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The Funnel Tree Algorithm for Finding Shortest Paths on Polyhedral Surfaces

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Abstract In this paper, we present an O(n^2)-time algorithm for finding the exact shortest paths from a fixed source point to all other vertices on a triangulated polyhedral surface of a convex polytope in three-dimensional space using the concept of funnels on the surface. Our algorithm builds a funnel tree to compute such shortest paths. The funnel tree is built by recursively splitting funnels. Because their left borders are straightest geodesics, funnels are determined explicitly by the law of cosines. Known approaches such as the planar unfolding technique, source images and the projections of ones that take a lot of operations are avoided in our algorithm. Therefore, our algorithm outperforms the others on reduction of running time.

Our algorithm is implemented in Python. Comparing with Kaneva and O'Rourke's implementation of Chen and Han's algorithm, our algorithm runs significantly faster while the number of nodes in the tree is relatively small.

Keywords: Exact algorithm, funnel, planar unfolding, polytope, shortest path, straightest geodesic, tree.

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1 Introduction

Computing exact shortest paths from a fixed source point to all other points on a polyhedral surface is a well-studied problem in computational geometry. Chen and Han [2] provided an $O(n^2)$-time algorithm based on a key observation of "one angle one split" for determining the globally shortest path joining a given source to any point on the surface. The algorithm was implemented by Kaneva and O'Rourke [4]. Also, Kapoor [5] announced a further improvement using the wavefront method. Xin and Wang [11] presented an efficient visibility-based algorithm for determining a locally exact shortest path from a source point to a destination point on a (triangulated) polyhedral surface. Although Xin and Wang [12] improved Chen and Han's algorithm [2] with respect to the running time, finding exact solutions still requires a high time cost for very large models. Then, Xin and Wang [13] applied the improved Chen and Han's algorithm to different versions of the shortest path problems. They showed that the improved Chen and Han's algorithm can be naturally extended to the "multiple sources, any destination" version. Also, introducing a well-chosen heuristic factor into the improved Chen and Han's algorithm induced an exact solution to the "single source, single destination" version. All mentioned shortest paths are L_2 shortest paths.

In [1], we introduced the concept of funnels along a sequence of adjacent triangles in three-dimensional space and straightest geodesics inside a sequence of adjacent triangles. The straightest geodesics along a sequence of adjacent triangles are modified from the concept of straightest geodesics on a polyhedral surface of Polthier and Schmies ([9] and [10]). These concepts are used to compute the exact shortest path between two points along a sequence of adjacent triangles, by constructing a sequence of funnels without using the planar unfolding technique. Each funnel is determined by a final curve and some orienting curves according to Phu's method of orienting curves ([7] and [8]).

In this paper, the funnels on the surface of a polytope are defined and put in a tree to compute all globally shortest paths from the fixed source point to all destination points on a polyhedral surface. The main differences between Chen and Han's algorithm and our funnel tree algorithm are that

– We do not use the planar unfolding technique of Chen and Han's algorithm that costs many operations.
– We do not use the technique of source images and the projections of ones of Chen and Han's algorithm that also cost many operations, instead compare local angles.
– The left borders of our funnels are straightest geodesics that reduce many operations.

Therefore, the number of operations in our funnel tree algorithm is relatively small compared to the number of operations in Chen and Han's algorithm.

The funnel tree algorithm and Chen and Han's algorithm have the same $O(n^2)$ time complexity. We implemented our funnel tree algorithm in Python 3 and compared with Chen and Han's algorithm (implemented by Kaneva and O'Rourke [4]). The funnel tree algorithm runs significantly faster by a factor of 1 to 12 on instances of 100–800 vertices in an experimental study on 10 data sets (see Table 1) while the number of nodes in the funnel tree is relatively small (see Table 2 and Figures 14, 15, and 16).

2 Funnels and Funnel Trees on a Polytope

Consider a sequence of adjacent triangles $S := \bigcup_{i=1}^{m} f_i$, $m \geq 2$, where f_i are faces, $e_i = f_i \cap f_{i+1}$ are edges of a polytope. For $a, b \in S$, denote by $SP_S(a, b)$ the shortest path joining a and b in S.

In this paper, we define funnels on the surface of a polytope. Comparing with [1], the definition of funnels has a slight change because a such sequence S of adjacent triangles is not given and the number of adjacent triangles $m \geq 2$ is not fixed.

Let s be a vertex of f_1, p be a vertex of some f_i ($1 \leq i \leq m$) and q be a vertex of f_m (the last triangle of S) such that $q \neq p, s \notin e_1, i = 2, \ldots, m$ and if $i = m$ then $SP_S(p, q) = [p, q] = e_m$. If we view the polytope from the outside and from f_1 to f_m on S then $SP_S(s, p)$ is on the left of $SP_S(s, q)$. Let u be the furthest common vertex of $SP_S(s, p)$ and $SP_S(s, q)$ with respect to s. Let $F_{p,q,S}(s)$ denote the region of S bounded by $SP_S(s, p)$, $SP_S(s, q)$ and $SP_S(p, q)$.

Definition 1 The region $F_{p,q,S}(u)$ is called a *funnel* (of the polytope) associated with the vertex u and the path $SP_S(p, q)$. u is called the *cusp* of the funnel, $SP_S(u, p)$ is called the *left border* and $SP_S(u, q)$ is called the *right border* of the funnel. If $[v, q]$ is an edge of the polytope, v is called the *direct destination* of the funnel $F_{p,q,S}$.

In this paper, the polytope is assumed to be convex. Then, $SP(s, p) \cap SP(s, q) = \{s\}$ if $p \notin SP(s, q)$ and $q \notin SP(s, p)$. Hence, in Definition 1, $u \equiv s$. Therefore in the notations of funnels, the cusp u of funnels is omitted and we also use the notation $F_{p,q}$ for $F_{p,q,S}(u)$ and omit "the corresponding sequence S and the cusp u" if there is no confusion.

We recall the notion of straightest geodesics on polyhedral surface which was introduced by Polthier and Schimes ([9] and [10]). Let γ be a path on the surface of the polytope. Let w be a vertex of $\mathcal{K}$ and $\{g_1, \ldots, g_k\}$ be the set of faces of the polytope containing w as a vertex and α_i be the interior angle of the face f_i at the vertex w. Then the total angle α_w is given by $\alpha_w = \sum_{i=1}^{m} \alpha_i$. Because the polytope is convex, we have $\alpha(w) \leq 2\pi$. If $\alpha(w) < 2\pi$ ($\alpha(w) = 2\pi$, respectively), then w is called a spherical (an Euclidean, respectively) vertex. If w is a relative interior point of a face or an edge of the polytope, then w is an Euclidean vertex. We denote by α^l and α^r the left and right angles of the path at a point in which $\alpha^l + \alpha^r = \alpha$, where α is the total vertex angle of the point. γ is a straightest geodesic on the polytope if $\alpha^l = \alpha^r$ at w, for each point $w \in \gamma$.

As a shortest path joining s and a vertex is a straightest geodesic, such a path is the left border of some funnel with cusp s. From now on, we consider only funnels such that *those left borders are straightest geodesics*. If S consists of only one $\triangle spq$ then $F_{p,q,S}$ itself is the triangle. In Fig. 1, the funnel $F_{5,0,S_1}$ is triangle $\triangle(7,5,0)$, where $S_1 = \triangle(7,5,0)$. In this example, $SP(7,5) = SP(p,q) = [p,q]$ is an edge of the cube. In Fig. 2, the shaded regions are funnels, but $SP(p,q)$ is not an edge of the polytope.

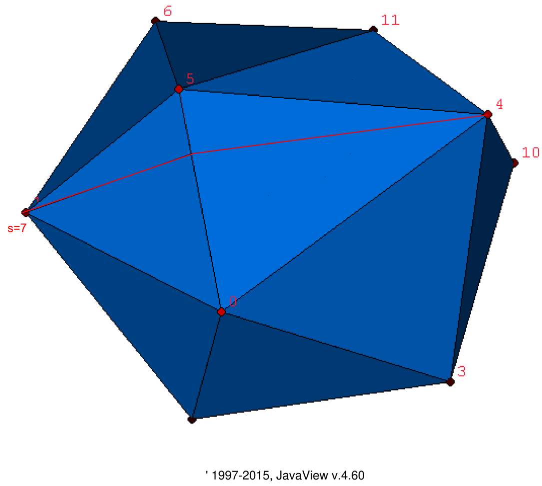

Fig. 1 Two funnels $F_{5,0,S_1}$ and $F_{0,4,S_2}$ on the surface of a cube, where $s := 7$, $S_1 = \triangle(7,5,0)$, $S_2 = \triangle(7,5,0) \cup \triangle(0,5,4)$.

Consider a funnel $F_{p,q,S}$ and suppose that $e_m = [p,q]$. Let $v \notin S$ be a vertex of the polytope such that $[q,v]$ is an edge of the polytope. Let $\overline{\triangle pqv}$ be the sequence of adjacent triangles of the polytope between e_i and $[q,v]$ having two edges incident to the vertex q and $S \cap \overline{\triangle pqv} = e_i$. $S' = S \cup \overline{\triangle pqv}$ is called a sequence of adjacent triangles of the polytope between two edges $[q,p]$ and $[q,v]$.

Definition 2 Consider a funnel $F_{p,q,S}$, its direct destination v and $S' = S \cup \overline{\triangle pqv}$. If $F_{p,v,S'}$ or $F_{v,q,S'}$ exist (i.e. $F_{p,q,S}$ and $F_{p,v,S'}$ or $F_{v,q,S'}$ have the same cusp s), it is called a *child* of $F_{p,q,S}$.

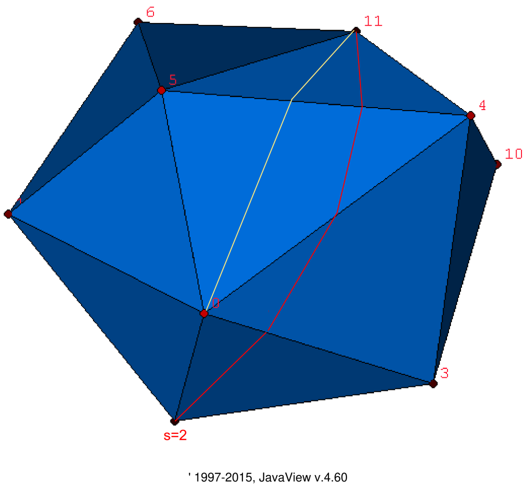

Fig. 2 The funnel $F_{0,11,S}$ (the region formed by the edge $[s, 0]$ and white and red paths) is associated with the vertex $s = 2$ and the path $SP_S(0, 11)$, where $S = \triangle(2, 0, 3) \cup \triangle(0, 4, 3) \cup \triangle(4, 0, 5) \cup \triangle(5, 11, 4)$.

In Fig. 1, vertex 1 is a direct destination of $F_{5,4,S_1}$ and funnel $F_{1,4,S'_1}$ is a child of $F_{5,4,S_1}$, where $S_1 = \triangle(7, 5, 4)$ and $S'_1 = S_1 \cup \triangle(1, 5, 4)$.

Let s be a point on the polyhedral surface. In [2], Chen and Han built a tree, called a sequence tree, that contains the shortest paths from s to all vertices of the surface, using the planar unfolding technique and the source images and the projections of them. Without loss of generality, assume that s is a vertex of the surface. In this section, we build a funnel tree using the concept of funnels. As can be seen below that a node of the funnel tree is a funnel with some direction. Let a funnel $F_{p,q,S}$ be defined as in Definition 1.

Definition 3 A *funnel tree* is a tree with s, where each node other than the root s is a triple $(F_{p,q,S}, SP_S(p, q), s)$, denoted by $F_{p,q,S}$. A node $F_{p,v,S'}$ is a child of the node $F_{p,q,S}$ if the funnel $F_{p,v,S'}$ is a child of the funnel $F_{p,q,S}$.

Two nodes (or two funnels with the same cusp) $F_{p,q,S}$ and F_{p,q,S_1} occupy a vertex v if v is the direct destination of both funnels $F_{p,q,S}$ and F_{p,q,S_1}.

3 Determining Children of a Funnel

Let $\overline{\triangle pqv}g$ be the sequence of adjacent triangles of the polytope between two edges $[q, p]$ and $[q, v]$. Let $S' = S \cup \overline{\triangle pqv}$. The existance of funnels $F_{p,v,S'}$ and $F_{v,q,S'}$ can be checked by comparing angles of triangles as follows.

Let $[v_1, q], \ldots, [v_j, q]$ be edges incident to q. Set

$$\beta_v = \beta_{v_j} := \angle spq + \angle v_j pq. \tag{1}$$

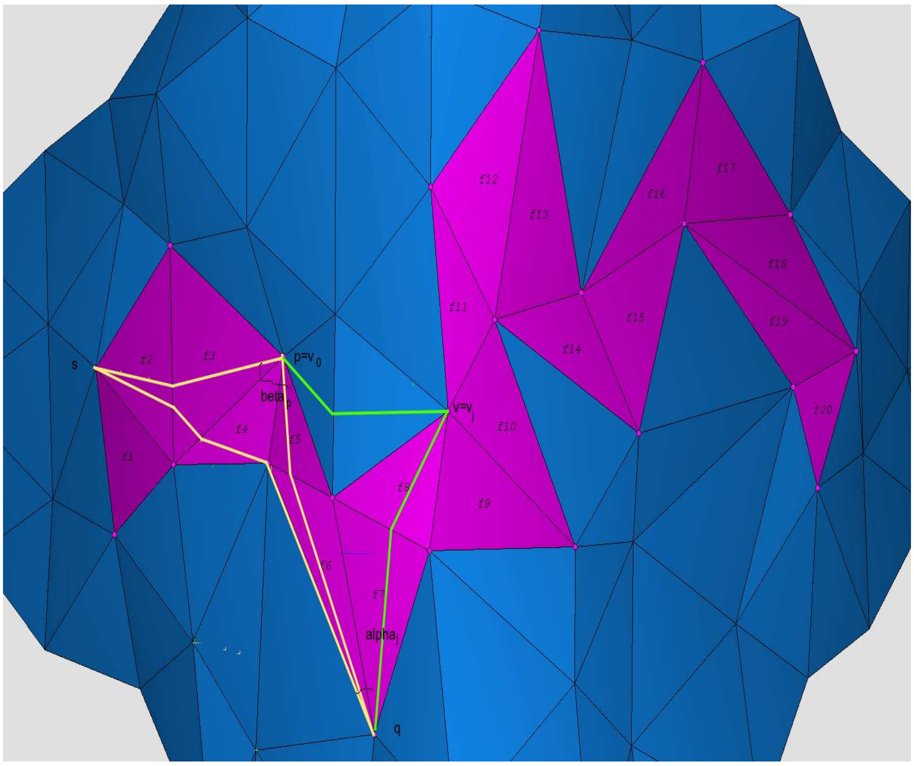

Fig. 3 F is formed by the yellow paths. As $\beta_p < \pi$, the funnel F has a child.

The edge pv_j can be computed from the relation

$$l([p,v_j]) = \sqrt{l([v_j,q])^2 + l([p,q])^2 - 2l([v_j,q])\cdot l([p,q])\cos\sum_{i=0}^{j}\angle v_i qp}, \tag{2}$$

(where $v_0 := p$) and then the angle $\angle v_j pq$ can be computed from the relation

$$\frac{l([v_j,p])}{\sin\angle v_j qp} = \frac{l([v_j,q])}{\sin\angle v_j pq} \tag{3}$$

(see Fig. 3). It follows from (1), (2), and (3) that

$$\begin{aligned}\beta_v &= \beta_{v_j}\\ &:= \angle spq + \arcsin\left(\frac{l([v_j,q])\sin\angle v_j qp}{\sqrt{l([v_j,q])^2 + l([p,q])^2 - 2l([v_j,q])\cdot l([p,q])\cos\sum_{i=0}^{j}\angle v_i qp}}\right).\end{aligned} \tag{4}$$

Let $v = v_j$ be a direct destination of $F_{p,q,S}$ and B_1, B_2 denote the left border and the right border of $F_{p,q,S}$, respectively. B_1 is a straightest geodesic. If $\beta \geq \pi$ then the relative interior of the funnel $F_{p,v,S'}(s)$ is empty, where s is a cusp of the funnel $F_{p,v,S'}(s)$. Hence $F_{p,q,S}$ has no child, see Fig. 4 (1). Otherwise, i.e.,

$$\beta_v < \pi, \tag{5}$$

let w be the first vertex on B_2 right after s. If

$$\angle psv < \angle psw \tag{6}$$

then $F_{p,q,S}$ has two children $F_{p,v,S'}$ and $F_{v,q,S'}$ (see Fig. 4 (2)). If $\angle psv \geq \angle psw$, then $F_{p,q,S}$ has one child $F_{vp,S'}$, see Fig. 4 (3).

In the case of (5), the triangle $\triangle pqv$ is well-defined by two edges $[q, v]$, $[q, p]$ and the angle $\sum_{i=0}^{j} \angle v_i qp$ between these edges. Let $\overline{\triangle pqv}$ be the sequence of adjacent triangles of the polytope between two edges $[q, p]$ and $[q, v]$.

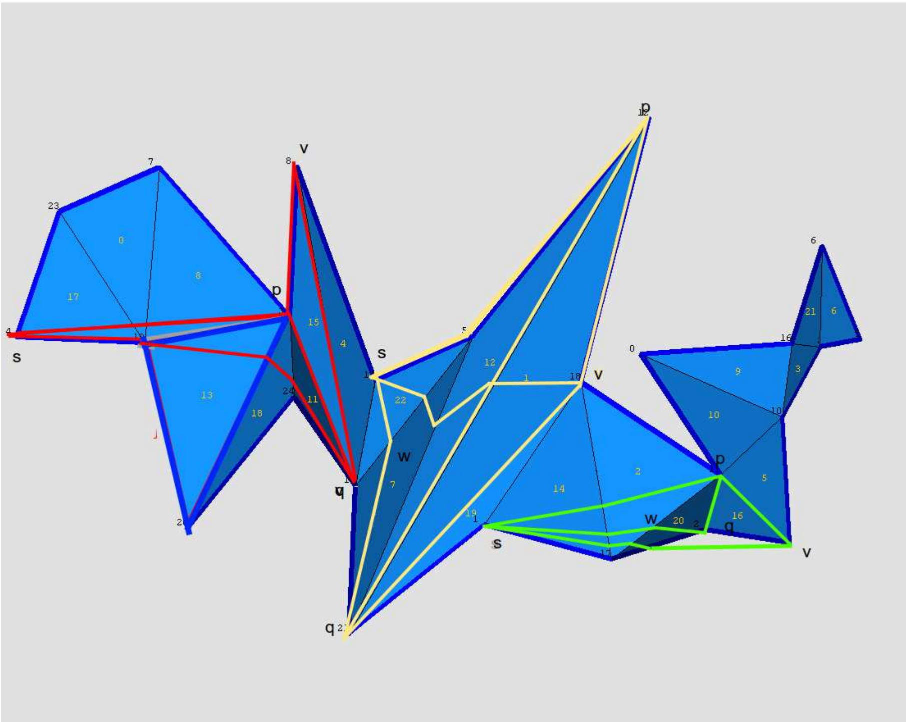

Fig. 4 (1) Funnel $F_{p,q,S}$ coloured in red, has no child (i.e., (5) does not hold), (2) Funnel $F_{p,q,S}$ coloured in yellow, has two children F_{pv} and $F_{v,q}$, (3) Funnel $F_{p,q,S}$ coloured green, in has one child F_{pv} (right boundary of the corresponding funnel includes the segment $[s, w]$).

Consider funnel $F_{p,t}$ and destination vertex v such that $SP(p, t)$ is not a line segment and some angles at t are not angles of faces of the polytope (in

Fig. 5, $SP(p,t) \neq [p,t]$ and the angle α_t at t is not the sume of angles of the polytope).

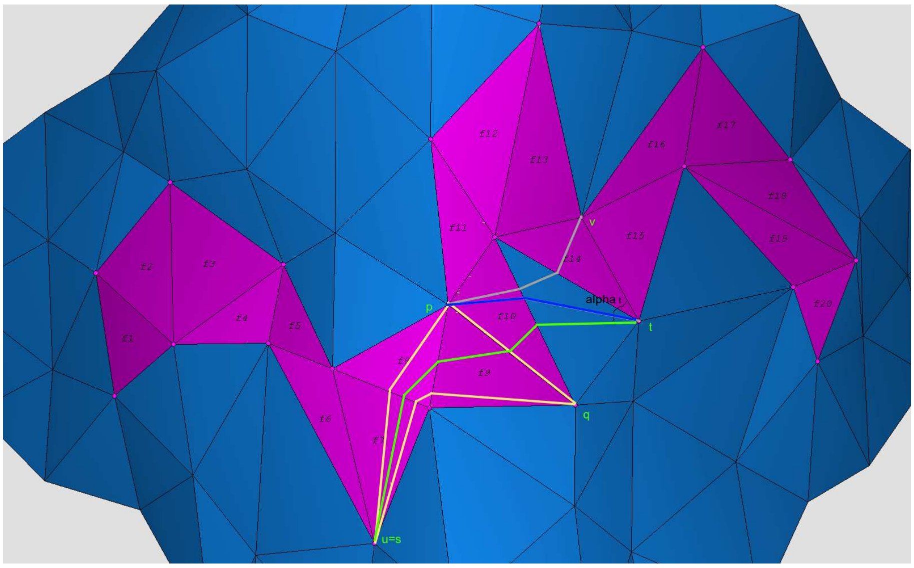

Fig. 5 Funnel $F_{p,q}$ is coloured in yellows. $SP(p,t)$ is not a line segment. Funnel $F_{p,t}$ formed by the yellow path $SP(s,p)$, the green path $SP(s,t)$, and the blue path $SP(p,v)$, has two children.

To find children of $F_{p,t}$, we use the law of cosines as follows:

– Using law of cosines for $\triangle ptq$ to get the length of $SP(p,v)$ and $\angle ptq$,
– Using law of cosines for $\triangle ptv$ to get the length of $SP(p,t)$ and $\angle vpt$,
– Using law of cosines for $\triangle puv$ to get the length of $SP(u,v)$, angles $\angle upv$ and $\angle tup$,

here $\triangle ptq$, $\angle vpt$, and $\angle puv$ indicate the corresponding triangles that have edges such that whose lengths are of the paths $SP(p,t), SP(t,q), SP(p,q), SP(v,t), SP(v,p), SP(u,v), SP(p,u)$.

4 The Funnel Tree Algorithm

The Algorithm 1 below builds the funnel tree by recursively splitting funnels. For both cases when determining children of one funnel, if the next triangle belong to sequence triangles S, then the funnel has no child.

Procedure Clip off Funnels($\overline{\triangle pqv}, S, S_1$) executes the three cases 2)-4) of Lemma 3 of the Appendix. It compares two funnels $F_{p,q,S}$ and F_{p,q,S_1} that occupy a vertex v of the sequence $\overline{\triangle pqv}$ and their children to determine which children will be deleted.

Algorithm 1 FUNNEL TREE FOR FINDING SHORTEST PATHS

Input: s is a vertex of the polyhedral surface.
Output: A funnel tree that contains the shortest paths from s to all vertices of the surface.

$root := s$
For all the edge $[p, q]$ opposite to s:
  Set $S = \triangle spq$
  Insert $F_{p,q,S}$ as root's children.
While $k \leq n$ and the k^{th} level has nodes: ▷ *n is the number of faces*
  For each funnel (node) $F_{p,q,S}$ at the k^{th} level:
    Let $v = v_j$ = direct destination of the funnel $F_{p,q,S}$ and β_v be determined by (4).
    While $\beta_v < \pi$ ▷ *i.e., (5) holds*
      Take the sequence S' of adjacent triangles of the polytope between $[p, q]$ and $[q, v]$ having two edges incident at q.
      Set $S' := S \cup \overline{\triangle pqv}$.
      If the funnels $F_{p,v,S'}(t_1)$ and $F_{v,q,S'}(t_2)$ has the same cusp s (i.e., $s = t_1 = t_2$) and (6) holds
        Then $F_{p,q,S}$ has two children $F_{p,v,S'}$, $F_{v,q,S'}$
        Insert the children that $F_{p,q,S}$ can have as follows
          If $\angle pvq$ is previously marked by another funnel called F_{p,q,S_1}
            Then call Procedure CLIP OFF FUNNELS$(\overline{\triangle pqv}, S, S_1)$
            ▷ *This procedure allows to implement the conclusions 2)-4) of Lemma 3 determines that child*
        Insert the two both children of $F_{p,q,S}$ and mark $\angle pvq$
      Else, $F_{p,q,S}$ has one child, then insert the child.
  $k+ = 1$

Procedure 2 CLIP OFF FUNNELS$(\overline{\triangle pqv}, S, S_1)$

Let l (l_1, respectively) be the length of the path $SP_S(s, v)$ ($SP_{S_1}(s, v)$, respectively).
  if $l < l_1$ ▷ applying Lemma 3 2)
    if $\angle pvz > \angle pvz_1$ ▷ z and z_1, respectively are the intersections of paths $SP_{S \cup \overline{\triangle pqv}}(s, v)$ and $SP_{S_1 \cup \overline{\triangle pqv}}(s, v)$, respectively with the line segment $[p, q]$.
      then delete child $F_{v,q,S_1 \cup \overline{\triangle pqv}}$ of F_{p,q,S_1}
    else if $\angle pvz < \angle pvz_1$
      then delete child $F_{p,v,S_1 \cup \overline{\triangle pqv}}$ of F_{p,q,S_1}
  if $l_1 < l$ ▷ applying Lemma 3 3)
    if $\angle pvz > \angle pvz_1$
      then delete child $F_{p,v,S \cup \overline{\triangle pqv}}$ of $F_{p,q,S}$
    else if $\angle pvz < \angle pvz_1$
      then delete child $F_{v,q,S \cup \overline{\triangle pqv}}$ of $F_{p,q,S}$
  else ▷ applying Lemma 3 4)
    if $\angle pvz > \angle pvz_1$
      then delete child $F_{p,v,S \cup \overline{\triangle pqv}}$ of $F_{p,q,S}$ and child $F_{v,q,S_1 \cup \overline{\triangle pqv}}$ of F_{p,q,S_1}
    else if $\angle pvz < \angle pvz_1$
      then delete child $F_{v,q,S \cup \overline{\triangle pqv}}$ of $F_{p,q,S}$ and child $F_{p,v,S_1 \cup \overline{\triangle pqv}}$ of F_{p,q,S_1}

Theorem 1 *Algorithm 1 computes all shortest paths on a triangulated polyhedron surface $\mathcal{P}$ from s to the other vertices of $\mathcal{P}$ in $O(n^2)$ time.*

Proof. Clearly, the left border of the funnel $F_{p,q,S}$ is a locally shortest path from s to p. Then a children which is not deleted, determines a locally shortest path from s to v. Thus a funnel tree determines the shortest paths from the vertex s to some vertices. By Lemma 4, Algorithm 1 determines all shortest paths from s to all other vertices.

For $k = 1$, the tree consists of just triangles incident to s, which is in fact the shortest paths from s to all vertices of those triangles. The tree grows by exploring a new vertex and updating shorter path lengths. Incrementing i for the next iteration of **for** loop and determing new funnels then preserve the loop invariant. The **for** terminate when no new funnel can be found, then there is no shortest path can be found.

As checking (4), (6), and (6) takes constant, time complexity can be determined if we determine the number of nodes of the funnel tree at the fixed k^{th}-level.

Take a fixed $k \leq n$. It follows from Lemma 3 that Procedure CLIP OFF FUNNELS($\overline{\triangle pqv}, S, S_1$) compares two funnels $F_{p,q,S}$ and F_{p,q,S_1} that occupy a vertex v of the sequence $\overline{\triangle pqv}$ and their children to determine which children will be deleted. Hence, the number of children is scalar with the number of funnels. Therefore, Algorithm 1 runs in $O(n^2)$. □

Note that Algorithm 1 computes all globally shortest paths from s to the other vertices of $\mathcal{P}$.

4.1 Examples

We now illustrate Algorithm 1. By writing $F_{p,q}$, we omit "the corresponding sequence S" in the notation of the funnel $F_{p,q,S}$ in the next example if there is no confusion.

Example 4.1.1: Given a cube with 8 vertices numbered from 0 to 7. The cube is triangulated as in Fig. 6.

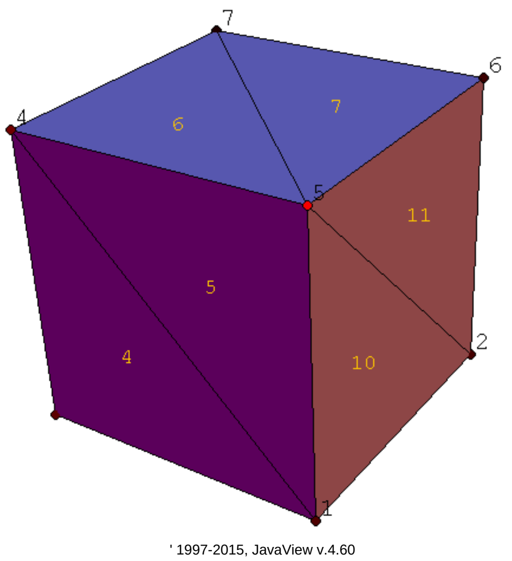

Fig. 6 A triangulated cube with 8 vertices and 12 faces.

The vertex 4 is chosen as the source point ($s = 4$ is the root of the funnel tree). There are 5 faces adjacent to the vertex 4 then the root of the funnel tree has 5 children: $F_{7,5}, F_{5,1}, F_{1,0}, F_{0,3}$, and $F_{3,7}$.

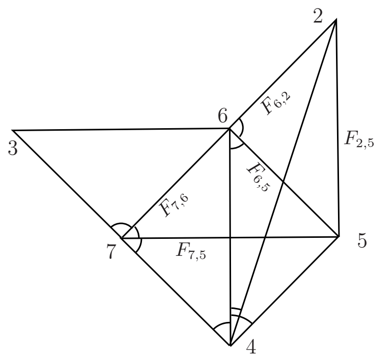

Fig. 7 For more intutive, funnel $F_{7,5}$ and other faces are planar unfoled.

Consider the funnel $F_{7,5}$ (Fig. 7).

$$l(SP(4,7)) = l([4,7]), \quad \angle 475 = \frac{\pi}{4}, \quad \angle 745 = \frac{\pi}{2}.$$

The direct destination of $F_{7,5}$ is vertex 6. We have $\angle 675 = \frac{\pi}{4}$, then $\angle 476 = \angle 475 + \angle 675 = \frac{\pi}{2} < \pi$. By (2), we get

$$l([4,6]) = \sqrt{l([4,7])^2 + l([7,6])^2 - 2l([4,7])l([7,6])\cos\angle 476}.$$

By (3), we have $\sin\angle 746 = \frac{l([7,6])\sin\angle 476}{l([4,6])}$. It implies that $\angle 746 = \frac{\pi}{4}$. Since $\angle 746 < \angle 745$, we obtain that $F_{7,5}$ has two children $F_{7,6}$ and $F_{6,5}$.

Funnel $F_{7,6}$ is determined by

$$l(SP(4,7)) = l([4,7]), \quad \angle 476 = \frac{\pi}{2}, \quad \angle 746 = \frac{\pi}{4}.$$

Funnel $F_{6,5}$ is determined by

$$\begin{aligned}
&l(SP(4,6)) = l([4,6]),\\
&\angle 465 = \angle 765 - \angle 764 = \angle 765 - (\pi - \angle 476 - \angle 746) = \frac{\pi}{4},\\
&\angle 645 = \angle 745 - \angle 746 = \frac{\pi}{4}.
\end{aligned}$$

Consider funnel $F_{7,6}$, its direct destination is vertex 3. We have $\angle 376 = \frac{\pi}{2}$. Then $\angle 476 + \angle 376 = \pi$. In this case, funnel F_{76} has no child with the direct destination point 3, but the funnel will have a child at some direct destination point. In this case, such a funnel and the corresponding node are called as abnormal funnel and abnormal node, respectively.

Consider funnel $F_{6,5}$, its direct destination is vertex 2. From (2)-(3), we check that $F_{6,5}$ has two children $F_{6,2}$ and $F_{2,5}$. The other funnels of funnel tree are found in the same way. This tree has 106 nodes including 41 abnormal nodes (abnormal funnel). Figure 9 shows three levels of this tree.

For example, to find the shortest path joining vertex 4 to vertex 2, we see that funnels $F_{6,5}, F_{1,3}$ and $F_{3,6}$ occupy vertex 2. The shortest path $SP(4,2)$ along $F_{2,5}$ passes through the interior of edges 65 and 75. Our problem changes to find the shortest path $SP(4,2)$ passes through the interior of edges 75 and 65. We use JavaView function to show the shortest path (Fig. 8). This is exactly the shortest path we are looking for.

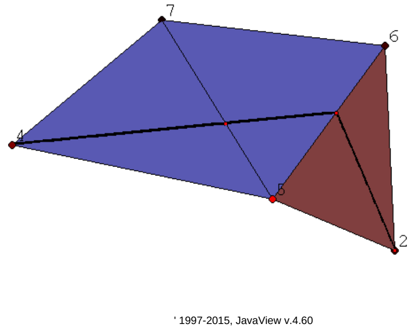

Fig. 8 The shortest path joining vertex 4 to vertex 2 passes through edges 75 and 65.

Example 4.1.2: Given a polytope has 12 vertices with 20 faces as in Fig. 10. Vertex 0 is choosen as the source point. In this example the needed information of a funnel $F_{p,q}$ is stored in a 3-tuple (x, y, z) where x is the angle the cusp in degree, $y = [s, p]$ is the length of its left boundary (length from the source point s to point p), and z is the angle $\angle spq$.

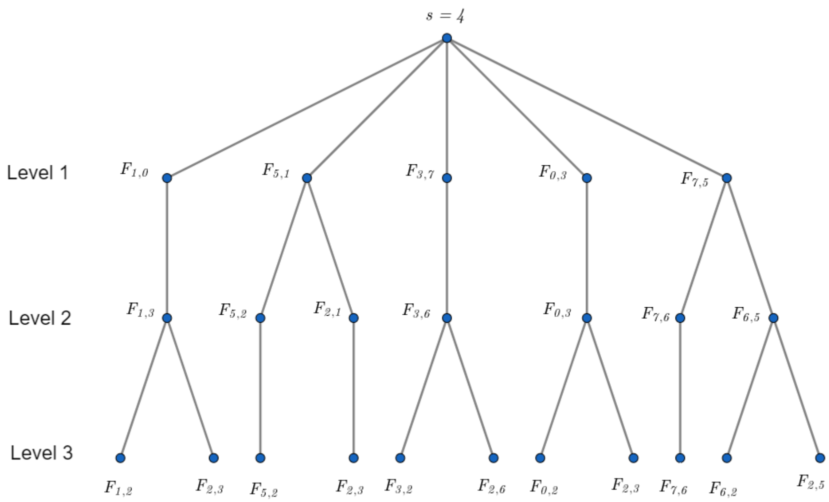

Fig. 9 The first three level of funnel tree for example 4.1.1

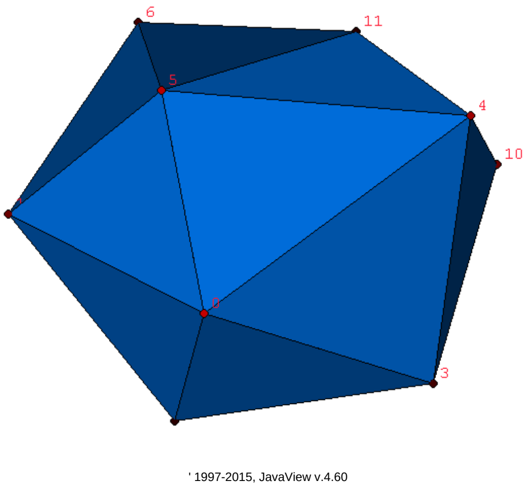

Fig. 10 A polytope with 12 vertices and 20 faces.

For $k = 1$, the source point 0 has five children:

1. $F_{5,4} = (65.95, 0.77, 68.71)$
2. $F_{4,3} = (60.20, 1.01, 59.09)$
3. $F_{3,2} = (60, 0.99, 60)$
4. $F_{2,1} = (59.66, 0.99, 60.67)$
5. $F_{1,5} = (49.39, 1, 49.39)$.

For $k = 2$, the children at nodes are as follows

1. $F_{5,4}$ has two children: $F_{5,11}$ and $F_{11,4}$. $F_{5,11} = (29.52, 0.77, 128.25)$, $F_{11,4} = (36.43, 1.59, 37.30)$.
2. $F_{4,3}$ has two children $F_{4,10}$ and $F_{10,3}$. $F_{4,10} = (30.91, 1.01, 118.18)$, $F_{10,3} = (29.30, 1.73, 29.30)$.
3. $F_{3,2}$ has two children $F_{3,9}$ and $F_{9,2}$. $F_{3,9} = (29.66, 0.99, 120.67)$, $F_{9,2} = (30.34, 1.72, 30)$.
4. $F_{2,1}$ has two children $F_{2,7}$ and $F_{7,1}$. $F_{2,7} = (29.66, 0.99, 120.67)$, $F_{7,1} = (30, 1.72, 30.34)$.
5. $F_{1,5}$ has two children $F_{1,6}$ and $F6, 5$. $F_{1,6} = (35.43, 1, 107, 62)$, $F_{6,5} = (13.96, 1.59, 12.44)$.

For $k = 3$, we have 14 nodes. For $k = 4$, we have 23 nodes. The other levels can be calculated in the same way. This tree has 196 nodes. Figure 11 shows the shortest paths for Example 4.1.2.

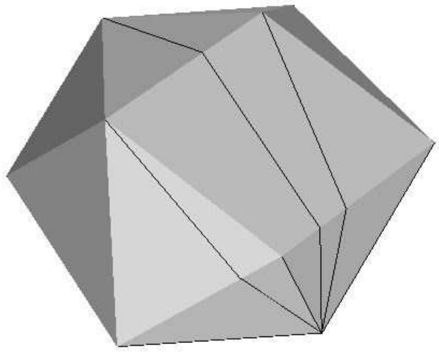

Fig. 11 The shortest paths from vertex 0 to other vertices.

4.2 Advantages of the Funnel Tree Algorithm over Chen and Hans's one

Chen and Han's algorithm used the planar unfolding technique and projection method. Our algorithm uses funnel trees. In concrete,

– Using the conditions (1)-(5)-(6), we build funnel trees without using the planar unfolding technique that costs many operations.
– In our algorithm, we do not need to check if a vertex belongs to the shadow of a node, rotating operations, two sign-area operations.
– In our algorithm, the children of a funnel can be determined by comparing the angles with a small number of operations.

Experimental results given the the next section also show that our funnel tree algorithm runs faster in clock time for several instances of spheres, cubes, and spirals.

4.3 Implementation and Experimental Results

To build a funnel tree, we find children of a given funnel, namely, determine funnels $F_{p,v,S'}$, $F_{v,q,S'}$ from a given funnel $F_{p,q,S}$, where its direct destination is v and $S' = S \cup \overline{\triangle pqv}$. (5)-(6) are enough to determine such funnels with a small number of operations.

Indeed, the funnels $F_{p,v,S'}$ and $F_{v,q,S'}$ can be determined by comparing the angle $\angle spv$ and π then comparing angles $\angle psv$ and $\angle psw$. It takes 2 operations to compute the angle $\angle spv$ and compare this angle with π, 9 operations to compute edge sv in (2) and 4 operations to compute the angle $\angle psv$ in (3). Therefore, for the present funnel $F_{p,q,S}$, the new funnel $F_{p,v,S'}$ can be determined in 15 operations. If $\angle psv < \angle psq$ then $F_{p,q,S}$ has child $F_{v,q,S'}$ and this child can be determined in 5 more operations.

Experimental results.

We implemented our funnel tree algorithm in Python 3 and compared with Chen and Han's algorithm (implemented by Kaneva and O'Rourke [4]). Computer configurations: Ubuntu 16.04, CPU Core i7 2.6 GHz, RAM 16 GB.

• Table 1 shows the running time of two algorithms. We can see in most cases that the Funnel Tree algorithm is faster than Chen and Han's algorithm, especially in the case where the polytope has a cube shape.

• Table 2 shows the number of nodes on the trees generated by two algorithms. The number of nodes on the funnel tree is smaller than one of the Chen and Han's algorithm.

• Figures 14, 15, and 16 show the trend of the number of nodes generated by the funnel tree algorithm. Based on this trend we can see that the number of nodes does not increase too fast and will decrease when it reaches the peak.

Dataset	Number of vertices	Number of faces	Running time Funnel tree (ms)	Running time Chen & Han (ms)
cube1	115	226	35	32
cube2	132	260	40	45
cube3	170	336	31	80
cube4	192	380	51	111
sphere1	195	898	46	79
sphere2	347	1076	75	283
sphere3	523	658	121	857
sphere4	807	394	159	2037
spiral1	300	196	101	208
spiral2	400	596	112	373

Table 1 Running time of the funnel tree algorithm, and Kaneva & O'Rourke's implementation for Chen and Han's algorithm

Dataset	Number of vertices	Number of faces	Number of nodes Funnel tree	Number of nodes Chen & Han
cube5	141	278	1492	3809
cube6	298	592	4377	9456
cube7	229	454	3742	6828
cube8	276	548	4703	9165
sphere3	451	898	3554	13447
sphere4	540	1076	3959	17649
sphere5	331	658	3293	9104
sphere6	199	394	1797	6177
spiral3	100	196	1663	3399
spiral4	300	596	3805	9321

Table 2 The number of nodes of the funnel tree algorithm, and Kaneva & O'Rourke's implementation for Chen and Han's algorithm

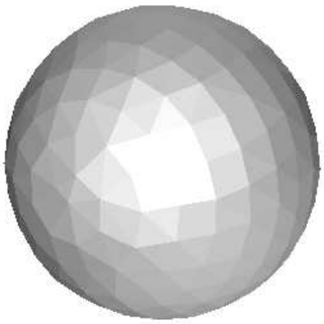

Fig. 12 The polytope named "spiral1' given in Table 1 and having 300 vertices.

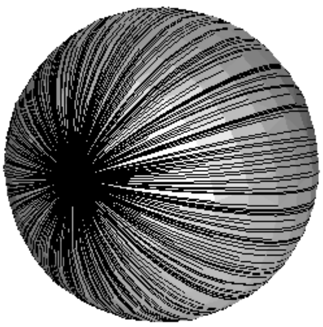

Fig. 13 The shortest paths joining a vertex with all other 299 vertices of the polytope "spiral1" determined by the funnel tree algorithm.

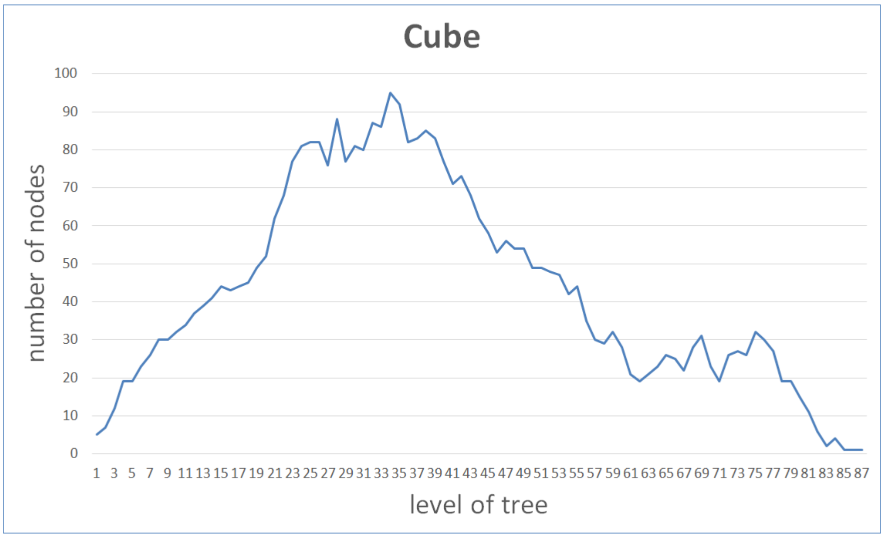

Fig. 14 The trend of the number of nodes when running algorithm which a polytope has a cube shape given in Table 2

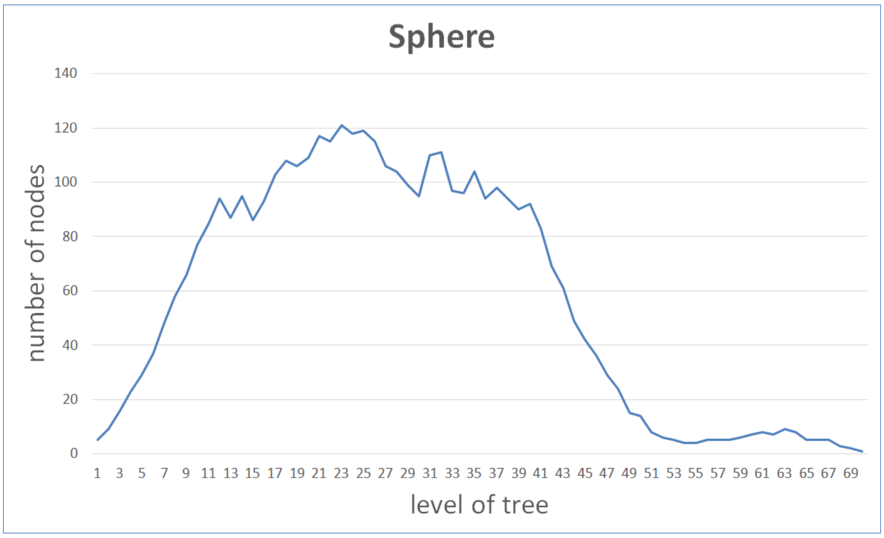

Fig. 15 The trend of the number of nodes when running algorithm which a polytope has a sphere shape given in Table 2

5 Concluding Remarks

In this paper, for a clear exposition, we restrict our dicussion to the case of convex polytope and build a funnel tree. The case of non-convex polytope is done similarly as in Sect. 5 [2].

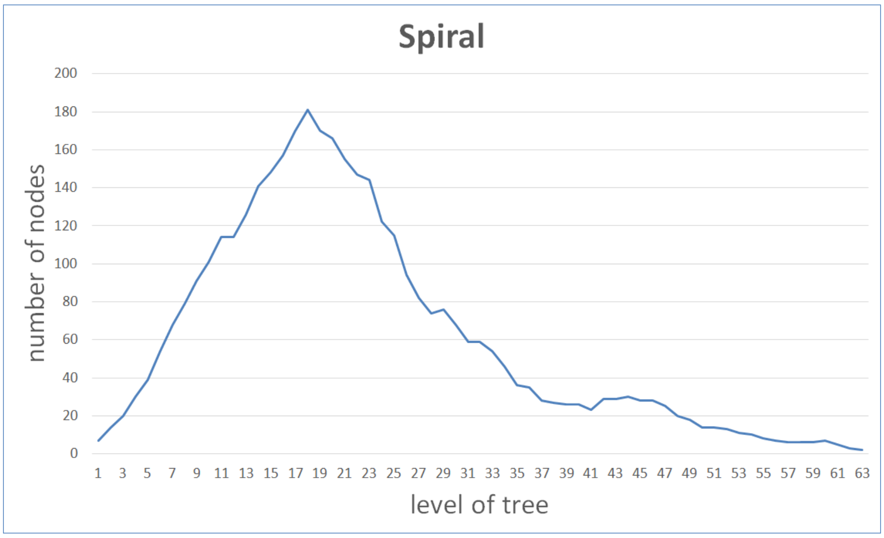

Fig. 16 The trend of the number of nodes when running algorithm which a polytope has a spiral shape given in Table 2

The concept of funnels associated with an edge and having the cusp u can be extended to the concept of segment-funnels associated with two edges. Similar to Algorithm 1, we can build a funnel tree of segment-funnels and then get the shortest paths from the source edge e to all destination edges on a polyhedral surface.

Given a number of circular obstacles in a plane, the problem of computing the shortest path between two points can be solved approximately by approximating circular obstacles with convex polygonal obstacles, then compute the shortest path avoiding polygonal obstacles. Therefore, the result is an approximate solution. We want to find the exact solution of the fowlling problem: "Given a set of disks with different radii on the faces of a terrain (each face may have some such disks), given an object s on terrain (s may be points, segments, faces), but outside the disks, our aim is to present efficient algorithms for computing disk-avoiding shortest paths from s to all destonation objects on the terrain". The special case when the terrain is in a plane, s and t are points was considered by Kim et al. in [6]. Hopefully, the funnel technique still is used successfully for this problem.

The problem of finding all shortest paths from a source point s to all destination vertices t can be used to solve the problem of finding all shortest paths from a source point s to all destination points t on the polytopes with the same complexity, where n is the number of vertices of the polytope. Indeed, let t be an arbitrary point in a triangle face abc of the polytope. Then $SP(s,t)$ belongs to one of funnels $F_{a,b}, F_{b,c}$, or $F_{c,a}$.

6 Appendix

Lemma 1 *Take s_1, s_2, p, q, v are on the plane such that $l([v, s_1]) \geq l([v, s_2])$ ($l([v, s_1]) \leq l([v, s_2])$, respectively), $[v, s_1] \cap [p, q] \neq \emptyset$, $[v, s_2] \cap [p, q] \neq \emptyset$, q and s_2 are on the same side with the line vs_1. Then $l([x, s_1]) > l([x, s_2])$ for all $x \in [v, q]$ ($l([x, s_1]) < l([x, s_2])$ for all $x \in [p, v]$, respectively).*

Proof. Since $[v, s_1] \cap [p, q] \neq \emptyset$, $[v, s_2] \cap [p, q] \neq \emptyset$, q and s_2 are on the same side with the line vs_1, we conclude that v and q are on the same side with the perpendicular bisector of the line segment $[s_1, s_2]$ (see Fig. 17). $l([v, s_1]) > l([v, s_2])$ implies that $l([x, s_1]) > l([x, s_2])$ for all $x \in [v, q]$.

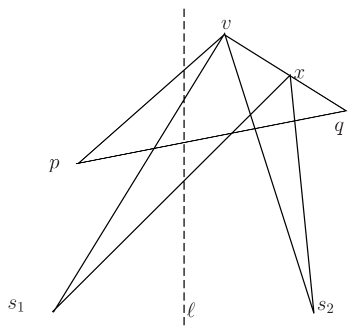

Fig. 17 As v and q are on the same side with the perpendicular bisector m of the line segment $[s_1, s_2]$, we have $l([x, s_1]) > l([x, s_2])$ for all $x \in [v, q]$.

□

Lemma 2 *Given on the same path $SP_S(p, q)$ two funnels $F_{p,q,S}$ and F_{p,q,S_1}, assume that $S \neq S_1$. Then $\angle pvz \neq \angle pvz_1$, where z and z_1, respectively are the intersections of paths $SP_{S \cup \overline{\triangle xqv}}(s, v)$ and $SP_{S_1 \cup \overline{\triangle xqv}}(s, v)$, respectively with the path $SP_S(p, q)$, where $\overline{\triangle xqv}$: the sequence of adjacent triangles of the polytope between two edges $[q, v]$ and $[q, x]$.*

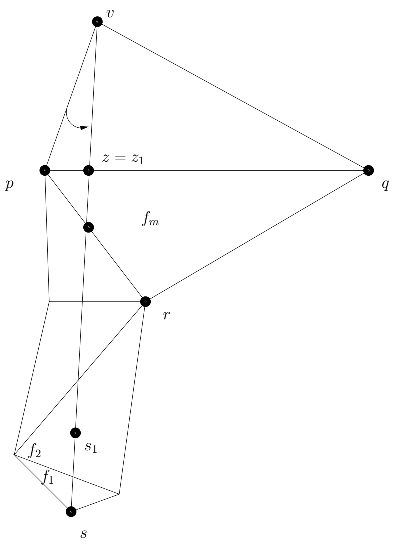

Fig. 18 $\angle pvz = \angle pvz_1$ implies that s, s_1 and v are collinear. Hence, if $s, s_1 \notin \triangle pq\bar{r}$ then $S_1 \subset S$ or $S \subset S_1$.

Proof. For simplicity, for the funnel $F_{a,b}$ and its direct destination c, we assume that $\overline{\triangle abc}$ has only one triangle, i.e., $\overline{\triangle abc} = \triangle abc$.

Assume the contrary that $\angle pvz = \angle pvz_1$. Suppose that I and I_1 are the images of s via the unfolds S and S_1 on the plane of $SP_S(p,q)$ and v. $\angle pvz = \angle pvz_1$ implies that I, I_1 and v are collinear on the plane of $SP_S(p,q)$ and v. Assume that $\triangle pqr$ is an adjacent triangle with $\triangle pqv$ (see Fig. 18). Then, $\bar{r}$ is an image of r on the plane $\triangle pqv$ via both two unfolds as $\triangle pqr \subset S \cap S_1$. There are two cases:

i) $I \in \triangle pq\bar{r}$ or $I_1 \in \triangle pq\bar{r}$. Then $S = S_1 = \triangle pqr$, a contradiction.

ii) $I, I_1 \notin \triangle pq\bar{r}$. Then the line segment vI intersects the ege $p\bar{r}$ or the edge $q\bar{r}$ of $\triangle pq\bar{r}$. Assume that vI intersects the ege $p\bar{r}$. Because I, I_1 and v are collinear and vI and vI_1 lie on the images of S and S_1 via the unfold, $S_1 \subset S$ or $S \subset S_1$. Hence, these two unfolds coincide and therefore, $S = S_1$, a contradiction. □

On the same path $SP_S(p,q)$, assume that $F_{p,q,S}$ and F_{p,q,S_1} $(S \neq S_1)$ have the same direct destination v. Then at most one of these funnels can have two children which can be used to define the funnel tree. This can be seen in the following.

Lemma 3 *Given on the same path $SP_S(p,q)$ two funnels $F_{p,q,S}$ and F_{p,q,S_1} $(S \neq S_1)$ which occupy vertex v of the sequence $\overline{\triangle xqv}$, let l (l_1, respectively) be the length of the path $SP_S(s,v)$ ($SP_{S_1}(s,v)$, respectively), z and z_1, respectively* *be the intersections of paths $SP_S(s,v)$ and $SP_{S_1}(s,v)$, respectively with the path $SP_S(p,q)$. Then*

1) If $l < l_1$ then $F_{p,q,S}$ has two children $F_{p,v,S\cup\overline{\triangle xqv}}$ and $F_{v,q,S\cup\triangle xqv}$.

If $\angle pvz > \angle pvz_1$ then the child $F_{v,q,S_1\cup\overline{\triangle xqv}}$ of F_{p,q,S_1} is deleted.

If $\angle pvz < \angle pvz_1$ then the child $F_{p,v,S_1\cup\overline{\triangle xqv}}$ of F_{p,q,S_1} is deleted.

2) If $l > l_1$ then F_{p,q,S_1} has two children $F_{p,v,S_1\cup\overline{\triangle xqv}}$ and $F_{v,q,S_1\cup\overline{\triangle xqv}}$ and one child of $F_{p,q,S}$ is deleted similar to case 2), namely,

If $\angle pvz > \angle pvz_1$ then the child $F_{p,v,S\cup\overline{\triangle xqv}}$ of $F_{p,q,S}$ is deleted.

If $\angle pvz < \angle pvz_1$ then the child $F_{v,q,S\cup\overline{\triangle xqv}}$ of $F_{p,q,S}$ is deleted.

3) If $l = l_1$ then $F_{p,q,S}$ ties F_{p,q,S_1} and one child of $F_{p,q,S}$ and one child of F_{p,q,S_1} are deleted similar to case 2), namely,

If $\angle pvz > \angle pvz_1$ then the child $F_{p,v,S\cup\overline{\triangle xqv}}$ of $F_{p,q,S}$ and the child $F_{v,q,S_1\cup\triangle xqv}$ of F_{p,q,S_1} are deleted.

If $\angle pvz < \angle pvz_1$ then the child $F_{v,q,S\cup\overline{\triangle xqv}}$ of $F_{p,q,S}$ and the child $F_{p,v,S_1\cup\triangle xqv}$ of F_{p,q,S_1} are deleted.

Proof. For simplicity, for the funnel $F_{a,b}$ and its direct destination c, we assume that $\overline{\triangle abc}$ has only one triangle, i.e., $\overline{\triangle abc} = \triangle abc$.

We indicate that which funnel can have two children. Set $S' = S \cup \triangle xqv$ and $S'_1 = S_1 \cup \triangle xqv$. We consider the followings:

i) If $l < l_1$ then we prove that F_{p,q,S_1} has one child and we let $F_{p,q,S}$ have two children. Indeed, we unfold S and S_1 on the plane of $SP_S(p,q)$ and v, and suppose thatI, I_1 are images of s on the plane of $SP_S(p,q)$ and v. It follows from $S \neq S_1$ and Lemma 2 that $\angle pvz \neq \angle pvz_1$.

i1) If $\angle pvz > \angle pvz_1$ then by Lemma 1, $l([s,k]) < l([s_1,k])$ for all $k \in [v,q]$. We are in position to prove that the funnel F_{v,q,S'_1} is deleted. Assume that F_{v,q,S'_1} and $F_{v,q,S'}$ occupy some vertex y of the polytope and $\triangle vqy$ and $\triangle xqv$ have the same $SP_S(v,q)$. It follows that

$$l(SP_{S'\cup\triangle vqy}(s,y)) < l(SP_{S'_1\cup\triangle vqy}(s,y)).$$

Then, the funnel F_{v,q,S'_1} is deleted.

i2) If $\angle pvz < \angle pvz_1$ then again, by Lemma 1, $l([s,k]) < l([s_1,k])$ for all $k \in [p,v]$. We are in position to prove that the funnel F_{p,v,S_1} is deleted. Assume that F_{p,v,S'_1} and $F_{p,v,S'}$ occupy some vertex w of the polytope and $\triangle xvw$ and $\triangle xqv$ have the same common $SP_S(p,v)$. It follows that

$$l(SP_{S'\cup\triangle vpy}(s,w)) < l(SP_{S'_1\cup\triangle vpy}(s,w)).$$

Then, the funnel F_{p,v,S'_1} is deleted.

ii) Similarly, if $l_1 < l$, F_{p,q,S_1} has two children while $F_{p,q,S}$ has only one child which can be used to define the funnel tree.

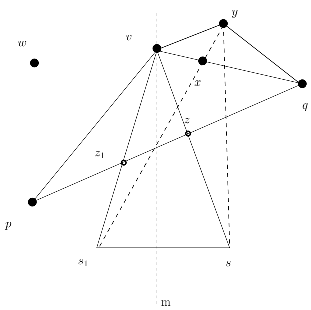

Fig. 19 $l_1 = l$. $\angle pvz > \angle pvz_1$ implies that $F := F_{v,q,S}$ and $F_1 := F_{v,p,S_1}$ are not overlap then $F_{v,q,S}$ and F_{v,p,S_1} can be used to define the funnel tree.

iii) Finally, if $l = l_1$ then $F_{p,q,S}$ ties F_{p,q,S_1}. If $\angle pvz > \angle pvz_1$ ($\angle pvz < \angle pvz_1$, respectively) then, as seen in Fig. 19, $F := F_{v,q,S'}$ and $F_1 := F_{p,v,S_1'}$ ($F_{p,v,S'}$ and $F_{v,q,S_1'}$, respectively) are not overlap then $F_{v,q,S'}$ and $F_{p,v,S_1'}$ ($F_{p,v,S}$ and F_{v,q,S_1}, respectively) can be used to define the funnel tree.

We now consider whether $F_{v,q,S_1'}$ ($F_{p,v,S'}$, respectively) is not a child of the funnel F_{p,q,S_1} ($F_{p,q,S}$, respectively).

iii1) If $\angle pvz > \angle pvz_1$ then by Lemma 1, $l([s,k]) < l([s_1,k])$ for all $k \in [v,q]$ and $l([s,\bar{k}]) > l([s_1,\bar{k}])$ for all $k \in [v,p]$. Assume that $F_{v,q,S_1'}$ and $F_{v,q,S'}$ occupy some vertex y of the polytope and $\triangle vqy$ and $\triangle xqv$ have the same $SP_S(v,q)$ and $F_{p,v,S'}$ and $F_{p,v,S_1'}$ occupy some vertex w of the polytope and $\triangle vxw$ and $\triangle xqv$ have the same $SP_S(v,p)$. It follows that

$$l(SP_{S'\cup\triangle vqy}(s,y)) < l(SP_{S_1'\cup\triangle vqy}(s,y))$$

and

$$l(SP_{S_1'\cup\triangle vxw}(s,w)) < l(SP_{S'\cup\triangle vxw}(s,w)).$$

Then, the funnels $F_{v,q,S_1'}$ and $F_{p,v,S'}$ are deleted.

iii2) If $\angle pvz < \angle pvz_1$ then again, by Lemma 1, $l([s,k]) > l([s_1,k])$ for all $k \in [v,q]$ and $l([s,\bar{k}]) < l([s_1,\bar{k}])$ for all $k \in [v,p]$. Assume that $F_{v,q,S_1'}$ and $F_{v,q,S'}$ occupy some vertex y of the polytope and $\triangle vqy$ and $\triangle xqv$ have the same $SP_S(v,q)$ and $F_{p,v,S'}$ and $F_{p,v,S_1'}$ occupy some vertex w of the polytope and $\triangle vxw$ and $\triangle xqv$ have the same $SP_S(v,p)$. It follows that

$$l(SP_{S'\cup\triangle vqy}(s,y)) > l(SP_{S_1'\cup\triangle vqy}(s,y))$$

and

$$l(SP_{S_1'\cup\triangle vxw}(s,w)) > l(SP_{S'\cup\triangle vxw}(s,w)).$$

Then, the funnels $F_{v,q,S'}$ and F_{p,v,S_1} are deleted. □

Lemma 4 *All funnels that contain shortest paths starting from s are among the funnels computed in Algorithm 1.*

Proof. Let v be an arbitrary vertex on the surface of the polyhedron. Because the polyhedron is convex, there always exists at least on shortest path $SP(s, v)$ from s to v on the polyhedral surface. Let S be the sequences of triangles that contains $SP(s, v)$. Because the number of faces is finite, the number of triangles in S and the number of direct destinations are also finite. For a funnel F_{p,q,S^*} ($S^* \subset S$), and for a direct destination $u \in S$ of F_{p,q,S^*}, if F_{p,q,S^*} has no child as in line 8 of Algorithm 1, the algorithm skips vertex u, updates $S^* = S^* \cup \overline{\triangle pqu}$, and then moves on the next direct destination in the next level. Therefore, after finite steps when vertex v is reached, a funnel $F_{v,q,S}$ having $SP(s, v)$ as the left border is formed. Assume for the purpose of contradiction that $F_{v,q,S}$ is deleted by Lemma 3, then there exists a shorter path from s to v along the surface of the polyhedron. This contradicts the fact that $SP(s, v)$ is the shortest path joining s and v on the polyhedral surface. The proof of the lemma is completed.

□

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