Necessary and sufficient conditions for assignability of dichotomy spectrum of one-sided discrete time-varying linear systems

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April 5, 2021

Abstract

We consider a version of the pole placement problem for one-sided linear discrete time-varying linear systems. Our purpose is to prove that uniform complete controllability is equivalent to possibility of arbitrary assignment of the dichotomy spectrum.

Keywords: discrete time-varying linear systems, pole placement theorem, dichotomy spectrum, uniform complete controllability, uniform complete stabilization.

1 Introduction

One of the primary methods of designing of a control for linear systems with time-invariant coefficients is the pole placement method, also known as the pole-shifting or the spectrum assignment method [22], idea of which is to construct of a feedback in such a way that the eigenvalues of the closed loop system have a priory given location. A theoretical basis for this method is a theorem proved by R. Kalman in [14], which shows that it is possible if and only if the system under consideration is controllable.

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In order to cope with growing requirements formulated for control systems in the process of the model building we would like to generalize this methodology to time-varying systems. Attempts of generalization of this methodology to the time-varying systems faced many difficulties (see [5]). One of them is that there is no single equivalent of poles for time-varying systems. Their role to a certain extent is played by Lyapunov, Bohl, and Perron exponents or by dichotomy spectrum. Problem of generalization of pole placement theorem to the time-varying systems has been so far considered mainly for continuous-time systems and Lyapunov exponents as a counterpart of poles [16]. Results for discrete-time systems and Lyapunov exponents are summarized in [4], [6] and [3].

The main drawback of Lyapunov exponents is that they are not continuous function of the system coefficients and therefore W. A. Coppel in the preface to his book [9] wrote: "Several years ago I formed the view that dichotomies, rather than Lyapunov's characteristic exponents, are the key to questions of asymptotic behavior for non-autonomous differential equations" and in this book he effectively proved this statement, among others, by showing one of the most important properties of dichotomies which is its roughness under a wide class of perturbations. In general, the dichotomy spectrum is also not a continuous function of the coefficients (see [19] for the formulation of the problem of continuity and conditions for it) but, in contrast to Lyapunov spectrum, it does not change under perturbations tending to zero if we consider the system on half line (see [20]).

When considering the concept of the dichotomy, one can observe significant differences in properties and research methodologies depending on whether we consider this concept on the whole time axis, or on the half line. On the one hand, the dichotomy spectrum on the whole time axis is important for persistence, as well as bifurcation problems of bounded solutions (see [18] for a survey). On the other hand, the dichotomy spectra for linear systems on the half line are crucial in stability theory and more general for the ordering of invariant manifolds in the stable hierarchy (see e.g. [17, Corollary 4.2.12)]). It is also known that, a half line exponential dichotomy is a much weaker assumption than exponential dichotomy on the whole line (see e.g. [20]). Accordingly, the related dichotomy spectrum on half line is a subset of dichotomy spectrum on whole time axis and has simpler fine structure. Moreover the dichotomy spectrum on half line is more robust than in the full line case see [20]) and the dichotomy spectra of upper-triangular linear systems are fully determined by the diagonal elements (see [20]). The last property fails on the full time axis.

Due to the usefulness of the pole placement method in designing control for systems with constant coefficients, it is natural to ask about possibility of assignment of the dichotomy spectrum. This question was first raised at work [10], where it was shown that for two-sided discrete time-varying linear systems uniform complete controllability is a sufficient condition for assignability of the dichotomy spectrum. The present paper continues this line of research. The first aim is to extend the result of [10] to one-sided discrete time-varying linear systems. Next, we show that for this system uniform complete controllability is not only sufficient but also necessary condition for assignability of the dichotomy spectrum . To prove this result we use the concept of uniform complete stabilizability introduced for continuous-time system in [13].

The note is organized as follows: In Section 2, we introduce the setting and state the main result. The proof of the main result is presented in Section 3. Section 4 is devoted to reexamine the Example 1 from [4] to illustrate the theoretical result of the paper. To conclude the introductory section, we introduce some notations which are used throughout this paper:

Denote by $\mathbb{N}$ the set of natural numbers. By $\mathbb{R}^n$ we denote the n-dimensional Euclidean space with Euclidean norm $\|\cdot\|$ and by $\mathbb{R}^{n\times m}$ the set of matrices of size n by m with real entries. For a matrix $A\in\mathbb{R}^{n\times n}$, $\|A\|$ denotes the spectral norm. $GL_n(\mathbb{R})$ is the subset of $\mathbb{R}^{n\times n}$ consisting of invertible matrices. The identity matrix of size n by n is denoted by I_n. The set of all bounded sequences $B:\mathbb{N}\to\mathbb{R}^{n\times m}$, $B=(B(k))_{k\in\mathbb{N}}$ is denoted by $\mathcal{L}^\infty(\mathbb{N},\mathbb{R}^{n\times m})$, the set of all sequences $A:\mathbb{N}\to\mathbb{R}^{n\times n}$, $A=(A(k))_{k\in\mathbb{N}}$ such that $A(k)\in GL_n(\mathbb{R})$, $A\in\mathcal{L}^\infty(\mathbb{N},\mathbb{R}^{n\times n})$ and $A^{-1}=\left(A^{-1}(k)\right)_{k\in\mathbb{N}}\in\mathcal{L}^\infty(\mathbb{N},\mathbb{R}^{n\times n})$ is denoted by $\mathcal{L}^{\mathrm{Lya}}(\mathbb{N},\mathbb{R}^{n\times n})$ and the elements of $\mathcal{L}^{\mathrm{Lya}}(\mathbb{N},\mathbb{R}^{n\times n})$ will be called Lyapunov sequences.

2 Setting and the statement of the main result

Consider an one-sided linear time-varying system described by the following equation

$$x(k+1)=A(k)x(k)+B(k)u(k), \tag{1}$$

where $A\in\mathcal{L}^{\mathrm{Lya}}(\mathbb{N},\mathbb{R}^{n\times n})$, $B\in\mathcal{L}^\infty(\mathbb{N},\mathbb{R}^{n\times m})$ and $u:\mathbb{N}\to\mathbb{R}^m$ is the control. For $(k_0,x_0)\in\mathbb{N}\times\mathbb{R}^n$ the solution of system (1) satisfying $x(k_0)=x_0$, will be denoted by $x(\cdot,k_0,x_0,u)$. We now recall the notion of uniform complete controllability of system (1), see e.g. [11] and also [25].

Definition 1 (Uniform complete controllability) *System (1) is called uniformly completely controllable if there exist a positive α and a natural* *number K such that for all $(k_0, \xi) \in \mathbb{N} \times \mathbb{R}^n$ there exists a control sequence $u(\ell)$, $\ell = k_0, k_0+1, \ldots, k_0+K-1$ such that $x(k_0+K, k_0, 0, u) = \xi$ and*

$$\|u(\ell)\| \leq \alpha \|\xi\|, \qquad \ell = k_0, k_0+1, \ldots, k_0+K-1.$$

If in system (1) we apply a control of the form

$$u(k) = F(k)x(k), \qquad k \in \mathbb{N}$$

where $F : \mathbb{N} \to \mathbb{R}^{m\times n}$, $F = (F(k))_{k\in\mathbb{N}}$ is a linear feedback, we obtain a so called closed loop system

$$x(k+1) = (A(k) + B(k)F(k))\, x(k). \tag{2}$$

A linear feedback $F \in \mathcal{L}^\infty(\mathbb{N}, \mathbb{R}^{m\times n})$ such that $A + BF \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{N}, \mathbb{R}^{n\times n})$ will be called *admissible*. The main interest of this note is to determine possibility of the assignability of the dichotomy spectrum associated with (2) under admissible linear feedbacks. To formulate the main result of this note, we recall now the notion of dichotomy spectrum for an arbitrary linear discrete time-varying system of the form

$$x(k+1) = M(k)x(k), \tag{3}$$

where $M = (M(k))_{k\in\mathbb{N}} \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{N}, \mathbb{R}^{n\times n})$. Let $\Phi_M : \mathbb{N}\times\mathbb{N} \to \mathbb{R}^{n\times n}$, $\Phi_M = (\Phi_M(k,\ell))_{k,\ell\in\mathbb{N}}$ denote the transition matrix of (3), i.e.

$$\Phi_M(k,\ell) = \begin{cases} M(k-1)\cdots M(\ell), & \text{if } k > \ell, \\ I_n, & \text{if } k = \ell, \\ \Phi_M^{-1}(\ell,k) & \text{if } k < \ell. \end{cases}$$

System (3) is said to admit an *exponential dichotomy* (ED), if there exist C, $\alpha > 0$ and an invariant family of projections $(P(k))_{k\in\mathbb{N}}$ in $\mathbb{R}^{n\times n}$, i.e. $P(k+1)M(k) = M(k)P(k)$ for $k \in \mathbb{N}$, such that for all $k, \ell \in \mathbb{N}$ we have

$$\begin{aligned} \|\Phi_M(k,\ell)P(\ell)\| &\leq Ce^{-\alpha(k-\ell)} && \text{for } k \geq \ell, \\ \|\Phi_M(k,\ell)\,(I_n - P(\ell))\,\| &\leq Ce^{\alpha(k-\ell)} && \text{for } k \leq \ell, \end{aligned}$$

see e.g. [8, 12]. Based on the notion of exponential dichotomy, we have the following definition of the dichotomy spectrum associated with (3), see e.g. [2].

Definition 2 (Dichotomy spectrum) *The dichotomy spectrum of (3) is defined by*

$$\Sigma_{\rm ED}(M) := \left\{\gamma \in \mathbb{R} : x(k+1) = e^{-\gamma} M(k)x(k) \textit{ has no ED}\right\}.$$

The structure of the dichotomy spectrum is described by the following theorem from [2, Theorem 3.4]

Theorem 3 *The dichotomy spectrum $\Sigma_{\rm ED}(M)$ consists of at most n disjoint closed intervals i.e.*

$$\Sigma_{\rm ED}(M) = [\alpha_1, \beta_1] \cup \cdots \cup [\alpha_\ell, \beta_\ell],$$

where $\alpha_1 \leq \beta_1 < \alpha_2 \leq \beta_2 < \cdots < \alpha_\ell \leq \beta_\ell$ and $\ell \leq n$.

In what follows, we recall the definition of assignability of dichotomy spectrum, see [10].

Definition 4 (Assignability of dichotomy spectrum) *The dichotomy spectrum of (1) is called assignable if for arbitrary $1 \leq \ell \leq n$ and arbitrary closed disjoint intervals $[a_1, b_1], \ldots, [a_\ell, b_\ell]$, there exists an admissible linear feedback F such that $\Sigma_{\rm ED}(A + BF) = \cup_{i=1}^{\ell} [a_i, b_i]$.*

We are now in a position to state the main result of this paper.

Theorem 5 *The dichotomy spectrum of (1) is assignable if and only if system (1) is uniformly completely controllable.*

3 Proof of the Main Result

3.1 Preparatory results

This subsection is devoted to recall some facts about dichotomy spectrum which are useful for the proof of the main result. Firstly, we remind the reader that system (3) is called *kinematically equivalent* to a system

$$y(k+1) = N(k)y(k), \tag{4}$$

provided that there exists a sequence $L \in \mathcal{L}^{\rm Lya}(\mathbb{N}, \mathbb{R}^{n\times n})$ such that

$$L(k+1)N(k) = M(k)L(k) \quad \text{ for all } k \in \mathbb{N}.$$

The following result (see [20, Remark 4.33]) describes an important property of the dichotomy spectrum.

Theorem 6 *The dichotomy spectrum is invariant under kinematic equivalence, i.e. if (3) and (4) are kinematically equivalent, then* $\Sigma_{\rm ED}(M) = \Sigma_{\rm ED}(N)$.

Finally, in our further consideration a crucial role is played by the following result known as diagonal significance of dichotomy spectrum (see [20, Corollary 3.25]).

Theorem 7 *If the matrices* $M = (M(k))_{k\in\mathbb{N}} \in \mathcal{L}^{\rm Lya}(\mathbb{N}, \mathbb{R}^{n\times n})$ *in (3) are upper triangular and* $m_{ii} = (m_{ii}(k))_{k\in\mathbb{N}}$ *,* $i = 1, ..., n$ *are the diagonal elements of* $M = (M(k))_{k\in\mathbb{N}}$*, then*

$$\Sigma_{\rm ED}(M) = \bigcup_{i=1}^{n} \Sigma_{\rm ED}(m_{ii}),$$

where $\Sigma_{\rm ED}(m_{ii})$ *is the dichotomy spectrum of the scalar system*

$$x_i(k+1) = m_{ii}(k)x_i(k).$$

It should be emphasized that the above theorem is no longer true when we consider the dichotomy spectrum along the entire straight line (see [21, Example 2.7]).

3.2 Proof of the necessity part of Theorem 5

In this subsection, we always assume that the dichotomy spectrum of (1) is assignable. To prove that system (1) is uniformly completely controllable, we will verify the following relations:

(R1) Assignability of dichotomy spectrum implies uniform complete stabilization,

(R2) Uniform complete stabilization implies uniform complete controllability.

Recall that the concept of uniform complete stabilizability was used for the first time in the paper of Ikeda et al. [13] in the context of continuous time-varying systems. For continuous time-invariant systems this is understood as the possibility to assign arbitrary exponential decay rates (and this is indeed the problem studied in [24]). Also for continuous time-varying systems the concept is discussed in [1, Remark 3.10]. The discrete counterparts of this concept may be formulated as follows.

Definition 8 (Uniform complete stabilization) *System (1) is called uniformly completely stabilizable if for any $w > 0$ there exist an admissible linear feedback F and $C > 0$ such that*

$$\|\Phi_{A+BF}(k, \ell)\| \leq Ce^{-w(k-\ell)} \tag{5}$$

for all $k, \ell \in \mathbb{N}$, $k \geq \ell$.

We first verify the relation (R1):

Theorem 9 *If the dichotomy spectrum of* (1) *is assignable then* (1) *is uniformly completely stabilizable.*

Proof. Let $w > 0$ be arbitrary but fixed. Choose and fix Γ satisfying that $\Gamma < -w$. By the assumption on arbitrary assignability of dichotomy spectrum, there exists an admissible linear feedback $F = (F(n))_{n\in\mathbb{N}}$ such that the dichotomy spectrum $\Sigma_{\rm ED}(A + BF)$ of the system

$$x(n + 1) = (A(n) + B(n)F(n))x(n)$$

is given by

$$\Sigma_{\rm ED}(A + BF) = \{\Gamma\}. \tag{6}$$

Thus, $(\Gamma, \infty) \subset \rho_{\rm ED}(A + BF)$, where $\rho_{\rm ED}(A + BF) := \mathbb{R} \setminus \Sigma_{\rm ED}(A + BF)$. Then for each $\gamma \in (\Gamma, \infty)$ the system

$$x(k + 1) = e^{-\gamma}(A(k) + B(k)F(k))x(k),$$

exhibits an exponential dichotomy with an invariant family of projections $(P_\gamma(k))_{k\in\mathbb{N}}$. Thanks to [2, Lemma 3.2], we have

$$\mathrm{im}P_{\gamma_1}(k) = \mathrm{im}P_{\gamma_2}(k) \quad \text{for } \gamma_1, \gamma_2 \in (\Gamma, \infty). \tag{7}$$

Since $A + BF \in \mathcal{L}^{\rm Lya}(\mathbb{N}, \mathbb{R}^{n\times n})$, there exists $\beta > \Gamma$ such that

$$\|\Phi_{A+BF}(k, \ell)\| \leq e^{\beta(k-\ell)} \quad \text{for } k \geq \ell$$

and therefore for any $\beta' > \beta$ we have

$$\|e^{-\beta'(k-\ell)}\Phi_{A+BF}(k, \ell)\| \leq e^{(\beta-\beta')(k-\ell)} \quad \text{for } k \geq \ell.$$

The last inequality means that $\beta' \in \rho_{\rm ED}(A + BF)$ and $P_{\beta'}(\ell) = I_n$ for all $\ell \in N$. Furthermore, it is known that $\rho_{\rm ED}(A+BF)$ consists of several open intervals (also called spectral gaps) and the projection associated with the largest gap is the identity I_n, c.f. [15, Lemma 5.4]. Consequently, from (7) we arrive that $P_{-w}(\ell) = I_n$ for all $\ell \in \mathbb{N}$. Then, there exist $C, \alpha > 0$ such that

$$\|\Phi_{A+BF}(k,\ell)\| \leq Ce^{(-w-\alpha)(k-\ell)} \quad \text{for } k \geq \ell,$$

which completes the proof. ■

Next, we verify the relation (R2):

Theorem 10 *If system (1) is uniformly completely stabilizable then it is uniformly completely controllable.*

Proof. Since $A \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{N}, \mathbb{R}^{n\times n})$, there exists $C_1 > 0$ such that

$$\|\Phi_A(k_1,k_2)\| \leq e^{C_1|k_1-k_2|} \tag{8}$$

for all $k_1, k_2 \in \mathbb{N}$. According to the assumption about uniform complete stabilizability, there exists a linear feedback $F \in \mathcal{L}^{\infty}(\mathbb{N}, \mathbb{R}^{m\times n})$ and a constant $C_2 > 0$ such that

$$\|F(k)\| \leq C_2 \tag{9}$$

$$\|\Phi_{A+BF}(k_2,k_1)\| \leq Ce^{-2C_1(k_2-k_1)} \tag{10}$$

for all $k_1, k_2 \in \mathbb{N}$, $k_2 > k_1$ and certain $C > 0$. Since the transition matrix Φ_{A+BF} of the closed loop system satisfies

$$\Phi_{A+BF}(k_2,k_1) = \Phi_A(k_2,k_1) + \sum_{j=k_1}^{k_2-1} \Phi_A(k_2,j+1)B(j)F(j)\Phi_{A+BF}(j,k_1),$$

for all $k_1, k_2 \in \mathbb{N}$ or equivalently

$$\Phi_A(k_1,k_2)\Phi_{A+BF}(k_2,k_1) = I_n + \sum_{j=k_1}^{k_2-1} \Phi_A(k_1,j+1)B(j)F(j)\Phi_{A+BF}(j,k_1),$$

then

$$\begin{aligned} x^{\mathrm{T}} &= x^{\mathrm{T}}\Phi_A(k_1,k_2)\Phi_{A+BF}(k_2,k_1) \\ &\quad - \sum_{j=k_1}^{k_2-1} x^{\mathrm{T}}\Phi_A(k_1,j+1)B(j)F(j)\Phi_{A+BF}(j,k_1) \end{aligned}$$

for any nonzero $x \in \mathbb{R}^n$ and therefore

$$\|x\| \|\Phi_A(k_1, k_2)\| \|\Phi_{A+BF}(k_2, k_1)\|$$

$$+ \sum_{j=k_1}^{k_2-1} \|x^{\mathrm{T}} \Phi_A(k_1, j+1) B(j)\| \|F(j)\| \|\Phi_{A+BF}(j, k_1)\| \geq \|x\|.$$

Using (8)-(10) and the fact that $e^{-C_1(k_2-k_1)} < 1$ for $k_2 > k_1$ we obtain

$$Ce^{-C_1(k_2-k_1)} \|x\| \geq \|x\| - CC_2 \sum_{j=k_1}^{k_2-1} \|x^{\mathrm{T}} \Phi_A(k_1, j+1) B(j)\| \qquad (11)$$

for all $k_1, k_2 \in \mathbb{N}$, $k_2 > k_1$.

Suppose now that (1) is not uniformly completely controllable. As it is known ([11] Definition 1, pp. 33 and Proposition 3, pp. 34, [25]) uniform complete controllability of (1) is equivalent to the following: there exist numbers $\alpha > 0$, and $T \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ the following inequality holds

$$\alpha I_n \leq W(k+T, k),$$

where

$$W(k_1, k_0) := \sum_{k=k_0}^{k_1-1} \Phi_A(k_0, k+1) B(k) B^{\mathrm{T}}(k) \Phi_A^{\mathrm{T}}(k_0, k+1).$$

Therefore if (1) is not uniformly completely controllable, then for each $\alpha > 0$ and each $T \in \mathbb{N}$, there is $y \in \mathbb{R}^n$ such that inequality

$$y^{\mathrm{T}} W(k_0+T, k_0) y \leq \alpha \|y\|^2$$

satisfies for certain $k_0 \in \mathbb{N}$. From the last inequality we get

$$\sum_{j=k_0}^{k_0+T-1} \|y^{\mathrm{T}} \Phi_A(k_0, j+1) B(j)\|^2 \leq \alpha \|y\|^2. \qquad (12)$$

Using Cauchy-Schwarz inequality (see [23]) we get

$$\sum_{j=k_0}^{k_0+T-1} \|y^{\mathrm{T}} \Phi_A(k_0, j+1) B(j)\|$$

$$\leq \sqrt{T \sum_{j=k_0}^{k_0+T-1} \|y^{\mathrm{T}} \Phi_A(k_0, j+1) B(j)\|^2}.$$

Combining the last inequality with (12) we obtain

$$\sum_{j=k_0}^{k_0+T-1} \|y^{\mathrm{T}}\Phi_A(k_0, j+1)B(j)\| \le \sqrt{T\alpha}\|y\|.$$

Now, we set $C_3 = \sup\left\{\left\|A^{-1}(j)\right\| : j \in \mathbb{N}\right\}$, $T = \min\left\{l \in \mathbb{N} : l \ge \frac{\ln 3C_3}{C_1}\right\}$ and $\alpha = \left(9C^2C_2^2T\right)^{-1}$, and choose y and k_0 satisfying (12). Then, if we take $x = y$, $k_1 = k_0$ and $k_2 = k_0 + T$, in (11), we get

$$\frac{1}{3}\|y\| \ge Ce^{-C_1T}\|y\| \ge \|y\| - CC_2\sqrt{T\alpha}\|y\| = \frac{2}{3}\|y\|.$$

This is a contradiction. Therefore, system (1) is uniformly completely controllable. ■

Proof of the necessity part of Theorem 5. The proof follows immediately from Theorem 9 and Theorem 10. ■

3.3 Proof of the sufficiency part of Theorem 5

The fact that uniform complete controllability implies assignability of the dichotomy spectrum of two-sided discrete time-varying systems has been proved in [10]. The proof consists of two ingredients. The first one is that for a uniformly completely controllable discrete time-varying control system and a given diagonal discrete time-varying system, there is a bounded linear feedback control such that the corresponding closed-loop system is kinematically equivalent to an upper-triangular system whose diagonal part coincides with the given diagonal system (see [6, Theorem 4.6]). The second one is a presentation of the dichotomy spectrum of a special classes of upper-triangular systems ([10, Proposition 3.4]). Although these things can be extended to one-sided system by several slight modifications, to make the paper self-contained we give below a brief proof of the sufficiency part of Theorem 5.
Proof of the sufficiency part of Theorem 5. Let $[a_1, b_1], \ldots, [a_\ell, b_\ell]$, where $1 \le \ell \le n$, be arbitrary disjoint closed intervals. For $1 \le i \le \ell$, we construct an arbitrary positive scalar sequence $(p^i(k))_{k\in\mathbb{N}}$ such that the dichotomy spectrum $\Sigma_{\mathrm{ED}}(p^i)$ of

$$z(k+1) = p^i(k)z(k) \qquad \text{for } k \in \mathbb{N}.$$

is the interval $[a_i, b_i]$. For instance, we can choose $(p^i(k))_{k\in\mathbb{N}}$ as

$$p^i(k) = \begin{cases} e^{a_i}, & \text{for } k \in [2^{2m}, 2^{2m+1}), m \in \mathbb{N}\cup\{0\}, \\ e^{b_i}, & \text{for } k \in [2^{2m+1}, 2^{2m+2}), m \in \mathbb{N}\cup\{0\}. \end{cases}$$

For $\ell + 1 \le i \le n$, let $p^i(k) = p^1(k)$. According to [6, Theorem 4.6], there exist an admissible linear feedback control and a sequence of upper triangular matrices $(C(k))_{k\in\mathbb{N}} \in \mathcal{L}^{\mathrm{Lya}}(\mathbb{N}, \mathbb{R}^{n\times n})$, where $C(k) = (c_{ij}(k))_{1\le i,j\le n}$ with $c_{ii}(k) = p^i(k)$ such that

$$x(k+1) = (A(k) + B(k)F(k))x(k), \quad y(k+1) = C(k)y(k)$$

are kinematically equivalent and therefore have the same dichotomy spectra (see Theorem 6). `By Theorems 6 and 7,` we arrive at

$$\Sigma_{\mathrm{ED}}(A + BF) = \Sigma_{\mathrm{ED}}(C) = \bigcup_{i=1}^{n} \Sigma_{\mathrm{ED}}(p^i) = \bigcup_{i=1}^{\ell} [a_i, b_i].$$

The proof is complete. ∎

4 Example

Example 11 *Let us define a sequence $(n_k)_{k\in\mathbb{N}}$ by the recurrent formulae*

$$n_1 = 1, \quad n_{2m} = mn_{2m-1}, \quad n_{2m+1} = m + n_{2m}$$

for all $m \in \mathbb{N}$. The sequence $(n_k)_{k\in\mathbb{N}}$ is strictly increasing for $k \ge 2$ and tends to $+\infty$

Put

$$b(k) = \begin{cases} 1 & \text{for} \quad k = 1, \\ 1 & \text{for} \quad k \in [n_{2m-1}, n_{2m} - 1], \\ 0 & \text{for} \quad k \in [n_{2m}, n_{2m+1} - 1], \end{cases}$$

for $m = 2, 3, \ldots,$ and consider the scalar linear control equation

$$x(k+1) = x(k) + b(k)u(k). \tag{13}$$

This example has been considered in [4], where it has been shown that system (13) is not uniformly completely controllable but it has assignable Lyapunov spectrum. From Theorem 5 it follows that system (13) does not have the dichotomy spectrum assignable. We will show that only intervals of the form $[c, d]$, where $0 \in [c, d]$ may be a dichotomy spectrum of system

$$x(k+1) = (1 + b(k)f(k))\, x(k), \tag{14}$$

where $(f(k))_{k\in\mathbb{N}}$ is an admissible linear feedback and that each such a interval is a dichotomy spectrum of system (14) for certain admissible linear feedback $(f(k))_{k\in\mathbb{N}}$. To proof of this we will use Proposition 2.4 from [21] which says that the dichotomy spectrum of scalar equation

$$x(k+1) = a(k)x(k)$$

has the form of interval $\left[\underline{\beta}(a), \overline{\beta}(a)\right]$, where $\underline{\beta}(a)$ and $\overline{\beta}(a)$ are the lower and upper Bohl exponents of the sequence $a = (a(k))_{k\in\mathbb{N}}$, respectively, given by

$$\underline{\beta}(a) = \lim_{j\to\infty} \frac{1}{j}\left(\inf_{k=1,2,\dots} \ln\left(\prod_{i=k}^{k+j-1} |a(i)|\right)\right)$$

and

$$\overline{\beta}(a) = \lim_{j\to\infty} \frac{1}{j}\left(\sup_{k=1,2,\dots} \ln\left(\prod_{i=k}^{k+j-1} |a(i)|\right)\right).$$

Suppose that for certain admissible linear feedback $(f(k))_{k\in\mathbb{N}}$ an interval $[c,d]$ is the dichotomy spectrum of (14). Since $n_{2m+1} - n_{2m} \to \infty$ when $m\to\infty$, then

$$\begin{aligned} d &= \lim_{m\to\infty} \frac{\sup_{k=1,2,\dots} \ln\left(\prod_{i=k}^{k+n_{2m+1}-n_{2m}-1} |1+b(i)f(i)|\right)}{n_{2m+1}-n_{2m}} \\ &\geq \lim_{m\to\infty} \frac{\ln\left(\prod_{i=n_{2m}}^{n_{2m+1}-1} |1+b(i)f(i)|\right)}{n_{2m+1}-n_{2m}} \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} c &= \lim_{m\to\infty} \frac{\inf_{k=1,2,\dots} \ln\left(\prod_{i=k}^{k+n_{2m+1}-n_{2m}-1} |1+b(i)f(i)|\right)}{n_{2m+1}-n_{2m}} \\ &\leq \lim_{m\to\infty} \frac{\ln\left(\prod_{i=n_{2m}}^{n_{2m+1}-1} |1+b(i)f(i)|\right)}{n_{2m+1}-n_{2m}} \\ &= 0. \end{aligned}$$

Therefore $0 \in [c,d]$. *Now let us fix an interval* $[c,d]$ *such that* $0 \in [c,d]$ *and define the linear feedback gain* $(f(k))_{k\in\mathbb{N}}$ *as follows*

$$f(k) = \begin{cases} e^d - 1 & \text{for } k \in [n_{4m-1}, n_{4m} - 1], \\ e^c - 1 & \text{for } k \notin [n_{4m-1}, n_{4m} - 1], \end{cases}$$

for $m \in \mathbb{N}$. *On the one hand, by the definition of* f *and by the inequality* $e^d - 1 \geq 0$, *we have*

$$|1 + b(i)f(i)| \leq e^d$$

and therefore

$$\overline{\beta}\,(1 + bf) \leq d. \tag{15}$$

On the other hand, since $n_{4m} - n_{4m-1} = (2m-1)\, n_{4m-1} \to \infty$ *when* $m \to \infty$, *then*

$$\begin{aligned} & \overline{\beta}\,(1+bf) \\ = \;& \lim_{j\to\infty} \frac{1}{j} \sup_{k=1,2,\ldots} \ln\left(\prod_{i=k}^{k+j-1} |1 + b(i)f(i)|\right) \\ \geq \;& \lim_{m\to\infty} \frac{\sup\limits_{k=1,2,\ldots} \ln\left(\prod\limits_{i=k}^{k+n_{4m}-n_{4m-1}-1} |1 + b(i)f(i)|\right)}{n_{4m} - n_{4m-1}} \\ \geq \;& \lim_{m\to\infty} \frac{\ln\left(\prod\limits_{i=n_{4m-1}}^{n_{4m}-1} |1 + b(i)f(i)|\right)}{n_{4m} - n_{4m-1}} \\ = \;& d, \end{aligned}$$

which together with (15) implies that

$$\overline{\beta}\,(1 + bf) = d.$$

In a similar way we may show that $\underline{\beta}\,(1 + bf) = c$.

5 Conclusions

In this paper we investigated a problem of assignability of dichotomy spectrum by time-varying bounded linear feedback. We have shown that the dichotomy spectrum is assignable if and only if the system is uniformly completely controllable. We proved this by using the concept of uniform complete stabilizability and by showing that this property implies uniform complete controllability. The last result is interesting by itself and an open question is if the opposite implication is also true as for continuous-time systems (see [1]). Observe also that our proof of sufficiency of uniform complete controllability for assignability of the dichotomy spectrum is constructive and allow one to design a feedback loop that places the dichotomy spectrum in a predetermined location. In this way, these proofs can be used as a basis of a design method for time-varying control system similar to the pole placement method.

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