An analysis on solutions to fractional neutral differential equations with a delay

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Abstract

This paper discusses some qualitative properties of solutions to fractional delay neutral differential equations. By combining a new weighted norm, the Banach fixed point theorem and an elegant technique for extending solutions, results on existence, uniqueness, and growth rate of global solutions under a might Lipschitz continuous condition of the vector field are first established. Then, the exact solution of linear delay fractional neutral differential equations are derived and the stability of two equations of this kind are studied by using use Rouché's theorem to describe the position of poles of the characteristic polynomials and the Final value theorem to detect the asymptotic behavior of solutions. Numerical simulations are finally presented to illustrate the theoretical findings.

Key words: Caputo fractional derivative, Fractional delay neutral differential equations, Existence and uniqueness of solutions, Exponential boundedness, Stability

1 Introduction

The interconnection between two (or more) physical systems is always accompanied by transfer phenomena (of material, energy, or information), such as transport and propagation, which can be represented mathematically by delay elements. This is a cause why delay differential equations are usually used in modeling problems coming from physics. Delay differential equations also play an important role in describing various phenomena in biosciences, chemistry, or economics. For more applications of these equations, see [3, 12] and references therein.

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Recently, delay fractional-order systems have received considerable research attention because they provide models of practical systems in which the fractional rate of change depends on the influence of their present and hereditary effects. In [13], by the Final value theorem for Laplace transforms, the well-known method of steps, and the Argument principle, the authors have presented several analytical and numerical approaches for the stability analysis of linear fractional-order delay differential equations. In [4], the authors have obtained a general results on the existence, uniqueness and growth rate of solutions to fractional-order systems with delays based on the Banach fixed point theorem and a weighted norm. In [21], the authors have proposed a necessary and sufficient condition for the stability of the system via eigenvalues of the system matrix and their location in a specific area of the complex plane. By the linearization method and generalized Mittag-Leffler functions, in [19, 18] the authors have proved the stability of nonlinear fractional-order delay systems. Furthermore, using Lyapunov applicant functional, in [8] the authors also obtained a sufficient condition for stability. In [9], the authors have discussed the initialization of fractional delay differential equations and they have investigated the effects of the initial condition not only on the solution but also on the fractional operator as well and they discussed the difference between solutions obtained by incorporating or not the initial function in the memory of the fractional derivative.

Neutral delay differential equation is a kind of delay differential equation containing the derivative of the unknown function both with and without delays. Hence, the theory of neutral delay differential equations is even more complicated than the theory of their non-neutral counterpart. To the best of our knowledge, up to now, there are only few works on fractional neutral delay differential equation (FNDDEs) published in the literature. Below we review briefly some contributions to this topic.

In [1], based on Krasnoselskii's fixed point theorem, the author proved the existence of at least one solution to a class of fractional neutral functional differential equations with bounded delay. The existence of mild solutions for a class of abstract fractional neutral integro-differential equations with state-dependent delay is studied in [7] by the Leray–Schauder alternative fixed point theorem. Recently, in [22], the authors have derived a new fractional Halanay-like inequality, which is used to characterize the long-term behavior of solutions to fractional neutral functional differential equations of Hale type. Conditions for contractivity and dissipativity of these equations have been established under almost the same assumptions for the classical integer-order case. They have also proposed a numerical scheme based on the L_1-method coupled with linear interpolation to illustrate the theoretical results. In [2], the authors have studied the robust stability of uncertain fractional order nonlinear systems having neutral-type delay and input saturation; by combining Lyapunov–Krasovskii functional, sufficient criteria on asymptotic robust stability of such systems with the help of linear matrix inequalities are specified to compute the gain of state-feedback controller. An optimization is also derived using the cone complementarity linearization method for finding the controller gains subject to maximizing the domain of attraction.

This paper is devoted to discussing some qualitative properties of solutions to FNDDEs. The paper is organized as follows. In Section 2, we recall briefly some basic notations concerning fractional derivatives and delay fractional differential equations. In Section 3, we give a result on the existence and uniqueness of global solutions to FNDDEs and in Section 4 we prove their exponential boundedness. In Section 5 we derive an explicit representation, based on generalized three-parameter Mittag-Leffler functions, of the solution of some linear FNDDEs. In Section 6 we discuss in details the stability of two classes of linear FNDDEs and some numerical simulations are presented in Section 7 to illustrate the theoretical results obtained in the paper.

2 Preliminaries

In this section we recall some definitions and a result on the integral representation of solutions of fractional-order equations that will be used in the sequel. For $0 < \alpha < 1$, $[a, b] \subset \mathbb{R}$ and a measurable function $x : [a, b] \to \mathbb{R}$ such that $\int_a^b |x(\tau)|d\tau < \infty$, the *Riemann-Liouville (RL) integral* of order α is defined by

$$I_{a+}^{\alpha} x(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} x(s) \mathrm{d}s, \quad t \in (a, b),$$

where $\Gamma(\cdot)$ is the Gamma function. The *Riemann–Liouville fractional derivative* ${}^{\mathrm{RL}}D_{a+}^{\alpha} x$ of a integrable function $x : [a, b] \to \mathbb{R}$ is defined by

$${}^{\mathrm{RL}}D_{a+}^{\alpha} x(t) = D I_{a+}^{1-\alpha} x(t) \text{ for almost } t \in (a, b],$$

with $D = \mathrm{d}/\mathrm{d}t$ the usual integer-order derivative. The *Caputo fractional derivative* ${}^{\mathrm{C}}D_{a+}^{\alpha} x$ of a continuous function $x : [a, b] \to \mathbb{R}$ is defined

$${}^{\mathrm{C}}D_{a+}^{\alpha} x)(t) := {}^{\mathrm{RL}}D_{a+}^{\alpha}(x(t) - x(a)) \text{ for almost } t \in (a, b].$$

For more details on fractional calculus, we would like to introduce the reader to the monographs [5, 14, 16] and to the interesting work by G. Vainikko [20]. Let τ and N be arbitrary real constants such that $\tau > 0, N \neq 0$, and $\phi \in C^1\big([-\tau, 0]; \mathbb{R}\big)$ be a given function. In this paper we consider the following FNDDE

$${}^{\mathrm{C}}D_{0+}^{\alpha}\big[x(t) + Nx(t-\tau)\big] = f(t, x(t), x(t-\tau)), \quad t \geq 0, \tag{1}$$

subject to the initial condition

$$x(t) = \phi(t), \quad \forall t \in [-\tau, 0], \tag{2}$$

where $x : [0, \infty) \to \mathbb{R}$ is a unknown function and $f : [0, \infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous. To prove the existence of solutions to the initial condition problem (1)–(2), we need to convert it into an equivalent delay integral equation. This is stated in the following lemma.

Lemma 2.1. A function $x \in C\big([-\tau, \infty); \mathbb{R}\big)$ is a solution of the problem (1)-(2) on $[-\tau, \infty)$ if and only if it is a solution of the delay integral equation

$$\begin{aligned} x(t) = \phi(0) &+ N\phi(-\tau) - Nx(t-\tau) \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), x(s-\tau)) \mathrm{d}s, \ \ \forall t \in [0, \infty), \end{aligned} \tag{3}$$

and satisfies

$$x(t) = \phi(t), \ \ \forall t \in [-\tau, 0].$$

Proof. The proof of this lemma is similar to the one of [5, Lemma 2] and thus we omit it. □

3 Existence and uniqueness of global solutions of FNDDEs

Let $T > 0$ be arbitrary. Consider the following initial value problem on a finite interval $[-\tau, T]$:

$$
{}^{\mathrm{C}}D^{\alpha}_{0+}\big[x(t) + Nx(t-\tau)\big] = f(t, x(t), x(t-\tau)),\ \ t \in (0, T], \tag{4}
$$

$$
x(t) = \phi(t), \quad t \in [-\tau, 0]. \tag{5}
$$

Here $f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the following assumptions:

(A1) f is continuous on $[0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$;

(A2) there exists a continuous function $L : [0, T] \times \mathbb{R} \to \mathbb{R}_{\geq 0}$ such that for any $t \in [0, T]$, $x, \hat{x}, y \in R$, it is

$$
|f(t, x, y) - f(t, \hat{x}, y)| \leq L(t, y)|x - \hat{x}|.
$$

By proposing a new weighted norm and modifying the approach in the proof of [4, Theorem 3.1], we are able to obtain the following result on the existence and uniqueness of a global solution to the system (4)–(5).

Theorem 3.1. Assume that conditions ($A1$) and ($A2$) hold. Then, the fractional delay neutral differential equation (4) with the initial condition (5) has a unique solution on the interval $[-\tau, T]$.

Proof. By the same arguments as in the proof of [5, lemma 6.2], the system (4)–(5) is equivalent to the integral equation

$$
\begin{aligned}
x(t) = \phi(0) &+ N\phi(-\tau) - Nx(t-\tau) \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), x(s-\tau))\mathrm{d}s,\ \ \forall t \in [0, T],
\end{aligned}
$$

with the initial condition

$$
x(t) = \phi(t),\ \ \forall t \in [-\tau, 0]. \tag{6}
$$

First, we consider the case $0 < T \leq \tau$. In this case, the equation (6) becomes

$$
\begin{aligned}
x(t) = \phi(0) &+ N\phi(-\tau) - N\phi(t-\tau) \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), \phi(s-\tau))\mathrm{d}s,\ t \in [0, T].
\end{aligned} \tag{7}
$$

Let $\beta := \max_{t\in[0,T]} L(t, \phi(t-\tau))$ and λ be a large positive constant which will be chosen later. On the space $C([0, \tau]; \mathbb{R})$, we define the metric

$$d_\lambda(\xi,\hat{\xi}) := \underset{t\in[0,r]}{sup} \frac{|\xi(t)-\hat{\xi}(t)|}{\mathrm{e}^{\lambda t}}, \quad \forall \xi,\hat{\xi} \in C([0,\tau];\mathbb{R}).$$

It is obvious that $C([0,r];\mathbb{R})$ equipped the metric d_λ is complete. We now consider the operator $\mathcal{T}_\phi : C([0,\tau];\mathbb{R}) \to C([0,\tau];\mathbb{R})$ defined as

$$\begin{aligned}(\mathcal{T}_\phi\,\xi)(t) :=& \phi(0) + N\phi(-\tau) - N\phi(t-\tau)\\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,\xi(s),\phi(s-\tau))\mathrm{d}s,\ \forall t\in[0,\tau].\end{aligned}$$

For any $\xi,\hat{\xi} \in C([0,r];\mathbb{R})$ and any $t\in[0,T]$, we have

$$\begin{aligned}|(\mathcal{T}_\phi\,\xi)(t) - (\mathcal{T}_\phi\,\hat{\xi})(t)| &\le \frac{\max_{s\in[0,t]} L(s,\phi(s-\tau))}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|\xi(s)-\hat{\xi}(s)|\mathrm{d}s\\ &\le \frac{\mathrm{e}^{\lambda t}\beta}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mathrm{e}^{-\lambda(t-s)}\frac{|\xi(s)-\hat{\xi}(s)|}{\mathrm{e}^{\lambda s}}\mathrm{d}s\\ &\le \frac{\mathrm{e}^{\lambda t}\beta}{\lambda^\alpha} d_\lambda(\xi,\hat{\xi}).\end{aligned}$$

This implies that

$$\frac{|(\mathcal{T}_\phi\,\xi)(t) - (\mathcal{T}_\phi\,\hat{\xi})(t)|}{\mathrm{e}^{\lambda t}} \le \frac{\beta}{\lambda^\alpha} d_\lambda(\xi,\hat{\xi}),\ \ \forall t\in[0,T].$$

Therefore,

$$d_\lambda(\mathcal{T}_\phi\,\xi, \mathcal{T}_\phi\,\hat{\xi}) \le \frac{\beta}{\lambda^\alpha} d_\lambda(\xi,\hat{\xi}),\ \ \forall \xi,\hat{\xi}\in C([0,T];\mathbb{R}).$$

Take $\lambda > 0$ large enough, for example, $\lambda^\alpha > \beta$. Then, the operator $\mathcal{T}_\phi$ is contractive on $(C([0,T];\mathbb{R}), d_\lambda)$. By virtue of Banach fixed point theorem, there exist a unique fixed point $\xi^*_\tau(\cdot)$ of $\mathcal{T}_\phi$ in $C([0,T];\mathbb{R})$. Put

$$\Phi_T(t,\phi) := \begin{cases}\phi(t) & \text{if } t\in[-\tau,0],\\ \xi^*_\tau(t) & \text{if } t\in[0,T].\end{cases}$$

Then, $\Phi_T(\cdot,\phi)$ is the unique solution of the problem (4)–(5) on $[-\tau,T]$. For the case $T>\tau$, using the approach as in [4], we divide the interval $[0,T]$ into subintervals $[0,\tau]\cup\cdots\cup[(k-1)\tau,T]$, where $k\in\mathbb{N}$ satisfying $0\le T-k\tau<\tau$. The existence and uniqueness of solutions to (4)–(5) on $[-\tau,k\tau]$ will be showed by induction. Suppose that (4)–(5) has a unique solution denoted by $\Phi_{\ell\tau}(\cdot)$ on $[-\tau,\ell\tau]$ with $\ell\in\mathbb{Z}_{\ge 0}$ and $0\le\ell<k$. On the space $C([\ell\tau,(\ell+1)\tau];\mathbb{R})$, let

$$
\begin{aligned}
\mathcal{T}_{(\ell+1)\tau}\xi(t) := \phi(0) &+ N\phi(-\tau) - N\Phi_{\ell\tau}(t-\tau) \\
&+ \frac{1}{\Gamma(\alpha)}\int_0^{\ell\tau}(t-s)^{\alpha-1}f(s,\Phi_{\ell\tau}(s),\Phi_{\ell\tau}(s-\tau))\mathrm{d}s \\
&+ \frac{1}{\Gamma(\alpha)}\int_{\ell\tau}^{t}(t-s)^{\alpha-1}f(s,\xi(s),\Phi_{\ell\tau}(s-\tau))\mathrm{d}s, \quad t\in[\ell\tau,(\ell+1)\tau].
\end{aligned}
$$

Take $\beta_\ell := \max_{t\in[\ell\tau,(\ell+1)\tau]} L(t,\Phi_{\ell\tau}(t-\tau))$. Then,

$$
\begin{aligned}
|(\mathcal{T}_{(\ell+1)\tau}\xi)(t) - (\mathcal{T}_{(\ell+1)\tau}\hat{\xi})(t)| &\le \frac{\beta_\ell}{\Gamma(\alpha)}\int_{\ell\tau}^{t}(t-s)^{\alpha-1}|\xi(s)-\hat{\xi}(s)|\mathrm{d}s \\
&\le \frac{\mathrm{e}^{\lambda t}\beta_\ell}{\Gamma(\alpha)}\int_{\ell\tau}^{t}(t-s)^{\alpha-1}\mathrm{e}^{-\lambda(t-s)}\frac{|\xi(s)-\hat{\xi}(s)|}{\mathrm{e}^{\lambda s}}\mathrm{d}s \\
&\le \frac{\mathrm{e}^{\lambda t}\beta_\ell}{\lambda^\alpha}d_{\ell,\lambda}(\xi,\hat{\xi}), \quad \forall t\in[\ell\tau,(\ell+1)\tau].
\end{aligned}
$$

Here, $d_{\ell,\lambda}(\xi,\hat{\xi}) := \max_{t\in[\ell\tau,(\ell+1)\tau]}\frac{|\xi(t)-\hat{\xi}(t)|}{\mathrm{e}^{\lambda t}}$ for any $\xi,\hat{\xi}\in C([\ell\tau,(\ell+1)\tau];\mathbb{R})$. Choose $\lambda > \beta_\ell^{1/\alpha}$, the the operator $\mathcal{T}_{(\ell+1)\tau}$ is contractive on the Banach space $(C([\ell\tau,(\ell+1)\tau];\mathbb{R}),d_{\ell,\lambda})$. Hence, $\mathcal{T}_{(\ell+1)\tau}$ has a unique fixed point $\xi^*_{\ell\tau}$ in $C([\ell\tau,(\ell+1)\tau];\mathbb{R})$. Define a new function $\Phi_{(\ell+1)\tau}(\cdot)$ by

$$
\Phi_{(\ell+1)\tau}(t) := \begin{cases} \Phi_{\ell\tau}(t) \text{ if } t\in[-\tau,\ell\tau], \\ \xi^*_{\ell\tau}(t) \text{ if } t\in[\ell\tau,(\ell+1)\tau]. \end{cases}
$$

Then, $\Phi_{(\ell+1)\tau}(\cdot)$ is the unique solution of (4)–(5) on $[-\tau,(\ell+1)\tau]$. Finally, let $\Phi_{k\tau}(\cdot)$ be the unique solution to (4)–(5) on $[-\tau,k\tau]$. We construct an operator $\mathcal{T}_f$ on $C([k\tau,T];\mathbb{R})$ by

$$
\begin{aligned}
\mathcal{T}_f\xi(t) := \phi(0) &+ N\phi(-\tau) - N\Phi_{k\tau}(t-\tau) \\
&+ \frac{1}{\Gamma(\alpha)}\int_0^{k\tau}(t-s)^{\alpha-1}f(s,\Phi_{k\tau}(s),\Phi_{k\tau}(s-\tau))\mathrm{d}s \\
&+ \frac{1}{\Gamma(\alpha)}\int_{k\tau}^{t}(t-s)^{\alpha-1}f(s,\xi(s),\Phi_{k\tau}(s-\tau))\mathrm{d}s, \quad t\in[k\tau,T].
\end{aligned}
$$

Using the estimates shown as above, $\mathcal{T}_f$ has a unique fixed point ξ^*_f in $C([k\tau,T];\mathbb{R})$. Take

$$
\Phi(t,\phi) := \begin{cases} \Phi_{k\tau}(t) \text{ if } t\in[-\tau,k\tau], \\ \xi^*_f(t) \text{ if } t\in[k\tau,T]. \end{cases}
$$

This function is the unique solution of the original system on $[-\tau,T]$. The proof is complete. □

Corollary 3.2. Consider the system (1)–(2). Suppose that the function f satisfies assumptions $(A1)$ and $(A2)$ for $t\in[0,\infty)$. Then, this system has a unique global solution on $[-\tau,\infty)$.

Proof. The proof of this corollary is similar to [4, Corollary 3.2]. Hence, we omit it. □

4 Exponential boundedness of FNDDEs

Let $\phi \in C^1([-\tau,0],\mathbb{R})$ be an arbitrary function. Consider the system

$$
{}^{\mathrm{C}}D^{\alpha}_{0+}\big[x(t)+Nx(t-\tau)\big] = f(t,x(t),x(t-\tau)),\ \ t\in(0,\infty), \tag{8}
$$

$$
x(t)=\phi(t),\quad t\in[-\tau,0]. \tag{9}
$$

Suppose that f is continuous and satisfies the following condition:

(H1) there exits a positive constant L such that

$$
|f(t,x,y)-f(t,\hat{x},\hat{y})| \le L\big(|x-\hat{x}|+|y-\hat{y}|\big),\ \ \forall t\ge 0, x,y,\hat{x},\hat{y}\in\mathbb{R};
$$

(H2) there exits a positive constant λ such that

$$
\sup_{t\ge 0}\frac{\int_0^t (t-s)^{\alpha-1}|f(s,0,0)|\mathrm{d}s}{\mathrm{e}^{\lambda t}} < \infty.
$$

We now show a bound of growth rate of solutions to the system (8)–(9).

Theorem 4.1. Assume that the conditions (H1) and (H2) are satisfied. Then, the global solution $\Phi(\cdot,\phi)$ on the interval $[-\tau,\infty)$ of (8)–(9) is exponentially bounded.

Proof. Let $\lambda>0$ be the constant satisfying the condition $(H2)$. Denote by $C_\lambda([-\tau,\infty);\mathbb{R})$ the set of all continuous functions $\xi:[-\tau,\infty)\to\mathbb{R}$ such that

$$
||\xi||_\lambda := \sup_{t\ge 0}\frac{\xi^*(t)}{\exp(\lambda t)} < \infty,\ \ \xi^*(t) := \sup_{-\tau\le\theta\le t}|\xi(\theta)|.
$$

It is obvious that $(C_\lambda([-\tau,\infty);\mathbb{R}); ||\cdot||_\lambda)$ is a Banach space. We construct an operator $\mathcal{T}_\phi$ on this space as follows:

$$
\begin{aligned}
&(\mathcal{T}_\phi\xi)(t) := \phi(t), t\in[-\tau,0],\\
&(\mathcal{T}_\phi\xi)(t) := \phi(0)+N\phi(-\tau)-N\xi(t-\tau)\\
&\qquad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,\xi(s),\xi(s-\tau))\mathrm{d}s,\quad t\ge 0.
\end{aligned}
$$

It is easily to see that $\mathcal{T}_\phi\xi\in C([-\tau,\infty);\mathbb{R})$ for all $\xi\in C_\lambda([-\tau,\infty);\mathbb{R})$. Now we will show that $\mathcal{T}_\phi\xi\in C_\lambda([-\tau,\infty);\mathbb{R})$ for all $\xi\in C_\lambda([-\tau,\infty);\mathbb{R})$. Indeed, let $\xi\in C_\lambda([-\tau,\infty);\mathbb{R})$ be arbitrary, for any $t\ge\tau$, we have

$$
\begin{aligned}
|(\mathcal{T}_\phi\xi)(t)| \le\ & |\phi(0)| + |N||\phi(-\tau)| + |N||\xi(t-\tau)| \\
& + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,\xi(s),\xi(s-\tau)) - f(s,0,0)|\mathrm{d}s \\
& + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,0,0)|\mathrm{d}s \\
& \le C_1 + |N|\xi^*(t) + \frac{L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(|\xi(s)| + |\xi(s-\tau)|)\mathrm{d}s \\
& + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,0,0)|\mathrm{d}s.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
|(\mathcal{T}_\phi\xi)(t)| \le\ & C_1 + |N|\mathrm{e}^{\lambda t}\frac{\xi^*(t)}{\mathrm{e}^{\lambda t}} + \frac{2L\mathrm{e}^{\lambda t}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mathrm{e}^{-\lambda(t-s)}\frac{\xi^*(s)}{\mathrm{e}^{\lambda s}}\mathrm{d}s \\
& + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,0,0)|\mathrm{d}s \\
& \le C_1 + |N|\mathrm{e}^{\lambda t}||\xi||_\lambda + \frac{2L\mathrm{e}^{\lambda t}}{\lambda^\alpha}||\xi||_\lambda \\
& + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,0,0)|\mathrm{d}s, \quad \forall t \ge \tau.
\end{aligned}
$$

Hence, for any $t \ge \tau$,

$$
\begin{aligned}
\frac{(\mathcal{T}_\phi\xi)^*(t)}{\mathrm{e}^{\lambda t}} \le\ & \frac{C_1}{\mathrm{e}^{\lambda t}} + |N|||\xi||_\lambda + \frac{2L}{\lambda^\alpha}||\xi||_\lambda \\
& + \frac{1}{\Gamma(\alpha)}\sup_{t\ge 0}\frac{\int_0^t (t-s)^{\alpha-1}|f(s,0,0)|\mathrm{d}s}{\mathrm{e}^{\lambda t}}.
\end{aligned}
$$

Thus,

$$
\sup_{t\ge 0}\frac{(\mathcal{T}_\phi\xi)^*(t)}{\mathrm{e}^{\lambda t}} < \infty.
$$

Next, we will show that operator $\mathcal{T}_\phi$ is contractive on $(C_\lambda([-\tau,\infty);\mathbb{R}); ||\cdot||_\lambda)$. Let $\xi, \hat{\xi} \in (C_\lambda([-\tau,\infty);\mathbb{R}); ||\cdot||_\lambda)$ be arbitrary, we have the following estimates on intervals $[-\tau, 0]$, $[0,\tau]$ and $[\tau,\infty)$. Consider $t \in [-\tau, 0]$, we see that

$$
|(\mathcal{T}_\phi\xi)(t) - (\mathcal{T}_\phi\hat{\xi})(t)| = 0.
$$

On the interval $[0,\tau]$, then

$$
\begin{aligned}
|(\mathcal{T}_\phi\xi)(t)-(\mathcal{T}_\phi\hat{\xi})(t)| &\le \frac{L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(|\xi(s)-\hat{\xi}(s)|+|\xi(s-\tau)-\hat{\xi}(s-\tau)|)\mathrm{d}s\\
&\le \frac{2L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(\xi-\hat{\xi})^*(s)\mathrm{d}s\\
&\le \frac{2L\mathrm{e}^{\lambda t}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mathrm{e}^{-\lambda(t-s)}\frac{(\xi-\hat{\xi})^*(s)}{\mathrm{e}^{\lambda s}}\mathrm{d}s\\
&\le \frac{2L\mathrm{e}^{\lambda t}}{\lambda^\alpha}||\xi-\hat{\xi}||_\lambda.
\end{aligned}
$$

For $t\in[\tau,\infty)$, then

$$
\begin{aligned}
|(\mathcal{T}_\Phi\xi)(t)-(\mathcal{T}_\Phi\hat{\xi})(t)| &\le |N||\xi(t-\tau)-\hat{\xi}(t-\tau)|\\
&\quad+\frac{L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(|\xi(s)-\hat{\xi}(s)|+|\xi(s-\tau)-\hat{\xi}(s-\tau)|)\mathrm{d}s\\
&\le |N|\mathrm{e}^{\lambda t}\frac{(\xi-\hat{\xi})^*(t-\tau)}{\mathrm{e}^{\lambda(t-\tau)}\mathrm{e}^{\lambda\tau}}+\frac{2L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(\xi-\hat{\xi})^*(s)\mathrm{d}s\\
&\le \mathrm{e}^{\lambda t}\frac{|N|}{\mathrm{e}^{\lambda\tau}}||\xi-\hat{\xi}||_\lambda+\mathrm{e}^{\lambda t}\frac{2L}{\lambda^\alpha}||\xi-\hat{\xi}||_\lambda.
\end{aligned}
$$

Hence,

$$
|(\mathcal{T}_\phi\xi)(t)-(\mathcal{T}_\phi\hat{\xi})(t)| \le \mathrm{e}^{\lambda t}\left(\frac{|N|}{\mathrm{e}^{\lambda\tau}}+\frac{2L}{\lambda^\alpha}\right)||\xi-\hat{\xi}||_\lambda,\ \ \forall t\in[-\tau,\infty).
$$

It implies that

$$
(\mathcal{T}_\phi\xi-\mathcal{T}_\phi\hat{\xi})^*(t)\le \mathrm{e}^{\lambda t}\left(\frac{|N|}{\mathrm{e}^{\lambda\tau}}+\frac{2L}{\lambda^\alpha}\right)||\xi-\hat{\xi}||_\lambda,\ \ \forall t\ge 0.
$$

Thus,

$$
||(\mathcal{T}_\phi\xi-\mathcal{T}_\phi\hat{\xi})||_\lambda \le \left(\frac{|N|}{\mathrm{e}^{\lambda\tau}}+\frac{2L}{\lambda^\alpha}\right)||\xi-\hat{\xi}||_\lambda.
$$

Choose λ large enough such that

$$
\frac{|N|}{\mathrm{e}^{\lambda\tau}}+\frac{2L}{\lambda^\alpha}<1,
$$

then $\mathcal{T}_\phi$ is contractive on $(C_\lambda([-\tau,\infty);\mathbb{R});||\cdot||_\lambda)$. The unique fixed point ξ^* of $\mathcal{T}_\phi$ is the unique solution to (8)–(9) in $C_\lambda([-\tau,\infty);\mathbb{R})$. Moreover, this solution is exponentially bounded. The proof is complete. □

5 Explicit representation of the solution of linear FNDDEs

For $a, b, N \in \mathbb{R}$ and an arbitrary continuous function $\phi(t) : [-\tau, 0] \to \mathbb{R}$, we now consider the special case of the linear FNDDE

$$\left\{ \begin{array}{l} {}^{\mathrm{C}}D_0^{\alpha}\big[x(t) + Nx(t-\tau)\big] = ax(t) + bx(t-\tau) \\ x(t) = \phi(t), \quad t \in [-\tau, 0] \end{array} \right. \tag{10}$$

for which we are interested in providing an explicit representation of its solution. Since assumptions (H1) and (H2) introduced in Section 4 are trivially satisfied, the solution $x(t)$ possesses the LT, say $X(s)$, and from well-known results on the LT of the fractional Caputo derivative we have

$$\mathcal{L}\Big({}^{\mathrm{C}}D_0^{\alpha} x(r)\,,s\Big) = s^{\alpha}X(s) - s^{\alpha-1}\phi(0).$$

Hence, by taking the LT to both sides of (10), we obtain

$$s^{\alpha}X(s) + Ns^{\alpha}\mathcal{L}\Big(x(t-\tau)\,,s\Big) - s^{\alpha-1}\big[\phi(0) + N\phi(-\tau)\big] = aX(s) + b\mathcal{L}\Big(x(t-\tau)\,,s\Big). \tag{11}$$

We know (see, for instance, [21, Eq (3.2)] or [9, Proposition 4.2]) that

$$\mathcal{L}\Big(x(t-\tau)\,,s\Big) = \mathrm{e}^{-s\tau}X(s) + \mathrm{e}^{-s\tau}\hat{X}_{\tau}(s), \quad \hat{X}_{\tau}(s) = \int_{-\tau}^{0} \mathrm{e}^{-st}\phi(t)\mathrm{d}t \tag{12}$$

and, therefore, one immediately obtains

$$\Big(1 - \frac{b - Ns^{\alpha}}{s^{\alpha} - a}\mathrm{e}^{-s\tau}\Big)X(s) = \frac{s^{\alpha-1}}{s^{\alpha} - a}\big[\phi(0) + N\phi(-\tau)\big] + \frac{b - Ns^{\alpha}}{s^{\alpha} - a}\mathrm{e}^{-s\tau}\hat{X}_{\tau}(s).$$

For sufficiently large $|s|$ the use of the series expansion

$$\Big(1 - \frac{b - Ns^{\alpha}}{s^{\alpha} - a}\mathrm{e}^{-s\tau}\Big)^{-1} = \sum_{k=0}^{\infty} \frac{(b - Ns^{\alpha})^k}{(s^{\alpha} - a)^k}\mathrm{e}^{-s\tau k}$$

leads to

$$X(s) = \frac{s^{\alpha-1}}{s^{\alpha} - a}\sum_{k=0}^{\infty} \frac{(b - Ns^{\alpha})^k}{(s^{\alpha} - a)^k}\mathrm{e}^{-s\tau k}\big[\phi(0) + N\phi(-\tau)\big] + \sum_{k=1}^{\infty} \frac{(b - Ns^{\alpha})^k}{(s^{\alpha} - a)^k}\mathrm{e}^{-s\tau k}\hat{X}_{\tau}(s)$$

and, hence, after exploiting standard rules for powers of binomials

$$(b - Ns^{\alpha})^k = \sum_{\ell=0}^{k} \binom{k}{\ell}(-1)^{\ell}N^{\ell}b^{k-\ell}s^{\alpha\ell}$$

we obtain the following representation of the LT of the solution of the linear FNDDE (10)

$$\begin{aligned} X(s) = & \sum_{k=0}^{\infty}\sum_{\ell=0}^{k} \binom{k}{\ell}(-1)^{\ell}N^{\ell}b^{k-\ell}\frac{s^{\alpha+\alpha\ell-1}}{(s^{\alpha} - a)^{k+1}}\mathrm{e}^{-s\tau k}\big[\phi(0) + N\phi(-\tau)\big] \\ & + \sum_{k=1}^{\infty}\sum_{\ell=0}^{k} \binom{k}{\ell}(-1)^{\ell}N^{\ell}b^{k-\ell}\frac{s^{\alpha\ell}}{(s^{\alpha} - a)^{k}}\mathrm{e}^{-s\tau k}\hat{X}_{\tau}(s) \end{aligned} \tag{13}$$

An explicit representation of the solution of (10) in the time domain can be obtained by inversion of its LT (13) only once the the initial function $\phi(t)$ has been specified. The following preliminary results are however necessary.

Let $\alpha > 0$ and $\beta, \gamma \in \mathbb{R}$ be some parameters, and consider the three-parameter Mittag-Leffler function (also known as the Prabhakar function) [11, 15]

$$E_{\alpha,\beta}^{\gamma}(z) = \frac{1}{\Gamma(\gamma)} \sum_{j=0}^{\infty} \frac{\Gamma(\gamma + j) z^j}{j!\Gamma(\alpha j + \beta)},$$

for which, when $t \geq 0$ and a is any real or complex value, we have the following result concerning the LT

$$\mathcal{L}\Big(e_{\alpha,\beta}^{\gamma}(t;a)\,,s\Big) = \frac{s^{\alpha\gamma-\beta}}{(s^{\alpha}-a)^{\gamma}}, \quad e_{\alpha,\beta}^{\gamma}(t;a) = t^{\beta-1}E_{\alpha,\beta}^{\gamma}(at^{\alpha}), \quad Re(s) > 0 \text{ and } |s| > |a|^{\frac{1}{\alpha}}. \tag{14}$$

Furthermore, whenever $\tau \geq 0$ it is a basic fact in the theory of LT (see, for instance, [17, Theorem 1.31]) that

$$\mathcal{L}^{-1}\Big(\frac{s^{\alpha\gamma-\beta}}{(s^{\alpha}-a)^{\gamma}}\mathrm{e}^{-sp}\,,s\Big) = \begin{cases} e_{\alpha,\beta}^{\gamma}(t-\tau;a) & t \geq \tau \\ 0 & t < \tau \end{cases} \tag{15}$$

We are now able to provide an explicit representation of the solution of linear FNDDEs for some examples of initial functions $\phi(t)$ (we consider here the same function $\phi(t)$ which will be used later on in the Section devoted to present numerical simulations; The solution for further functions $\phi(t)$ can be however obtained in a very similar way). In the following, for any real value x, with $\lfloor x \rfloor$ we will denote the greatest integer less than x.

Proposition 5.1. If $\phi(t) = x_0$, $\forall t \in [-\tau, 0]$, the exact solution of the linear FNDDE (10) is

$$\begin{aligned} x(t) = &\sum_{k=0}^{\lfloor t/\tau \rfloor} \sum_{\ell=0}^{k} \binom{k}{\ell} (-1)^{\ell} N^{\ell} b^{k-\ell} e_{\alpha,\alpha(k-\ell)+1}^{k+1}(t-\tau k;a)(1+N)x_0 \\ &- \sum_{k=1}^{\lfloor t/\tau \rfloor} \sum_{\ell=0}^{k} \binom{k}{\ell} (-1)^{\ell} N^{\ell} b^{k-\ell} e_{\alpha,\alpha(k-\ell)+1}^{k}(t-\tau k;a)x_0 \\ &+ \sum_{k=1}^{\lfloor t/\tau \rfloor+1} \sum_{\ell=0}^{k} \binom{k}{\ell} (-1)^{\ell} N^{\ell} b^{k-\ell} e_{\alpha,\alpha(k-\ell)+1}^{k}(t-\tau k+\tau;a)x_0. \end{aligned}$$

Proof. Since $\phi(t) = x_0$, $\forall t \in [-\tau, 0]$, it is immediate to compute

$$\hat{X}_{\tau}(s) = -\frac{1}{s}\big(1 - \mathrm{e}^{s\tau}\big)x_0$$

and, hence, the LT $X(s)$ obtained in (13) becomes

$$
\begin{aligned}
X(s) = & \sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^{\ell}N^{\ell}b^{k-\ell}\frac{s^{\alpha+\alpha\ell-1}}{(s^{\alpha}-a)^{k+1}}\mathrm{e}^{-s\tau k}(1+N)x_0 \\
& -\sum_{k=1}^{\infty}\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^{\ell}N^{\ell}b^{k-\ell}\frac{s^{\alpha\ell-1}}{(s^{\alpha}-a)^{k}}\mathrm{e}^{-s\tau k}x_0 \\
& +\sum_{k=1}^{\infty}\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^{\ell}N^{\ell}b^{k-\ell}\frac{s^{\alpha\ell-1}}{(s^{\alpha}-a)^{k}}\mathrm{e}^{-s\tau(k-1)}x_0.
\end{aligned}
$$

The proof now follows after recognizing the presence in each summation of the LT (14) of the three-parameter ML function, applying Eq. (15), and, for any t, truncating each summation at the maximum index k such that $t \geq \tau k$ (first and second summation) or $t \geq \tau(k-1)$ (third summation). $\square$

Proposition 5.2. If $\phi(t) = x_0 + mt$, $\forall t \in [-\tau, 0]$, the exact solution of the linear FNDDE (10) is

$$
\begin{aligned}
x(t) = & \sum_{k=0}^{\lfloor t/\tau\rfloor}\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^{\ell}N^{\ell}b^{k-\ell}e^{k+1}_{\alpha,\alpha(k-\ell)+1}(t-\tau k;a)\big[x_0+N\phi(-\tau)\big] \\
& -\sum_{k=1}^{\lfloor t/\tau\rfloor}\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^{\ell}N^{\ell}b^{k-\ell}e^{k}_{\alpha,\alpha(k-\ell)+1}(t-\tau k;a)x_0 \\
& +\sum_{k=1}^{\lfloor t/\tau\rfloor+1}\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^{\ell}N^{\ell}b^{k-\ell}e^{k}_{\alpha,\alpha(k-\ell)+1}(t-\tau k+\tau;a)\phi(-\tau) \\
& -\sum_{k=1}^{\lfloor t/\tau\rfloor}\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^{\ell}N^{\ell}b^{k-\ell}e^{k}_{\alpha,\alpha(k-\ell)+2}(t-\tau k;a)m \\
& +\sum_{k=1}^{\lfloor t/\tau\rfloor+1}\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^{\ell}N^{\ell}b^{k-\ell}e^{k}_{\alpha,\alpha(k-\ell)+2}(t-\tau k+\tau;a)m.
\end{aligned}
$$

Proof. When $\phi(t) = x_0 + mt$, $t \in [-\tau, 0]$, a standard computation allows to evaluate

$$
\begin{aligned}
\hat{X}_{\tau}(s) = \int_{-\tau}^{0}\mathrm{e}^{-st}\phi(t)\mathrm{d}t &= -\frac{1}{s}\big(1-\mathrm{e}^{s\tau}\big)x_0 + m\Big[-\frac{1}{s^2}-\frac{\tau}{s}\mathrm{e}^{s\tau}+\frac{1}{s^2}\mathrm{e}^{s\tau}\Big] \\
&= -\frac{1}{s}x_0 + \frac{1}{s}\mathrm{e}^{s\tau}\phi(-\tau) - \frac{1}{s^2}m + \frac{1}{s^2}\mathrm{e}^{s\tau}m
\end{aligned}
$$

and, after inserting the above expression for $\hat{X}_{\tau}(s)$ in the formula (13) for the LT of the solution of (10), we obtain

$$
\begin{aligned}
X(s) = & \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \binom{k}{\ell} (-1)^{\ell} N^{\ell} b^{k-\ell} \frac{s^{\alpha+\alpha\ell-1}}{(s^{\alpha}-a)^{k+1}} \mathrm{e}^{-s\tau k} \bigl(x_0 + N\phi(-\tau)\bigr) \\
& - \sum_{k=1}^{\infty} \sum_{\ell=0}^{k} \binom{k}{\ell} (-1)^{\ell} N^{\ell} b^{k-\ell} \frac{s^{\alpha\ell-1}}{(s^{\alpha}-a)^{k}} \mathrm{e}^{-s\tau k} x_0 \\
& + \sum_{k=1}^{\infty} \sum_{\ell=0}^{k} \binom{k}{\ell} (-1)^{\ell} N^{\ell} b^{k-\ell} \frac{s^{\alpha\ell-1}}{(s^{\alpha}-a)^{k}} \mathrm{e}^{-s\tau (k-1)} \phi(-\tau) \\
& - \sum_{k=1}^{\infty} \sum_{\ell=0}^{k} \binom{k}{\ell} (-1)^{\ell} N^{\ell} b^{k-\ell} \frac{s^{\alpha\ell-2}}{(s^{\alpha}-a)^{k}} \mathrm{e}^{-s\tau k} m \\
& + \sum_{k=1}^{\infty} \sum_{\ell=0}^{k} \binom{k}{\ell} (-1)^{\ell} N^{\ell} b^{k-\ell} \frac{s^{\alpha\ell-2}}{(s^{\alpha}-a)^{k}} \mathrm{e}^{-s\tau (k-1)} m
\end{aligned}
$$

and the proof is concluded in the same way as the proof of Proposition 5. □

The above explicit representations of exact solutions is of interest since it allows to accurately evaluate the solutions of linear FNDDEs once a procedure for the computation of the three-parameter ML functions $e^{k}_{\alpha,\beta}(t;a)$ is available. To this purpose the method devised in [10] to compute k-the order derivatives $E^{(k)}_{\alpha,\beta}(z)$ of the two-parameter ML function $E_{\alpha,\beta}(z)$ can be exploited since three-parameter ML functions are related to derivatives of two-parameter ML functions by the relationship $E^{k}_{\alpha,\beta}(z) = E^{(k)}_{\alpha,\beta-\alpha k+\alpha}(z)/(k-1)$.

Anyway, this approach does not seems suitable for computation on intervals of large size since it could require the evaluation of a considerable number of three-parameter ML functions. Moreover, a specific explicit representation of the exact solution must be derived in dependence of the selected initial function $\phi(t)$. For this reason, in the Section devoted to present numerical simulations we will derive a specific numerical scheme.

6 Asymptotic behavior of solutions of linear FNDDEs

This section is devoted to discuss the asymptotic behavior of solutions to linear FNDDEs (10). Our approach is to use the Final value theorem for Laplace transforms (see, e.g., [5, Theorem D. 13, p. 232]). We will focus on two different cases.

6.1 Case (C1): $a < 0$, $b = 0$

In this case, the linear FNDDE becomes

$$
{}^{\mathrm{C}}D^{\alpha}_{0+}\bigl[x(t) + Nx(t-\tau)\bigr] = ax(t), \quad t \geq 0, \tag{16}
$$

and, thanks to (11) and (12), the LT $X(s)$ of the solution $x(t)$ is

$$X(s) = \frac{s^{\alpha-1}(\phi(0) + N\phi(-\tau)) - Ns^{\alpha}\mathrm{e}^{-s\tau}\int_{-\tau}^{0} \mathrm{e}^{-su}\phi(u)\mathrm{d}u}{s^{\alpha} + Ns^{\alpha}\mathrm{e}^{-s\tau} - a}. \tag{17}$$

To investigate the asymptotic behavior of $x(t)$ it is necessary to locate possible poles of $X(s)$ in the complex plane. Denote the denominator of $X(s)$ by

$$Q(s) := s^{\alpha} + Ns^{\alpha}\mathrm{e}^{-s\tau} - a.$$

Due to the fact that $X(s)$ and $Q(s)$ have the same non-zero poles and $X(s)$ only has just a further single pole at the origin, we can restrict to study the roots of the equation $Q(s) = 0$.

Lemma 6.1. Let $a < 0$. The following statements hold:

(i) if $|N| \leq 1$, then $Q(s)$ has no pole in the closed right half plane $\{z \in \mathbb{C} : \Re(z) \geq 0\}$;

(ii) if $|N| > 1$, then $Q(s)$ has at least one pole in the open right half plane $\{z \in \mathbb{C} : \Re(z) > 0\}$.

Proof. (i) Since $Q(0) \neq 0$, the equation $Q(s) = 0$ is equivalent to

$$1 + N\mathrm{e}^{-\tau s} = as^{-\alpha}, \; s \neq 0. \tag{18}$$

We will show that (18) has no root in $\{z \in \mathbb{C} : \Re(z) \geq 0\}$. Indeed, on the contrary, assume that (18) has a root $s_0 \neq 0$ with $\Re(s_0) \geq 0$. Note that $1 + N\mathrm{e}^{-\tau s_0} \in D_1 := \{z \in \mathbb{C} : |z - (-1)| \leq |N|\}$ and $as_0^{-\alpha} \in D_2 := \{z \in \mathbb{C} : |\arg(z)| \leq \frac{\alpha\pi}{2}\}$. Furthermore, for $|N| \leq 1$, two domains D_1 and D_2 intersect at most one point at the origin which implies a contradiction.

(ii) To prove this point, we only have to show that the equation (18)

$$1 + N\mathrm{e}^{-\tau s} - as^{-\alpha} = 0 \tag{19}$$

has at last one root in the open right half plane $\{z \in \mathbb{C} : \Re(z) > 0\}$. Let $Q_1 := 1 + N\mathrm{e}^{-\tau s} - as^{-\alpha}$, $f(s) := 1 + N\mathrm{e}^{-s\tau}$ and $g(s) := -as^{-\alpha}$. First, we find roots of the equation $f(s) = 0$ in $\{z \in \mathbb{C} : \Re(z) > 0\}$. To determine, we consider the case $N > 1$. The case where $N < -1$ is proved similarly. It is easy to see that $s_k := \frac{\log N}{\tau} + \mathrm{i}\frac{(2k+1)\pi}{\tau}$, $k \in \mathbb{Z}$ are roots of the equation $f(s) = 0$. Let R be a fixed positive constant which will be chosen later. Define $C := C_1 \cup C_2 \cup C_3 \cup C_4$, where

$$C_1 := \left\{ z \in \mathbb{C} : \, z = s_1 + \mathrm{i}R, \; \frac{\log N}{2\tau} \leq s_1 \leq \frac{3\log N}{2\tau} \right\},$$
$$C_2 := \left\{ z \in \mathbb{C} : \, z = \frac{3\log N}{2\tau} + \mathrm{i}s_2, \; R \leq s_2 \leq R + \frac{2\pi}{\tau} \right\},$$
$$C_3 := \left\{ z \in \mathbb{C} : \, z = s_1 + \mathrm{i}(R + \frac{2\pi}{\tau}), \; \frac{\log N}{2\tau} \leq s_1 \leq \frac{3\log N}{2\tau} \right\},$$
$$C_4 := \left\{ z \in \mathbb{C} : \, z = \frac{\log N}{2\tau} + \mathrm{i}s_2, \; R \leq s_2 \leq R + \frac{2\pi}{\tau} \right\},$$

and let D be the domain bounded by the contour C. On C, we obtain the estimates

$$|f(s)| > 1 - \frac{N}{\mathrm{e}^{-\tau R}} > \frac{1}{2}, \text{ for R is large enough,}$$

$$|g(s)| < \frac{|a|}{R^\alpha} \to 0 \text{ as } R \to \infty.$$

Thus, by choosing R large, then

$$|f(s)| > |g(s)|, \ \forall s \in C.$$

On the other hand, as shown above, there is at least one zero point of f inside C. By Rouché's theorem (see, e.g., [3, Theorem 12.2, p. 398]), there is at least one zero point of Q_1 in $\{z \in \mathbb{C} : \Re(z) > 0\}$. The proof is complete. $\square$

We are now in a position to state the main result of this part.

Theorem 6.2. Let $a < 0$ and consider the linear FNDDE (16). Then, the following statements hold:

(i) if $|N| \leq 1$, then this equation is asymptotically stable;

(ii) if $|N| > 1$, then the equation is unstable.

Proof. (i) As shown above $X(s)$ does not have any poles in the closed right half-plane $\{s \in \mathbb{C} : \Re s \geq 0\}$ except for a simple pole at the origin. Hence, since from (17) it is $\lim_{s\to 0} sX(s) = 0$, by The final value theorem for Laplace transforms [5, Theorem D. 13, p. 232], we have

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s) = 0$$

which implies that (16) is asymptotically stable.
(ii) The proof of this part is obvious from the property of a function that its Laplace transform has at least one pole in the open right half-plane of the complex domain. $\square$

6.2 Case (C2): $a < 0$, $|b| < |a|$

The linear FNDDE is now

$$^{\mathrm{C}}D^\alpha_{0+}\big[x(t) + Nx(t-\tau)\big] = ax(t) + bx(t-\tau), \quad t \geq 0 \tag{20}$$

and, again, by exploiting (11) and (12), the LT $X(s)$ of the solution $x(t)$ is

$$X(s) = \frac{s^{\alpha-1}(\phi(0) + N\phi(-\tau)) + (-Ns^\alpha + b)\mathrm{e}^{-s\tau}\int_{-\tau}^0 \mathrm{e}^{-su}\phi(u)\mathrm{d}u}{s^\alpha + Ns^\alpha \mathrm{e}^{-s\tau} - a - b\mathrm{e}^{-\tau s}} \tag{21}$$

It is easy to see that $s = 0$ is only a simple pole of $X(s)$. Put

$$P(s) := s^\alpha + Ns^\alpha \mathrm{e}^{-s\tau} - a - b\mathrm{e}^{-\tau s}.$$

The following lemma gives information about zero points of P.

Lemma 6.3. Assume that $a < 0$ and $|b| < |a|$.

(i) If $|N| \le 1$, then the equation $P(s) = 0$ has no root in the closed right half-plane $\{s \in \mathbb{C} : \Re(s) \ge 0\}$.

(ii) If $|N| > 1$, then the equation above has at least one root in the open right half-plane $\{s \in \mathbb{C} : \Re(s) > 0\}$.

Proof. (i) Denote $D_1 := \{z \in \mathbb{C} : \Re(z) \ge 0\}$. Due to $P(0) \neq 0$, there exists $\varepsilon > 0$ which is small enough such that $P(s) \neq 0$ in the ball $B \coloneqq \{s \in \mathbb{C} : |s| \le \varepsilon\}$. On the other hand,

$$|P(s)| \ge |s|^\alpha(1 - |N|) - (|a| + |b|) \to \infty$$

as $s \in D_1$ and $|s| \to \infty$. Thus, there is $R > 0$ such that $P(s) \neq 0$ for all $s \in D_1 \cap \{z \in \mathbb{C} : |z| \ge R\}$. Denote $C_1 := \{z \in \mathbb{C} : z = \varepsilon(\cos\varphi + \mathrm{i}\sin\varphi),\ -\pi/2 \le \varphi \le \pi/2\}$, $C_3 := \{z \in C : z = R(\cos\varphi + \mathrm{i}\sin\varphi),\ -\pi/2 \le \varphi \le \pi/2)\}$, $C_2 := \{z \in \mathbb{C} : z = r(\cos\pi/2 - \mathrm{i}\sin\pi/2\}$ and $C_4 := \{z \in \mathbb{C} : z = r(\cos\pi/2 + \mathrm{i}\sin\pi/2)\}$. Put $f(s) := s^\alpha - a$, $g(s) := Ns^\alpha \mathrm{e}^{-s\tau} - b\mathrm{e}^{-\tau s}$. On C_1 and C_3, let $s = s_1 + \mathrm{i}s_2 = r(\cos\varphi + \mathrm{i}\sin\varphi)$, where $s_1 > 0$, $r = \varepsilon$ or $r = R$ and $\varphi \in [-\pi/2, \pi/2]$. We have

$$f(s) = s^\alpha - a = r^\alpha\cos(\alpha\varphi) - a + \mathrm{i}r^\alpha\sin(\alpha\varphi),$$

$$\begin{aligned}
g(s) &= Nr^\alpha \mathrm{e}^{\mathrm{i}\alpha\varphi}\mathrm{e}^{-\tau(s_1+is_2)} - b\mathrm{e}^{-\tau(s_1+is_2)}\\
&= Nr^\alpha \mathrm{e}^{-\tau s_1}(\cos(\alpha\varphi - \tau s_2) + \mathrm{i}\sin(\alpha\varphi - \tau s_2))\\
&\quad - b\mathrm{e}^{-\tau s_1}(\cos(\tau s_2) - \mathrm{i}\sin(\tau s_2))\\
&= Nr^\alpha \mathrm{e}^{-\tau s_1}\cos(\alpha\varphi - \tau s_2) - b\mathrm{e}^{-\tau s_1}\cos(\tau s_2)\\
&\quad + \mathrm{i}(Nr^\alpha \mathrm{e}^{-\tau s_1}\sin(\alpha\varphi - \tau s_2) + b\mathrm{e}^{-\tau s_1}\sin(\tau s_2)).
\end{aligned}$$

This implies that

$$|f(s)|^2 = r^{2\alpha} + a^2 - 2ar^\alpha\cos(\alpha\varphi), \tag{22}$$
$$|g(s)|^2 = N^2r^{2\alpha}\mathrm{e}^{-2\tau s_1} + b^2\mathrm{e}^{-2\tau s_1} - 2bNr^\alpha \mathrm{e}^{-2\tau s_1}\cos(\alpha\varphi). \tag{23}$$

From (22), (23) and the assumptions that $s_1 \ge 0$, $|N| \le 1$ and $|b| < |a|$, we see that

$$|f(s)| > |g(s)| \text{ on } C_1 \text{ and } C_3. \tag{24}$$

Now, we will compare $|f|$ and $|g|$ on C_4. For any $s \in C_4$, we describe $s = ir = r(\cos\pi/2 + i\sin\pi/2)$, where $r \in [\varepsilon, R]$. By a simple computation, we obtain the estimates

$$\begin{aligned}
|f(s)|^2 &= r^{2\alpha} + a^2 - 2ar^\alpha\cos\frac{\alpha\pi}{2},\\
|g(s)|^2 &= N^2r^{2\alpha} + b^2 - 2Nr^\alpha b\cos\frac{\alpha\pi}{2} \le N^2r^{2\alpha} + b^2 + 2|N|r^\alpha|b|\cos\frac{\alpha\pi}{2},
\end{aligned}$$

which implies that

$$|f(s)| > |g(s)| \text{ on } C_4. \tag{25}$$

Similarly, on C_2, we also have

$$|f(s)| > |g(s)|.$$

This together with (24), (25) imply that

$$|f(s)| > |g(s)| \text{ on } C := C_1 \cup C_2 \cup C_3 \cup C_4. \tag{26}$$

From (26), by Rouché's theorem, P has no zero in the domain D bounded by the contour C defined as above. Thus, P has no zero point in the closed right half-plane of the complex plane.

(ii) As in the proof of Lemma 6.1 (ii), we only need to show that the following equation has at least one root in the open right half-plane $\{z \in \mathbb{C} : \Re(z) > 0\}$:

$$1 + N\mathrm{e}^{-\tau s} - \frac{a}{s^\alpha} - \frac{b\mathrm{e}^{-\tau s}}{s^\alpha} = 0. \tag{27}$$

To do this, we set $f(s) := 1 + N\mathrm{e}^{-\tau s}$ and $g(s) := -\frac{a}{s^\alpha} - \frac{b\mathrm{e}^{-\tau s}}{s^\alpha}$. Take the contour C as in the proof of Lemma 6.1 (ii) with R is large enough. It is known that f has one zero in the domain bounded by C and $f(s) \neq 0$ on this contour. On the other hand,

$$|g(s)| \to 0$$

as $s \in \{z \in \mathbb{C} : \Re(z) > 0\}$ and $|s| \to \infty$. Thus, for R is large, we have

$$|g(s)| < \min_{s \in C} |f(s)| \leq |f(s)| \text{ for all } s \in C,$$

which together with Rouché's theorem imply that (27) has one root in the domain bounded by C, that is, this equation has at least one root in $\{z \in \mathbb{C} : \Re(z) > 0\}$. The proof is complete. □

Based on Lemma 6.3 and arguments as in the proof of Theorem 6.2, we obtain the following result.

Theorem 6.4. Consider the linear FNDDE (20). Assume that $a < 0$, $|b| < |a|$. Then,

- (i) if $|N| < 1$, then this equation is asymptotically stable;
- (ii) if $|N| > 1$, then it is unstable.

7 Numerical simulations

With the aim of verify the theoretical findings on the asymptotic behavior of solutions of linear FNDDEs, we consider here a numerical scheme based on the application of a standard product-integration rule of rectangular rule to the integral representation (3). Methods of this kind are widely employed to solve fractional differential equations (see, for instance [6]) and they can be easily adapted to solve FNDDEs as well.

Let $h > 0$ and consider an equispaced grid $t_n = nh$, $n = 0, 1, \ldots$, thanks to which the integral in (3) can be rewritten in a piece-wise way

$$\begin{aligned} x(t_n) &= \phi(0) + N\phi(-\tau) - Nx(t_n - \tau) \\ &\quad + \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t_n - s)^{\alpha-1} f(s, x(s), x(s-\tau)) \mathrm{d}s, \end{aligned}$$

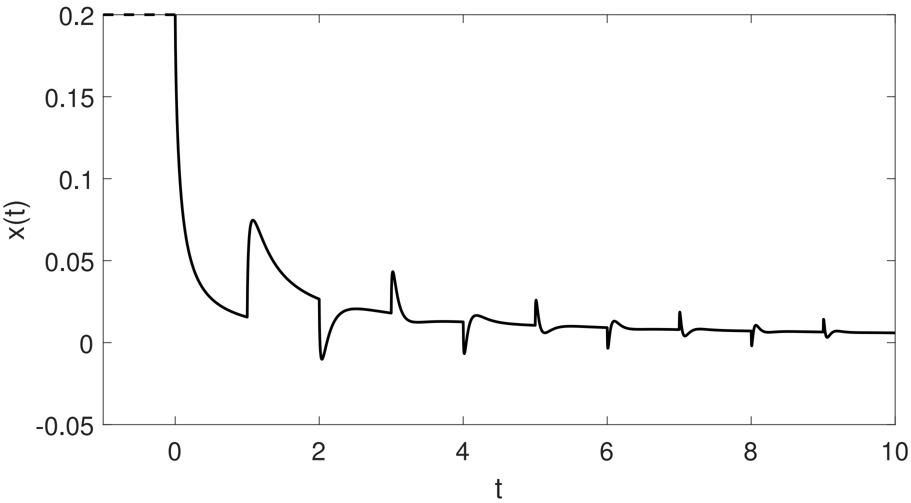

Figure 1: Trajectory of the solution $\Phi(\cdot,\phi)$ to system (28) when $\phi(t)=0.2$ on $[-1,0]$.

and hence the vector field $f(s,x(s),x(s-\tau))$ is approximated in each interval $[t_k,t_{k+1}]$ by the constant values assumed in one of the endpoints of $[t_k,t_{k+1}]$. For stability reasons, and avoid to introduce in the simulations spurious instabilities, we prefer to device an implicit method and adopt the approximation $f(s,x(s),x(s-\tau)) \approx f(t_{k+1},x(t_{k+1}),x(t_{k+1}-\tau))$, $s\in[t_k,t_{k+1}]$. After integrating in an exact way each integral we obtain the approximations $x_n \approx x(t_n)$

$$x_n = \phi(0) + N\phi(-\tau) - Nx(t_n-\tau) + h^{\alpha}\sum_{k=1}^{n} b_{n-k}^{(\alpha)} f(t_k,x_k,x_{k-\tau/h}),$$

where convolution weights b_n are defined as $b_n^{(\alpha)} = ((n+1)^{\alpha}-n^{\alpha})/\Gamma(\alpha+1)$. The approximation $x_{k-\tau/h}$ of $x(t_k-\tau)$ is obtained by interpolation of the two closest available approximations of the solution when $t_k-\tau$ is not a grid point and when it does not belong to $[-\tau,0]$. First-degree polynomial interpolation is clearly sufficient to preserve the first-order convergence of the PI rule. Finally, Newton-Raphson iterations are used to determine x_n from the above implicit scheme.

We now apply the above scheme to present some numerical examples illustrating the main results proposed in this paper.

Example 7.1. Consider the equation

$$\begin{aligned} {}^{\mathrm{C}}D^{0.7}_{0+}\big[x(t)+x(t-1)\big] &= -5x(t),\ t>0 \\ x(\cdot)\in C([-1,0];\mathbb{R}). \end{aligned} \tag{28}$$

This equation is stable. In Figure 1, we simulate a trajectory of the solution $\Phi(\cdot,\phi)$ to (28) with the initial condition $\phi(t)=0.2$ on $[-1,0]$.

Example 7.2. Consider the equation

$$\begin{aligned} {}^{\mathrm{C}}D^{0.7}_{0+}\big[x(t)-1.5x(t-1)\big] &= -5x(t),\ t>0 \\ x(\cdot)\in C([-1,0];\mathbb{R}). \end{aligned} \tag{29}$$

The equation (29) is unstable. We depict the trajectory of its solution $\Phi(\cdot,\phi)$ when $\phi(t)=0.2$ on $[-1,0]$ in Figure 2 below.

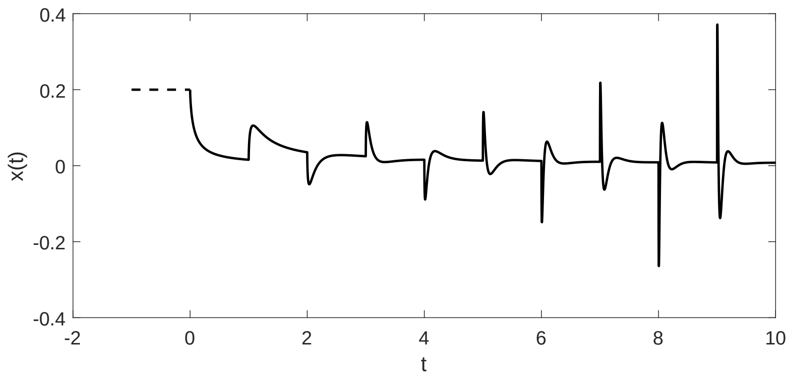

Figure 2: Trajectory of the solution $\Phi(\cdot, \phi)$ to system (29) when $\phi(t) = 0.2$ on $[-1, 0]$.

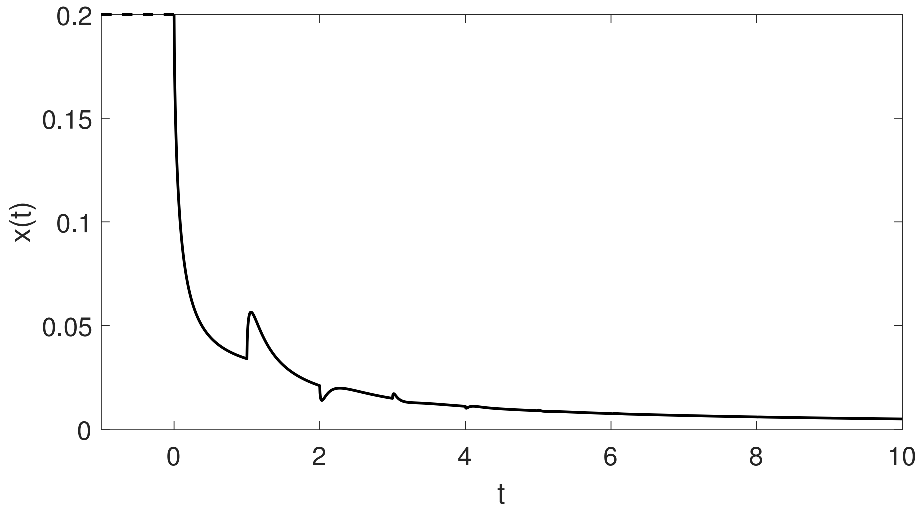

Figure 3: Trajectory of the solution $\Phi(\cdot,\phi)$ to system (30) when $\phi(t)=0.2$ on $[-1,0]$.

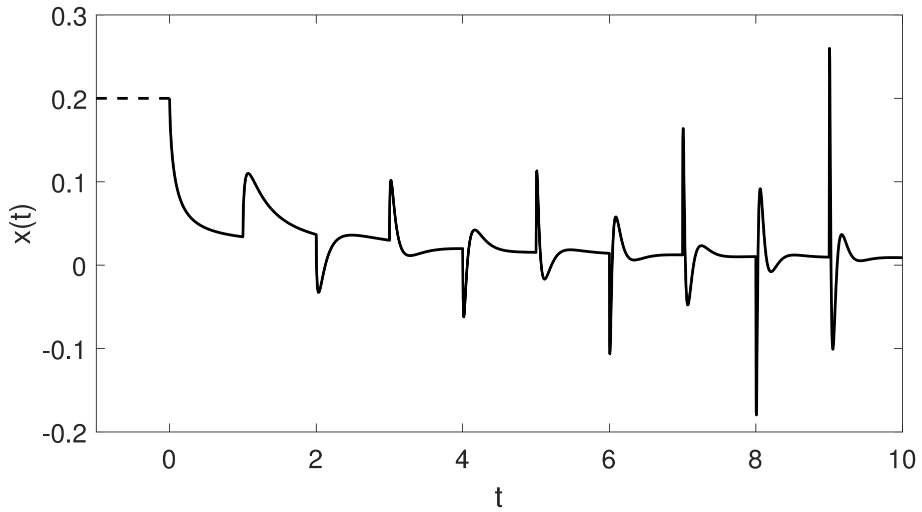

Figure 4: Trajectory of the solution $\Phi(\cdot,\phi)$ to system (31) when $\phi(t)=0.2$ on $[-1,0]$.

Example 7.3. Consider the equation

$$\begin{aligned} {}^{\mathrm{C}}D^{0.7}_{0+}\big[x(t)+0.5x(t-1)\big] &= -5x(t)+0.5x(t-1),\ t>0 \\ x(\cdot)\in C([-1,0];\mathbb{R}). \end{aligned} \tag{30}$$

As shown in Theorem 6.4 (i), this equation is asymptotically stable. In Figure 3, we simulate the trajectory of the solution $\Phi(\cdot,\phi)$ to (30) with the initial condition $\phi(t)=0.2$ on $[-1,0]$.

Example 7.4. Consider the equation

$$\begin{aligned} {}^{\mathrm{C}}D^{0.7}_{0+}\big[x(t)+1.5x(t-1)\big] &= -5x(t)+0.5x(t-1),\ t>0 \\ x(\cdot)\in C([-1,0];\mathbb{R}). \end{aligned} \tag{31}$$

As shown in Theorem 6.4 (ii), this equation is unstable. We simulate the trajectory of the solution $\Phi(\cdot,\phi)$ to (31) with the initial condition $\phi(t)=0.2$ on $[-1,0]$.

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