Limit theorems for the one dimensional random walk with random resetting to the maximum

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Abstract

The first part of this paper is devoted to study a model of one dimensional random walk with memory to the maximum position described as follows at each step the walker resets to the rightmost visited site with probability $r \in (0,1)$ and moves as the simple random walk with the remaining probability. Using the approach of renewal theory, we prove the laws of large numbers and the central limit theorems for the random walk. These results reprove and significantly enhance the analysis of asymptotic behavior of the mean value and variance of the process established in [13]. In the second part of this paper, we expand the analysis to the situation when the resetting rate to the maximum position, $r_n = \min\{rn^{-a}, \frac{1}{2}\}$ with r, a positive parameters, decreases over time. For this model, we first establish the asymptotic behavior of the mean values of X_n-the current position and M_n-the maximum position of the random walk. As a consequence, we observe an interesting phase transition in the asymptotic of $\mathbb{E}[X_n]/\mathbb{E}[M_n]$ when a varies, precisely it converges to 1 in the subcritical phase $a \in (0,1)$, to a constant $c \in (0,1)$ in the critical phase $a = 1$, to 0 in the supercritical phase $a > 1$. Finally, in the supercritical phase, $a > 1$, we show that the model behaves closely to the simple random walk in the sense that $\frac{X_n}{\sqrt{n}} \xrightarrow{(d)} \mathcal{N}(0,1)$, $\frac{M_n}{\sqrt{n}} \xrightarrow{(d)} \max_{0 \le t \le 1} B_t$, where $\mathcal{N}(0,1)$ is the standard normal distribution and $(B_t)_{t \ge 0}$ is the standard Brownian motion.

1 Introduction

Searching is one of the most basis and important processes and hence, there has been a consistent effort by scientists to optimize the task. Naturally, by the convenience the searching processes contain mainly the local steps in which the engine finds the better position in the neighborhood of current location. Furthermore, both empirical and theoretical researches show that the combination of the local steps with several long range moves would accelerate the process, see e.g. [2, 12, 15]. For instance, as illustrated in the Metropolis algorithms, if one use only the local steps to find the minimum of a function, the process may get very fast to some local minimums and then get stuck around there. Instead, if one sometime restart the searching to a

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position far away from the current area with some small but positive probability, the process could escape the local minimums and eventually reach the global one. The strategies of combining the local searches, usually modeled by diffusion of random walks, with the long moves modeled by the random resetting have been applied in a board of range disciplines, such as enzymatic reactions [16], biological processes [17, 18], foraging ecology [1, 8, 4, 19]. We refer the reader to the survey by Evan, Majumdar and Schehr [6] for more applications and the development of the stochastic resetting algorithms.

In this paper, we focus on a simple model, so-called *random walk with random resetting to the maximum position*, introduced by Majumdar, Sabhapandit and Schehr in [13]. In this model, the walker moves the one dimensional lattice and sometime reset to the maximum location visited so far. Intuitively, it can be viewed as a simulation of the animals looking for food. During the foraging period, the animals explore the territory locally (as a random walk) to find the food [3]. For smart animals, it can remember the already reached sites and thus occasionally relocate to the boundary between visited and unvisited regions to get higher chance of finding food. In the setting of one dimensional case, the border positions are simply the maximum and minimum sites.

More concretely, consider a random walk on $\mathbb{Z}$, starting from the origin. Let X_n and M_n denote the current position and the maximum one of the walker at the time n, respectively. In particular,

$$M_n = \max\{X_1, X_2, \ldots, X_n\}.$$

The process evolves as follows. Let $(r_n)_{n\geq 1}$ be a sequence of real numbers in $(0,1)$. At the step n, if the position X_{n-1} is strictly less than the maximum one M_{n-1}, then the walker is reset to the maximum position with probability r_n. With the remaining probability $1 - r_n$, the walker moves as the simple random walk. That means it goes to the left or right position with equal probability $\frac{1-r_n}{2}$. In the other case, if $X_n = M_n$, the walker move either to the left or right with the same probability $\frac{1}{2}$. This dynamics is summarized as follows:
if $X_{n-1} < M_{n-1}$ then

$$(X_n, M_n) = \begin{cases} (X_{n-1}+1, M_{n-1}), & \text{with probability } (1-r_n)/2, \\ (X_{n-1}-1, M_{n-1}), & \text{with probability } (1-r_n)/2, \\ (M_{n-1}, M_{n-1}), & \text{with probability } r_n, \end{cases}$$

otherwise

$$(X_n, M_n) = \begin{cases} (M_{n-1}+1, M_{n-1}+1), & \text{with probability } 1/2, \\ (M_{n-1}-1, M_{n-1}), & \text{with probability } 1/2. \end{cases}$$

Our interest is to establish the limit theorems for X_n and M_n. To state these theorems and to compare them with the existing results in the literature, we distinguish two cases:

(i) *Random walks with constant resetting probability to the maximum*, i.e. the resetting probability to the maximum is a constant, $r_n = r$ for all n.

(ii) *Random walk with decreasing resetting probability to the maximum*, i.e. the resetting probability decreases to 0 and is of the form $r_n = \min\{rn^{-a}, \frac{1}{2}\}$.

The resetting probability to the maximum is fixed

Consider the case that $r_n = r \in (0,1)$ for all n. The authors in [13], by using the generating function method, showed that the expectations of X_n and M_n grow linearly, and its variances grow diffusively with the same scaling coefficients. In this paper, by using the renewal theory we revisit the above results and prove the laws of large numbers and central limit theorems for X_n and M_n.

Theorem 1.1 (Laws of large numbers and central limit theorems for the random walk with constant resetting probability to the maximum)**.** *Assume that the resetting probability to the maximum is fixed, i.e. $r_n = r \in (0,1)$ for all $n \geq 1$. Define*

$$v(r) = \frac{r(1-r)}{r - 2r^2 + \sqrt{2r - r^2}}, \tag{1}$$

and

$$D(r) = \Big[(2 - 2r - 5r^2 + 3r^3) + (2 - r - r^2 + 2r^3)\sqrt{r(2-r)}\Big] \times \frac{(1-r)r^2}{\sqrt{r(2-r)}[r - 2r^2 + \sqrt{r(2-r)}]^3}. \tag{2}$$

Then, the following statements hold:

(i) The strong laws of large numbers for X_n and M_n hold: as $n \to \infty$,

$$\frac{M_n}{n} \xrightarrow{a.s} v(r), \quad \frac{X_n}{n} \xrightarrow{a.s} v(r),$$

and

$$\lim_{n\to\infty} \frac{\mathbb{E}[M_n]}{n} = \lim_{n\to\infty} \frac{\mathbb{E}[X_n]}{n} = v(r).$$

(ii) The central limit theorems for X_n and M_n hold: as $n \to \infty$,

$$\frac{M_n - v(r)n}{\sqrt{n}} \xrightarrow{(d)} \mathcal{N}(0, D(r)), \quad \frac{X_n - v(r)n}{\sqrt{n}} \xrightarrow{(d)} \mathcal{N}(0, D(r)),$$

and

$$\lim_{n\to\infty} \frac{\mathrm{Var}[M_n]}{n} = \lim_{n\to\infty} \frac{\mathrm{Var}[X_n]}{n} = D(r),$$

where $\mathcal{N}(0,\sigma^2)$ denotes the normal distribution of mean 0 and variance σ^2.

The key to the proof of Theorem 1.1 is the observation that the maximum process increases by 1, say $M_n = M_{n-1} + 1$, when $X_{n-1} = M_{n-1}$ and $X_n = M_n$ and does not change otherwise. In other words, M_n count the number of two consecutive zeroes of the the difference process $\{Y_n := M_n - X_n,\ n \geq 0\}$. Moreover, $(Y_n)_{n\geq 0}$ evolves as a memoryless random walk. By this observation, we can view $(M_n)_{n\geq 0}$ as a renewal reward process and then the limit theorems for M_n follow from the standard renewal theory. Then the limit theorems for X_n are derived by controlling $(Y_n)_{n\geq 0}$.

The resetting probability to the maximum is of the form $r_n = \min\{rn^{-a}, \frac{1}{2}\}$

Consider the case that the resetting probability decreases to 0 as the time tends to infinity and of the form $r_n = \min\{\frac{1}{2}, \frac{r}{n^a}\}$, where a, r are positive constants. Note that the model with decreasing resetting probability is quite natural, for e.g. the animal memory usually diminishes, and the factor $\frac{1}{2}$ can be replaced by any constant taking value in $(0,1)$. Our first result is to establish the asymptotic behavior of the mean values of X_n and M_n.

Theorem 1.2 (Asymptotic behavior of the mean values of the current position and the maximal position of random walk with decreasing resetting probability to the maximum)**.** *Consider the case $r_n = \min\{\frac{r}{n^a}, \frac{1}{2}\}$ for $n \geq 1$ with parameters $a, r > 0$.*

(i) For $a > \frac{3}{2}$ and $r > 0$, we have $\mathbb{E}[X_n] = \mathcal{O}(1)$. For $0 < a \leq \frac{3}{2}$ and $r > 0$,

$$\lim_{n\to\infty} \frac{\mathbb{E}[X_n]}{\varphi_a(n)} = F_X(r,a), \tag{3}$$

where

$$\varphi_a(n) = \begin{cases} n^{1-\frac{a}{2}} & \text{if } 0 < a \leq 1, \\ n^{\frac{3}{2}-a} & \text{if } 1 < a < \frac{3}{2}, \\ \log n & \text{if } a = \frac{3}{2}, \end{cases}$$

and

$$F_X(a,r) = \begin{cases} \dfrac{\sqrt{2r}}{2-a}, & \text{if } 0 < a < 1, \\ 2r^2\sqrt{\dfrac{2}{\pi}}B(3/2,r), & \text{if } a = 1, \\ \lambda_2(a,r) - \lambda_3(a,r) + \lambda_4(a,r) & \text{if } 1 < a \leq \frac{3}{2}, \end{cases}$$

with $\lambda_2(a,r), \lambda_3(a,r), \lambda_4(a,r)$ given in Lemma 3.8 and $B(\cdot,\cdot)$ the Beta function defined by $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$.

(ii) For $a, r > 0$,

$$\lim_{n\to\infty} \frac{\mathbb{E}[M_n]}{\psi_a(n)} = F_M(r,a),$$

where $\psi_a(n) = n^{\max\{1-\frac{a}{2},\frac{1}{2}\}}$ and

$$F_M(a,r) = \begin{cases} \dfrac{\sqrt{2r}}{2-a} & \text{if } 0 < a < 1, \\ (2r^2 + r)\sqrt{\dfrac{2}{\pi}}B(3/2,r) & \text{if } a = 1, \\ \sqrt{\dfrac{2}{\pi}} & \text{if } a > 1. \end{cases}$$

Remark 1.3. Theorem 1.2 indicates an interesting phase transition:

- *Subcritical phase* $0 < a < 1$: Both $\mathbb{E}[X_n]$ and $\mathbb{E}[M_n]$ grow as $n^{1-\frac{a}{2}}$ with same scaling factor $\frac{\sqrt{2r}}{2-a}$.

- *Critical phase* $a = 1$: $\mathbb{E}[X_n]$ and $\mathbb{E}[M_n]$ grow at the same level as $n^{1/2}$ but at different scaling coefficients.
- *Supercritical phase* $a > 1$: $\mathbb{E}[X_n]$ and $\mathbb{E}[M_n]$ have different orders of magnitude, and $\mathbb{E}[X_n] = o(\mathbb{E}[M_n])$.

Finally, we establish a limit result for X_n and M_n in the supercritical phase. Note that in this phase ($a > 1$) the resetting events rarely occur, since the number of resetting times has finite mean value. Hence, we can expect that the behavior of the model is similar to the one of the simple random walk. We indeed prove this prediction in the following result.

Theorem 1.4 (Limit theorem for random walk with decreasing resetting probability to the maximum in the supercritical phase)**.** *Assume that $a > 1$. Then for all $r > 0$,*

$$\lim_{n\to\infty} \frac{\mathrm{Var}[X_n]}{n} = 1, \qquad \lim_{n\to\infty} \frac{\mathrm{Var}[M_n]}{n} = 1 - \frac{2}{\pi},$$

and as $n \to \infty$,

$$\frac{X_n}{\sqrt{n}} \xrightarrow{(d)} \mathcal{N}(0,1), \qquad \frac{M_n}{\sqrt{n}} \xrightarrow{(d)} \max_{0\le t\le 1} B_t,$$

where $(B_t)_{t\ge 0}$ is the standard Brownian motion.

Remark 1.5. (i) Notice that limit theorems for the simple random walk take place with the same scaling factors and limiting distributions as in Theorem 1.4. It is worth noting also that the case $a = 1$ has been studied by the authors in [13] by using non rigorous arguments. We shall show in Remark 3.10 that their analysis is not correct. The continuity of the scaling factors F_X and F_M at critical points will be discussed in Remark 3.11.

(ii) At this moment, limit theorems for X_n and M_n in the subcritical and critical phases are still missing. In fact, our numerical simulation indicates that the variances of X_n and M_n also grow linearly, but we do not know how to prove and leave it as an open question, see Section 4.2 for more discussion.

The paper is organized as follows. In Sections 2 and 3, we give the proof of Theorems 1.1 and 1.2 respectively. In Section 4, we prove Theorem 1.4 and provide some numerical simulation and discussion on the asymptotic behavior of the variance in the case $0 < a \le 1$.

Finally, we summarize some notation frequently used in the paper. If f and g are two real functions, we write $f = \mathcal{O}(g)$ if there exists a constant $C > 0$, such that $f(x) \le Cg(x)$ for all x; $f \asymp g$ if $f = \mathcal{O}(g)$ and $g = \mathcal{O}(f)$; $f = o(g)$ if $g(x)/f(x) \to 0$ as $x \to \infty$.

2 Laws of large numbers and central limit theorems for the random walk with constant resetting probability to the maximum

The proof of Theorem 1.1 is divided in two parts. In the first one, we study the properties of the difference process $\{Y_n := M_n - X_n,\, n \ge 0\}$. In particular, we can rewrite M_n as the count of the number of two consecutive zeroes of $(Y_n)_{n\ge 0}$ and then represent $(M_n)_{n\geq 0}$ as a renewal reward process. In the second part, we apply the standard renewal theory to prove the limit theorems for $(M_n)_{n\geq 0}$ and $(X_n)_{n\geq 0}$. In this section, we assume that

$$r_n = r \in (0,1) \qquad \text{for all } n \geq 1.$$

2.1 The difference process $(Y_n)_{n\geq 0}$

Although the two processes $(X_n)_{n\geq 0}$ and $(M_n)_{n\geq 0}$ are not Markovian, the difference $(Y_n)_{n\geq 0}$ is. Indeed, it directly follows from the dynamics of the pair $(X_n, M_n)_{n\geq 0}$ that regardless the value of X_n and M_n, the process $(Y_n)_{n\geq 0}$ evolves as follows:
• if $Y_n > 0$, then

$$Y_{n+1} = \begin{cases} Y_n - 1, & \text{with probability } (1-r)/2, \\ Y_n + 1, & \text{with probability } (1-r)/2, \\ 0, & \text{with probability } r. \end{cases}$$

• if $Y_n = 0$, then

$$Y_{n+1} = \begin{cases} 0, & \text{with probability } 1/2, \\ 1, & \text{with probability } 1/2. \end{cases}$$

Hence, $(Y_n)_{n\geq 0}$ is a random walk in $\mathbb{N} = \{0, 1, \ldots\}$ starting at 0. At time n, if $Y_n > 0$, then in the next step, it moves to the left or right, with probability $(1-r)/2$, and jumps to 0 with probability r. On the other hand, if $Y_n = 0$, then in the next step, it stays at 0 with probability $1/2$ or move to 1 with probability $1/2$.

We now observe that $M_{i+1} = M_i + 1$ when $Y_i = Y_{i+1} = 0$ and $M_{i+1} = M_i$ otherwise. That means M_n counts the number of two consecutive zeroes of the sequence $(Y_n)_{n\geq 0}$, i.e.

$$M_n = \sum_{i=0}^{n-1} \mathbb{I}(Y_i = Y_{i+1} = 0).$$

Starting from this observation, we define a sequence of stopping time $(T_i)_{i\geq 0}$ with $T_0 = 0$ and for $i \geq 1$,

$$T_i = \inf\{j \geq T_{i-1} + 1 : Y_j = 0\}.$$

Then $(T_i)_{i\geq 0}$ forms a renewal process and

$$M_n = \sum_{i=1}^{K_n} \mathbb{I}(T_i = T_{i-1} + 1), \qquad K_n = \max\{i \geq 1 : T_i \leq n\}.$$

Moreover, for $i \geq 1$, we define

$$\tau_i = T_i - T_{i-1},$$

which is indeed the i^{th} waiting time of the process. Then $(\tau_i)_{i\geq 1}$ are i.i.d. random variables with the same distribution as

$$\tau = T_1 = \inf\{i \geq 1 : Y_i = 0\},$$

and

$$M_n = \sum_{i=1}^{K_n} \mathbb{I}(\tau_i = 1), \quad K_n = \max\{i : \tau_1 + \ldots + \tau_i \leq n\}. \tag{4}$$

In other words, $(M_n)_{n\geq 1}$ is a renewal reward process based on the renewal process $(T_i)_{i\geq 1}$ with the reward function given by $f(\tau_i) = \mathbb{I}(\tau_i = 1)$ for $i \geq 1$.

Lemma 2.1 (Generating function of τ)**.** *For any $k \geq 1$,*

$$\mathbb{P}(\tau > k) = \frac{1}{2}(1-r)^{k-1}\mathbb{P}(T_{1,0} > k-1), \tag{5}$$

where $T_{0,1} = \inf\{i \geq 1 : Z_i = 0\}$ is first time that the simple random walk $(Z_i)_{i\geq 0}$ starting at 1 touches 0. As a consequence, the generating function of τ is given by

$$K(s) := \mathbb{E}[s^\tau] = s + \frac{s-1}{1-r}\frac{1-\sqrt{1-a^2}}{(a+\sqrt{1-a^2}-1)}, \quad \text{for } s \in (0, \tfrac{1}{1-r}), \tag{6}$$

where $a = (1-r)s$.

Proof. We have

$$\{\tau > k\} = \{Y_1 = 1, Y_2 \neq 0, \dots, Y_k \neq 0\}.$$

Let us define

$$\mathcal{A}_k = \{Y_1 = 1\} \cap \{(Y_i)_{i\geq 2} \text{ doesn't reset at any step } i = 2, \dots, k\}.$$

Then

$$\mathbb{P}(\mathcal{A}_k) = \frac{1}{2}(1-r)^k. \tag{7}$$

Moreover,

$$\begin{aligned}\mathbb{P}(\tau > k \mid \mathcal{A}_k) &= \mathbb{P}(Y_2 \neq 0, \dots, Y_k \neq 0 \mid \mathcal{A}_k) \\ &= \mathbb{P}(Z_1 \neq 0, \dots, Z_{k-1} \neq 0 \mid Z_0 = 1).\end{aligned} \tag{8}$$

Indeed, for every step $2 \leq i \leq k$, we see that

$$\mathbb{P}(Y_i = Y_{i-1} \pm 1 \mid Y_{i-1} \neq 0\,; i \text{ is not a resetting time}) = \frac{(1-r)/2}{1-r} = \frac{1}{2}.$$

Hence, if $Y_{i-1} \neq 0$ and if i is not a resetting time, in the next step, Y_i moves up or down, with probability 1/2, like the simple random walk. Therefore, we obtain (8). Combining (7) and (8), we get the first result of Lemma 2.1.

To achieve the generating function of τ, we observe that

$$\begin{aligned}K(s) = \mathbb{E}[s^\tau] = \sum_{k=0}^{\infty}\mathbb{P}(\tau = k)s^k &= s + (s-1)\sum_{k=1}^{\infty}\mathbb{P}(\tau > k)s^k \\ &= s + \frac{s(s-1)}{2}\sum_{k=0}^{\infty}(1-r)^k\mathbb{P}(T_{1,0} > k)s^k \\ &= s + \frac{s(s-1)}{2}\left(1 + \sum_{k=1}^{\infty}\mathbb{P}(T_{1,0} > k)((1-r)s)^k\right).\end{aligned} \tag{9}$$

The generating function of $T_{1,0}$ is well known (see e.g. [11]) and is given as

$$F(s) = \mathbb{E}[s^{T_{1,0}}] = \sum_{k=0}^{\infty}\mathbb{P}(T_{1,0} = k)s^k = \frac{1-\sqrt{1-s^2}}{s},$$

which implies that

$$\sum_{k=1}^{\infty} \mathbb{P}(T_{1,0} > k)s^k = \frac{F(s)-1}{s-1},$$

and hence,

$$\sum_{k=1}^{\infty} \mathbb{P}(T_{1,0} > k)a^k = \frac{F(a)-1}{a-1},$$

with $a = (1-r)s$. Combining the above equation with (9), we derive the generating function of τ as

$$\begin{aligned} K(s) &= s + \frac{s(s-1)}{2}\left(1 + \frac{F(a)-1}{a-1}\right) = s + \frac{s-1}{(1-r)} \frac{1-\sqrt{1-a^2}-a}{2(a-1)} \\ &= s + \frac{s-1}{(1-r)} \frac{(1-\sqrt{1-a^2}-a)(a+\sqrt{1-a^2}-1)}{2(a-1)(1-\sqrt{1-a^2})} \\ &\qquad\qquad\qquad\qquad \times \frac{1-\sqrt{1-a^2}}{(a+\sqrt{1-a^2}-1)} \\ &= s + \frac{s-1}{1-r} \frac{1-\sqrt{1-a^2}}{(a+\sqrt{1-a^2}-1)}, \end{aligned}$$

as the statement of the lemma. □

Corollary 2.2. *For $r \in (0,1)$, we have*

$$\mathbb{E}[\tau] = \frac{1}{2v(r)}, \qquad \mathbb{E}[\tau^2] = E[\tau] + K''(1),$$

where $v(r)$ is as in Theorem 1.1, and $K''(1)$ is given by (12).

Proof. We compute the first derivative of generating function $K(s)$ as

$$\begin{aligned} K'(s) = 1 &+ \frac{(1-r)(s-1)s}{\sqrt{1-(1-r)^2s^2}\left(\sqrt{1-(1-r)^2s^2}+(1-r)s-1\right)} \\ &\qquad + \frac{1-\sqrt{1-(1-r)^2s^2}}{(1-r)\left(\sqrt{1-(1-r)^2s^2}+(1-r)s-1\right)} \\ &- \frac{(s-1)\left(-\frac{(1-r)^2s}{\sqrt{1-(1-r)^2s^2}}-r+1\right)\left(1-\sqrt{1-(1-r)^2s^2}\right)}{(1-r)\left(\sqrt{1-(1-r)^2s^2}+(1-r)s-1\right)^2}. \end{aligned}$$

Thus,

$$\begin{aligned} \mathbb{E}[\tau] = K'(1) &= \frac{(1-\sqrt{2r-r^2})(\sqrt{2r-r^2}+r)}{(1-r)(\sqrt{2r-r^2}-r)(\sqrt{2r-r^2}+r)} + 1 \\ &= \frac{r(1-2r)+\sqrt{2r-r^2}}{2r(1-r)} = \frac{1}{2v(r)}. \end{aligned} \tag{10}$$

Similarly, we can also compute the second moment of τ as

$$\mathbb{E}[\tau^2] = \frac{d}{ds}\Big(sK'(s)\Big)_{s=1} = K'(1) + K''(1) = \mathbb{E}[\tau] + K''(1). \tag{11}$$

After some algebraic calculation, we yield that

$$K''(1) = 2\Bigg[\frac{(1-r)}{\sqrt{2r-r^2}\left(\sqrt{2r-r^2}-r\right)} - \left(1-\sqrt{2r-r^2}\right)\frac{\sqrt{2r-r^2}+r-1}{\sqrt{2r-r^2}(\sqrt{2r-r^2}-r)^2}\Bigg], \qquad (12)$$

as desired. □

2.2 Proof of Theorem 1.1

First, we recall that $(M_n)_{n\geq 0}$ is a renewal reward process with $M_0 = 0$ and

$$M_n = \sum_{i=1}^{K_n} f(\tau_i), \qquad f(x) = \mathbb{I}(x=1),$$

where $(\tau_i)_{i\geq 1}$ is a sequence of i.i.d. copies of τ and $K_n = \max\{i \geq 1 : \tau_1 + \ldots + \tau_i \leq n\}$.

Then it follows from the standard theory of renewal reward process (see e.g. Chapter 4 in [7]) that as $n \to \infty$,

$$\frac{M_n}{n} \xrightarrow{\text{a.s.}} \mu, \qquad \lim_{n\to\infty} \frac{\mathbb{E}[M_n]}{n} \xrightarrow{\text{a.s.}} \mu, \qquad (13)$$

and

$$\frac{M_n - \mu n}{\sqrt{n}} \xrightarrow{(d)} \mathcal{N}(0, \sigma^2), \qquad \lim_{n\to\infty} \frac{\text{Var}[M_n]}{n} = \sigma^2, \qquad (14)$$

where

$$\mu = \frac{\mathbb{E}[f(\tau)]}{\mathbb{E}[\tau]}, \qquad \sigma^2 = \frac{\mathbb{E}[(f(\tau) - \tau\mu)^2]}{\mathbb{E}[\tau]}.$$

Since $\mathbb{P}(\tau = 1) = \frac{1}{2}$ and $\mathbb{E}[\tau] = 1/(2v(r))$, by Corollary 2.2, we have

$$\mu = \frac{\mathbb{E}[f(\tau)]}{\mathbb{E}[\tau]} = \frac{\mathbb{P}(\tau = 1)}{\mathbb{E}[\tau]} = v(r), \qquad (15)$$

and

$$\begin{aligned}\sigma^2 = \frac{\mathbb{E}[(f(\tau)-\tau\mu)^2]}{\mathbb{E}[\tau]} &= 2v(r)\mathbb{E}[\mathbb{I}(\tau=1) - 2\mu\mathbb{I}(\tau=1) + \tau^2\mu^2] \\ &= v(r)\left(1 - 2\mu + 2\mu^2\mathbb{E}[\tau^2]\right).\end{aligned}$$

Using the formula $\mathbb{E}[\tau^2]$ obtained in Corollary 2.2 and some computations, we can show that

$$\sigma^2 = D(r), \qquad (16)$$

with $D(r)$ given Theorem 1.1. The limit theorems for $(M_n)_{n\geq 0}$ have been proved.

Finally, we prove the limit theorems for $(X_n)_{n\geq 0}$. We first claim that

$$\sup_{n\geq 1} \mathbb{E}[Y_n^4] \leq \sup_{n\geq 1} \mathbb{E}\big[(T_{K_n} - n)^4\big] < \infty, \qquad (17)$$

and thus as $n \to \infty$,

$$\frac{T_{K_n} - n}{\sqrt{n}} \xrightarrow{a.s} 0, \qquad \frac{Y_n}{\sqrt{n}} \xrightarrow{a.s} 0. \tag{18}$$

Indeed, since $0 \le Y_n \le n - T_{K_n}$, it is sufficient to prove these claims for the sequence $(n - T_{K_n})_{n\ge 1}$. From the definition of K_n, we observe that for any $k \ge 1$

$$\mathbb{P}(n - T_{K_n} \ge k) \le \mathbb{P}(T_{K_n+1} - T_{K_n} \ge k) = \mathbb{P}(\tau_{K_n+1} \ge k) = \mathbb{P}(\tau \ge k).$$

Therefore, by Lemma 2.1,

$$\begin{aligned}\mathbb{E}[(n - T_{K_n})^4] \le \mathbb{E}[\tau^4] &\le 16 + \sum_{k\ge 1}(k+1)^4\mathbb{P}(\tau > k)\\ &\le 16 + \sum_{k\ge 1}(k+1)^4(1-r)^k < \infty.\end{aligned}$$

Then the claim (18) follows from standard arguments using Markov's inequality and the Borel-Cantelli lemma.

We know that $X_n = M_n - Y_n$ and by the above claims, $\frac{Y_n}{\sqrt{n}} \xrightarrow{a.s} 0$ as $n \to \infty$, and $\sup_{n\ge 1} \mathbb{E}[Y_n^4] < \infty$. Hence, the limit theorems (13) and (14) still hold when we replace M_n by X_n. The proof Theorem 1.1 is completed. □

3 Asymptotic behavior of the mean values of the current position and the maximal position of random walk with decreasing resetting probability to the maximum

The strategy for the proof of Theorem 1.2 is as follows. First, in the subsection 3.1, we adjust the dynamic of $(X_n)_{n\ge 0}$ to get a simpler random walk $(\hat{X}_n)_{n\ge 0}$ in which at any time the resetting events can occur independently of the relative position between $(\hat{X}_n)_{n\ge 0}$ and the maximum process $(\hat{M}_n)_{n\ge 0}$. We eventually show that the modification does not affect the asymptotic behavior of the mean value of $(X_n)_{n\ge 0}$ and $(M_n)_{n\ge 0}$. Next, we investigate the behavior of $\mathbb{E}[\hat{X}_n]$ and $\mathbb{E}[\hat{M}_n]$ in three cases $0 < a < 1$, $a = 1$ and $a > 1$ in the subsections 3.2, 3.3 and 3.4 respectively. Finally, we conclude the proof of Theorem 1.2 in the subsection 3.5

3.1 A modification of $(X_n)_{n\ge 0}$ and its properties

In the definition of random walk $(X_n)_{n\ge 0}$, the resetting events depend on the relation between the current position and the maximum one. Indeed, at the time n, the walker has to check whether $X_{n-1} < M_{n-1}$ before deciding to reset or not. We now adjust this rule to obtain a simpler model where the resetting steps can appear freely. That means at any time n, even while being at the maximum position, the walker makes a reset with probability r_n. More precisely, we consider the random walk $(\hat{X}_n)_{n\ge 0}$ starting at 0 with the following transition probabilities

$$(\hat{X}_n, \hat{M}_n) = \begin{cases}(\hat{X}_{n-1}+1, \max\{\hat{M}_{n-1},\, \hat{X}_{n-1}+1\}) & \text{with probability } \frac{1-r_n}{2},\\ (\hat{X}_{n-1}-1, \hat{M}_{n-1}) & \text{with probability } \frac{1-r_n}{2},\\ (\hat{M}_{n-1}, \hat{M}_{n-1}) & \text{with probability } r_n,\end{cases}$$

where for $k \geq 0$,

$$\hat{M}_k = \max\{\hat{X}_i,\, i = 0, \ldots, k\}.$$

The following lemma shows that the difference between the two random walks can be controlled via the number of resetting events.

Lemma 3.1 (Coupling lemma)**.** *Let $(Z_n)_{n\geq 0}$ be the simple random walk stating from 0. Let $\mathcal{J}$ be the random subset of $\{1, 2, \ldots\}$, which is independent of $(Z_n)_{n\geq 0}$ and defined by letting $j \in \mathcal{J}$ with probability r_j independently. Then we can construct the two processes $(X_i)_{i\geq 0}$ and $(\hat{X}_n)_{n\geq 0}$ from $(Z_n)_{n\geq 0}$ and $\mathcal{J}$ in such a way that*

$$\max_{0\leq i\leq n} \{|M_i - \hat{M}_i|, |X_i - \hat{X}_i|\} \leq |\mathcal{J} \cap [1, n]|, \tag{19}$$

for any $n \geq 0$, where $|A|$ is the cardinality of A.

Proof. The law of $(X_n)_{n\geq 0}$ and $(\hat{X}_n)_{n\geq 0}$ implies that between two consecutive resetting times, both processes move as simple random walks. Notice also that for every n, $(Z_{n+i} - Z_n)_{i\geq 0}$ is a simple random walk starting at 0, and is independent of $(Z_j)_{j\leq n}$. Hence, we can constitute two random walks $(X_n)_{n\geq 0}$ and $(\hat{X}_n)_{n\geq 0}$ using the path of $(Z_n)_{n\geq 0}$ as follows.

- First, we call $\mathcal{J} = \{t_1, t_2, \ldots\}$, which plays the role as the set of resetting times.
- For $0 \leq t \leq t_1 - 1$, we set $X_t = \hat{X}_t = Z_t$. At the resetting time t_1, put $\hat{X}_{t_1} = \hat{M}_{t_1-1} = \max_{0\leq t\leq t_1-1} Z_t$. The value of X_{t_1} depends also on X_{t_1-1} and M_{t_1-1}. If $X_{t_1-1} < M_{t_1-1}$, then $X_{t_1} = M_{t_1-1} = \max_{0\leq t\leq t_1-1} Z_t$. Otherwise, $X_{t_1} = Z_{t_1}$.
- For every $i \geq 1$, we set

$$\begin{cases} \hat{X}_{t_i} = \hat{X}_{t_{i-1}} + \max\{Z_s - Z_{t_{i-1}},\, t_{i-1} \leq s \leq t_i - 1\}, \\ \hat{X}_t = \hat{X}_{t_i} + Z_t - Z_{t_i}, \quad \text{for } t_i + 1 \leq t \leq t_{i+1} - 1. \end{cases}$$

Meanwhile, the construction of $(X_n)_{n\geq 0}$ is slightly complicated.

(i) If $X_{t_i-1} = M_{t_i-1}$, then $X_t = X_{t_i-1} + Z_t - Z_{t_i-1}$, for $t_i \leq t \leq t_{i+1} - 1$.

(ii) If $X_{t_i-1} < M_{t_i-1}$,

$$\begin{cases} X_{t_i} = X_{t_{i-1}} + \max_{t_{i-1}\leq s\leq t_i-1}\{Z_s - Z_{t_{i-1}}\} \\ \qquad\qquad + \mathbb{I}(X_{t_{i-1}} = X_{t_{i-1}-1} - 1, \max_{t_{i-1}\leq s\leq t_i-1}\{Z_s - Z_{t_{i-1}}\} = 0), \\ X_t = X_{t_i} + Z_t - Z_{t_i}, \quad \text{for } t_i + 1 \leq t \leq t_{i+1} - 1. \end{cases}$$

In summary, for any interval (t_i, t_{i+1}), the path $\{\hat{X}_t,\, t_i < t < t_{i+1}\}$ is obtained by translating $\{Z_t,\, t_i < t < t_{i+1}\}$ from Z_{t_i} to $\hat{X}_{t_i}$, and at the resetting steps, $\hat{X}_{t_i}$ is set to be the sum of $\hat{X}_{t_{i-1}}$ and $\max_{t_{i-1}\leq s\leq t_i-1}\{Z_s - Z_{t_{j-1}}\}$.
On the other hand, if $X_{t_i-1} = M_{t_i-1}$ (resp. $X_{t_i-1} < M_{t_i-1}$) then the path $\{X_t,\, t_i \leq t < t_{i+1}\}$ is derived by lifting the path $\{Z_t,\, t_i \leq t < t_{i+1}\}$ from Z_{t_i-1} to X_{t_i-1} (resp. X_{t_i}). For the value at resetting times, basically, we also set $X_{t_i} = X_{t_{i-1}} + \max_{t_{i-1}\leq s\leq t_i-1}\{Z_s - Z_{t_{i-1}}\}$. However, we have to consider more carefully the case that $X_{t_{i-1}} = X_{t_{i-1}-1} - 1$. If this happens, the walker does not reset at t_{i-1} and $X_{t_{i-1}-1} = M_{t_{i-1}}$, and thus $X_{t_{i-1}} = X_{t_{i-1}-1} - 1$. Assuming further that $\max_{t_{i-1}\leq s\leq t_i-1}\{Z_s - Z_{t_{i-1}}\} = 0$, the maximum value in the interval $t_{i-1} \leq s \leq t_i - 1$ is $X_{t_{i-1}}$ which is indeed $X_{t_{i-1}-1} - 1 = M_{t_{i-1}-1} - 1$. Therefore, at the time t_i, the walker resets to

the position of $X_{t_{i-1}-1} = M_{t_{i-1}-1} = X_{t_{i-1}} + 1$. That explains the appearance of the indicator function in the definition of X_{t_i}.

Next, we give an upper bound for the difference between the two processes. First, observe that if $t \in (t_i, t_{i+1})$,

$$X_t = X_{t_i} + Z_t - Z_{t_i}, \quad \hat{X}_t = \hat{X}_{t_i} + Z_t - Z_{t_i}. \tag{20}$$

Indeed, the equation for $\hat{X}_t$ directly follows from the definition. If $X_{t_i-1} < M_{t_i-1}$, the equation for X_t is already given in (ii). For the case $X_{t_i-1} = M_{t_i-1}$, we also have

$$X_t = X_{t_i-1} + Z_t - Z_{t_i-1} = X_{t_i} + Z_t - Z_{t_i},$$

since $X_{t_i} = X_{t_i-1} + Z_{t_i} - Z_{t_i-1}$.

It follows from (20) that

$$\max\{|X_j - \hat{X}_j|, |M_j - \hat{M}_j|\, j = 1 \ldots, n\} = \max\{|X_{t_i} - \hat{X}_{t_i}|, i \in \mathcal{J} \cap [1, n]\}. \tag{21}$$

We claim that that for all $i \geq 1$,

$$|(X_{t_i} - \hat{X}_{t_i}) - (X_{t_{i-1}} - \hat{X}_{t_{i-1}})| = |(X_{t_i} - X_{t_{i-1}}) - (\hat{X}_{t_i} - \hat{X}_{t_{i-1}})| \leq 1, \tag{22}$$

where $X_{t_0} = \hat{X}_{t_0} = 0$. Then this claim implies that

$$|X_{t_i} - \hat{X}_{t_i}| \leq i$$

for all $i \geq 1$. Combining this with (21) we get (19).

Hence, it remains to prove (22). We have

$$\hat{X}_{t_i} - \hat{X}_{t_{i-1}} = \max_{t_{i-1} \leq s \leq t_i - 1} \{Z_s - Z_{t_{i-1}}\}. \tag{23}$$

If $X_{t_i-1} = M_{t_i-1}$, then $Z_{t_i-1} - Z_{t_{i-1}} = \max_{t_{i-1} \leq s \leq t_i-1}\{Z_s - Z_{t_{i-1}}\}$ and

$$\begin{aligned} X_{t_i} &= X_{t_i-1} + Z_{t_i} - Z_{t_i-1} = X_{t_{i-1}} + Z_{t_i-1} - Z_{t_{i-1}} + Z_{t_i} - Z_{t_i-1} \\ &= X_{t_{i-1}} + \max_{t_{i-1} \leq s \leq t_i - 1}\{Z_s - Z_{t_{i-1}}\} + Z_{t_i} - Z_{t_i-1}. \end{aligned}$$

Therefore,

$$|(X_{t_i} - X_{t_{i-1}}) - (\hat{X}_{t_i} - \hat{X}_{t_{i-1}})| = |Z_{t_i} - Z_{t_i-1}| = 1.$$

If $X_{t_i-1} < M_{t_i-1}$, by (ii)

$$0 \leq (X_{t_i} - X_{t_{i-1}}) - \max_{t_{i-1} \leq s \leq t_i - 1}\{Z_s - Z_{t_{i-1}}\} \leq 1.$$

Combining the above estimates with (23), we obtain (22). □

Now we investigate the behavior of the modified process $(\hat{X}_n)_{n \geq 0}$. First, we call the elements of the random set $\mathcal{J}$ in Lemma 3.1 as

$$\mathcal{J} = \{t_1, t_2, \ldots\},$$

and define

$$k_n = \max\{i \geq 1 : t_i \leq n\}, \qquad t_0 = 0,$$

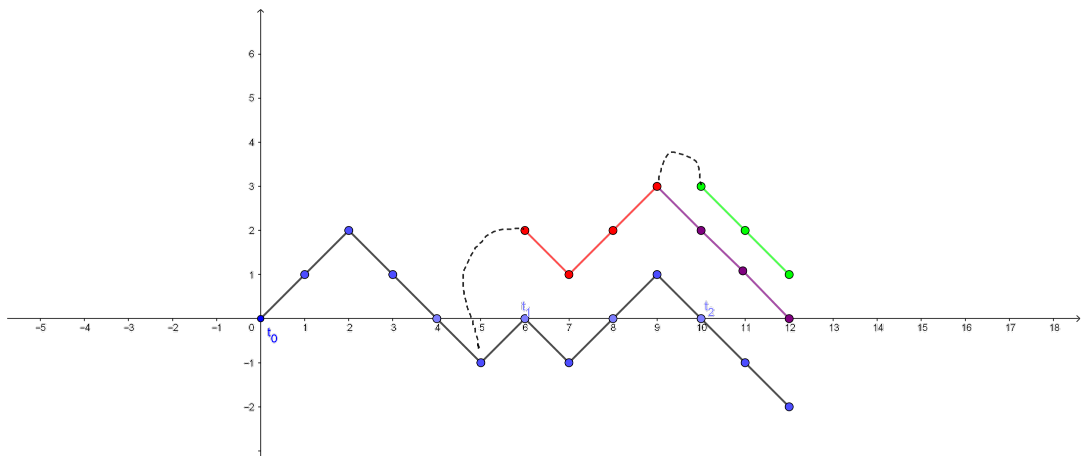


Figure 1: An illustration of two random walks $(X)_{n\geq 0}$ and $(\hat{X}_n)_{n\geq 0}$ constructed from the simple random walk $(Z)_{n\geq 0}$ (blue line) and a set $\mathcal{J}$ consisting of two resetting times t_1, t_2. Initially, three processes are identical until the time t_1-1. At t_1, both X_{t_1} and $\hat{X}_{t_1}$ reset to the maximum position, then the two walkers move as the transition by Z until t_2-1 (red line). At t_2, while $\hat{X}_{t_2}$ reset to the maximum position (green line), X_{t_2} follows the random walk Z to go down one step (purple line). After that, the two processes copy the path of Z until t_3-1.

with the convention that $k_n = 0$ if $\mathcal{J} \cap [1, n] = \varnothing$. For every $i, k \geq 0$, we define

$$m_k^{(i)} = \max_{0\leq s\leq k}\{Z_{t_i+s} - Z_{t_i}\}.$$

Then the mean value of $m_k^{(i)}$ is independent of i, so we can set

$$g(k) = \mathbb{E}[m_{k-1}^{(i)}] = \mathbb{E}[\max_{0\leq s\leq k-1} Z_s]. \tag{24}$$

The following lemma summarizes some properties of the asymptotic behavior of the simple random walk $(Z_n)_{n\geq 0}$.

Lemma 3.2. *[11] We have*

$$\lim_{k\to\infty}\frac{g(k)}{\sqrt{k}} = \sqrt{\frac{2}{\pi}}, \qquad \lim_{k\to\infty}\frac{\mathbb{E}[Z_k^2]}{k} = \lim_{k\to\infty}\frac{\mathbb{E}[(\max_{0\leq i\leq k-1} Z_i)^2]}{k} = 1.$$

We define the function $h : \mathbb{N} \to \mathbb{R}$ as follows. If $i = 1$ then $h(1) = 0$ and for $j \geq 2$,

$$h(j) = \sum_{k=1}^{j-1} g(k) P_{j,k}, \tag{25}$$

where

$$P_{j,k} \;:=\; \mathbb{P}[\sup\{i < j : i \in \mathcal{J}\} = j-k] = r_{j-k}\prod_{l=1}^{k-1}(1 - r_{j-l}). \tag{26}$$

We now give expressions of the mean values of $\hat{X}_n$ and $\hat{M}_n$ in term of two functions g and h and the random set $\mathcal{J}$.

Lemma 3.3. *For any $n \geq 1$,*

$$\begin{aligned}\mathbb{E}[\hat{X}_n] &= \mathbb{E}[g(t_1)\mathbb{I}(t_1 \leq n)] - \mathbb{E}[h(t_1)\mathbb{I}(t_1 \leq n)] + \sum_{j=1}^{n} h(j) r_j, \\ \mathbb{E}[\hat{M}_n] &= \mathbb{E}[\hat{X}_n] + \mathbb{E}[g(n+1-t_{k_n})].\end{aligned}$$

Proof. By the construction of $(\hat{X}_n)_{n\geq 0}$,

$$\hat{X}_{t_{k_n}} = \sum_{i=0}^{k_n-1} m^{(i)}_{t_{i+1}-t_i-1}.$$

By the strong Markov's property of the simple random walk $(Z_n)_{n\geq 0}$

$$\mathbb{E}[m^{(i)}_{t_{i+1}-t_i-1} \mid \mathcal{F}_{t_i}] = \mathbb{E}[g(t_{i+1}-t_i)],$$

where $\mathcal{F}_{t_i}$ is the sigma field $\sigma(Z_t, t \leq t_i)$. Therefore,

$$\mathbb{E}[\hat{X}_{t_{k_n}}] = \mathbb{E}\Big[\sum_{i=0}^{k_n-1} g(t_{i+1}-t_i)\Big] = \sum_{i=0}^{\infty} \mathbb{E}\big[g(t_{i+1}-t_i)\mathbb{I}(t_{i+1} \leq n)\big]. \tag{27}$$

We observe that for any $i \geq 1$,

$$\begin{aligned}\mathbb{E}\big[g(t_{i+1}-t_i) \mid t_{i+1} = j\big] &= \sum_{k=1}^{j-1} g(k)\mathbb{P}(t_i = j-k \mid t_{i+1} = j) \\ &= \sum_{k=1}^{j-1} g(k) P_{j,k} = h(j).\end{aligned}$$

Furthermore, the condition $i \geq 1$ is equivalent to $t_i > 0$, and hence

$$h(t_{i+1}) = \mathbb{E}[g(t_{i+1}-t_i)\mathbb{I}(t_i > 0) \mid t_{i+1}]. \tag{28}$$

Thus,

$$\begin{aligned}\mathbb{E}[g(t_{i+1}-t_i)\mathbb{I}(0 < t_i < t_{i+1} \leq n)] &= \mathbb{E}\Big[\mathbb{E}[g(t_{i+1}-t_i)\mathbb{I}(0 < t_i < t_{i+1}) \mid t_{i+1}]\mathbb{I}(t_{i+1} \leq n)\Big] \\ &= \mathbb{E}\Big[h(t_{i+1})\mathbb{I}(t_{i+1} \leq n)\Big].\end{aligned}$$

Now, combining the last equation with (27) we obtain

$$\begin{aligned}\mathbb{E}[\hat{X}_{t_{k_n}}] &= \mathbb{E}[g(t_1)\mathbb{I}(t_1 \leq n)] + \sum_{i=1}^{\infty} \mathbb{E}[h(t_{i+1})\mathbb{I}(t_{i+1} \leq n)] \\ &= \mathbb{E}[g(t_1)\mathbb{I}(t_1 \leq n)] + \mathbb{E}\Big[\sum_{i=2}^{\infty} h(t_i)\mathbb{I}(t_i \leq n)\Big] \\ &= \mathbb{E}[g(t_1)\mathbb{I}(t_1 \leq n)] + \mathbb{E}\Big[\sum_{i=2}^{k_n} h(t_i)\mathbb{I}(k_n \geq 2)\Big] \\ &= \mathbb{E}[g(t_1)\mathbb{I}(t_1 \leq n)] - \mathbb{E}[h(t_1)\mathbb{I}(t_1 \leq n)] + \mathbb{E}\Big[\sum_{i=1}^{k_n} h(t_i)\Big].\end{aligned}$$

Moreover,

$$\mathbb{E}\Big[\sum_{i=1}^{k_n} h(t_i)\Big] = \mathbb{E}\Big[\sum_{j\in\mathcal{J}\cap[1,n]} h(j)\Big] = \sum_{j=1}^{n} \mathbb{E}[h(j)\mathbb{I}(j\in\mathcal{J})] = \sum_{j=1}^{n} h(j)r_j.$$

Combining the two above equations, we get

$$\mathbb{E}[\hat{X}_{t_{k_n}}] = \mathbb{E}[g(t_1)\mathbb{I}(t_1 \le n)] - \mathbb{E}[h(t_1)\mathbb{I}(t_1\le n)] + \sum_{j=1}^{n} h(j)r_j. \tag{29}$$

By the definition of $\hat{M}_n$,

$$\hat{M}_n = \max_{t_{k_n}\le j\le n} \hat{X}_{t_{k_n}} + Z_j - Z_{t_{k_n}} = \hat{X}_{t_{k_n}} + m_{n-t_{k_n}}^{(k_n)},$$

and thus

$$\mathbb{E}[\hat{M}_n] = \mathbb{E}[\hat{X}_{t_{k_n}}] + \mathbb{E}[g(n+1-t_{k_n})].$$

On the other hand, since $\hat{X}_n = \hat{X}_{t_{k_n}} + Z_n - Z_{t_{k_n}}$,

$$\mathbb{E}[\hat{X}_n] = \mathbb{E}[\hat{X}_{t_{k_n}}].$$

Combining the last two equations with (29) we achieve the desired results. □

From the above lemma, to get $\mathbb{E}[\hat{M}_n]$ and $\mathbb{E}[\hat{X}_n]$, we have to study the function $h(\cdot)$. In next subsections, we will show that the asymptotic behavior of $h(\cdot)$ changes when the parameter a crosses the cirical value 1, and as consequence the behavior of $\mathbb{E}[\hat{M}_n]$ and $\mathbb{E}[\hat{X}_n]$ changes as well.

3.2 The subcritical phase $0<a<1$

Lemma 3.4. *Assume that* $0<a<1$. *Then*

$$\lim_{j\to\infty} \frac{h(j)}{j^{a/2}} = \frac{1}{\sqrt{2r}}.$$

Proof. To prove this lemma, it is sufficient to show that as $j\to\infty$,

$$h_c(j) := \sum_{k=[j^a/\log^2 j]}^{[j^a\log^2 j]} g(k)P_{j,k} = \left(\frac{1}{\sqrt{2r}} + o(1)\right) j^{a/2}, \tag{30}$$

and

$$h(j) - h_c(j) = o(j^{a/2}). \tag{31}$$

We start by proving (30). Observe that if $j^a/\log^2 j \le k \le j^a\log^2 j$ and j is large enough, $r_{j-l} = r(j-l)^{-a} = rj^{-a}(1+o(1))$ for any $l=1,\dots,k$. Hence, using the approximation $\log(1-x) = -x(1+o(1))$, we get

$$\begin{aligned} P_{j,k} &= r_{j-k}\prod_{l=1}^{k-1}(1-r_{j-l}) = r_{j-k}\exp\left(\sum_{l=1}^{k-1}\log(1-r_{j-l})\right) \\ &= r_{j-k}\exp\left(-(1+o(1))\sum_{l=1}^{k-1} r_{j-l}\right) = (1+o(1))rj^{-a}\exp\left(-(r+o(1))kj^{-a}\right). \end{aligned}$$

By Lemma 3.2, $g(k) = \sqrt{\frac{2+o(1)}{\pi}}\sqrt{k}$. Therefore, when $j \to \infty$,

$$\begin{aligned}
h_c(j) &= (1+o(1))\sqrt{\frac{2}{\pi}\frac{r}{j^a}} \sum_{k=[j^a/\log^2 j]}^{[j^a \log^2 j]} \sqrt{k}\exp\left(-(r+o(1))k/j^a\right) \\
&= (1+o(1))\sqrt{\frac{2}{\pi}\frac{r}{j^a}} \int_{j^a/\log^2 j}^{j^a \log^2 j} \sqrt{x}\exp\left(-(r+o(1))x/j^a\right) dx \\
&= (1+o(1))\sqrt{\frac{2}{\pi r}} j^{a/2} \int_{r/\log^2 j}^{r \log^2 j} \sqrt{y}\exp(-(1+o(1))y) dy \\
&= (1+o(1))\sqrt{\frac{2}{\pi r}} j^{a/2} \int_0^\infty \sqrt{y}\exp(-y) dy = \frac{1+o(1)}{\sqrt{2r}} j^{a/2}.
\end{aligned}$$

Here we have used the integral approximation and the change of variable $y = \frac{rx}{j^a}$ and the equation $\int_0^\infty \sqrt{y}e^{-y}dy = \frac{\sqrt{\pi}}{2}$.

Next we prove (31). The case $k \le j^a/\log^2 j$ can be treated by the same argument as above, since we still have

$$P_{j,k} = (1+o(1))rj^{-a}\exp\left(-(r+o(1))kj^{-a}\right),$$

and $g(k) = \mathcal{O}(\sqrt{k}) = \mathcal{O}(j^{a/2}/\log j)$. Hence

$$\begin{aligned}
\sum_{k=1}^{\lceil j^a/\log^2 j\rceil} g(k)P_{j,k} &= \mathcal{O}(1)\frac{j^{a/2}}{\log j} \times \frac{r}{j^a} \sum_{k=1}^{\lceil j^a/\log^2 j\rceil} \exp(-(1+o(1))rk/j^a) \\
&= \frac{\mathcal{O}(j^{a/2})}{j^a \log j} \int_1^{j^a/\log^2 j} \exp(-rx/j^a)dx = o(j^{a/2}), \qquad (32)
\end{aligned}$$

as $j \to \infty$. Finally, for the case $j^a \log^2 j \le k \le j-1$,

$$\begin{aligned}
P_{j,k} &= r_{j-k}\prod_{l=1}^{k-1}(1-r_{j-l}) \le \prod_{l=1}^{\lceil k/2\rceil}(1-r_{j-l}) \le (1-r_j)^{k/2} \\
&= (1-rj^{-a})^{k/2} \le \exp(-kr/(2j^a)) \le \exp(-r\log^2 j/2).
\end{aligned}$$

Hence, using $g(k) = \mathcal{O}(\sqrt{k}) = \mathcal{O}(\sqrt{j})$, we obtain that

$$\sum_{k=\lceil j^a \log^2 j\rceil}^{j-1} g(k)P_{j,k} = \mathcal{O}(j^{3/2})\exp(-r\log^2 j/2) = o(1), \qquad (33)$$

as $j \to \infty$. Combining (32) and (33) yields (31). □

Corollary 3.5. *Assume that* $0 < a < 1$. *Then it holds that*

$$\lim_{n\to\infty}\frac{\mathbb{E}[\hat{M}_n]}{n^{1-a/2}} = \lim_{n\to\infty}\frac{\mathbb{E}[\hat{X}_n]}{n^{1-a/2}} = \frac{\sqrt{2r}}{2-a}.$$

Proof. First, it is not hard to check that

$$\mathbb{E}[g(t_1)] + \mathbb{E}[h(t_1)] = \mathcal{O}(1). \qquad (34)$$

According to Lemma 3.3, we need to compute the sum $\sum_{j=1}^{n} h(j)r_j$. Observe that

$$\begin{aligned}\sum_{j=1}^{[n/\log n]} h(j)r_j &= \sum_{j=1}^{[n/\log n]} h(j)\left(\tfrac{r}{j^a}\wedge\tfrac{1}{2}\right) \le \frac{1}{2}\sum_{j=1}^{\lceil (2r)^{1/a}\rceil} h(j) + r\sum_{j=1}^{n/\log n} h(j)j^{-a} \\ &\le \mathcal{O}(1)+\mathcal{O}(1)\sum_{j=1}^{[n/\log n]} j^{-a/2} = o(n^{1-a/2}),\end{aligned}$$

where we have used $h(j)=\mathcal{O}(j^{a/2})$.

Moreover, by Lemma 3.4, $h(j) = (\frac{1}{\sqrt{2r}}+o(1))j^{a/2}$ for j large enough. Hence,

$$\begin{aligned}\sum_{j=\lceil n/\log n\rceil}^{n} h(j)r_j = \sum_{\lceil j=n/\log n\rceil}^{n} \frac{rh(j)}{j^a} &= (1+o(1))\sqrt{\frac{r}{2}}\sum_{j=\lceil n/\log n\rceil}^{n} j^{-a/2} \\ &= (1+o(1))\sqrt{\frac{r}{2}}\int_{n/\log n}^{n} x^{-a/2}dx = (1+o(1))n^{1-a/2}\frac{\sqrt{2r}}{2-a},\end{aligned}$$

as $n\to\infty$. Therefore,

$$\lim_{n\to\infty}\frac{\sum_{j=1}^{n} h(j)r_j}{n^{1-a/2}} = \frac{\sqrt{2r}}{2-a}. \tag{35}$$

Combining (34), (35) and Lemma 3.3, we obtain

$$\lim_{n\to\infty}\frac{\mathbb{E}[\hat{X}_n]}{n^{1-a/2}} = \frac{\sqrt{2r}}{2-a}. \tag{36}$$

On the other hand,

$$\mathbb{E}[g(n+1-t_{k_n})] \le g(n+1) = \mathcal{O}(\sqrt{n}) = o(n^{1-a/2}). \tag{37}$$

Combining the last two equations with Lemma 3.3 we get the result for $\mathbb{E}[\hat{M}_n]$. □

3.3 The critical phase $a = 1$

Similarly to subcritical case, we first investigate the order of magnitude of $h(j)$ as j tends to infinity.

Lemma 3.6. *Assume that $a=1$. Then*

$$\lim_{j\to\infty}\frac{h(j)}{\sqrt{j}} = \sqrt{\frac{2}{\pi}}rB\left(\tfrac{3}{2},r\right), \tag{38}$$

where we recall that $B(\cdot,\cdot)$ is the Beta function defined by $B(a,b)=\int_0^1 t^{a-1}(1-t)^{b-1}dt$.

Proof. To prove this lemma, we will show that as $j\to\infty$,

$$h_c(j) := \sum_{k=[j/\log^2 j]}^{[j-\log^2 j]} g(k)P_{j,k} = (1+o(1))\sqrt{\frac{2}{\pi}}rB\left(\tfrac{3}{2},r\right)\sqrt{j}, \tag{39}$$

and

$$h(j) - h_c(j) = o(\sqrt{j}). \tag{40}$$

We first prove (39). Let $\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx$ be the Gamma function and recall the Stirling's approximation, as $z \to \infty$

$$\Gamma(z) = \sqrt{\frac{2\pi}{z}}\left(\frac{z}{e}\right)^z (1 + o(1)). \tag{41}$$

Moreover, using $\Gamma(z+1) = z\Gamma(z)$, we rewrite $P_{j,k}$ as

$$\begin{aligned} P_{j,k} &= \frac{r}{j-k}\prod_{l=1}^{k-1}\left(1 - \frac{r}{j-l}\right) = \frac{r}{j-k}\prod_{i=j-k+1}^{j-1}\left(1 - \frac{r}{i}\right) \\ &= \frac{r}{j-k}\frac{\Gamma(j-r)}{\Gamma(j)}\frac{\Gamma(j-k+1)}{\Gamma(j-k-r+1)}. \end{aligned} \tag{42}$$

Hence, when $k \le j - \log^2 j$ and j is large enough, plugging (41) into (42) yields that

$$\begin{aligned} P_{j,k} &= (1+o(1))\frac{r}{j-k}\sqrt{\frac{j}{j-r}}\sqrt{\frac{j-k-r+1}{j-k+1}}\left(1 - \frac{r}{j}\right)^j \\ &\quad \times \left(1 + \frac{r}{j-k-r+1}\right)^{j-k-r+1}\frac{(j-k+1)^r}{(j-r)^r} \\ &= (r + o(1))\exp(r + o(1))\exp(-r - o(1))\frac{(j-k+1)^{r-1}}{(j-r)^r} \\ &= (1+o(1))r\frac{(j-k+1)^{r-1}}{(j-r)^r}. \end{aligned} \tag{43}$$

Combining this with $g(k) = \sqrt{\frac{2+o(1)}{\pi}}\sqrt{k}$ (by Lemma 3.2), we have as $j \to \infty$,

$$\begin{aligned} h_c(j) &= \sum_{k=[j/\log^2 j]}^{[j-\log^2 j]}(1+o(1))\sqrt{\frac{2}{\pi}}r\sqrt{k}\frac{(j-k+1)^{r-1}}{(j-r)^r} \\ &= (1+o(1))\sqrt{\frac{2}{\pi}}r\int_{j/\log^2 j}^{j-\log^2 j}\sqrt{x}\frac{(j-x+1)^{r-1}}{(j-r)^r}dx \\ &= (1+o(1))\sqrt{\frac{2}{\pi}}r\frac{\sqrt{j}j^r}{(j-r)^r}\int_{1/\log^2 j}^{1-\log^2 j/j}\sqrt{y}(1-y)^{r-1}dy \\ &= (1+o(1))\sqrt{\frac{2}{\pi}}\sqrt{j}\int_0^1\sqrt{y}(1-y)^{r-1}dy = (1+o(1))\sqrt{\frac{2}{\pi}}rB(3/2,r)\sqrt{j}. \end{aligned}$$

Here we have used the integral approximation, the change of variable $y = jx$, and $\int_0^1\sqrt{t}(1-y)^{r-1}dy = B(3/2,r)$.

We turn to prove (40). Using (43) and $g(k) = \mathcal{O}(\sqrt{k})$, we derive that as $j \to \infty$,

$$\begin{aligned} \sum_{k=1}^{\lceil j/\log^2 j\rceil} g(k)P_{j,k} &= \mathcal{O}(1)r\frac{\sqrt{j}}{\log j}\sum_{k=1}^{\lceil j/\log^2 j\rceil}\frac{(j-k+1)^{r-1}}{(j-r)^r} \\ &= \mathcal{O}(1)r\frac{\sqrt{j}}{\log j}\left(\frac{j}{j-r}\right)^r\int_0^{1/\log^2 j}(1-t)^{r-1}dt = o(\sqrt{j}). \end{aligned} \tag{44}$$

Finally, observe that if $j - \log^2 j \le k \le j-1$ then

$$P_{j,k} \le r \prod_{l=1}^{k-1} (1 - r_{j-l}) \le r \left(1 - \frac{r}{\log^2 j}\right)^{k/2} \le r \exp(-rk/(2\log^2 j))$$
$$\le r \exp(r/2) \exp(-rj/(2\log^2 j)).$$

Hence,

$$\sum_{k=j-\log^2 j}^{j-1} g(k) P_{j,k} = \mathcal{O}(\sqrt{j}) \log^2 j \exp(-rj/(2\log^2 j)) = o(1), \tag{45}$$

as $j \to \infty$. Combining (45) and (44) we get (40). □

Corollary 3.7. *Assume that $a = 1$. Then it holds that*

$$\lim_{n\to\infty} \frac{\mathbb{E}[\hat{X}_n]}{\sqrt{n}} = 2r^2 \sqrt{\frac{2}{\pi}} B(3/2, r), \qquad \lim_{n\to\infty} \frac{\mathbb{E}[\hat{M}_n]}{\sqrt{n}} = (2r^2 + r)\sqrt{\frac{2}{\pi}} B(3/2, r).$$

Proof. Analogously to Corollary 3.5, to get the mean values of $\mathbb{E}[\hat{M}_n]$ and $\mathbb{E}[\hat{X}_n]$, we compute the sum $\sum_{j=1}^n h(j) r_j$. First, we consider

$$\sum_{j=1}^{[n/\log n]} h(j) r_j = \sum_{j=1}^{[n/\log n]} h(j) \left(\frac{r}{j} \wedge \frac{1}{2}\right) \le \frac{1}{2} \sum_{j=1}^{\lceil 2r \rceil} h(j) + r \sum_{j=1}^{[n/\log n]} h(j) j^{-1}$$
$$= \mathcal{O}(1) + \mathcal{O}(1) \sum_{j=1}^{[n/\log n]} j^{-1/2} = o(\sqrt{n}),$$

since $h(j) = \mathcal{O}(\sqrt{j})$ by Lemma 3.6.

Moreover, using $h(j) = \big(r\sqrt{\frac{2}{\pi}} B(3/2, r) + o(1)\big)\sqrt{j}$, we obtain that as $n \to \infty$,

$$\sum_{j=\lceil n/\log n\rceil}^{n} h(j) r_j = (1 + o(1)) r^2 \sqrt{\frac{2}{\pi}} B(3/2, r) \sum_{\lceil n/\log n\rceil}^{n} \frac{1}{\sqrt{j}}$$
$$= (1 + o(1)) r^2 \sqrt{\frac{2}{\pi}} B(3/2, r) \int_{n/\log n}^{n} \frac{1}{\sqrt{x}} dx$$
$$= 2(1 + o(1)) r^2 \sqrt{\frac{2}{\pi}} B(3/2, r) \sqrt{n}.$$

Thus,

$$\sum_{j=1}^{n} h(j) r_j = (2 + o(1)) r^2 \sqrt{\frac{2}{\pi}} B(3/2, r) \sqrt{n}. \tag{46}$$

On the other hand,

$$\mathbb{E}[g(t_1)\mathbb{I}(t_1 \le n)] = \sum_{k=1}^{n} g(k) \mathbb{P}(t_1 = k) = \sum_{k=1}^{n} k^{-(r+\frac{1}{2})} = o(\sqrt{n}), \tag{47}$$

since $g(k) = \mathcal{O}(\sqrt{k})$ and

$$\mathbb{P}(t_1 = k) = \prod_{j=1}^{k-1} \left(1 - \frac{r}{j}\right) \frac{1}{k} \leq k^{-1} \exp(-\sum_{j=1}^{k-1} r/j) = \mathcal{O}(k^{-r-1}).$$

Similarly,

$$\mathbb{E}[h(t_1)\mathbb{I}(t_1 \leq n)] = o(\sqrt{n}). \tag{48}$$

Using (47), (48) with (46) and Lemma 3.3, we get

$$\lim_{n\to\infty} \frac{\mathbb{E}[\hat{X}_n]}{\sqrt{n}} = 2r^2 \sqrt{\frac{2}{\pi}} B(3/2, r). \tag{49}$$

Finally, to compute $\mathbb{E}[\hat{M}_n] = \mathbb{E}[\hat{X}_{t_{k_n}}] + \mathbb{E}[g(n + 1 - t_{k_n})]$. Observe that

$$\mathbb{E}[g(n - t_{k_n})] = \sum_{k=1}^{n} g(k)\mathbb{P}(n + 1 - t_{k_n} = k) + g(n + 1)\mathbb{P}(t_1 > n)$$

$$= \sum_{k=1}^{n} g(k)P_{n+1,k} + g(n + 1)\mathbb{P}(t_1 > n) = h(n + 1) + g(n + 1)\mathbb{P}(t_1 > n).$$

Therefore,

$$\mathbb{E}[\hat{M}_n] = \mathbb{E}[\hat{X}_{t_{k_n}}] + h(n + 1) + g(n + 1)\mathbb{P}(t_1 > n).$$

Notice that

$$g(n + 1)\mathbb{P}(t_1 > n) = g(n + 1) \prod_{j=1}^{n} \Big(1 - (r/j \wedge 1/2)\Big) = o(g(n + 1)) = o(\sqrt{n})$$

and by Lemma 3.6

$$h(n + 1) = (1 + o(1))r\sqrt{\frac{2}{\pi}} B(3/2, r)\sqrt{n}. \tag{50}$$

Combining the last three equations and Lemma 3.3 yields

$$\lim_{n\to\infty} \frac{\mathbb{E}[\hat{M}_n]}{\sqrt{n}} = (2r^2 + r)\sqrt{\frac{2}{\pi}} B(3/2, r),$$

which completes the proof of lemma. □

3.4 The suppercritical phase $a > 1$

In the case $a > 1$, we observe that the resetting rate r_n decays very fast and the number of reset points is quite small (with finite expectation). Thus we expect that when $n \to \infty$, the asymptotic behavior of the random walk gets closer to the one of the simple random walk. In particular, we will show that $\mathbb{E}[M_n]$ is of order $\sqrt{n}$ and $\mathbb{E}[X_n] = o(\mathbb{E}[M_n])$. Moreover, in the next section, we also prove that the variance of X_n and M_n is asymptotically the same as the one of the simple random walk. Recall that for $a > 1$, we set

$$\varphi_a(n) = \begin{cases} n^{\frac{3}{2}-a} & \text{if } 1 < a < \frac{3}{2}, \\ \log n & \text{if } a = \frac{3}{2}, \\ 1 & \text{if } a > \frac{3}{2}. \end{cases} \tag{51}$$

Lemma 3.8. *Assume that $a > 1$. Then the following statements hold.*

(i)
$$\lim_{j\to\infty} \frac{h(j)}{\sqrt{j}} = \lambda(a,r) := \sqrt{\frac{2}{\pi}}\Big(1 - \prod_{i=1}^{\infty}(1-r_i)\Big).$$

(ii)
$$\lim_{n\to\infty} \frac{\mathbb{E}[g(t_1)\mathbb{I}(t_1 \le n)]}{\varphi_a(n)} = \lambda_1(a,r),$$

where $\lambda_1(a,r)$ is as in (58).

(iii)
$$\lim_{n\to\infty} \frac{\mathbb{E}[h(t_1)\mathbb{I}(t_1 \le n)]}{\varphi_a(n)} = \lambda_2(a,r),$$

where $\lambda_2(a,r)$ is as in (59).

(iv)
$$\lim_{n\to\infty} \frac{\sum_{j=1}^{n} h(j) r_j}{\varphi_a(n)} = \lambda_3(a,r),$$

where $\lambda_3(a,r)$ is as in (60).

(v)
$$\lim_{n\to\infty} \frac{\mathbb{E}[g(n+1-t_{k_n})]}{\sqrt{n}} = \sqrt{\frac{2}{\pi}}.$$

Proof. Similarly to the previous cases, to prove (i) we shall show that as $j \to \infty$,

$$h_c(j) := \sum_{k=[j-\log^2 j]}^{j-1} g(k) P_{j,k} = (1+o(1))\lambda_1(a,r)\sqrt{j}, \tag{52}$$

and

$$h - h_c(j) = o(\sqrt{j}). \tag{53}$$

By Lemma 3.2, $g(k) = (\sqrt{2/\pi} + o(1))\sqrt{k}$. Thus as $j \to \infty$,

$$\begin{aligned}
h_c(j) &= (\sqrt{2/\pi} + o(1))\sqrt{j} \sum_{k=[j-\log^2 j]}^{j-1} P_{j,k} \\
&= (\sqrt{2/\pi} + o(1))\sqrt{j} \sum_{k=[j-\log^2 j]}^{j-1} r_{j-k} \prod_{l=1}^{k-1}(1-r_{j-l}) \\
&= (\sqrt{2/\pi} + o(1))\sqrt{j} \sum_{i=1}^{[\log^2 j]} r_i \prod_{s=i+1}^{j-1}(1-r_s) \\
&= (\sqrt{2/\pi} + o(1))\sqrt{j} \sum_{i=1}^{\infty} r_i \prod_{s=i+1}^{\infty}(1-r_s) = (\lambda(a,r) + o(1))\sqrt{j}.
\end{aligned}$$

Here for the last equation, we have used $\sum_{i=1}^{\infty} r_i \prod_{s=i+1}^{\infty}(1-r_s) = \mathbb{P}(\mathcal{J} \neq \varnothing) = 1 - \mathbb{P}(\mathcal{J} = \varnothing) = 1 - \prod_{i=1}^{\infty}(1-r_i)$, and for the fourth equation, we have used the following approximation

$$\begin{aligned}
&\Big| \sum_{i=1}^{[\log^2 j]} r_i \prod_{s=i+1}^{j-1} (1-r_s) - \sum_{i=1}^{\infty} r_i \prod_{s=i+1}^{\infty} (1-r_s) \Big| \\
\leq\ & \sum_{i=1}^{[\log^2 j]} r_i \left(\prod_{s=i+1}^{j-1} (1-r_s) - \prod_{s=i+1}^{\infty} (1-r_s) \right) + \sum_{i=j}^{\infty} r_i \prod_{s=i+1}^{\infty} (1-r_s) \\
\leq\ & \sum_{i=1}^{[\log^2 j]} r_i \left(1 - \prod_{s=j}^{\infty} (1-r_s) \right) + \sum_{i=j}^{\infty} r_i \\
\leq\ & \sum_{i=1}^{[\log^2 j]} r_i \left(\sum_{s=j}^{\infty} r_s \right) + \sum_{i=j}^{\infty} r_i = \left(1 + \sum_{i=1}^{[\log^2 j]} r_i \right) \left(\sum_{s=j}^{\infty} r_s \right) = o(1),
\end{aligned}$$

as $j \to \infty$, since the sum $\sum_{s \geq 1} r_s$ is convergent when $a > 1$. Hence, (52) is proved.

Next, we prove (53). By Lemma 3.2, $g(k) = \mathcal{O}(\sqrt{k}) = \mathcal{O}(\sqrt{j})$. Moreover, if $1 \leq k \leq j - \log^2 j$ then $P_{j,k} \leq r_{j-k} = r(j-k)^{-a}$. Therefore,

$$\begin{aligned}
h(j) - h_c(j) = \sum_{k=1}^{[j-\log^2 j]} g(k) P_{j,k} &\leq r\mathcal{O}(\sqrt{j}) \sum_{k=1}^{[j-\log^2 j]} (j-k)^{-a} \\
&= \mathcal{O}(\sqrt{j}(\log^2 j)^{1-a}) = o(\sqrt{j}).
\end{aligned}$$

We turn to prove (ii). First, recall that

$$\mathbb{E}[g(t_1)\mathbb{I}(t_1 \leq n)] = \sum_{k=1}^{n} g(k)\mathbb{P}(t_1 = k).$$

The case $1 < a < 3/2$. Using $g(k) = (\sqrt{2/\pi} + o(1))\sqrt{k}$ and the definition of $P_{j,k}$, it follows that as $n \to \infty$,

$$\begin{aligned}
\sum_{k=1}^{[n/\log n]} g(k)\mathbb{P}(t_1 = k) &= \sum_{k=1}^{[(2r)^{1/a}]} g(k)\mathbb{P}(t_1 = k) + \sum_{k=\lceil (2r)^{1/a} \rceil}^{[n/\log n]} g(k)\mathbb{P}(t_1 = k) \\
&= \mathcal{O}(1) + \mathcal{O}(1) \sum_{k=\lceil (2r)^{1/a} \rceil}^{[n/\log n]} \frac{\sqrt{k}}{k^a} \prod_{l=1}^{k-1} (1-r_l) \\
&= \mathcal{O}(1) + \mathcal{O}(1) \left(\frac{n}{\log n} \right)^{3/2-a} = o(n^{3/2-a}). \qquad (54)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k=\lceil n/\log n \rceil}^{n} g(k)\mathbb{P}(t_1 = k) &= (\sqrt{2/\pi} + o(1)) r \int_{n/\log n}^{n} \frac{\sqrt{k}}{k^a} \prod_{l=1}^{\infty} (1-r_l) \\
&= \frac{(\sqrt{2/\pi} + o(1)) \prod_{l=1}^{\infty}(1-r_l) r}{3/2 - a} n^{3/2-a}, \qquad (55)
\end{aligned}$$

by using the integral approximation. Combine (54) and (55), we get that

$$\lim_{n\to\infty} \frac{\mathbb{E}[g(t_1)\mathbb{I}(t_1 \le n)]}{n^{3/2-a}} = \frac{\sqrt{2/\pi}\prod_{l=1}^{\infty}(1-r_l)r}{3/2-a}. \tag{56}$$

The case $a = 3/2$. We have

$$\sum_{k=1}^{[\log n]} g(k)\mathbb{P}(t_1 = k) = \mathcal{O}(1)\sqrt{\log n}\sum_{k=1}^{[\log n - 1]} r_k = \mathcal{O}(\sqrt{\log n}),$$

since the series $\sum_{k\ge 1} r_k$ converges for $a = 3/2$. Moreover, using the integral approximation and Lemma 3.2 we obtain

$$\begin{aligned}\sum_{k=\lceil \log n\rceil}^{n} g(k)\mathbb{P}(t_1 = k) &= (\sqrt{2/\pi} + o(1))r \sum_{k=\lceil \log n\rceil}^{n} k^{-1}\prod_{l=1}^{k-1}(1-r_l)\\ &= (\sqrt{2/\pi} + o(1))\prod_{l=1}^{\infty}(1-r_l)r\log n.\end{aligned}$$

Hence, if $a = 3/2$ then

$$\lim_{n\to\infty} \frac{\mathbb{E}[g(t_1)\mathbb{I}(t_1 \le n)]}{\log n} = \sqrt{2/\pi}\prod_{l=1}^{\infty}(1-r_l)r. \tag{57}$$

The case $a > 3/2$. It is not hard to show that

$$\lim_{n\to\infty}\mathbb{E}[g(t_1)\mathbb{I}(t_1 \le n)] = \lim_{n\to\infty}\sum_{k=1}^{n} g(k)r_k\prod_{l=1}^{k-1}(1-r_l) = \sum_{k=1}^{\infty} g(k)r_k\prod_{l=1}^{k-1}(1-r_l) < \infty.$$

In summary, we yield the statement of (ii) saying that

$$\lim_{n\to\infty} \frac{\mathbb{E}[g(t_1)\mathbb{I}(t_1 \le n)]}{\varphi_a(n)} = \lambda_1(a,r),$$

where

$$\lambda_1(a,r) = \begin{cases} \dfrac{\sqrt{2/\pi}r\prod_{l=1}^{\infty}(1-r_l)}{3/2-a} & \text{if } 1 < a < \frac{3}{2},\\ \sqrt{2/\pi}r\prod_{l=1}^{\infty}(1-r_l) & \text{if } a = \frac{3}{2},\\ \sum_{k=1}^{\infty} g(k)r_k\prod_{l=1}^{k-1}(1-r_l) & \text{if } a > \frac{3}{2}. \end{cases} \tag{58}$$

The proof of (iii) and(iv) is analogous. First, recall that

$$\mathbb{E}[h(t_1)\mathbb{I}(t_1 \le n)] = \sum_{k=2}^{n} h(k)\mathbb{P}(t_1 = k).$$

Using (i) and similar arguments as for (ii), we can deduce the following

$$\lim_{n\to\infty} \frac{\mathbb{E}[h(t_1)\mathbb{I}(t_1 \le n)]}{\varphi_a(n)} = \lambda_2(a,r), \qquad \lim_{n\to\infty} \frac{\sum_{j=1}^{n} h(j)r_j}{\varphi_a(n)} = \lambda_3(a,r),$$

where

$$\lambda_2(a,r) = \begin{cases} \dfrac{\lambda(a,r)r\prod_{l=1}^{\infty}(1-r_l)}{3/2-a} & \text{if } 1<a<\frac{3}{2}, \\ \lambda(a,r)r\prod_{l=1}^{\infty}(1-r_l) & \text{if } a=\frac{3}{2}, \\ \sum_{k=1}^{\infty} h(k)r_k\prod_{l=1}^{k-1}(1-r_l) & \text{if } a>\frac{3}{2}; \end{cases} \tag{59}$$

and

$$\lambda_3(a,r) = \begin{cases} \dfrac{\lambda(a,r)r}{3/2-a} & \text{if } 1<a<\frac{3}{2}, \\ \lambda(a,r)r & \text{if } a=\frac{3}{2}, \\ \sum_{j=1}^{\infty} h(j)r_j & \text{if } a>\frac{3}{2}. \end{cases} \tag{60}$$

To prove (v), we observe that

$$g(n+1) \geq \mathbb{E}[g(n+1-t_{k_n})] \geq g(n+1-[n^{1/2}])\mathbb{P}(t_{k_n} \leq [n^{1/2}]).$$

Moreover, $g(k) = (\sqrt{2/\pi}+o(1))\sqrt{k}$ and $\mathbb{P}(t_{k_n} \leq [n^{1/2}]) \geq \prod_{k\geq[n^{1/2}]}(1-r/j^a) \to 1$ as $n\to\infty$, since $a>1$. Therefore, $g(n+1-t_{k_n}) = (\sqrt{2/\pi}+o(1))\sqrt{n}$. □

Combining Lemma 3.8 and Lemma 3.3, we derive the asymptotic behavior of $\mathbb{E}[\hat{X}_n]$ and $\mathbb{E}[\hat{M}_n]$.

Corollary 3.9. *Assume that $a>1$. Then it holds that*

$$\lim_{n\to\infty}\frac{\mathbb{E}[\hat{X}_{t_{k_n}}]}{\varphi_a(n)} = \lim_{n\to\infty}\frac{\mathbb{E}[\hat{X}_n]}{\varphi_a(n)} = F_X(a,r), \qquad \lim_{n\to\infty}\frac{\mathbb{E}[\hat{M}_n]}{\sqrt{n}} = \sqrt{\frac{2}{\pi}},$$

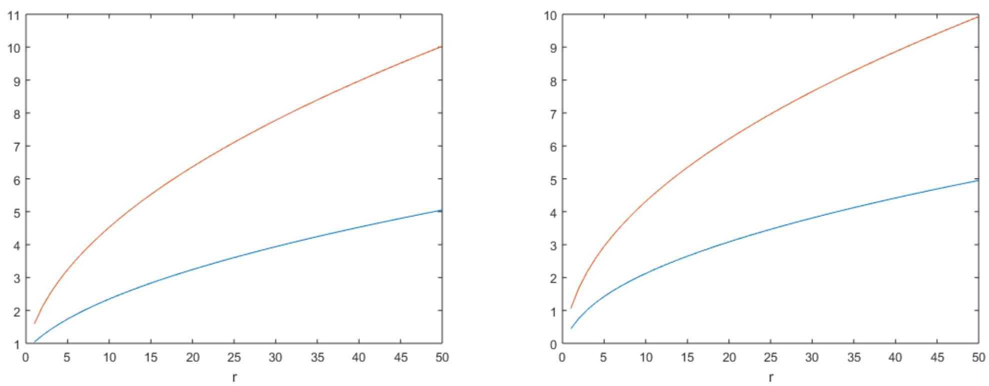
where $\varphi_a(n)$ is given in (51), and $F_X(a,r) = \lambda_1(a,r)-\lambda_2(a,r)+\lambda_3(a,r)$, with $\lambda_1(1,r),\ldots,\lambda_3(a,r)$ as in Lemma 3.8.

3.5 Conclusions

We now give the proof of Theorem 1.2 by using the previous results. It follows from Lemma 3.1 that

$$\max\{|\mathbb{E}[X_n]-\mathbb{E}[\hat{X}_n]|, |\mathbb{E}[M_n]-\mathbb{E}[\hat{M}_n]|\} \leq \max\{\mathbb{E}[|X_n-\hat{X}_n|], \mathbb{E}[|M_n-\hat{M}_n|]\}$$
$$\leq \mathbb{E}[|\mathcal{J}\cap[1,n]|] =: R_a(n) = \sum_{j=1}^{n} r_j = \begin{cases} \mathcal{O}(n^{1-a}) & \text{if } 0<a<1, \\ \mathcal{O}(\log n) & \text{if } a=1, \\ \mathcal{O}(1) & \text{if } a>1. \end{cases}$$

Hence, $R_a(n) = o(\mathbb{E}[\hat{M}_n])$ for any $a>0$, and $R_a(n) = o(\mathbb{E}[\hat{X}_n])$ for $0<a\leq\frac{3}{2}$. Thus, the limit theorems for $\mathbb{E}[M_n]$ with $a>0$ and for $\mathbb{E}[X_n]$ with $0<a\leq\frac{3}{2}$ follow from Corollaries 3.5, 3.7 and 3.9. When $a>\frac{3}{2}$, we have $\mathbb{E}[X_n]=\mathcal{O}(1)$, since $R_a(n)=\mathcal{O}(1)$ and $\mathbb{E}[\hat{X}_n]=\mathcal{O}(1)$. □



(a) The graphs of $F_M(1,r)$ (red line) and $f_M(1,r)$ (blue line).

(b) The graphs of $F_X(1,r)$ (red line) and $f_X(1,r)$ (blue line).

Figure 2: Comparison of our scaling factors with results in [13].

Remark 3.10 (Comparison with mean field results). In [13], the authors also studied the case $a = 1$. By using non rigorous arguments, they gave the asymptotic behavior of $\mathbb{E}[X_n]$ and $\mathbb{E}[M_n]$ as

$$f_M(r) = \lim_{n\to\infty} \frac{\mathbb{E}[M_n]}{\sqrt{n}} = \frac{1}{\sqrt{2r}}[(r+1/2)\mathrm{erf}(\sqrt{r}) + \sqrt{r/\pi}\exp(-r)],$$
$$f_X(r) = \lim_{n\to\infty} \frac{\mathbb{E}[X_n]}{\sqrt{n}} = \frac{1}{\sqrt{2r}}[(r-1/2)\mathrm{erf}(\sqrt{r}) + \sqrt{r/\pi}\exp(-r)],$$

where $\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z \exp(-u^2)du$. These results are different from ours, see in particular Figure 2. A gap of the mean field arguments in [13] is that the authors used the analysis for the case $a = 0$, i.e. $r_n = r \in (0,1)$ for all $n \geq 1$ (the resetting rate is fixed) to apply to the case $a = 1$ (the rate decays as n^{-1}). In fact, this approach gives the correct order of magnitude of the expectation values of X_n and M_n, but it offers incorrect scaling factors.

Remark 3.11 (Smooth interpolation of scaling factors). We consider the continuities of the two scaling functions $F_X(a,r)$ and $F_M(a,r)$. We observe that the two functions are not continuous in the usual sense at the critical points $a = 0$, $a = 1$ and $a = 3/2$ (the last point only for F_X). This fact is reasonable since the order magnitude of $\mathbb{E}[X_n]$ and $\mathbb{E}[M_n]$ changes when a crosses these points. However, considering these functions in the broad sense, when $a \nearrow 1$ for example, we should compare $F(a \nearrow 1, r)$ with $F(1,r)$ at $r \nearrow \infty$, here and below F stands for both F_X and F_M. Indeed, denoting $r_{n,a} = \min\{\frac{r}{n^a}, \frac{1}{2}\}$, we have $r := nr_{n,a} \to \infty$ as $n \to \infty$, for any $a < 1$. Similarly, as $a \searrow 1$, we compare $F(a \searrow 1, r)$ with $F(1,r)$ at $r \searrow 0$, and consider the same analysis for other critical points $a = 0$ and $a = 3/2$. First, for the case $a \searrow 0$, $F(a \searrow 0, r) = \sqrt{r/2}$. Moreover, as $r \to 0$, $F(0,r) = v(r) = (\frac{1}{\sqrt{2}} + o(1))\sqrt{r}$. As a consequence,

$$\lim_{r\to 0} \frac{F(a \searrow 0, r)}{F(0,r)} = 1. \tag{61}$$

For $a \nearrow 1$, we have $F(a \nearrow 1, r) = \sqrt{2r}$. In addition, as $r \to \infty$, $F(1,r) = (2\sqrt{2/\pi} +$

$o(1))r^2B(3/2,r) = (\sqrt{2}+o(1))\sqrt{r}$, since $B(3/2,r) = (\frac{\sqrt{\pi}}{2}+o(1))r^{-3/2}$. Hence,

$$\lim_{r\to\infty} \frac{F(a\nearrow 1,r)}{F(1,r)} = 1. \tag{62}$$

When $a \searrow 1$, we have $\prod_{i=1}^{\infty}(1-r_i) \to 0$. Thus $\lambda(a\searrow 1,r) = \sqrt{2/\pi}$, and $\lambda_1(a\searrow 1,r) = \lambda_2(a\searrow 1,r) = 0$, and $\lambda_3(a\searrow 1,r) = \sqrt{8/\pi}r$. So $F_X(a\searrow 1,r) = \sqrt{8/\pi}r$. On the other hand, $F_X(1,r) = \sqrt{8/\pi}r^2B(3/2,r) = (\sqrt{8/\pi}+o(1))r$ as $r\to 0$, since $B(3/2,r) = (1+o(1))r^{-1}$. Therefore,

$$\lim_{r\to 0} \frac{F_X(a\searrow 1,r)}{F_X(1,r)} = 1. \tag{63}$$

Moreover, $F_M(a\searrow 1,r) = \sqrt{2/\pi}$ and $F_M(1,r) = \sqrt{\frac{2}{\pi}}(2r^2+r)B(3/2,r) = \sqrt{2/\pi} + o(1)$, since $B(3/2,r) = (1+o(1))r^{-1}$ as $r\to 0$. Thus

$$\lim_{r\to 0} \frac{F_M(a\searrow 1,r)}{F_M(1,r)} = 1. \tag{64}$$

These equations (61) – (64) imply that the functions F_X and F_M are continuous in the broad sense at $a=0$ and $a=1$. We say that the two functions have the smooth interpolation property at these points. When $a \nearrow \frac{3}{2}$, the function $F_X(\nearrow \frac{3}{2}, r)$ blows up and so the smooth interpolation is not true.

4 Limit theorems for random walk with decreasing resetting probability to the maximum in the supercritical phase

In this section, we study the variance and limit theorems of X_n and M_n in the supercritical phase $a>1$. Then we present the numerical results for the asymptotic behavior of the variances of X_n and M_n in the other case $0<a\le 1$.

4.1 Proof of Theorem 1.4

Recall the coupling of two random walks $(X_n)_{n\ge 0}$ and $(\hat{X}_n)_{n\ge 0}$ via the simple random walk $(Z_n)_{n\ge 0}$ and the random set $\mathcal{J}$ as in Lemma 3.1. We write

$$\mathcal{J} = \{t_1, t_2, \ldots\}, \qquad k_n = |\mathcal{J}\cap[1,n]|, \tag{65}$$

with the convention that $k_n = 0$ when $\mathcal{J}\cap[1,n] = \varnothing$, and $t_0 = 0$. Then for any $l\in\mathbb{N}$,

$$\begin{aligned}\mathbb{E}[k_n^l] &= \mathbb{E}\Big[\Big(\sum_{j=1}^{n}\mathbb{I}(j\in\mathcal{J})\Big)^l\Big] = \sum_{j_1,\ldots,j_l=1}^{n}\mathbb{P}\Big(j_1,\ldots,j_l\in\mathcal{J}\Big)\\ &\le 2^l\Big[\sum_{i=1}^{l}\Big(\sum_{j=1}^{n} r_j\Big)^i\Big] = \mathcal{O}(1),\end{aligned} \tag{66}$$

since $\sum_{j\ge 1} r_j$ converges. In addition, by Lemma 3.1

$$\max\{|X_n-\hat{X}_n|, |M_n-\hat{M}_n|\} \le k_n. \tag{67}$$

Hence, $|\mathbb{E}[X_n] - \mathbb{E}[\hat{X}_n] = \mathcal{O}(1)$, and thus

$$|\mathbb{E}[X_n]^2 - \mathbb{E}[\hat{X}_n]^2| = \mathcal{O}(\mathbb{E}[|\hat{X}_n|]). \tag{68}$$

Moreover, by the Cauchy-Schwarz inequality and (66)

$$\begin{aligned} |\mathbb{E}[X_n^2] - \mathbb{E}[\hat{X}_n^2]| \leq \mathbb{E}[k_n(|X_n| + |\hat{X}_n|)] &\leq \sqrt{2}\mathbb{E}[k_n^2]^{1/2}(\mathbb{E}[X_n^2] + \mathbb{E}[\hat{X}_n^2])^{1/2} \\ &= \mathcal{O}((\mathbb{E}[X_n^2] + \mathbb{E}[\hat{X}_n^2])^{1/2}). \end{aligned} \tag{69}$$

We observe that if $|a-b| \leq \alpha\sqrt{a+b}$ for some $\alpha > 0$ then

$$|a-b| \leq 8(\alpha^2+1)\sqrt{b+1}. \tag{70}$$

Indeed, assume that $|a-b| > 8(\alpha^2+1)\sqrt{b+1}$. Then $\alpha\sqrt{a+b} > 8(\alpha^2+1)\sqrt{b+1}$. Thus $a \geq \max\{2b, 8(\alpha^2+1)\}$. Therefore,

$$\frac{a}{2} \leq a - b \leq \alpha\sqrt{a+b} \leq \alpha\sqrt{2a},$$

and thus $a \leq 8\alpha^2$, which is a contradiction. Hence, (70) is proved.

It follows from (69) and (70) that

$$|\mathbb{E}[X_n^2] - \mathbb{E}[\hat{X}_n^2]| = \mathcal{O}(\mathbb{E}[\hat{X}_n^2])^{1/2}).$$

Combining this with (68), we yield that

$$|\operatorname{Var}[X_n] - \operatorname{Var}[\hat{X}_n]| \leq |\mathbb{E}[X_n]^2 - \mathbb{E}[\hat{X}_n]^2| + |\mathbb{E}[X_n^2] - \mathbb{E}[\hat{X}_n^2]| = \mathcal{O}(\mathbb{E}[\hat{X}_n^2])^{1/2}), \tag{71}$$

since $\mathbb{E}[|\hat{X}_n|] \leq (\mathbb{E}[\hat{X}_n^2])^{1/2}$. Similarly,

$$|\operatorname{Var}[M_n] - \operatorname{Var}[\hat{M}_n]| = \mathcal{O}(\mathbb{E}[\hat{M}_n^2])^{1/2}). \tag{72}$$

The two estimates (71) and (72) suggest the variances of X_n and M_n can be approximated by the counterparts of $\hat{X}_n$ and $\hat{M}_n$. We now focus on computing $\operatorname{Var}[\hat{X}_n]$ and $\operatorname{Var}[\hat{M}_n]$. By the construction of $(\hat{X}_n)_{n\geq 0}$,

$$\hat{X}_n = \hat{X}_{t_{k_n}} + Z_n - Z_{t_{k_n}}, \qquad \hat{M}_n = \hat{X}_{t_{k_n}} + m^{(i)}_{n-t_{k_n}}, \quad \hat{X}_{t_{k_n}} = \sum_{i=0}^{k_n-1} m^{(i)}_{t_{i+1}-t_i-1}, \tag{73}$$

where $t_0 = 0$ and $t_1, \ldots, t_{k_n}$ are elements of $\mathcal{J}$ as in (65), and $(Z_n)_{n\geq 0}$ is the simple random walk and for any $i, k \geq 0$

$$m^{(i)}_k = \max_{0\leq j\leq k}\{Z_{j+t_i} - Z_{t_i}\}.$$

Notice that given t_{k_n}, the variables $Z_n - Z_{n-t_{k_n}}$ and $m^{(k_n)}_{n-t_{k_n}}$ are independent of $\hat{X}_{t_{k_n}}$. Therefore, since $\mathbb{E}[Z_n - Z_{t_{k_n}} \mid t_{k_n}] = 0$,

$$\mathbb{E}[\hat{X}_n^2] = \mathbb{E}[(Z_n - Z_{t_{k_n}})^2] + \mathbb{E}[(\hat{X}_{t_{k_n}})^2] = \mathbb{E}[(Z_{n-t_{k_n}})^2] + \mathbb{E}[(\hat{X}_{t_{k_n}})^2]. \tag{74}$$

Moreover, sine $\mathbb{E}[m^{(k_n)}_{n-t_{k_n}} \mid t_{k_n}] = g(n+1-t_{k_n})$

$$\mathbb{E}[\hat{M}_n^2] = \mathbb{E}[(\hat{X}_{t_{k_n}})^2] + 2\mathbb{E}[\hat{X}_{t_{k_n}} g(n+1-t_{k_n})] + \mathbb{E}[(m^{(k_n)}_{n-t_{k_n}})^2], \tag{75}$$

here we recall that $g(k) = \mathbb{E}[m_{k-1}]$. We observe that $\hat{X}_{t_{k_n}} \geq 0$, since it is the sum of maximum values of the simple random walk in some intervals. Therefore,

$$0 \leq \mathbb{E}[\hat{X}_{t_{k_n}} g(n+1-t_{k_n})] \leq \mathbb{E}[\hat{X}_{t_{k_n}}] g(n+1) = \mathcal{O}(n^{1/2}) \mathbb{E}[\hat{X}_{t_{k_n}}] = o(n), \tag{76}$$

since $\mathbb{E}[\hat{X}_{t_{k_n}}] = \mathcal{O}(\varphi_a(n)) = o(n^{1/2})$, by Corollary 3.9.

Next, we deal with $\mathbb{E}[(\hat{X}_{t_{k_n}})^2]$. Let $\varepsilon = \min\{\frac{a-1}{2}, \frac{1}{4}\}$ and $\ell = \lceil \frac{2}{\varepsilon} \rceil$. Then by Markov's inequality and (66),

$$\mathbb{P}[k_n \geq n^{\varepsilon}] \leq \frac{\mathbb{E}[k_n^{\ell}]}{n^{\varepsilon \ell}} = \mathcal{O}(n^{-\ell\varepsilon}) = \mathcal{O}(n^{-2}).$$

Thus $\mathbb{E}[(\hat{X}_{t_{k_n}})^2 \mathbb{I}(k_n \geq n^{\varepsilon})] \leq n^2 \mathbb{P}(k_n \geq n^{\varepsilon}) = \mathcal{O}(1)$, and hence

$$\mathbb{E}[(\hat{X}_{t_{k_n}})^2] = \mathcal{O}(1) + \mathbb{E}[(\hat{X}_{t_{k_n}})^2 \mathbb{I}(k_n \leq n^{\varepsilon})]. \tag{77}$$

By using (73) and the inequality that $(x_1 + \ldots + x_k)^2 \leq k(x_1^2 + \ldots + x_k^2)$,

$$\mathbb{E}[(\hat{X}_{t_{k_n}})^2 \mathbb{I}(k_n \leq n^{\varepsilon})] \leq \mathbb{E}\Bigg[k_n \mathbb{I}(k_n \leq n^{\varepsilon}) \sum_{i=0}^{k_n-1} \left(m^{(i)}_{t_{i+1}-t_i-1}\right)^2\Bigg] \leq n^{\varepsilon} \mathbb{E}\Bigg[\sum_{i=0}^{k_n-1} \left(m^{(i)}_{t_{i+1}-t_i-1}\right)^2\Bigg]. \tag{78}$$

Define

$$q(k) = \mathbb{E}[m_{k-1}^2] = \mathbb{E}[(\max_{0 \leq i \leq k-1} Z_i)^2].$$

By Lemma 3.2,

$$\lim_{k \to \infty} \frac{q(k)}{k} = 1. \tag{79}$$

In particular, $q(k) \leq Ck$ for all $k \geq 1$ with C an universal constant. Hence,

$$\mathbb{E}\Bigg[\sum_{i=0}^{k_n-1} \left(m^{(i)}_{t_{i+1}-t_i-1}\right)^2\Bigg] = \mathbb{E}\Bigg[\sum_{i=0}^{k_n-1} q(t_{i+1}-t_i-1)\Bigg] \leq C\mathbb{E}[t_{k_n}].$$

Moreover,

$$\mathbb{E}[t_{k_n}] = \sum_{k=1}^{n} k r_k \prod_{i \geq k+1} (1-r_i) \leq \sum_{k=1}^{n} r k^{1-a} = \mathcal{O}(n^{2-a}).$$

Combining the last two equations with (77) and (78) gives that

$$\mathbb{E}[(\hat{X}_{t_{k_n}})^2] = \mathcal{O}(1) + \mathcal{O}(n^{\varepsilon+2-a}) = o(n), \tag{80}$$

since $\varepsilon \leq (a-1)/2$ and $a > 1$.

We are now in the position to compute the variances of X_n and M_n. Observe that

$$q(n+1) \geq \mathbb{E}[(m^{(k_n)}_{n-t_{k_n}})^2] = \mathbb{E}[q(n-t_{k_n})] \geq q(n-[n^{1/2}]) \mathbb{P}(t_{k_n} \leq [n^{1/2}]).$$

Since $q(n+1) = (1+o(1))n$ and $\mathbb{P}(t_{k_n} \leq [n^{1/2}]) \to 1$ as $n \to \infty$, we deduce that

$$\mathbb{E}[(m^{(k_n)}_{n-t_{k_n}})^2] = n + o(n). \tag{81}$$

It follows from (75), (76), (80) and (81) that

$$\mathbb{E}[\hat{M}_n^2] = n + o(n). \tag{82}$$

Therefore, using $\mathbb{E}[\hat{M}_n] = (\sqrt{2/\pi} + o(1))\sqrt{n}$ (by Corollary 3.9),

$$\mathrm{Var}[\hat{M}_n] = \mathbb{E}[\hat{M}_n^2] - (\mathbb{E}[\hat{M}_n])^2 = \Big(1 - \frac{2}{\pi}\Big)n + o(n). \tag{83}$$

On the other hand, since $\mathbb{E}[Z_k^2] = (1 + o(1))k$ and $\mathbb{P}(t_{k_n} \geq [n^{1/2}]) = o(1)$,

$$\begin{aligned}
\mathbb{E}[Z_{n-t_{k_n}}^2] &= \sum_{k=0}^{n} \mathbb{E}[Z_{n-k}^2]\mathbb{P}(t_{k_n} = k) \\
&= \sum_{k=0}^{[n^{1/2}]} \mathbb{E}[Z_{n-k}^2]\mathbb{P}(t_{k_n} = k) + \mathcal{O}(n)\mathbb{P}(t_{k_n} \geq [n^{1/2}]) = n + o(n).
\end{aligned}$$

Combining the above equation with (74), (80) yields

$$\mathbb{E}[\hat{X}_n^2] = n + o(n), \tag{84}$$

which together with $\mathbb{E}[\hat{X}_n] = \mathcal{O}(\varphi_a(n)) = o(\sqrt{n})$ (by Corollary 3.9) implies that

$$\mathrm{Var}[\hat{X}_n] = n + o(n). \tag{85}$$

Finally, using (71), (84) and (85) we arrive at

$$\mathrm{Var}[X_n] = n + o(n).$$

Similarly, it follows from (72), (82) and (83) that

$$\mathrm{Var}[M_n] = \Big(1 - \frac{2}{\pi}\Big)n + o(n).$$

We now prove limit theorems for X_n and M_n. By (73),

$$\frac{X_n}{\sqrt{n}} = \frac{X_n - \hat{X}_n}{\sqrt{n}} + \frac{\hat{X}_{t_{k_n}}}{\sqrt{n}} - \frac{Z_{t_{k_n}}}{\sqrt{n}} + \frac{Z_n}{\sqrt{n}} =: U_n + \frac{Z_n}{\sqrt{n}}. \tag{86}$$

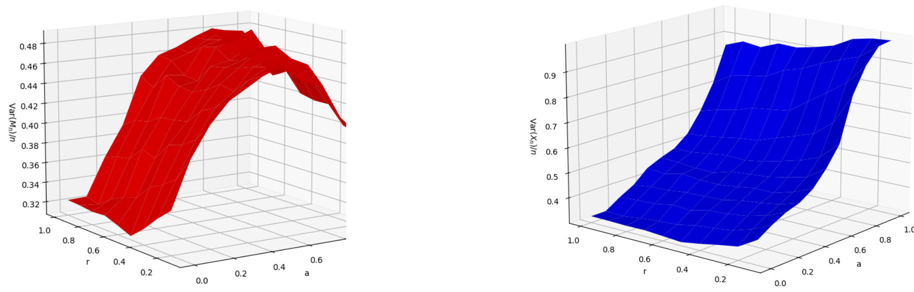
We observe that $\mathbb{E}[|X_n - \hat{X}_n|] \leq \mathbb{E}[k_n] = \mathcal{O}(1)$, and $\mathbb{E}[|\hat{X}_{t_{k_n}}|] = \mathbb{E}[\hat{X}_{t_{k_n}}] = \mathcal{O}(\varphi_a(n)) = o(\sqrt{n})$, and since $\mathbb{E}[|Z_k|] = \mathcal{O}(k^{1/2})$,

$$\mathbb{E}[|Z_{t_{k_n}}|] = \sum_{k=0}^{n} \mathbb{E}[|Z_k|]\mathbb{P}(t_{k_n} = k) = \mathcal{O}(1)\sum_{k=1}^{n} k^{1/2}k^{-a} = \mathcal{O}(\varphi_a(n)) = o(\sqrt{n}).$$

Therefore, $\mathbb{E}[|U_n|] = o(1)$ and thus (86) implies that

$$\frac{X_n}{\sqrt{n}} \xrightarrow{(d)} \mathcal{N}(0,1), \tag{87}$$

since $(n^{-1/2}Z_n)_{n\geq 1}$ converges in distribution to $\mathcal{N}(0,1)$.



(a) The graph of $D_M(a, r)$ (b) The graph of $D_X(a, r)$

Figure 3: Scaling functions of the variances

For the limit theorem of M_n, using (73),

$$\frac{M_n}{\sqrt{n}} = \frac{M_n - \hat{M}_n}{\sqrt{n}} + \frac{\hat{X}_{t_{k_n}}}{\sqrt{n}} + \frac{m^{(k_n)}_{n-t_{k_n}}}{\sqrt{n}} =: V_n + \frac{m^{(k_n)}_{n-t_{k_n}}}{\sqrt{n}}. \tag{88}$$

By similar arguments as above, $\mathbb{E}[|V_n|] = o(1)$. Moreover,

$$m_n \geq m_{n-t_{k_n}} \geq m_{n-[n^{1/2}]}\mathbb{I}(k_n \leq [n^{1/2}]).$$

In addition, $\mathbb{P}(k_n \leq [n^{1/2}]) \to 1$ and $(n^{-1/2}m_n)_{n\geq 1}$ converges weakly to the random variable $\max_{0\leq t\leq 1} B_t$, as $n \to \infty$, and so does the sequence $(n^{-1/2}m^{(k_n)}_{n-t_{k_n}})_{n\geq 1}$. Hence,

$$\frac{M_n}{\sqrt{n}} \xrightarrow{(d)} \max_{0\leq t\leq 1} B_t. \tag{89}$$

We have finished the proof of Theorem 1.4. □

4.2 Simulation for the variances of X_n and M_n

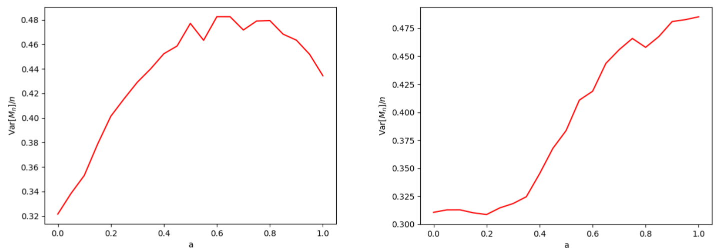
As we have proved, in both the cases $a = 0$ or $a > 1$ the variances of X_n and M_n grow linearly in n regardless the value of a. So, we predict the following.

Open question: Let $0 < a \leq 1$ and $r > 0$. Prove that the following limits exist:

$$D_X(a, r) := \lim_{n\to\infty} \frac{\mathrm{Var}[X_n]}{n}, \qquad D_M(a, r) := \lim_{n\to\infty} \frac{\mathrm{Var}[M_n]}{n}.$$

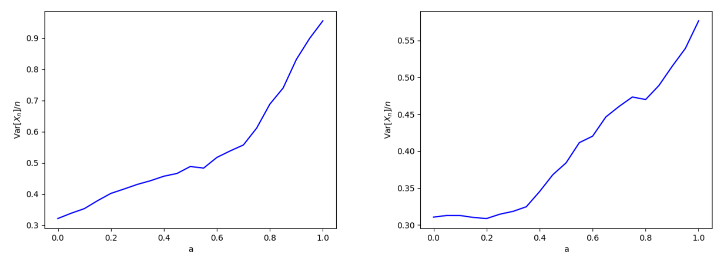
When a gets to 0 our modifying X_n by $\hat{X}_n$ might lead a large error in the variance approximation. In particular, when $a = 0$ the difference between the two processes grows linearly, and it would make a non-negligible error in the approximation. Furthermore, the computing for the variances of $\hat{X}_n$ and $\hat{M}_n$ is also highly nontrivial, so we leave the problem to future researches.

We have done some simulation supporting to this open question.



(a) The graph of $D_M(a, \frac{1}{2})$ (b) The graph of $D_M(a, 10)$

Figure 4: Scaling function of the variance of M_n



(a) The graph of $D_X(a, \frac{1}{2})$ (b) The graph of $D_X(a, 10)$

Figure 5: Scaling function of the variance of X_n

In Figure 3, we consider 10×10 values of (a, r) in $[0, 1] \times [0, 1]$. Corresponding to each pair (a, r), we make 15000 independent samples of (X_n, M_n) with $n = 10^4$ and then compute the empirical variances of X_n and M_n.

In Figures 4 and 5, for $r = \frac{1}{2}$ and $r = 10$, we consider 20 values of a ranging in $[0, 1]$ and make 20000 samples of (X_n, M_n) with $n = 10^4$ for each pair (a, r).

The numerical results show that both the variances of X_n and M_n is of order n. It seems that for all r, the scaling function $D_X(\cdot, r)$ increases in a and tends to 1 as a reaches to 1. Meanwhile, the function $D_M(\cdot, r)$ might not be increasing in the neighborhood of $a = 1$ when r is small, but it does increase for large r.

Acknowledgement

The work of Van Hao Can and Van Quyet Nguyen is supported by International Center for Research and Postgraduate Training in Mathematics under grant number ICRTM03-2020.02. The work of Thai Son Doan is funded by Vietnam Ministry of Education and Training under grant number B2021- TDV-01.

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