THE BIFURCATION SET OF A RATIONAL FUNCTION VIA NEWTON POLYTOPES

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Abstract. The bifurcation sets of polynomial functions have been studied by many mathematicians from various points of view. In particular, Némethi and Zaharia described them in terms of Newton polytopes. In this paper, we will show analogous results for rational functions.

1. Introduction

Let $f(z) \in \mathbb{C}[z_1, \dots, z_n]$ be a polynomial of n (≥ 2) variables. Then for the function $f\colon \mathbb{C}^n \longrightarrow \mathbb{C}$ defined by it there exists a finite subset $B \subset \mathbb{C}$ such that the restriction

$$\mathbb{C}^n \setminus f^{-1}(B) \longrightarrow \mathbb{C} \setminus B$$

of f is a C^∞ locally trivial fibration. We denote by B_f the smallest subset $B \subset \mathbb{C}$ satisfying this property. Let $\mathrm{Sing} f \subset \mathbb{C}^n$ be the set of the critical points of $f\colon \mathbb{C}^n \longrightarrow \mathbb{C}$. Then by the definition of B_f, obviously we have

$$f(\mathrm{Sing} f) \subset \mathrm{B}_f.$$

The elements of B_f are called bifurcation values of f. The description of the bifurcation set $\mathrm{B}_f \subset \mathbb{C}$ is a fundamental problem and was studied by many mathematicians e.g. [3], [4], [5], [9], [10], [12], [14], [19], [22], [26], [29], [31], [32] and [33] etc. The essential difficulty lies in the fact that in general f has a lot of singularities at infinity. In [19], Némethi and Zaharia succeeded in describing B_f in terms of the Newton polytope of f. For the generalizations to polynomial maps $f = (f_1, \dots, f_k) : \mathbb{C}^n \to \mathbb{C}^k$ for $n \geq k \geq 1$, see [4] and [22]. For the generalization to mixed polynomials, see [5].

In this paper, we will show that analogous results hold for rational functions. Let $P(z), Q(z) \in \mathbb{C}[z_1, \dots, z_n]$ be polynomials of n (≥ 2) variables. Assume that they are coprime each other. Let

$$f(z) = \frac{P(z)}{Q(z)} \qquad (z \in \mathbb{C}^n \setminus Q^{-1}(0))$$

be the rational function defined by them and consider the map $f\colon \mathbb{C}^n \setminus Q^{-1}(0) \longrightarrow \mathbb{C}$ associated to it. Then as in the case of polynomial maps we can define the bifurcation set $\mathrm{B}_f \subset \mathbb{C}$ of f such that $f(\mathrm{Sing} f) \subset \mathrm{B}_f$ (see [8]). After the pioneering paper [8] of Gusein-Zade, Luengo and Melle-Hernández, the local and global properties of rational functions were studied from various points of view by [1], [2], [21], [23] and [27] etc.

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In order to introduce our main results, from now we prepare some notations. Let $\mathrm{N}(P), \mathrm{N}(Q) \subset \mathbb{R}^n_{\geq 0}$ be the Newton polytopes of P, Q respectively and

$$\mathrm{N}(f) := \mathrm{N}(P) + \mathrm{N}(Q)$$

their Minkowski sum. Recall that for a vector u in the dual vector space of $\mathbb{R}^n$ we can define its supporting faces in $\mathrm{N}(f)$, $\mathrm{N}(P)$ and $\mathrm{N}(Q)$ (see Definition 2.4 for the details). Then for a face $\gamma \prec \mathrm{N}(f)$ there exist faces $\gamma(P) \prec \mathrm{N}(P)$ and $\gamma(Q) \prec \mathrm{N}(Q)$ such that

$$\gamma = \gamma(P) + \gamma(Q)$$

(see Section 2 for the details.). We shall say that a face $\gamma \prec \mathrm{N}(f)$ is of type I if it is supported by a vector $u \in \mathbb{R}^n \setminus \mathbb{R}^n_{\geq 0}$ and the affice span $\mathrm{Aff}(\gamma(P) - \gamma(Q)) \simeq \mathbb{R}^{\dim \gamma}$ of the polytope $\gamma(P) - \gamma(Q) \subset \mathbb{R}^n$ in $\mathbb{R}^n$ contains the origin $0 \in \mathbb{R}^n$. Clearly, if $Q(z) = 1$ and $f(z) = P(z)$ is a polynomial, this notion corresponds to that of bad faces of $\mathrm{N}(f) = \mathrm{N}(P)$ defined by Némethi and Zaharia [19] (cf. [28], [29] and [30] for a slightly different one). We denote the set of faces of $\mathrm{N}(f)$ of type I by $\mathscr{F}_I$. For $\gamma \in \mathscr{F}_I$ by using the Laurent polynomials $P_{\gamma(P)}(z)$ and $Q_{\gamma(Q)}(z)$ on the torus $T = (\mathbb{C}^*)^n$ we define a function $f_\gamma : T \setminus Q^{-1}_{\gamma(Q)}(0) \longrightarrow \mathbb{C}$ by

$$f_\gamma(z) = \frac{P_{\gamma(P)}(z)}{Q_{\gamma(Q)}(z)} \qquad (z \in T \setminus Q^{-1}_{\gamma(Q)}(0))$$

Then our main result is as follows.

Theorem 1.1. *Assume that the divisor $P^{-1}(0) \cup Q^{-1}(0) \subset \mathbb{C}^n$ is normal crossing in a neighborhood of $P^{-1}(0) \cap Q^{-1}(0)$ and $f(z) = \frac{P(z)}{Q(z)}$ is non-degenerate (see Definition 2.7). Then we have*

$$\mathrm{B}_f \subset f(\mathrm{Sing} f) \cup \{0\} \cup \Big(\bigcup_{\gamma \in \mathscr{F}_I} f_\gamma(\mathrm{Sing} f_\gamma) \Big). \tag{1}$$

Note that the first assumption of this theorem is satisfied by generic polynomials $P(x)$ and $Q(x)$ such that $P(0) \neq 0$ and $Q(0) \neq 0$. Moreover, in the two dimensional case $n = 2$ the same is true also for generic $P(z)$ and $Q(z)$. For $n \geq 2$, if the intersection of $\mathrm{N}(Q)$ and each coordinate axis of $\mathbb{R}^n$ is equal to $\{0\} \subset \mathbb{R}^n$ then the the first assumption of Theorem 1.1 is satisfied by generic $P(z)$ and $Q(z)$. Indeed, for such $Q(z)$ we have

$$Q^{-1}(0) \subset T = (\mathbb{C}^*)^n \subset \mathbb{C}^n.$$

This is the case when $Q(z) = 1$ and $f(z) = P(z)$ is a polynomial. If $f(z) = P(z)$ is non-degenerate (at infinity) and convenient, by a result of Broughton [3] the polynomial map $f : \mathbb{C}^n \to \mathbb{C}$ is tame at infinity and

$$\mathrm{B}_f = f(\mathrm{Sing} f).$$

However, for rational functions $f(z) = \frac{P(z)}{Q(z)}$, by Theorem 1.1 and the analogues of the results in [29] and [33] for rational functions (which can be proved by toric compactifications of $\mathbb{C}^n$), even if $P(z)$ and $Q(z)$ are convenient there might be some type I faces of $\mathrm{N}(f)$ and hence we do not have the equality $\mathrm{B}_f = f(\mathrm{Sing} f)$ in general. See Section 4 for the details. As in Gusein-Zade, Luengo and Melle-Hernández [8], our non-degeneracy condition in Definition 2.7 is inspired from the classical one for polynomial functions over complete intersection subvarieties in $\mathbb{C}^n$ used by many authors such as [15] and [25] etc. For the proof of Theorem 1.1 we also need to refine the methods of Némethi and Zaharia in [19]. Finally, note that the monodromies of rational functions over $\mathbb{C}^n$ were studied by [8] and [23].

2. Preliminary notions and results

Let $P(z), Q(z) \in \mathbb{C}[z_1, \ldots, z_n]$ be polynomials of $n(\geq 2)$-variables with coefficients in $\mathbb{C}$. We define a rational function $f(z)$ by

$$f(z) = \frac{P(z)}{Q(z)} \qquad (z \in \mathbb{C}^n \setminus Q^{-1}(0)).$$

We will study the map from $\mathbb{C}^n \setminus Q^{-1}(0)$ to $\mathbb{C}$ defined by f. Let us set $\mathrm{I}(f) = P^{-1}(0) \cap Q^{-1}(0) \subset \mathbb{C}^n$. If P and Q are coprime, then $\mathrm{I}(f)$ is nothing but the set of the indeterminacy points of f. In fact, the set $\mathrm{I}(f)$ depends on the pair $(P(z), Q(z))$ of polynomials representing $f(z)$. For example, if we take a non-zero polynomial $R(z)$ on $\mathbb{C}^n$ and set

$$g(z) = \frac{P(z)R(z)}{Q(z)R(z)} \qquad (z \in \mathbb{C}^n),$$

then the set $\mathrm{I}(g) = \mathrm{I}(f) \cup R^{-1}(0)$ might be bigger than $\mathrm{I}(f)$. In this way, we distinguish $f(z) = \frac{P(z)}{Q(z)}$ from $g(z) = \frac{P(z)R(z)}{Q(z)R(z)}$ even if their values coincide over an open dense subset of $\mathbb{C}^n$. This is the convention due to Gusein-Zade, Luengo and Melle-Hernández [8] etc. Hereafter, we assume that $P(z)$ and $Q(z)$ are coprime.

Definition 2.1. We say that $c \in \mathbb{C}$ is an atypical value of f if for any open neighborhood U of c, the restriction $f^{-1}(U) \cap (\mathbb{C}^n \setminus Q^{-1}(0)) \to U$ of f is not a C^∞ trivial fibration. The bifurcation set $\mathrm{B}_f \subset \mathbb{C}$ is the set of all the atypical values of f.

For a polynomial or rational function g on $\mathbb{C}^n$ as in [18], we set

$$\mathrm{grad}g(z) := \left(\overline{\frac{\partial g}{\partial z_1}(z)}, \ldots, \overline{\frac{\partial g}{\partial z_n}(z)} \right),$$

where $\overline{a}$ is the complex conjugate of $a \in \mathbb{C}$. For $z = (z_1, \ldots, z_n), w = (w_1, \ldots, w_n) \in \mathbb{C}^n$, $\langle z, w \rangle$ stands for the Hermite inner product of z and w, i.e. $\langle z, w \rangle = \sum_{i=1}^n z_i \overline{w_i}$. Moreover, for $z \in \mathbb{C}^n$ we set $\|z\| := \sqrt{\langle z, z \rangle} \in \mathbb{R}_{\geq 0}$.

Definition 2.2. (1) We define a subset $\mathrm{M}_f \subset \mathbb{C}^n$ by

$$\mathrm{M}_f := \{z \in \mathbb{C}^n \setminus Q^{-1}(0) \mid \text{there exists } \lambda \in \mathbb{C} \text{ such that } \mathrm{grad}f(z) = \lambda z\}$$

(2) We define a subset $\mathrm{S}_f \subset \mathbb{C}$ by

$$\mathrm{S}_f := \left\{ s_0 \in \mathbb{C} \;\middle|\; \begin{array}{l} \text{there exists a sequence } \{z^k\}_{k=0}^\infty \subset \mathrm{M}_f \text{ such that} \\ \lim_{k\to\infty} \|z^k\| = \infty \text{ and } \lim_{k\to\infty} f(z^k) = s_0. \end{array} \right\}$$

Definition 2.3. (1) Let $g(z) = \sum_{\alpha \in \mathbb{Z}^n} a_\alpha z^\alpha \in \mathbb{C}[z_1^\pm, \ldots, z_n^\pm]$ be a Laurent polynomial with coefficients in $\mathbb{C}$. Then the Newton polytope $\mathrm{N}(g) \subset \mathbb{R}^n$ of g is the convex full of the set $\mathrm{supp}(f) := \{\alpha \in \mathbb{Z}^n \mid a_\alpha \neq 0\}$ in $\mathbb{R}^n$.

(2) Let $P(z), Q(z) \in \mathbb{C}[z_1, \dots, z_n]$ be polynomials and $f(z)$ the rational function $\frac{P(z)}{Q(z)}$ defined by them on $\mathbb{C}^n$. Then the Newton polytope $\mathrm{N}(f) \subset \mathbb{R}^n$ of f is the Minkowski sum of $\mathrm{N}(P)$ and $\mathrm{N}(Q)$. Namely we set

$$\mathrm{N}(f) := \{x + y \in \mathbb{R}^n \mid x \in \mathrm{N}(P),\ y \in \mathrm{N}(Q)\}.$$

For real vectors $u = (u_1, \dots, u_n), v = (v_1, \dots, v_n) \in \mathbb{R}^n$, we set $\langle u, v\rangle := \sum_{i=1}^n u_i v_i$.

Definition 2.4. (1) Let S be a polytope in $\mathbb{R}^n$. For a vector $u \in \mathbb{R}^n$, we set $d_S^u := \min_{w\in S}\langle u, w\rangle \in \mathbb{R}$. Moreover, for a real vector $u \in \mathbb{R}^n$, the supporting face γ_S^u of S by u is a polytope defined by

$$\gamma_S^u := \{v \in S \mid \langle u, v\rangle = \min_{w\in S}\langle u, w\rangle\}.$$

(2) For a Laurent polynomial $g(z) \in \mathbb{C}[z_1^{\pm}, \dots, z_n^{\pm}]$ and a real vector $u \in \mathbb{R}^n$ we set $d_g^u := d_{\mathrm{N}(g)}^u$ and $\gamma_g^u := \gamma_{\mathrm{N}(g)}^u$.

(3) For a rational function $f(z) = \frac{P(z)}{Q(z)}$ on $\mathbb{C}^n$ and a real vector $u \in \mathbb{R}^n$, we set $d_f^u := d_P^u - d_Q^u$ and $\gamma_f^u := \gamma_{\mathrm{N}(f)}^u$.

Definition 2.5. Let $\frac{P(z)}{Q(z)}$ be a rational function on $\mathbb{C}^n$.

(1) We say that a face $\gamma \prec \mathrm{N}(f)$ of $\mathrm{N}(f)$ is of type I if there exists a vector $u \in \mathbb{R}^n \setminus \mathbb{R}^n_{\geq 0}$ such that $\gamma_f^u = \gamma$ and for any such u we have $d_f^u (= d_P^u - d_Q^u) = 0$. We denote the set of all the type I faces of $\mathrm{N}(f)$ by $\mathscr{F}_{\mathrm{I}}$.

(2) We say that a face $\gamma \prec \mathrm{N}(f)$ of $\mathrm{N}(f)$ is of type II, if it is not of type I but there exists $u \in \mathbb{R}^n \setminus \mathbb{R}^n_{\geq 0}$ such that $\gamma_f^u = \gamma$. We denote the set of all the type II faces of $\mathrm{N}(f)$ by $\mathscr{F}_{\mathrm{II}}$.

For a Laurent polynomial $g(z) = \sum_{\alpha\in\mathbb{Z}^n} a_\alpha z^\alpha \in \mathbb{C}[z_1^{\pm}, \dots, z_n^{\pm}]$ and a face $\gamma \prec \mathrm{N}(g)$, we set $g_\gamma(z) := \sum_{\alpha\in\gamma} a_\alpha z^\alpha$. We regard it as a function on $T = (\mathbb{C}^*)^n$. Let $f(z) = \frac{P(z)}{Q(z)}$ be a rational function. Then for a face $\gamma \prec \mathrm{N}(f)$ and a real vector $u \in \mathbb{R}^n$ such that $\gamma_f^u = \gamma$, the faces $\gamma_P^u \prec \mathrm{N}(P)$ and $\gamma_Q^u \prec \mathrm{N}(Q)$ do not depend on u. By taking such u we set

$$\gamma(P) = \gamma_P^u, \qquad \gamma(Q) = \gamma_Q^u.$$

Then we have

$$\gamma = \gamma(P) + \gamma(Q).$$

Let $\mathrm{Aff}(\gamma(P) - \gamma(Q)) \simeq \mathbb{R}^{\dim\gamma}$ be the affice span of the polytope $\gamma(P) - \gamma(Q) \subset \mathbb{R}^n$ in $\mathbb{R}^n$. Then the face $\gamma \prec \mathrm{N}(f)$ is of type I iff it is supported by a vector $u \in \mathbb{R}^n \setminus \mathbb{R}^n_{\geq 0}$ and

$$0 \in \mathrm{Aff}(\gamma(P) - \gamma(Q)).$$

Example 2.6. Let the Newton polytopes of $P(z)$ and $Q(z)$ be as in Figures 1 and 2. In this case, $\mathrm{N}(f)$ is a polytope as in Figure 3. Then, the lines $\overline{OA}$, $\overline{OD}$ and $\overline{AB}$ and the points O and A are of type I, and the other faces of $\mathrm{N}(f)$ are of type II.

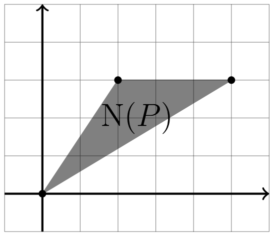

FIGURE 1. $\mathrm{N}(P)$

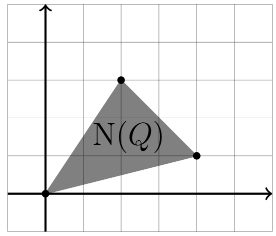

FIGURE 2. $\mathrm{N}(Q)$

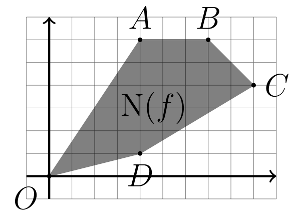

FIGURE 3. $\mathrm{N}(f)$

For a face $\gamma \prec \mathrm{N}(f)$ ($\gamma \neq \mathrm{N}(f)$) by using the Laurent polynomials $P_{\gamma(P)}(z)$ and $Q_{\gamma(Q)}(z)$ on the torus $T = (\mathbb{C}^*)^n$ we define a function $f_\gamma : T \setminus Q_{\gamma(Q)}^{-1}(0) \longrightarrow \mathbb{C}$ by

$$f_\gamma(z) = \frac{P_{\gamma(P)}(z)}{Q_{\gamma(Q)}(z)} \qquad (z \in T \setminus Q_{\gamma(Q)}^{-1}(0))$$

Definition 2.7. Let $f(z) = \frac{P(z)}{Q(z)}$ be a rational function on $\mathbb{C}^n$. Then we say that f is non-degenerate if $\mathrm{grad}P_{\gamma(P)}(z)$ (resp. $\mathrm{grad}Q_{\gamma(Q)}(z)$) does not vanish on $P_{\gamma(P)}^{-1}(0) \setminus Q_{\gamma(Q)}^{-1}(0) \subset T$ (resp. $Q_{\gamma(Q)}^{-1}(0) \setminus P_{\gamma(P)}^{-1}(0) \subset T$) for any face $\gamma \prec \mathrm{N}(f)$ of type II, and the two vectors $\mathrm{grad}P_{\gamma(P)}(z)$ and $\mathrm{grad}Q_{\gamma(Q)}(z)$ are linearly independent on $P_{\gamma(P)}^{-1}(0) \cap Q_{\gamma(Q)}^{-1}(0) \subset T$ for any face $\gamma \prec \mathrm{N}(f)$ of type I or II.

Lemma 2.8. *Let $f(z) = \frac{P(z)}{Q(z)}$ be a rational function. Assume that a face $\gamma \prec \mathrm{N}(f)$ is of type II. Then we have $f_\gamma(\mathrm{Sing}f_\gamma) \subset \{0\}$. If moreover f is non-degenerate in the sense of Definition 2.7, we have $f_\gamma(\mathrm{Sing}f_\gamma) = \emptyset$.*

Proof. By the definition of faces of type II, we can take a vector $u = (u_1, \ldots, u_n) \in \mathbb{R}^n \setminus \mathbb{R}^n_{\geq 0}$ such that $\gamma_f^u = \gamma$ and $d_f^u = d_P^u - d_Q^u \neq 0$. To prove the first assertion, assume that there exists non-zero $t_0 \in f_\gamma(\mathrm{Sing}f_\gamma)$, i.e. there is a point $z^0 \in \mathrm{Sing}f_\gamma(\subset T \setminus Q_{\gamma(Q)}^{-1}(0))$ such that $f_\gamma(z^0) = t_0 \neq 0$. Since we have

$$\frac{\partial P_{\gamma(P)}}{\partial z_i}(z^0) = \left(\frac{\partial P_{\gamma(P)}}{\partial z_i}(z^0) - t_0 \frac{\partial Q_{\gamma(Q)}}{\partial z_i}(z^0) \right) \cdot \frac{1}{Q_{\gamma(Q)}(z^0)} \quad (i = 1, \ldots, n),$$

we obtain

$$\frac{\partial P_{\gamma(P)}}{\partial z_i}(z^0) = t_0 \cdot \frac{\partial Q_{\gamma(Q)}}{\partial z_i}(z^0) \quad (i = 1, \ldots, n).$$

By Euler's theorem for quasi-homogeneous polynomials, we get

$$d_P^u \cdot P_{\gamma(P)}(z^0) = t_0 \cdot d_Q^u \cdot Q_{\gamma(Q)}(z^0).$$

Since $f_\gamma(z^0) = t_0$ and $t_0 \neq 0$, we have

$$d_P^u = d_Q^u,$$

which is a contradiction. The second assertion is now clear since if f is non-degenerate, the central fiber $f_\gamma^{-1}(0) = P_{\gamma(P)}^{-1}(0) \setminus Q_{\gamma(Q)}^{-1}(0)$ is also smooth. $\square$

We will use the following lemma.

Lemma 2.9 (Curve Selection Lemma, c.f. [20, Lemma 2]). *Let $f_1(x), \dots, f_s(x)$, $g_1(x), \dots, g_t(x)$, $h_1(z), \dots, h_u(x) \in \mathbb{R}[x_1, \dots, x_n]$ be polynomials with real coefficients. Let $U = \{x \in \mathbb{R}^m \mid f_i(x) = 0,\ 1 \le i \le s\}$ and $W = \{x \in \mathbb{R}^m \mid g_i(x) > 0,\ 1 \le i \le t\}$. Suppose that there exists a sequence $\{x^k\}_{k=0}^{\infty} \subset U \cap W$ such that $\lim_{k\to\infty} \|x^k\| = \infty$ and for all $1 \le i \le u$, $\lim_{k\to\infty} h_i(x^k) = 0$. Then, there exists a real analytic curve $p\colon (0,1) \to U \cap W$ of the form $p(t) = at^{\alpha} + a_1 t^{\alpha+1} + \dots$ with $a \in \mathbb{R}^m \setminus \{0\}$ and $\alpha < 0$ such that $\lim_{t\to 0} \|p(t)\| = \infty$ and $\lim_{t\to 0} h_i(p(t)) = 0$ for any $1 \le i \le u$.*

Remark 2.10. By the proof of the above lemma in [20], we see moreover that α is a half integer.

3. Main theorems

Theorem 3.1. *Let $f(z) = \frac{P(z)}{Q(z)}$ be a rational function $\mathbb{C}^n \setminus Q^{-1}(0) \to \mathbb{C}$. Assume that the divisor $P^{-1}(0) \cup Q^{-1}(0) \subset \mathbb{C}^n$ is normal crossing in a neighborhood of $P^{-1}(0) \cap Q^{-1}(0)$. Then we have*

$$\mathrm{B}_f \subset f(\mathrm{Sing} f) \cup \mathrm{S}_f.$$

Proof. First, let us consider the simplest case where $P^{-1}(0), Q^{-1}(0) \subset \mathbb{C}^n$ are smooth and intersect transversally. For $R > 0$ we set $S_R = \{z \in \mathbb{C}^n \mid \|z\| = R\}$. Let $\mathcal{S}$ be the coarsest Whitney stratification of the normal crossing divisor $P^{-1}(0) \cup Q^{-1}(0)$. Then there exists $R_0 \gg 0$ such that for any $R > R_0$ the sphere S_R intersects each stratum in $\mathcal{S}$ transversally. Now let $s_0 \in \mathbb{C}$ be a point such that $s_0 \notin f(\mathrm{Sing} f) \cup \mathrm{S}_f$ and $D \subset \mathbb{C}$ a small open disc centered at s_0 satisfying the condition

$$\overline{D} \subset \mathbb{C} \setminus \{f(\mathrm{Sing} f) \cup \mathrm{S}_f\}.$$

Then by an analogue of Némethi and Zaharia [19, Lemma 3] for rational functions, there exists $R_1 \ge R_0$ such that

$$f^{-1}(D) \cap \mathrm{M}_f \cap \{z \in \mathbb{C}^n \mid \|z\| > R_1\} = \emptyset.$$

This implies that for any $R > R_1$ the sphere S_R intersects the fiber $f^{-1}(s)$ transversally for any $s \in D$. Let $\pi : \widetilde{\mathbb{C}^n} \to \mathbb{C}^n$ be the blow-up of $\mathbb{C}^n$ along $P^{-1}(0) \cap Q^{-1}(0)$ and $E = \pi^{-1}\{P^{-1}(0) \cap Q^{-1}(0)\}$ the exceptional divisor in it. Then the meromorphic extension $g := f \circ \pi$ of f to $\widetilde{\mathbb{C}^n}$ has no point of indeterminacy and for any $s \in \mathbb{C}$ its fiber $g^{-1}(s)$ intersects E transversally. Moreover for $R > R_0$ we see that the closure

$$\widetilde{S_R} := \overline{\pi^{-1}[S_R \setminus \{P^{-1}(0) \cap Q^{-1}(0)\}]} \subset \widetilde{\mathbb{C}^n}$$

is a smooth real hypersurface of the complex manifold $\widetilde{\mathbb{C}^n}$. For $s \in \mathbb{C}$ let $\mathcal{S}_s$ be the coarsest Whitney stratification of the normal crossing divisor $g^{-1}(s) \cup E$. Then for any $R > R_1$ the real hypersurface $\widetilde{S_R}$ intersects each stratum in $\mathcal{S}_s$ transversally. This implies that for any point of $g^{-1}(s) \cap E \cap \widetilde{S_R}$ and a local coordinate system $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_n)$ of $\widetilde{\mathbb{C}^n}$ around it such that $E = \{\zeta_1 = 0\}$ we can find locally a smooth real vector field $v(\zeta)$ on $\widetilde{\mathbb{C}^n}$ such that

$$v(\zeta)\zeta_1 \equiv 0, \qquad v(\zeta)g(\zeta) \equiv 1$$

and $v(\zeta)$ is tangent to the real hypersurface $\widetilde{S_{\|\pi(\zeta)\|}}$ passing through the point ζ. By the first (resp. third) condition on $v(\zeta)$, its integral curves do not go into the exceptional divisor E (resp. at infinity) in finite time. Now by our choice of D and the construction of the blow-up π, the morphisim $g^{-1}(D) \to D$ induced by g is a (non-proper) holomorphic submersion. Moreover the boundary of the closure

$$\overline{g^{-1}(D)} = \overline{\pi^{-1}f^{-1}(D)} \subset \widetilde{\mathbb{C}^n}$$

is smooth and intersects E transversally. Then as in the proof of Némethi and Zaharia [19, Theorem 1], by using a partition of unity we can construct a smooth real vector field $\tilde{v}$ globally defined on $g^{-1}(D)$ such that

$$\tilde{v}g \equiv 1$$

whose integral curves do not go into the exceptional divisor E or at infinity in finite time. By the restriction u of $\tilde{v}$ to $f^{-1}(D) = g^{-1}(D) \setminus E \subset \mathbb{C}^n$ and its multiple iu ($i := \sqrt{-1}$) we can prove that the morphism $f^{-1}(D) \to D$ is a C^∞ trivial fibration over D. Finally, let us consider the general case. We can construct a composition $\pi : \widetilde{\mathbb{C}^n} \to \mathbb{C}^n$ of several blow-ups of $\mathbb{C}^n$ over $P^{-1}(0) \cap Q^{-1}(0)$ so that the meromorphic extension $g := f \circ \pi$ of f to $\widetilde{\mathbb{C}^n}$ has no point of indeterminacy (see e.g. the proof of [16, Theorem 3.6] and [17, Section 3]). Then the proof proceeds similarly to the one in the previous case. This completes the proof. □

Note that the assumption of this theorem are satisfied by generic polynomials $P(z)$ and $Q(z)$ such that $P(0) \neq 0$ and $Q(0) \neq 0$. Moreover, in the two dimensional case $n = 2$ the same is true also for generic $P(z)$ and $Q(z)$. For $n \geq 2$, if the intersection of $\mathrm{N}(Q)$ and each coordinate axis of $\mathbb{R}^n$ is equal to $\{0\} \subset \mathbb{R}^n$ then the assumption of Theorem 3.1 is satisfied by generic $P(z)$ and $Q(z)$. Indeed, for such $Q(z)$ we have

$$Q^{-1}(0) \subset T = (\mathbb{C}^*)^n \subset \mathbb{C}^n.$$

This is the case when $Q(z) = 1$ and $f(z) = P(z)$ is a polynomial.

Theorem 3.2. *Let $f(z) = \frac{P(z)}{Q(z)}$ be a rational function $\mathbb{C}^n \setminus Q^{-1}(0) \to \mathbb{C}$. Assume that f is non-degenerate in the sense of Definition 2.7. Then, we have*

$$\mathrm{S}_f \subset \{0\} \cup \Big(\bigcup_{\gamma \in \mathscr{F}_{\mathrm{I}}} f_\gamma(\mathrm{Sing} f_\gamma) \Big). \tag{2}$$

Proof. Our proof is inspired from that of [19, Theorem 2]. Assume that $s_0 \in \mathrm{S}_f$. Then, by the definition of S_f, there exists a sequence $\{z^k\}_{k=0}^{\infty}$ in M_f such that $\lim_{k\to\infty} \|z^k\| = \infty$ and $\lim_{k\to\infty} f(z^k) = s_0$. By the curve selection lemma (Lemma 2.9), we can take an analytic curve $h(t)\colon (0,1) \to \mathbb{C}^n$ of the form

$$h(t) = at^\alpha + a_1 t^{\alpha+1} + \cdots \quad (a \neq 0 \text{ and } \alpha < 0), \tag{3}$$

satisfying the conditions:

$$\begin{cases} h(t) \in \mathrm{M}_f \quad (t \in (0,1)), \\ \lim_{t\to 0} \left\| h(t) \right\| = \infty, \\ \lim_{t\to 0} f(h(t)) = s_0. \end{cases}$$

By the definition of M_f, there is an analytic function $\lambda(t)\colon (0,1) \to \mathbb{C}$ such that

$$\mathrm{grad} f(h(t)) = \lambda(t) h(t). \tag{4}$$

We will use the identities:

$$\frac{df(h(t))}{dt} = \left\langle \frac{dh}{dt}(t), \mathrm{grad} f(h(t)) \right\rangle. \tag{5}$$

If $\mathrm{grad} f(h(t)) \equiv 0$ $(t \in (0,1))$, the identity (5) implies that $\frac{df(h(t))}{dt} \equiv 0$ and $f(h(t))$ is a constant function. Hence $\sigma = \lim_{t\to 0} f(h(t))$ is in $\mathrm{Sing} f$. Therefore, we can assume $\mathrm{grad} f(h(t)) \not\equiv 0$. If $f(h(t)) \equiv 0$, the identities (5) and (4) imply that

$$\overline{\lambda(t)} \left\langle \frac{dh}{dt}(t), h(t) \right\rangle \equiv 0.$$

By (3), we have

$$\left\langle \frac{dh}{dt}(t), h(t) \right\rangle = |a|^2 \alpha t^{2\alpha - 1} + \cdots .$$

Here $\cdots$ stands for higher order terms. In particular, $\left\langle \frac{dh}{dt}(t), h(t) \right\rangle \not\equiv 0$ and we thus obtain $\lambda(t) \equiv 0$, which is in contradiction with $\mathrm{grad} f(h(t)) (= \lambda(t) h(t)) \not\equiv 0$. So, we will also assume $f(h(t)) \not\equiv 0$.

Let the expansions of $f(h(t)), \mathrm{grad}(f(h(t)))$ and $\lambda(t)$ be of the following forms:

$$\begin{cases} f(h(t)) = b t^{\beta} + \cdots\cdots , \\ \mathrm{grad} f(h(t)) = c t^{\rho} + \cdots\cdots , \\ \lambda(t) = \lambda_0 t^{\delta} + \cdots\cdots , \end{cases}$$

where $b \in \mathbb{C}, c \in \mathbb{C}^n, \lambda_0 \in \mathbb{C}$ are not zero. Note that the assumption $\lim_{t\to 0} f(h(t)) = s_0 \in \mathbb{C}$ implies $\beta \geq 0$. By considering the expansions of both sides of (4), we have

$$\rho = \delta + \alpha, \quad \text{and}$$
$$c = \lambda_0 a.$$

Hence, we have $\langle a, c \rangle \neq 0$. For an analytic function $g(t) = g_0 t^{\eta} + \cdots\cdots$ $(g_0 \neq 0)$, we denote by $\mathrm{deg} g(t)$ its degree with respect to t. Namely we set $\mathrm{deg} g(t) = \eta$. Then the degree of the right hand side of (5) is equal to $\alpha - 1 + \rho$. By (5), we thus obtain

$$\alpha - 1 + \rho \ (= \beta - 1) \geq -1,$$

which implies $\rho > 0$ since we have $\alpha < 0$. Moreover, we have

$$\delta = \rho - \alpha > 0.$$

We may assume that

$$h(t) = (w_1^0 t^{\nu_1} + w_1^1 t^{\nu_1+1} + \cdots, \dots, w_k^0 t^{\nu_k} + w_k^1 t^{\nu_k+1} + \cdots, 0, \dots, 0), \tag{6}$$

where $w_1^0 \neq 0, \dots, w_k^0 \neq 0$ and $\alpha = \nu_1 \leq \nu_2 \leq \cdots \leq \nu_k$. We identify

$$\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_{k+1} = \cdots = x_n = 0\}$$

with $\mathbb{R}^k$. Then, we will consider the supporting face $\gamma \subset \mathbb{R}^k (\subset \mathbb{R}^n)$ of $\mathrm{N}(f) \cap \mathbb{R}^k (= \mathrm{N}(P) \cap \mathbb{R}^k + \mathrm{N}(Q) \cap \mathbb{R}^k)$ by the vector $(\nu_1, \dots, \nu_k) \in \mathbb{R}^k$. Since $f(h(t)) \not\equiv 0$, we have $\mathrm{N}(P) \cap \mathbb{R}^k \neq \emptyset$ and $\mathrm{N}(Q) \cap \mathbb{R}^k \neq \emptyset$. Let $m (< 0)$ be a real number smaller than the (non-positive) integer

$$\begin{aligned}\min\{\nu_1 w_1 + \cdots + \nu_k w_k \in \mathbb{R} \mid (w_1, \dots, w_n) \in \mathrm{N}(f)\} \\ - \min\{\nu_1 w_1 + \cdots + \nu_k w_k \in \mathbb{R} \mid (w_1, \dots, w_k) \in \mathrm{N}(f) \cap \mathbb{R}^k\}\end{aligned}$$

and set

$$\nu := (\nu_1, \dots, \nu_k, -m, \dots, -m) \in \mathbb{R}^n.$$

Then γ is the supporting face of $\mathrm{N}(f) (\subset \mathbb{R}^n_{\geq 0})$ by $\nu \in \mathbb{R}^n$. Recall that by using the decomposition $\gamma = \gamma(P) + \gamma(Q)$ $(\gamma(P) \prec \mathrm{N}(P), \gamma(Q) \prec \mathrm{N}(Q))$ we defined $f_\gamma(z) := \frac{P_{\gamma(P)}(z)}{Q_{\gamma(P)}(z)}$ and $d_f^\nu = d_P^\nu - d_Q^\nu$. Set

$$w^0 := (w_1^0, \dots, w_k^0, 1, \dots, 1) \in T = (\mathbb{C}^*)^n.$$

Then, for $j = 1, \dots, k$ we have

$$\begin{cases} P(h(t)) = P_{\gamma(P)}(w^0) t^{d_P^\nu} + \cdots, \\ \dfrac{\partial P}{\partial z_j}(h(t)) = \dfrac{\partial P_{\gamma(P)}}{\partial z_j}(w^0) t^{d_P^\nu - \nu_j} + \cdots, \end{cases}$$

$$\begin{cases} Q(h(t)) = Q_{\gamma(Q)}(w^0) t^{d_Q^\nu} + \cdots, \\ \dfrac{\partial Q}{\partial z_j}(h(t)) = \dfrac{\partial Q_{\gamma(Q)}}{\partial z_j}(w^0) t^{d_Q^\nu - \nu_j} + \cdots. \end{cases}$$

We set

$$\begin{cases} e_P := \deg P(h(t)), \\ e_Q := \deg Q(h(t)). \end{cases}$$

Namely the expansions of $P(h(t))$ and $Q(h(t))$ are of the form:

$$\begin{cases} P(h(t)) = P_{e_P} t^{e_P} + \cdots\cdots, \\ Q(h(t)) = Q_{e_Q} t^{e_Q} + \cdots\cdots, \end{cases}$$

with $P_{e_P} \neq 0$ and $Q_{e_P} \neq 0$. Note that

$$\begin{cases} e_P \geq d_P^\nu, \\ e_Q \geq d_Q^\nu. \end{cases} \tag{$\bigstar$}$$

Since $\lim_{t\to 0} f(h(t)) = s_0 \in \mathbb{C}$, we have $e_P \geq e_Q$. If $e_P > e_Q$, the value $s_0 = \lim_{t\to 0} f(h(t))$ is 0 and contained in the right hand side of (2). So we will assume $e := e_P = e_Q$ in the following.

We set

$$l := \min\{d_P^\nu, d_Q^\nu\}.$$

We will use the obvious identity:

$$\overline{Q(h(t))}\mathrm{grad}P(h(t)) - \overline{P(h(t))}\mathrm{grad}Q(h(t)) = \overline{Q^2(h(t))}\mathrm{grad}f(h(t)). \tag{7}$$

By (4) and (6), the $j(>k)$-th entry of the right hand side of (7) is zero. Note also that for $1 \leq j \leq k$ the degree of the j-th entry of the left hand side of (7) is larger than or equal to $e + l - \nu_j$. We set

$$\widetilde{P_e} := \begin{cases} P_e & (\text{if } l = d_Q^\nu) \\ 0 & (\text{otherwise}). \end{cases} \tag{8}$$

$$\widetilde{Q_e} := \begin{cases} Q_e & (\text{if } l = d_P^\nu) \\ 0 & (\text{otherwise}). \end{cases} \tag{9}$$

Note that at least one of $\widetilde{P_e}$ and $\widetilde{Q_e}$ is not zero. For $1 \leq j \leq k$ let $A_j \in \mathbb{C}$ be the coefficient of $t^{e+l-\nu_j}$ in the j-th entry of the left hand side of (7). Then its complex conjugate $\overline{A_j}$ is expressed as

$$\overline{A_j} = \widetilde{Q_e}\frac{\partial P_{\gamma(P)}}{\partial z_j}(w^0) - \widetilde{P_e}\frac{\partial Q_{\gamma(Q)}}{\partial z_j}(w^0).$$

Namely we have

$$\begin{pmatrix} \overline{A_1} \\ \overline{A_2} \\ \vdots \\ \overline{A_k} \end{pmatrix} = \widetilde{Q_e}\begin{pmatrix} \frac{\partial P_{\gamma(P)}}{\partial z_1}(w^0) \\ \frac{\partial P_{\gamma(P)}}{\partial z_2}(w^0) \\ \vdots \\ \frac{\partial P_{\gamma(P)}}{\partial z_k}(w^0) \end{pmatrix} - \widetilde{P_e}\begin{pmatrix} \frac{\partial Q_{\gamma(Q)}}{\partial z_1}(w^0) \\ \frac{\partial Q_{\gamma(Q)}}{\partial z_2}(w^0) \\ \vdots \\ \frac{\partial Q_{\gamma(Q)}}{\partial z_k}(w^0) \end{pmatrix}. \tag{10}$$

We set

$$J := \{1 \leq j \leq k \mid A_j \neq 0\}, \text{ and} \tag{11}$$

$$j_0 := \min J \ (\text{when } J \neq \emptyset). \tag{12}$$

If $J \neq \emptyset$ and $A_j \neq 0$ for $j \in J$, by (4) and (7), we have

$$e + l - \nu_j = 2e + \delta + \nu_j, \text{ and} \tag{13}$$

$$\overline{A_j} = Q_e^2\overline{\lambda_0}\overline{w_j^0}. \tag{14}$$

Therefore, we have

$$\nu_j = \frac{1}{2}(-e + l - \delta)$$

and in particular $\nu_j = \nu_{j_0}$ $(j \in J)$. Moreover, since $e \geq l$ and $\delta > 0$, we have

$$\nu_j < 0 \tag{15}$$

for such j.

Lemma 3.3. *If $J \neq \emptyset$, we have the equality*

$$Q_e^2 \nu_{j_0} \overline{\lambda_0} \sum_{j \in J} |w_j^0|^2 = \widetilde{Q_e} d_P^\nu P_{\gamma(P)}(w^0) - \widetilde{P_e} d_Q^\nu Q_{\gamma(Q)}(w^0). \tag{16}$$

In particular, the right hand side of (16) is not 0.

Proof of Lemma 3.3. Assume $J \neq \emptyset$. By Euler's equality for quasi-homogeneous polynomials, we have

$$\sum_{1 \leq j \leq k} \nu_j w_j^0 \frac{\partial P_{\gamma(P)}}{\partial z_j}(w^0) = d_P^\nu P_{\gamma(P)}(w^0), \text{ and} \tag{17}$$

$$\sum_{1 \leq j \leq k} \nu_j w_j^0 \frac{\partial Q_{\gamma(Q)}}{\partial z_j}(w^0) = d_Q^\nu Q_{\gamma(Q)}(w^0). \tag{18}$$

Then we have

$$\begin{aligned}
\sum_{j \in J} w_j^0 \nu_j \overline{A_j} &= \sum_{1 \leq j \leq k} w_j^0 \nu_j \overline{A_j} \\
&= \sum_{1 \leq j \leq k} w_j^0 \nu_j \left\{ \widetilde{Q_e} \frac{\partial P_{\gamma(P)}}{\partial z_j}(w^0) - \widetilde{P_e} \frac{\partial Q_{\gamma(Q)}}{\partial z_j}(w^0) \right\} \\
&= \widetilde{Q_e} d_P^\nu P_{\gamma(P)}(w^0) - \widetilde{P_e} d_Q^\nu Q_{\gamma(Q)}(w^0) \quad \text{(by (17) and (18))}.
\end{aligned} \tag{19}$$

On the other hand, by (14), we have

$$\sum_{j \in J} w_j^0 \nu_j \overline{A_j} = Q_e^2 \nu_{j_0} \overline{\lambda_0} \sum_{j \in J} |w_j^0|^2. \tag{20}$$

Combining (19) and (20), we obtain the desired equality.

The second assertion follows from the facts: $Q_e \neq 0$, $\lambda_0 \neq 0$, $w_j^0 \neq 0$ and (15). □

Now, let us finish the proof of Theorem 3.2.

(**Case 1**) We first assume that $P_{\gamma(P)}(w^0) \neq 0$ and $Q_{\gamma(Q)}(w^0) \neq 0$. In this case, we have $e = e_P = d_P^\nu$ and $e = e_Q = d_Q^\nu$, and hence

$$l = d_P^\nu = d_Q^\nu \text{ and } d_f^\nu = 0.$$

Therefore, we have

$$\begin{aligned}
\text{(RHS of (16))} &= d_P^\nu \left\{ \widetilde{Q_e} P_{\gamma(P)}(w^0) - \widetilde{P_e} Q_{\gamma(Q)}(w^0) \right\} \quad (\text{since } d_P^\nu = d_Q^\nu) \\
&= 0 \quad (\text{since } \widetilde{P_e} = P_{\gamma(P)}(w^0) \text{ and } \widetilde{Q_e} = Q_{\gamma(Q)}(w^0)).
\end{aligned}$$

If $J \neq \emptyset$, this contradicts the second assertion of Lemma 3.3. Therefore, we have $J = \emptyset$ i.e. $A_j = 0$ $(1 \leq j \leq k)$. Moreover, for $1 \leq j \leq k$, we have

$$\overline{A_j} = Q_{\gamma(Q)}(w_0)\frac{\partial P_{\gamma(P)}}{\partial z_j}(w^0) - P_{\gamma(P)}(w^0)\frac{\partial Q_{\gamma(Q)}}{\partial z_j}(w^0), \text{ and hence}$$

$$\frac{\partial f_\gamma}{\partial z_j}(w^0) = \frac{\overline{A_j}}{Q^2_{\gamma(Q)}(w^0)} = 0.$$

Therefore, we have $w^0 \in \mathrm{Sing} f_\gamma$. Since $\nu \in \mathbb{R}^n \setminus \mathbb{R}^n_{\geq 0}$ the face γ is of type I or II. But Lemma 2.8 implies that γ is of type I and hence

$$s_0 = \lim_{t \to 0} f(h(t)) = f_\gamma(w^0) \in f_\gamma(\mathrm{Sing} f_\gamma)$$

is contained in the right hand side of (2).

(**Case 2**) Next, we assume that $P_{\gamma(P)}(w^0) = 0$ and $Q_{\gamma(Q)}(w^0) \neq 0$. In this case, we have $e = e_P > d^\nu_P$ and $e = e_Q = d^\nu_Q$ and hence

$$l = d^\nu_P < d^\nu_Q.$$

Moreover by $d^\nu_f \neq 0$ the face γ is of type II. Therefore, for $1 \leq j \leq k$ we have

$$\overline{A_j} = Q_{\gamma(Q)}(w^0)\frac{\partial P_{\gamma(P)}}{\partial z_j}(w^0).$$

Since $P_{\gamma(P)}(w^0) = 0$ and γ is of type II, by the non-degeneracy condition (Definition 2.7), $\frac{\partial P_{\gamma(P)}}{\partial z_j}(w^0) \neq 0$ for some $1 \leq j \leq k$. Hence, J is not empty. On the other hand, in this case we have

$$(\text{RHS of (16)}) = Q_{\gamma(Q)}(w^0) d^\nu_P P_{\gamma(P)}(w^0) = 0.$$

But, this contradicts the second assertion of Lemma 3.3.

(**Case 3**) Similarly, we assume that $P_{\gamma(P)}(w^0) \neq 0$ and $Q_{\gamma(Q)}(w^0) = 0$. In this case, we have $e = e_P = d^\nu_P$ and $e = e_Q > d^\nu_Q$ and hence

$$l = d^\nu_Q < d^\nu_P.$$

Moreover by $d^\nu_f \neq 0$ the face γ is of type II. Therefore, for $1 \leq j \leq k$ we have

$$\overline{A_j} = -P_{\gamma(P)}(w^0)\frac{\partial Q_{\gamma(Q)}}{\partial z_j}(w^0).$$

Since $Q_{\gamma(Q)}(w^0) = 0$ and γ is of type II, by the non-degeneracy condition, $\frac{\partial Q_{\gamma(Q)}}{\partial z_j}(w^0) \neq 0$ for some $1 \leq j \leq k$. Hence, J is not empty. On the other hand, we have

$$(\text{RHS of (16)}) = -P_{\gamma(P)}(w_0) d^\nu_Q Q_{\gamma(Q)}(w^0) = 0.$$

But, this contradicts the second assertion of Lemma 3.3.

(**Case 4**) Finally, we assume that $P_{\gamma(P)}(w^0) = 0$ and $Q_{\gamma(Q)}(w^0) = 0$. In this case, we have $e = e_P > d^\nu_P$ and $e = e_Q > d^\nu_Q$. Since $\nu \in \mathbb{R}^n \setminus \mathbb{R}^n_{\geq 0}$ the face γ is of type I or II. Then by $P_{\gamma(P)}(w^0) = 0$, $Q_{\gamma(Q)}(w^0) = 0$ and the non-degeneracy condition, the complex vectors $\mathrm{grad} P_{\gamma(P)}(w^0)$ and $\mathrm{grad} Q_{\gamma(Q)}(w^0)$ are linearly independent. Therefore, by (10) we get $J \neq \emptyset$. On the other hand, we have

$$(\text{RHS of (16)}) = \widetilde{Q_e} d_P^\nu P_{\gamma(P)}(w^0) - \widetilde{P_e} d_Q^\nu Q_{\gamma(Q)}(w^0) = 0.$$

But, this contradicts the second assertion of Lemma 3.3.

This completes the proof. □

Combining Theorems 3.1 and 3.2, we obtain Theorem 1.1. We will consider the following condition:

$$\text{For any vector } u \in \mathbb{R}^n \setminus \mathbb{R}^n_{\geq 0}, \text{ we have } d_Q^u \geq d_P^u. \tag{$*$}$$

It is satisfied if $P(0) \neq 0$, $Q(0) \neq 0$ and $\mathrm{N}(Q) \subset \mathrm{N}(P)$. This is the case in particular when $Q(z) = 1$ (i.e. $f(z) = P(z)$ is a polynomial) and $P(0) = f(0) \neq 0$.

Theorem 3.4. *In the situation in Theorem 1.1, assume moreover the condition* ($*$). *Then we have*

$$\mathrm{B}_f \subset f(\mathrm{Sing} f) \cup \Big(\bigcup_{\gamma \in \mathscr{F}_{\mathrm{I}}} f_\gamma(\mathrm{Sing} f_\gamma) \Big).$$

Proof. Assume that a point $s_0 \in \mathrm{S}_f \setminus f(\mathrm{Sing} f)$ is not contained in $\cup_{\gamma \in \mathscr{F}_{\mathrm{I}}} f_\gamma(\mathrm{Sing} f_\gamma)$. It is enough to get a contradiction only for $s_0 = 0$. Let us assume $s_0 = 0$. We will use the notations and the results before (★) in the proof of Theorem 3.2. Then, we have $e_P > e_Q$. Therefore, if $P_{\gamma(P)}(w^0) \neq 0$, we have $e_P = d_P^\nu$ and hence $d_P^\nu > e_Q \geq d_Q^\nu$, which contradicts the condition ($*$). Therefore, we have

$$P_{\gamma(P)}(w^0) = 0.$$

By the condition ($*$), for $1 \leq j \leq k$ the degree of the j-th entry of the left hand side of (7) is larger than or equal to $e_Q + d_P^\nu - \nu_j$. Let $A_j \in \mathbb{C}$ be the coefficient of $t^{e_Q + d_P^\nu - \nu_j}$ in it. Then its complex conjugate $\overline{A_j}$ is expressed as

$$\overline{A_j} = Q_{e_P} \frac{\partial P_{\gamma(P)}}{\partial z_j}(w^0).$$

We define J and j_0 as (11) and (12). Since $\nu \in \mathbb{R}^n \setminus \mathbb{R}^n_{\geq 0}$ the face γ is of type I or II. If γ is of type I, $Q_{\gamma(Q)}(w^0) \neq 0$ and $J = \emptyset$, we have $w^0 \in \mathrm{Sing} f_\gamma$ and

$$s_0 = 0 = f_\gamma(w^0) \in f_\gamma(\mathrm{Sing} f_\gamma).$$

This is a contradiction. So, in the case where γ is of type I and $Q_{\gamma(Q)}(w^0) \neq 0$, we have $J \neq \emptyset$. Also in the other cases (where γ is of type II or $P_{\gamma(P)}(w^0) = Q_{\gamma(Q)}(w^0) = 0$), by $P_{\gamma(P)}(w^0) = 0$ and the non-degeneracy condition we have $\frac{\partial P_{\gamma(P)}}{\partial z_j}(w^0) \neq 0$ for some $1 \leq j \leq k$ and hence $J \neq \emptyset$. Similarly to the argument in the proof of Theorem 3.2, by using $e_Q \geq d_Q^\nu \geq d_P^\nu$ we obtain $\nu_j = \nu_{j_0}$ for any $j \in J$ and $\nu_{j_0} < 0$. Moreover, in this situation, we have an equality similar to (16):

$$Q_{e_Q} \nu_{j_0} \sum_{j \in J} |w_j^0|^2 = d_P^\nu P_{\gamma(P)}(w^0).$$

The right hand side is 0. Since the left hand side is not zero, this is a contradiction. □

Corollary 3.5. *(Némethi and Zaharia* [19, Theorem 2]*) In the situation in Theorem 1.1, assume moreover that $Q(z) = 1$ (i.e. $f(z) = P(z)$ is a polynomial) and $P(0) = f(0) \neq 0$. Then we have*

$$\mathrm{B}_f \subset f(\mathrm{Sing} f) \cup \Big(\bigcup_{\gamma \in \mathscr{F}_{\mathrm{I}}} f_\gamma(\mathrm{Sing} f_\gamma) \Big).$$

In this corollary, for the face $\gamma = \{0\} \prec \mathrm{N}(f)$ of type I we have $\gamma(P) = \gamma(Q) = \{0\}$, $f_\gamma(z) = f(0) \neq 0$ and

$$f_\gamma(\mathrm{Sing} f_\gamma) = \{f(0)\}.$$

4. The two dimensional case and examples

In this section, we show that in the two dimensional case $n = 2$ the inclusion

$$\mathrm{B}_f \subset f(\mathrm{Sing} f) \cup \{0\} \cup \Big(\bigcup_{\gamma \in \mathscr{F}_{\mathrm{I}}} f_\gamma(\mathrm{Sing} f_\gamma) \Big)$$

in Theorem 1.1 is indeed an equality outside a finite subset of $\mathbb{C}$ and give some examples. Let $\gamma \prec \mathrm{N}(f)$ be a 0-dimensional face of type I. Then $\gamma(P) \prec \mathrm{N}(P)$ and $\gamma(Q) \prec \mathrm{N}(Q)$ are also 0-dimensional, $\gamma(P) = \gamma(Q)$ and

$$f_\gamma = \frac{P_{\gamma(P)}}{Q_{\gamma(Q)}} : T \setminus Q^{-1}_{\gamma(Q)}(0) \longrightarrow \mathbb{C}$$

is a non-zero constant function on $T \setminus Q^{-1}_{\gamma(Q)}(0) = T$ (here $Q_{\gamma(Q)}$ is a monomial). We denote its value by $c(\gamma) \in \mathbb{C}$. Then we define a subset $\mathrm{C}_f \subset \mathbb{C}$ by

$$\mathrm{C}_f := \{ c(\gamma) \in \mathbb{C} \mid \gamma \in \mathscr{F}_{\mathrm{I}},\ \dim \gamma = 0 \} \subset \mathbb{C}.$$

Theorem 4.1. *In the situation of Theorem 1.1, assume moreover that $n = 2$. Then we have an equality*

$$\mathrm{B}_f \setminus (\{0\} \cup \mathrm{C}_f) = \Big\{ f(\mathrm{Sing} f) \cup \Big(\bigcup_{\gamma \in \mathscr{F}_{\mathrm{I}}} f_\gamma(\mathrm{Sing} f_\gamma) \Big) \Big\} \setminus (\{0\} \cup \mathrm{C}_f). \tag{21}$$

Proof. We follow the proof of [29, Theorem 4.3]. Since $f(\mathrm{Sing} f) \subset \mathrm{B}_f$, it suffices to show the inclusion

$$\Big(\bigcup_{\gamma \in \mathscr{F}_{\mathrm{I}}} f_\gamma(\mathrm{Sing} f_\gamma) \Big) \setminus (f(\mathrm{Sing} f) \cup \{0\} \cup \mathrm{C}_f) \subset \mathrm{B}_f.$$

Let $s_0 \in \mathbb{C}$ be a point in the left hand side. We define a $\mathbb{Z}$-valued function $\chi_c : \mathbb{C} \longrightarrow \mathbb{Z}$ on $\mathbb{C}$ by

$$\chi_c(s) = \sum_{j \in \mathbb{Z}} (-1)^j \dim H^j_c(f^{-1}(s); \mathbb{C}) \qquad (s \in \mathbb{C})$$

and its jump $E_f(\sigma) \in \mathbb{Z}$ at $s_0 \in \mathbb{C}$ by

$$E_f(s_0) = -\{ \chi_c(s_0 + \varepsilon) - \chi_c(s_0) \} \in \mathbb{Z},$$

where $\varepsilon > 0$ is sufficiently small. Then it is enough to show that $E_f(s_0) \neq 0$. From now, we will use the terminologies in [6], [11] and [13] etc. For the point $s_0 \in \mathbb{C}$ define a function $h : \mathbb{C} \longrightarrow \mathbb{C}$ on $\mathbb{C}$ by $h(s) = s - s_0$ so that we have $h^{-1}(0) = \{s_0\}$. Then we have

$$E_f(s_0) = -\sum_{j\in\mathbb{Z}} (-1)^j \dim H^j \phi_h(Rf_!\mathbb{C}_{\mathbb{C}^2\setminus Q^{-1}(0)})_{s_0},$$

where $\phi_h : \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{C}) \longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\{s_0\})$ is Deligne's vanishing cycle functor associated to h. Now we introduce an equivalence relation $\sim$ on (the dual vector space of) $\mathbb{R}^2$ by $u \sim u' \iff \gamma_f^u = \gamma_f^{u'}$. We can easily see that for any face $\gamma \prec \mathrm{N}(f)$ of $\mathrm{N}(f)$ the closure of the equivalence class associated to it in $\mathbb{R}^2$ is an $(2-\dim\gamma)$-dimensional rational convex polyhedral cone $\sigma(\gamma)$ in $\mathbb{R}^2$. Moreover the family $\{\sigma(\gamma) \mid \gamma \prec \mathrm{N}(f)\}$ of cones in $\mathbb{R}^2$ thus obtained is a subdivision of $\mathbb{R}^2$. We call it the dual subdivision of $\mathbb{R}^2$ by $\mathrm{N}(f)$. If $\dim \mathrm{N}(f) = 2$ it satisfies the axiom of fans (see [7] and [24] etc.). We call it the dual fan of $\mathrm{N}(f)$. Let Σ_0 be a complete fan in $\mathbb{R}^2$ obtained by subdividing the dual subdivision. Note that all the cones in it are proper and convex. Let Σ be a smooth and complete fan in $\mathbb{R}^2$ containing all the 1-dimensional cones $\tau \simeq \mathbb{R}^1_{\geq 0}$ in Σ_0 such that $\tau \cap \mathbb{R}^2_{\geq 0} = \{0\}$ and satisfying the condition $\mathbb{R}^2_{\geq 0} \in \Sigma$. Let X_Σ be the toric variety associated to it. Then X_Σ is a smooth compactification of $\mathbb{C}^2$. This construction of X_Σ is inspired from the one in Zaharia [33]. Recall that the torus $T = (\mathbb{C}^*)^2$ acts on X_Σ and the T-orbits in it are parametrized by the cones τ in Σ. For a cone $\tau \in \Sigma$ denote by $T_\tau \simeq (\mathbb{C}^*)^{2-\dim\tau}$ the corresponding T-orbit. If $\tau \in \Sigma$ is not contained in $\mathbb{R}^2_{\geq 0}$ and its relative interior is contained in that of the cone $\sigma(\gamma)$ for a type II face γ of $\mathrm{N}(f)$, then by the non-degeneracy condition the closures $\overline{P^{-1}(0)}, \overline{Q^{-1}(0)} \subset X_\Sigma$ of $P^{-1}(0), Q^{-1}(0) \subset \mathbb{C}^2$ respectively in X_Σ intersect T_τ transversally. At such intersection points, (the meromorphic extension) of f to X_Σ may have indeterminacy. Moreover for $n = 2$ we have

$$(\overline{P^{-1}(0)} \cap T_\tau) \cap (\overline{Q^{-1}(0)} \cap T_\tau) = \emptyset.$$

If $\tau \in \Sigma$ is not contained in $\mathbb{R}^2_{\geq 0}$ and its relative interior is contained in that of the cone $\sigma(\gamma)$ for a type I face γ of $\mathrm{N}(f)$ such that $\dim\gamma = 1$, then the order of the meromorphic extension of f to X_Σ along the T-divisor $\overline{T_\tau} \subset X_\Sigma$ is zero. Moreover, by the non-degeneracy condition we have

$$(\overline{P^{-1}(0)} \cap T_\tau) \cap (\overline{Q^{-1}(0)} \cap T_\tau) = \emptyset.$$

As in [29, Section 3], by constructing a tower of blow-ups $\pi : \widetilde{X_\Sigma} \longrightarrow X_\Sigma$ of X_Σ to eliminate the indeterminacy of f we obtain a commutative diagram:

$$\begin{tikzcd} \mathbb{C}^2 \setminus Q^{-1}(0) \arrow[r, "\iota"] \arrow[d, "f"'] & \widetilde{X_\Sigma} \arrow[d, "g"] \\ \mathbb{C} \arrow[r, "j"'] & \mathbb{P}^1 \end{tikzcd}$$

of holomorphic maps, where $\iota : \mathbb{C}^2 \setminus Q^{-1}(0) \hookrightarrow \widetilde{X_\Sigma}$ and $j : \mathbb{C} \hookrightarrow \mathbb{P}^1$ are the inclusion maps and g is proper. By this construction, if $\tau \in \Sigma$ is not contained in $\mathbb{R}^2_{\geq 0}$ and its relative interior is contained in that of the cone $\sigma(\gamma)$ for a type I face γ of $\mathrm{N}(f)$, then π induced an isomorphism $\pi^{-1}(T_\tau) \simeq T_\tau$. So we regard T_τ as a subset of $\widetilde{X_\Sigma}$. Since

g is proper, by [6, Proposition 4.2.11] and [13, Exercise VIII.15] we thus obtain an isomorphism

$$\phi_h(Rf_!\mathbb{C}_{\mathbb{C}^2\setminus Q^{-1}(0)})_{s_0} \simeq R\Gamma(g^{-1}(s_0);\phi_{h\circ g}(\iota_!\mathbb{C}_{\mathbb{C}^2\setminus Q^{-1}(0)})).$$

By our choice of the point $s_0 \in \mathbb{C}$, the support of $\phi_{h\circ g}(\iota_!\mathbb{C}_{\mathbb{C}^2\setminus Q^{-1}(0)}) \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(g^{-1}(s_0))$ is contained in the (non-empty) finite subset of $g^{-1}(s_0) \subset \widetilde{X_\Sigma}$ consisting of the points $q \in T_{\sigma(\gamma)}$ for 1-dimensional type I faces γ of $\mathrm{N}(f)$ such that $q \in \mathrm{Sing} f_\gamma$ and $s_0 = f_\gamma(q)$. Here we naturally regard f_γ as a rational function on $T_{\sigma(\gamma)} \simeq \mathbb{C}^*$. In a neighborhood of the point $q \in T_{\sigma(\gamma)}$ it coincides with the restriction of g to $T_{\sigma(\gamma)} \subset \widetilde{X_\Sigma}$. For one $q \in T_{\sigma(\gamma)}$ of such points, let $\mu_q \geq 0$ be the Milnor number of the (possibly singular) complex hypersurface $g^{-1}(s_0)$ (in fact, it is an algebraic curve having at most an isolated singular point at q) of $\widetilde{X_\Sigma}$ at q. Denote by $m_q \geq 2$ the multiplicity of the zeros of the function $f_\gamma - s_0$ at q. Note that in a neighborhood of the point q in $\widetilde{X_\Sigma}$ the sequence

$$0 \to \mathbb{C}_{\mathbb{C}^2\setminus Q^{-1}(0)} \to \mathbb{C}_{\widetilde{X_\Sigma}} \to \mathbb{C}_{T_{\sigma(\gamma)}} \to 0$$

is exact. Then as in the final part of the proof of [29, Theorem 4.3] we obtain

$$\chi(\phi_{h\circ g}(\iota_!\mathbb{C}_{\mathbb{C}^2\setminus Q^{-1}(0)})_q) = -\mu_q - (m_q - 1) < 0.$$

Consequently, we get $E_f(s_0) > 0$. This completes the proof. □

By Theorems 3.4 and 4.1 we obtain the following result.

Corollary 4.2. *In the situation of Theorem 1.1, assume moreover the condition (∗) and that $n = 2$. Then we have an equality*

$$\mathrm{B}_f \setminus \mathrm{C}_f = \Big\{ f(\mathrm{Sing} f) \cup \Big(\bigcup_{\gamma\in\mathscr{F}_{\mathrm{I}}} f_\gamma(\mathrm{Sing} f_\gamma) \Big) \Big\} \setminus \mathrm{C}_f. \tag{22}$$

Similarly, also in higher dimensions $n \geq 3$ we obtain results similar to the ones in [29] and [33]. We leave their precise formulations to the readers. If $Q(z) = 1$ and $f(z) = P(z)$ is a polynomial which is non-degenerate (at infinity) and convenient, then by a result of Broughton [3] the polynomial map $f : \mathbb{C}^n \to \mathbb{C}$ is tame at infinity and

$$\mathrm{B}_f = f(\mathrm{Sing} f).$$

However, for rational functions $f(z) = \frac{P(z)}{Q(z)}$, by Theorems 1.1 and 4.1, even if $P(z)$ and $Q(z)$ are convenient there might be some type I faces of $\mathrm{N}(f)$ and hence we do not have the equality $\mathrm{B}_f = f(\mathrm{Sing} f)$ in general.

For the value 0, let us consider the following example.

Example 4.3. Let $f = \frac{x^2+y}{x+y}$. It is easy to check that f is non-degenerate in the sense of Definition 2.7. Let us consider the value $0 \in \mathbb{C}$. For a small disc $D \subset \mathbb{C}$ centered at it, we have

$$f^{-1}(D) = \left\{ (x, \frac{x^2 - tx}{1-t}) \mid x \in \mathbb{C} \setminus \{0, 1\}, t \in D \right\}.$$

It is easy to check that the restriction map $f : f^{-1}(D) \to D$ is a trivial fibration. This means $0 \notin \mathrm{B}_f$.

Moreover, by [21, Theorem 1.2] we have:

$$\mathrm{B}_f = \mathrm{B}_\infty \cup f(\mathrm{Sing} f) \cup \mathrm{K}_1(f),$$

see [21, Definition 2.2] for the definition of the set $\mathrm{K}_1(f)$, while $\mathrm{B}_\infty(f)$ is the set of critical value at infinity of f. One can easily check that in this example $f(\mathrm{Sing} f) = \mathrm{K}_1(f) = \emptyset$. Therefore $\mathrm{B}_f = \mathrm{B}_\infty(f)$. Let us consider the polynomial

$$g_t(x, y) := x^2 + y - t(x + y)$$

and $\delta(y, t)$ to be the discriminant of $g_t(x, y)$ with respect to the variable x. Then

$$\delta(y, t) = 4(1 - t)y - t^2.$$

Hence by [21, Corollary 3.7] we get $\mathrm{B}_f = \mathrm{B}_\infty(f) = \{1\}$.

On the other hand, for the set on the right hand side of the inclusion (1) in Theorem 1.1, the only non-empty set among those of $\bigcup_{\gamma \in \mathscr{F}_\mathrm{I}} f_\gamma(\mathrm{Sing} f_\gamma)$ comes from the face function $\frac{y}{y}$ which again provides us the value 1.

Regarding the set C_f, we will see from the example below that in general C_f is not a subset of B_f.

Example 4.4. Let $f := \frac{x+y}{x+2y}$. Then $\mathrm{C}_f = \{1/2, 1\}$. For any small neighborhood D of $1/2$ (such as D contains $1/2$ and does not contain 1), we have $f^{-1}(D) = \{(-\frac{1-2t}{1-t} y, y) : y \in \mathbb{C}^*\}$. Hence the restriction $f : f^{-1}(D) \to D$ is a locally trivial fibration. This means $1/2 \notin \mathrm{B}_f$. Similarly $1 \notin \mathrm{B}_f$.

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