the set S defined by

$$t_{d+1} \geq 0, \ldots, t_{d+k} \geq 0, t_{d+k+1} = 0, \ldots, t_n = 0.$$

Let λ_i and ν_j be the Lagrange multipliers with respect to the minimizer x^*. Define the Lagrangian function

$$L(x) := f(x) - \sum_{i \in I(x^*)} \lambda_i g_i(x) - \sum_{j=1}^{m} \nu_j h_j(x).$$

Note that $\nabla L(0) = 0$. In the t-coordinate system, define the functions

$$\begin{aligned} F(t) &:= f(\Phi^{-1}(t)), \\ \widehat{L}(t) &:= L(\Phi^{-1}(t)) = F(t) - \sum_{r=d+1}^{d+k} \lambda_{r-d} t_r - \sum_{r=d+k+1}^{n} \nu_{r-d-k} t_r. \end{aligned}$$

Clearly,

$$\nabla \widehat{L}(0) = \nabla L(0) D\Phi^{-1}(0) = \nabla L(0) = 0.$$

This implies that

$$\frac{\partial F}{\partial t_r}(0) = \begin{cases} 0 & \text{if } r = 1, \ldots, d, \\ \lambda_{r-d} & \text{if } r = d+1, \ldots, d+k, \\ \nu_{r-d-k} & \text{if } r = d+k+1, \ldots, n. \end{cases}$$

Furthermore, for $(t_1, \ldots, t_d)$ near $0 \in \mathbb{R}^d$, it holds that

$$\begin{aligned} F(t_1, \ldots, t_d, 0, \ldots, 0) &= \widehat{L}(t_1, \ldots, t_d, 0, \ldots, 0), \\ &= L(\Phi^{-1}(t_1, \ldots, t_d, 0, \ldots, 0)). \end{aligned}$$

Let $x(t) := \Phi^{-1}(t) = (\Phi_1^{-1}(t), \ldots, \Phi_n^{-1}(t))$. For all i, j, we have

$$\frac{\partial^2 \widehat{L}(t)}{\partial t_i \partial t_j} = \sum_{1 \le r, s \le n} \frac{\partial^2 L(x(t))}{\partial x_r \partial x_s} . \frac{\partial \Phi_r^{-1}(t)}{\partial t_i} . \frac{\partial \Phi_s^{-1}(t)}{\partial t_j} + \sum_{1 \le r \le n} \frac{\partial L(x(t))}{\partial x_r} . \frac{\partial^2 \Phi_r^{-1}(t)}{\partial t_i \partial t_j}.$$

Note that $\nabla L(0) = 0$ and $x(0) = \Phi^{-1}(0) = 0$. Hence

$$\frac{\partial^2 \widehat{L}}{\partial t_i \partial t_j}(0) = \sum_{1 \le r, s \le n} \frac{\partial^2 L}{\partial x_r \partial x_s}(0) . \frac{\partial \Phi_r^{-1}}{\partial t_i}(0) . \frac{\partial \Phi_s^{-1}}{\partial t_j}(0).$$

On the other hand, we have $D\Phi(0) = D\Phi^{-1}(0) = I_n$-the identity matrix. Therefore for all $i, j = 1, \ldots, d$,

$$\left.\frac{\partial^2 F}{\partial t_i \partial t_j}\right|_{t=0} = \left.\frac{\partial^2 \widehat{L}}{\partial t_i \partial t_j}\right|_{t=0} = \left.\frac{\partial^2 L}{\partial x_i \partial x_j}\right|_{x=0}.$$

Since the vector space $T_{x^*}S$ is defined by the vectors $e^1, \ldots, e^d$, the second-order sufficiency condition implies that the sub-Hessian

$$\left(\frac{\partial^2 L}{\partial x_i \partial x_j}(0)\right)_{1 \le i, j \le d}$$

is positive definite.

Define the definable C^p-function $A \colon (\mathbb{R}^d, 0) \to (\mathbb{R}, 0)$, by

$$A(t_1, \ldots, t_d) := F(t_1, \ldots, t_d, 0, \ldots, 0).$$

Then $A(0) = 0, \nabla A(0) = 0$ and the Hessian matrix $\nabla^2 A(0)$ is positive define. On the other hand, by Lemma 2.2, there exist definable C^{p-1}-functions $B_r \colon (\mathbb{R}^{d+k}, 0) \to (\mathbb{R}, 0)$ for $r = 1, \ldots, k$, and $C_r \colon (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ for $r = 1, \ldots, m$ such that

$$\begin{aligned} F(t_1, \ldots, t_{d+k}, 0, \ldots, 0) &= F(t_1, \ldots, t_d, 0, \ldots, 0) + \sum_{r=d+1}^{d+k} t_r B_{r-d}(t_1, \ldots, t_{d+k}), \\ F(t_1, \ldots, t_n) &= F(t_1, \ldots, t_{d+k}, 0, \ldots, 0) + \sum_{r=d+k+1}^{n} t_r C_{r-d-k}(t_1, \ldots, t_n). \end{aligned}$$

Then for all $t := (t_1, \ldots, t_n) \in \mathbb{R}^n$ near 0, we have

$$F(t_1, \ldots, t_n) \;=\; A(t_1, \ldots, t_d) + \sum_{r=d+1}^{d+k} t_r B_{r-d}(t_1, \ldots, t_{d+k}) + \sum_{r=d+k+1}^{n} t_r C_{r-d-k}(t_1, \ldots, t_n).$$

In view of Lemma 2.3 (applied to the function A), there is a definable C^{p-2}-diffeomorphism

$$\theta\colon (\mathbb{R}^d, 0) \to (\mathbb{R}^d, 0), \quad (t_1, \ldots, t_d) \mapsto (z_1, \ldots, z_d),$$

such that

$$A \circ \theta^{-1}(z_1, \ldots, z_d) \;=\; \sum_{r=1}^{d} z_r^2.$$

We extend θ to a definable C^{p-2}-diffeomorphism

$$\Theta\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0), \quad t \mapsto z := \Theta(t),$$

by putting

$$\Theta(t_1, \ldots, t_n) \;:=\; (\theta(t_1, \ldots, t_d), t_{d+1}, \ldots, t_n).$$

Next we put

$$\begin{aligned}
\widetilde{B}_i(z_1, \ldots, z_{d+k}) &:= B_i(\theta^{-1}(z_1, \ldots, z_d), z_{d+1}, \ldots, z_{d+k}), \quad i = 1, \ldots, k, \\
\widetilde{C}_j(z_1, \ldots, z_n) &:= C_j(\theta^{-1}(z_1, \ldots, z_d), z_{d+1}, \ldots, z_n), \quad j = 1, \ldots, m.
\end{aligned}$$

Clearly, $\widetilde{B}_i$ and $\widetilde{C}_j$ are definable C^{p-2}-functions, $\widetilde{B}_i(0) = B_i(0) = \lambda_i > 0$. In particular, in some neighbourhood of $0 \in \mathbb{R}^{d+k}$, the functions $\widetilde{B}_i$ are squares of definable C^{p-1}-functions. For all $z := (z_1, \ldots, z_n) \in \mathbb{R}^n$ near 0, we have

$$\begin{aligned}
f \circ (\Theta \circ \Phi)^{-1}(z) &= F \circ \Theta^{-1}(z) \\
&= \sum_{r=1}^{d} z_r^2 + \sum_{r=d+1}^{d+k} z_r \widetilde{B}_{r-d}(z_1, \ldots, z_{d+k}) + \sum_{r=d+k+1}^{n} z_r \widetilde{C}_{r-d-k}(z).
\end{aligned}$$

Let $x := (\Theta \circ \Phi)^{-1}(z)$; or equivalently,

$$z = (\theta(x_1, \ldots, x_d), g_1(x), \ldots, g_k(x), h_1(x), \ldots, h_m(x)).$$

Then it is easy to see that the functions

$$\begin{aligned}
\phi_0(x) &:= \|\theta(x_1, \ldots, x_d)\|^2, \\
\phi_i(x) &:= \widetilde{B}_i\big(\theta(x_1, \ldots, x_d), g_1(x), \ldots, g_k(x)\big), \quad i = 1, \ldots, k, \\
\phi_i(x) &:= 0, \quad i = k+1, \ldots, l, \\
\psi_j(x) &:= \widetilde{C}_j\big(\theta(x_1, \ldots, x_d), g_1(x), \ldots, g_k(x), h_1(x), \ldots, h_m(x)\big), \quad j = 1, \ldots, m,
\end{aligned}$$

have the desired properties. $\square$

We are now ready to prove our second main result.

Theorem 3.2 (Global Nichtnegativstellensätz)**.** *Assume that f is nonnegative on S. If the regularity, strict complementarity and second-order sufficiency conditions hold at every zeros of the restriction of f on S, then there are definable C^{p-2}-functions $\phi_i, \psi_j \colon \mathbb{R}^n \to \mathbb{R}$ for $i = 0, \ldots, l$ and $j = 1, \ldots, m$, where each ϕ_i is a sum of squares of definable C^{p-2}-functions, such that*

$$f = \phi_0 + \sum_{i=1}^{l} \phi_i g_i + \sum_{j=1}^{m} \psi_j h_j.$$

Proof. Our assumptions yield that the definable set $f^{-1}(0) \cap S$ is discrete (see, for example, [12, Corollary 3.2.30]). This, together with Lemma 2.1, implies that $f^{-1}(0) \cap S$ is a finite set, say $\{x_1^*, \ldots, x_N^*\}$. In view of Theorem 3.1, for each $k = 1, \ldots, N$, there exist a definable open neighbourhood U_k of x_k^* and definable C^{p-2}-functions $\phi_{k,i}, \psi_{k,j} \colon U_k \to \mathbb{R}$ for $i = 0, 1, \ldots, l$ and $j = 1, \ldots, m$, where each $\phi_{k,i}$ is a sum of squares of definable C^{p-2}-functions, such that on U_k we have

$$f = \phi_{k,0} + \sum_{i=1}^{l} \phi_{k,i} g_i + \sum_{j=1}^{m} \psi_{k,j} h_j.$$

Since f is positive on the set $S \setminus \{x_1^*, \ldots, x_N^*\}$, we can find a (definable) open set U_{N+1} containing $S \setminus \bigcup_{k=1}^{N} U_k$ such that f is positive on U_{N+1}. On the set U_{N+1}, define the definable C^p-functions $\phi_{k,i}$ and $\psi_{k,j}$ by

$$\begin{aligned}
\phi_{N+1,0} &:= f, \\
\phi_{N+1,i} &:= 0 \quad \text{for} \quad i = 1, \ldots, l, \\
\psi_{N+1,j} &:= 0 \quad \text{for} \quad j = 1, \ldots, m.
\end{aligned}$$

For $k = N+2, \ldots, N+l+1$, on the definable open set $U_k := \{x \in \mathbb{R}^n \mid g_{k-N-1}(x) < 0\}$, define the definable C^p-functions $\phi_{k,i}$ and $\psi_{k,j}$ by

$$\begin{aligned}
\phi_{k,0} &:= \left(\frac{f+1}{2}\right)^2, \\
\phi_{k,i} &:= \left(\frac{f-1}{2}\right)^2 \cdot \frac{1}{l(-g_i)} \quad \text{for} \quad i = 1, \ldots, l, \\
\psi_{k,j} &:= 0 \quad \text{for} \quad j = 1, \ldots, m.
\end{aligned}$$

For $k = N+l+2, \ldots, N+l+m+1$, on the definable open set $U_k := \{x \in \mathbb{R}^n \mid h_{k-N-l-1}(x) \neq 0\}$, define the definable C^p-functions $\phi_{k,i}$ and $\psi_{k,j}$ by

$$\begin{aligned}
\phi_{k,0} &:= 0 \quad \text{for} \quad i = 0, \ldots, l, \\
\psi_{k,j} &:= \frac{f}{mh_j} \quad \text{for} \quad j = 1, \ldots, m.
\end{aligned}$$

By definition, on the set U_k, $k = N+1, \ldots, N+l+m+1$, we have

$$f = \phi_{k,0} + \sum_{i=1}^{l} \phi_{k,i} g_i + \sum_{j=1}^{m} \psi_{k,j} h_j.$$

Since $\{U_k\}_{k=1,\ldots,N+l+m+1}$ is a the family of definable open sets covering $\mathbb{R}^n$, it follows from Lemma 2.4 that there exist definable C^p-functions $\theta_k \colon \mathbb{R}^n \to [0,1]$, $k = 1, \ldots, N+l+m+1$, such that $\operatorname{supp} \theta_k \subset U_k$ and

$$\sum_{k=1}^{N+l+m+1} \big[\theta_k(x)\big]^2 = 1 \quad \text{for all} \quad x \in \mathbb{R}^n.$$

For $k = 1, \ldots, N+l+m+1$, $i = 0, 1, \ldots, l$, and $j = 1, \ldots, m$, the functions $\theta_k^2 \phi_{k,i}$ and $\theta_k^2 \psi_{k,j}$ extend by 0 to (definable) C^{p-2}-functions over $\mathbb{R}^n$. By construction, the functions $\theta_k^2 \phi_{k,i}$ are sums of squares of definable C^{p-2}-functions.

Finally, on $\mathbb{R}^n$ we have

$$\begin{aligned} f &= 1 \cdot f = \left(\sum_{k=1}^{N+l+m+1} \theta_k^2 \right) f = \sum_{k=1}^{N+l+m+1} \theta_k^2 f \\ &= \sum_{k=1}^{N+l+m+1} \left(\theta_k^2 \phi_{k,0} + \sum_{i=1}^{l} \theta_k^2 \phi_{k,i} g_i + \sum_{j=1}^{m} \theta_k^2 \psi_{k,j} h_j \right) \\ &= \sum_{k=1}^{N+l+m+1} \theta_k^2 \phi_{k,0} + \sum_{i=1}^{l} \left(\sum_{k=1}^{N+l+m+1} \theta_k^2 \phi_{k,i} \right) g_i + \sum_{j=1}^{m} \left(\sum_{k=1}^{N+l+m+1} \theta_k^2 \psi_{k,j} \right) h_j. \end{aligned}$$

The proof is complete. □

4. Global optimality conditions

In this section, we derive global optimality conditions which generalize the KKT optimality conditions for nonlinear optimization.

Let $f, g_1, \ldots, g_l, h_1, \ldots, h_m \colon \mathbb{R}^n \to \mathbb{R}$ be definable C^p-functions ($p \geq 3$) and assume that

$$S := \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \ldots, g_l(x) \geq 0, h_1(x) = 0, \ldots, h_m(x) = 0\} \neq \emptyset.$$

Let $(x^*, \lambda, \nu) \in S \times \mathbb{R}^l_+ \times \mathbb{R}^m$ be a vector satisfying the KKT optimality conditions associated with the problem $\min_{x \in S} f(x)$, that is

$$\begin{aligned} &\nabla f(x^*) - \sum_{i=1}^{l} \lambda_i \nabla g_i(x^*) - \sum_{j=1}^{m} \nu_j \nabla h_j(x^*) = 0, \\ &\lambda_i g_i(x^*) = 0, \ \ \lambda_i \geq 0, \quad \text{for} \ \ i = 1, \ldots, l. \end{aligned}$$

It follows that x^* a stationary point of the Lagrangian function

$$L(x) := f(x) - \sum_{i=1}^{l} \lambda_i g_i(x) - \sum_{j=1}^{m} \nu_j h_j(x).$$

However, in general, x^* is not a global minimizer of L (and may not even be a local minimizer).

On the other hand, we have the following global optimality conditions, which is inspired by the work of Lasserre [14, Chapter 7] (see also [10, Subsection 7.4.5]).

Theorem 4.1. *Assume that $f_* := \min_{x\in S} f(x) > -\infty$. If the regularity, strict complementarity and second-order sufficiency conditions hold at every minimizers of the restriction of f on S, there are definable C^{p-2}-functions $\phi_i, \psi_j \colon \mathbb{R}^n \to \mathbb{R}$ for $i = 0, \ldots, l$ and $j = 1, \ldots, m$, where each ϕ_i is a sum of squares of definable C^{p-2}-functions, such that*

$$f - f_* \ = \ \phi_0 + \sum_{i=1}^{l} \phi_i g_i + \sum_{j=1}^{m} \psi_j h_j. \tag{1}$$

Let $x^* \in S$ be a global minimizer of f on S. The following statements hold:

(i) $\phi_0(x^*) = 0$, *and* $\phi_i(x^*) g_i(x^*) = 0$ *and* $\phi_i(x^*) \geq 0$ *for all* $i = 1, \ldots, l$.

(ii) $\nabla f(x^*) - \sum_{i=1}^{l} \phi_i(x^*) \nabla g_i(x^*) - \sum_{j=1}^{m} \psi_j(x^*) \nabla h_j(x^*) = 0$.

(iii) x^* *is a global minimizer of the (generalized) Lagrangian function*

$$\mathscr{L}(x) := f(x) - f_* - \sum_{i=1}^{l} \phi_i(x) g_i(x) - \sum_{j=1}^{m} \psi_j(x) h_j(x).$$

Proof. The existence of functions ϕ_i and ψ_j for which the equality (1) holds follows immediately from Theorem 3.2.

(i) From (1) and the fact that x^* is a global minimizer of f on S, we get

$$0 \ = \ f(x^*) - f_* \ = \ \phi_0(x^*) + \sum_{i=1}^{l} \phi_i(x^*) g_i(x^*) + \sum_{j=1}^{m} \psi_j(x^*) h_j(x^*),$$

which in turn implies (i) because $g_i(x^*) \geq 0$ for $i = 1, \ldots, l$, $h_j(x^*) = 0$ for $j = 1, \ldots, m$, and the functions ϕ_i are all sums of squares of definable C^{p-2}-functions, hence nonnegative.

(ii) Differentiating (1), using (i) and the fact that the functions ϕ_i are sums of squares of definable C^{p-2}-functions yield (ii).

(iii) Since ϕ_0 is a sum of squares of definable C^{p-2}-functions, we have for all $x \in \mathbb{R}^n$,

$$\mathscr{L}(x) = f(x) - f_* - \sum_{i=1}^{l} \phi_i(x) g_i(x) - \sum_{j=1}^{m} \psi_j(x) h_j(x) = \phi_0(x) \geq 0.$$

Note that $\mathscr{L}(x^*) = \phi_0(x^*) = 0$. Therefore, x^* is a global minimizer of $\mathscr{L}$. $\square$

Remark 4.1. Theorem 4.1 implies the following facts.

(i) The equality (1) can be interpreted as a *global optimality condition*.

(ii) The function $\mathscr{L}(x) := f(x) - f_* - \sum_{i=1}^{l} \phi_i(x) g_i(x) - \sum_{j=1}^{m} \psi_j(x) h_j(x)$ is a *generalized Lagrangian function*, with generalized Lagrange (definable functions) multipliers $((\phi_i), (\psi_j))$ instead of scalar multipliers $(\lambda, \nu) \in \mathbb{R}^l_+ \times \mathbb{R}^m$. It is a sum of squares of definable C^{p-2}-functions (hence nonnegative on $\mathbb{R}^n$), vanishes at every global minimizer $x^* \in S$, and so x^* is also a global minimizer of the generalized Lagrangian function.

(iii) The generalized Lagrange multipliers $((\phi_i), (\psi_j))$ provide a *certificate of global optimality* for $x^* \in S$ in the nonconvex case exactly as the Lagrange multipliers $(\lambda, \nu) \in \mathbb{R}^l_+ \times \mathbb{R}^m$ provide a certificate in the convex case.

We should also mention that in the KKT optimality conditions, only the constraints $g_i(x) \geq 0$ that are active at x^* have a possibly nontrivial associated Lagrange (scalar) multiplier λ_i. Hence the nonactive constraints do not appear in the Lagrangian function $\mathscr{L}$ defined above. In contrast, in the global optimality condition (1), every constraint $g_i(x) \geq 0$ has a possibly nontrivial Lagrange multiplier $\phi_i(x)$. But if $g_i(x^*) > 0$ then necessarily $\phi_i(x^*) = 0 = \lambda_i$, as in the KKT optimality conditions.

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