A survey on the stability of (Extended) Linear Sand Pile Model

PHAN Thi Ha Duong

Abstract We give a survey of our works on the natural extensions of the well-known Sand Pile Model. These extensions consist of adding outside grains on random columns, allowing sand grains to move from left to right and from right to left, considering cycle graphs and the extension to infinity. We study the reachable configurations and fixed points of each model and show how to compute the set of fixed points, the time of convergence and the distribution of fixed points.

1 Introduction

The Sand Piles Model (SPM) was introduced in 1987 by Bak, Tang and Wiesenfeld as a sample model of the Self organized criticality (SOC) phenomena [3]. The authors simulated the behavior of a sand pile which builds up when sand is dropped on a line. A configuration is modeled as a sequence of columns consisting of cubic sand grains such that the height of columns is decreasing from left to right. In this model, a sand grain can fall down from a column to its right neighbors if the difference of height of the two columns is at least two. This model is investigated in many works in physics, combinatorics and computer science [13, 22, 26, 33, 50]. Independtly, a similar model - the Chip Firing Game - was defined by Björner, Lovász and Shor in 1991 [8, 7]. Formally, a CFG is a model consisting of a directed (or undirected) multi-graph G (also called *support graph*), the set of *configurations* on G and an *evolution rule* on this set.

In this paper, as the support graph has a linear structure, for convenient, we call the model (Linear) Sand Pile Model. This model can be defined mathematically as follows.

Institute of Mathematics - Vietnam Academy of Science and Technology, 18 Hoang Quoc Viet, Hanoi, Vietnam. e-mail: phanhaduong@math.ac.vn

Definition 1 Sand pile model, with respect to a positive integer n, denoted by $SPM(n)$, is a model where configurations are partitions of n such that:

- Initial configuration: $(\underline{n})$ (that means the position 0 has value n, or equivalently $a_0 = n$ and $a_i = 0$ for all $i \neq 0$).
- Local right vertical rule $\mathcal{R}$: for all $i \geq 0$, $(\ldots, a_i, a_{i+1}, \ldots) \rightarrow (\ldots, a_i - 1, a_{i+1} + 1, \ldots)$ if $a_i \geq a_{i+1} + 2$.
- Global rule: at each step, we apply once the $\mathcal{R}$ rule.

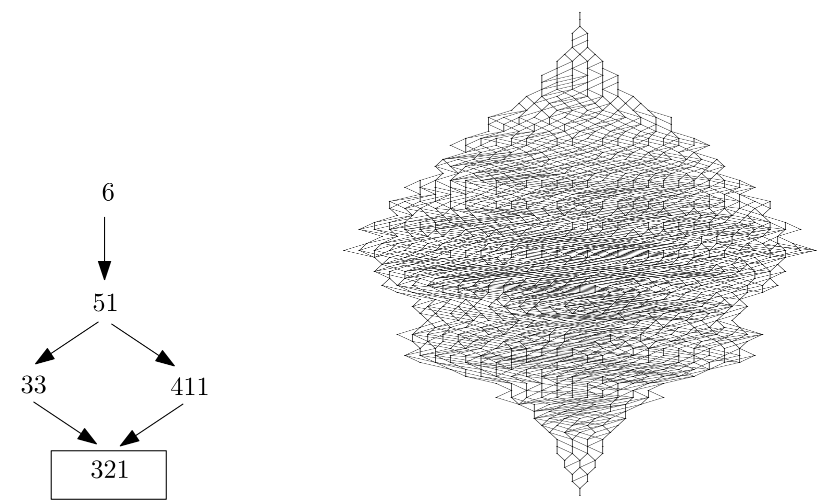


Fig. 1 Example of configuration spaces of Sand piles model: $SPM(6)$ and $SPM(30)$

On the other hand, in 1973, Brylawski described a model to generate all partitions of a given arbitrary integer n [9], this model can be considered as an extension of SPM because it is nothing but the SPM with an adding horizontal rule which allow grain to slide along a plateau.

Definition 2 Brylawski's model, with respect to a positive integer n, denoted by $L_B(n)$, is a model where configurations are partitions of n such that:

- Initial configuration: $(\underline{n})$.
- Local right vertical rule $\mathcal{R}$: $(\ldots, a_i, a_{i+1}, \ldots) \rightarrow (\ldots, a_i - 1, a_{i+1} + 1, \ldots)$ if $a_i \geq a_{i+1} + 2$.
- Local right horizontal rule $\mathcal{H}$: $(\ldots, p+1, p, \ldots, p, p-1, \ldots) \rightarrow (\ldots, p, p, \ldots, p, p, \ldots)$.
- Global rule: at each step, we apply once the $\mathcal{R}$ rule, or once the $\mathcal{H}$ rule.

From these first steps, SPM and Brylawski's model have been extended in many different directions [11, 14, 15, 16, 19, 22, 25, 26]. In particular, the problem is derived from real-life questions and focusing on the following issues.

- Reachability problem: study the necessary and sufficient conditions for a configuration to be reachable from another one by applying a sequence of transition rules. In most cases, based on the reachable relation, one can define an order relation.
- Configuration space: the set of all reachable configurations is called space configuration. This set equipped with the order relation can have many interesting structures.
- Stability problem: Determine if the model proceeds to stable configurations (called also "fixed points") or runs non-stop. This property is related to the structure of the configuration space.
- Convergence problem: Almost all models are non-deterministic then if we consider all the cases, it is possible that different sequences of transitions proceed to various fixed points. The uniqueness of the fixed point is usually proven when the configuration space has a lattice structure. Otherwise, the model can have many fixed points.
- Characterization of fixed points: If there is several fixed points (and even if it is unique), then it is important to find out their characterisation.
- Stabilization time of the model. If the model diverges, it is clear that the time to reach different fixed points are variety. But even if the model converges, there are different convergent times which depend on the local behavious of the model. Hence evaluate the upper and lower bounds for convergence time is also an object of study.

We define the order relation (if it exists) on the space configuration of a model by: a configuration a is greater than a configuration b if b is reachable from a by applying a sequence of transition rules (which is the inverse of the definition in Chapter 2).

Both configuration spaces of $SPM(n)$ and $L_B(n)$ have an order relation, moreover they have a lattice structure.

The following results were established for SPM [22, 26, 27].

- Reachability of SPM [22]. A partition of n is a reachable configuration of $SPM(n)$ if and only if it does not contains subsequences of the two following forms: (p, p, p) or $(p + 1, p + 1, p, p - 1, \ldots, p - q + 1, p - q, p - q)$ (we call this condition *SPM condition*).
- Fixed point of SPM. The model $SPM(n)$ has an unique fixed point, which is $(p, p - 1, \ldots, q, +1, q, q, q - 1, \ldots, 2, 1)$ where p and q are uniquely defined by $p(p+1)/2 \leq n < (p+1)(p+2)/2$ and $q = n - p(p+1)/2$ (q can be equal to 0).
- Lattice structure of the configuration space of SPM. The configuration space of SPM equipped with the reachability order is a lattice.
- Length of chains in SPM. Every chain between two configurations in $SPM(n)$ has the same length.

And results achieved for L_B ([9, 28]):

- The *exhaustive property* of the model: all partitions are reachable. This property is almost always true for all natural extensions of the model.

- Therefore, the unique fixed point of the model is easy to determine, it is $(1, 1, \ldots, 1)$.
- Lattice structure of the configuration space of $L_B(n)$. The reachability order is the dominance order, that mean b is reachable from a by L_B transition if and only if b is smaller than a by dominance ordering: for all $1 \leq i$, $\sum_{j=1}^{i} b_j \leq \sum_{j=1}^{i} a_j$.
- Length of chains in L_B. The property of the SPM that all chains between two given configurations have the same length is no longer true for L_B model. Therefore the problem of finding longest and shortest chains is particularly interesting. While it is pretty simple to find shortest chains, finding longest chains is quite complicated and requires a technical proof which is based on two energies, these energies are defined for the vertical and the horizontal rules respectively.
  Shortest chains. A shortest chain in L_B can be constructed as follows: applying the V-transition at the first position to obtain the partition $(n-1, 1)$. Then apply alternatively V-transition at the first position and H-transition at the second position $n-3$ times to obtain the partition $(2, 1, 1, \ldots, 1)$. At this state, apply the H-transition at the first position, and obtain the fixed point $(1, 1, \ldots, 1)$. This chain has length $2n-4$.
  Longest chains: in [28], the authors proved that longest chains are chains of a sequence of V-transitions followed by a sequence of H-transitions. And their length is $\theta(n^{3/2})$.

These two models were extended by different approaches. First, by parameterizing the horizontal rule, we defined the Ice pile model [26]. Then by considering that m sand grains can fall down in the same time (for a given m), we introduced the model $CFG(n, m)$ [27]. Independently, a similar model of $CFG(n, m)$ was studied by the physicist Kadanoff [31]. The enumeration of the number of reachable configurations of these models was widely studied [44, 20]. On the other hand, a parallel version of SPM was studied in [15, 16]. This model is deterministic and converges to the fixed point of the classical SPM. For this parallel model, the reachability problem was studied by mean of language theory.

In this paper, we give a survey of our works, in collaborations with Enrico Formenti, Kevin Perrot, Pham Van Trung and Tran Thi Thu Huong, on the following natural extensions of two models SPM and L_B.

- We investigate the study of all stable configurations when outside grains are added on random columns [47]
- A very natural extension of SPM is the symmetric SPM on which sand grains can move from left to right and from right to left. Here, the model lost its properties of existence and uniqueness of the fixed point, therefore this raises the question of how to compute the set of fixed points, the time of convergence and the distribution of fixed points [21, 46, 43].
- We explore the Sand Pile Model and Chip Firing Game on cycle graphs. We study the reachable configurations and fixed points of each model and the similarities between these models [10].
- Finally, we investigate the extension to infinity of Brylawski model, this model gives a method to generate partitions of all integers by rule of a dynamical model.

Moreover, this model has a recursive structure from which one can deduce some interesting enumerative formula on partitions [35, 38, 39].

2 The stability of SPM

In this section, we consider a more extended Sand Piles Model where outside grains are added on random columns. More precisely, each time the model reach a stable configuration, one grain is added to a random column, and the model evolves to reach another stable configuration, and so on. We investigate the study of all such stable configurations.

First, we give a formal definition of this model and prove that the set of all stable configurations has a lattice structure which is a sub-lattice of the well-known Young lattice. then we compute explicitly the smallest and greatest times to reach a stable configuration from the initial configuration, and the smallest and greatest times to reach a stable configuration from another stable configuration. These times illustrate the behaviour of the model under outside actions. The key idea of this computation is the introduction of the notion "energy". Indeed, for each configuration, we treat each of its grain by defining the energy of a grain being the greatest number of its possible moves.

2.1 Extended Sand Piles Model and its stable configurations

As well as in almost works of SPM, we represent configurations of this model by integer partitions. So let us first give some preliminary notions:

Definition 3 (i) A *partition* is an integer sequence $a = (a_1, a_2, \dots, a_k)$ such that $a_1 \geq a_2 \geq \dots \geq a_k > 0$ (by convention, $a_j = 0$ for all $j > k$ and $a_0 = \infty$). We call a_i *part* of partition a; and k *length* of a, and write $l(a) = k$. We say that a is a partition of n, or n is the *weight* of a, and write $w(a) = n$, if $\sum_{i=1}^{i=k} a_i = n$.

(ii) A *smooth partition* is a partition such that all differences between two consecutive parts are at most 1.

(iv) *Young's lattice* is the lattice of all partitions ordered by containment [51] (*i.e.* $a \leq b$ if and only if $a_i \geq b_i$ for all $i = 1, 2, \dots, \min\{l(a), l(b)\}$).

From this definition, one can see that a stable configuration is represented by a smooth partition. So, in the following, we say partition (resp. smooth partition) for configuration (resp. stable configuration).

The Extended Sand Piles Model (ESPM) is a discrete dynamical model where all configurations are partitions and the initial configuration being the partition (0). This model consists of two evolution (or transition) rules:

- *Falling rule (inside action)*: one grain on the column i can fall down to the column $i+1$ if the height difference between the column i and the column $i+1$ is greater than or equal to 2.
- *Adding rule (outside action)*: one grain can be added to one column of a smooth partition such that the obtained one is still a partition.

We denote also $ESPM$ the set of all reachable partitions from the initial (0). We call a *chain* in this model a sequence of transitions. By convention, a chain of one element (with no transitions) is of length 0. More particularly, a chain between two smooth partitions is called an *avalanche chain*. Finally, we denote by $a^{\downarrow i}$ the integer sequence obtained from a by increasing part i of a by 1.

Figure 2 shows first elements of $ESPM$. One can see that $ESPM$ does not contain all partitions. However, we prove that this model contains all smooth partitions.

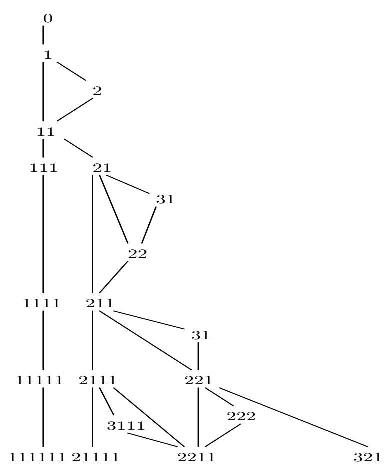


Fig. 2 First elements of the poset $ESPM$

Proposition 1 *All smooth partitions are reachable from the initial partition.*

In order to study the behaviour of the model under outside actions, we investigate the set of all stable configurations and the relations between them. We denote the induced subposet of all smooth partitions of the poset $ESPM$ by $(SESPM, \leq_S)$. We will describe the nature of order relation in $SESPM$.

First, we analyze the movement of a grain when it is added from outside to a stable configuration. So, let $a = (a_1, \ldots, a_k)$ be a smooth partition. One grain is added on column i of a with the condition that $a_i < a_{i-1}$. After that, if $a_i = a_{i+1}$, this grain stays at column i and does not move anymore. Otherwise, this grain move to a new position $j > i$ such that $a_i, a_{i+1}, \ldots, a_j$ is a consecutive decreasing integers and that $a_j = a_{j+1}$. Finally, this grain stays at column j and does not move anymore. The obtained configuration b of this sequence of moves of this grain is the same as the configuration obtained by only one move: adding a grain directly on position j. Hence, this analyze proves the following result: In the $SESPM$, an element b is an immediate successor of an element a if and only if b can be obtained from a by adding one grain at some column. Figure 3 shows some first elements of the poset $SESPM$. To finish, we discuss about the relation between $SESPM$ and the Young

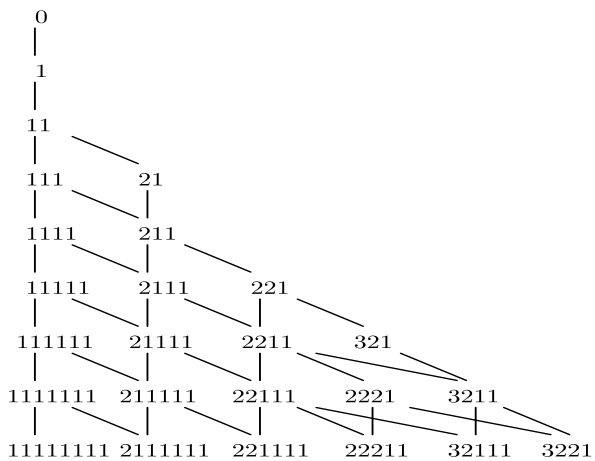


Fig. 3 First elements of the poset $SESPM$

lattice. Due to the characterization of the containment order, we know that the poset $SESPM$ is a suborder of the Young lattice. Futhermore we prove that this relation is in fact a sublattice relation.

Theorem 1 *The poset $SESPM$ is ordered by containment, moreover it is a sublattice of the Young lattice.*

2.2 Avalanche chains

The purpose of this subsection is to describe the needed time to reach a stable configuration in the Extended Sand Piles Model. We know that there are probably different sequences of evolutions to reach a stable configuration from another stable configuration. Their sizes may be quite different and depend on the columns in which evolution rules are applied. We next show that the smallest length of avalanche chain depends only on the weight of the considered stable configurations. Otherwise, the problem is much more complicated than for the greatest length.

Theorem 2 *Let a and b be two smooth partitions and $b <_S a$. Then*
(i) *The smallest length of avalanche chain from the initial configuration* (0) *to a is equal to $w(a)$.*
(ii) *The smallest length of avalanche chain from a to b is equal to $w(b) - w(a)$.*

To compute the greatest length of avalanche chains, we consider the movement of grains. We constate that, when one grain is added to a smooth partition, it slides down to a position until the obtained partition is smooth and after that this grain can not be moved. Hence the number of moves of a grain depends only the position (column) where it is added. We will define by energy of a grain its greatest number of possible moves. Then we will define energy of a configuration the summation of energy of all of its grains. The main result of this section is to prove that the greatest length of avalanche chain to reach a stable configuration is equal to its energy.

Let us recall that, in our model, each configuration is represented by a partition, or more precisely, by its Ferrers diagram, where each grain is represented by a case (i, j) where i (resp. j) is the column (resp. row) index. So, let us denoted by $F(a)$ the diagram of a partition $a = (a_1, a_2, \ldots, a_k)$, and we write $(i, j) \in F(a)$ for all case (i, j) such that $1 \leq i \leq k$ and $1 \leq j \leq a_i$ (see Figure 4 as an example). We say

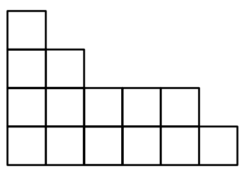


Fig. 4 The representation of the Ferrer diagram of the partition $a = (4, 3, 2, 2, 2, 1)$

that i is a *smooth column* of a if $i = 1$ or $a_i = a_{i-1}$ for $i > 1$. Moreover, for a case (i, j), we define *diagonal* $D(i, j)$ the set of all case (i', j') such that $i' + j' = i + j$ and $1 \leq j' \leq j$ (see Figure 5). We give now the formal definition and some properties

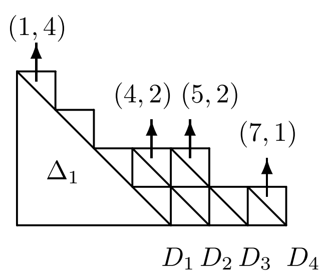


Fig. 5 Smooth columns $(1, 4, 5, 7)$ and conresponding diagonals D_1, D_2, D_3, D_4 of $b = (4, 3, 2, 2, 2, 1, 1)$

of energy.

Definition 4 Let $a = (a_1, a_2, \ldots, a_k)$ be a smooth partition.
(i) The energy $e_a(i, j)$ is the greatest possible moves that a grain can do to reach the position (i, j).
(ii) The energy $E(a)$ of a is $E(a) = \sum_{(i,j) \in F(a)} e_a(i, j)$.

Lemma 1 *Let $a = (a_1, a_2, \dots, a_k)$ be a smooth partition.*
(i) *We have: $e_a(i, j) = i + 1 - \min\{r : a_r < a_{r-1}$ and $a_r + r \geq j + i - 1\}$.*
(ii) *Moreover, if $(i, j) \in F(a)$ and $(i-1, j+1) \in F(a)$ then*

$$e_a(i-1, j+1) = e_a(i, j) - 1.$$

Now, we want to compute explicitly the energy of a smooth partition $a = (a_1, \dots, a_k)$. Let $1 = i_1 < i_2 < \dots < i_\ell$ be all smooth columns of a. And let D_i the diagonal (i, a_i). It is evident that we can decompose $F(a)$ as the following disjoint union:

$$F(a) = \Delta_1 \bigsqcup D_2 \bigsqcup \dots \bigsqcup D_\ell$$

where Δ_1 is the set of all case (i, j) such that $1 \leq i, j$ and $i + j \leq a_1 + 1$. We then compute the energy of a in each of such subset.

Proposition 2 *Let a be a smooth partition, and let $1 = i_1 < i_2 < \dots < i_\ell$ be all smooth columns of a. We have:*

$$E(a) = \frac{a_1(a_1+1)(a_1+2)}{6} + \sum_{r=2}^{\ell} i_r a_{i_r} - \sum_{r=3}^{\ell} i_{r-1} a_{i_r} + \sum_{r=2}^{\ell} \frac{a_{i_r}(a_{i_r} - 1)}{2}.$$

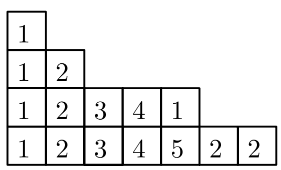


Fig. 6 Representation of the partition $b = (4, 3, 2, 2, 2, 1, 1)$, the number in each case is the energy of the corresponding grain. The greatest length from (0) to (b) is 34

We state now the main result of this subsection.

Theorem 3 *Let a be a smooth partition. Then the greatest length of avalanche chains from (0) to a is equal to $E(a)$.*

From this theorem we can study avalanche chain between two stable configurations.

Corollary 1 *Let $b \leq_S a$ be two smooth partitions. Then the greatest length of avalanche chains from a to b is $\sum_{(i,j) \in F(b) - F(a)} e_b(i, j)$.*

Nevertheless, it is important to note that the greatest length from a to b is not equal to $E(b) - E(a)$ because for $(i, j) \in F(b)$, we have not $e_a(i, j) = e_b(i, j)$ (see Figure 9 for an encounter example). Moreover, it is easy to see that $e_b(i, j) \geq e_a(i, j)$. This implies that the greatest length of avalanche chains from a to b is smaller than or equal to the difference of the one from (0) to b and the one from (0) to a. This result is opposite to the result in the case of smallest length where the egality is hold.

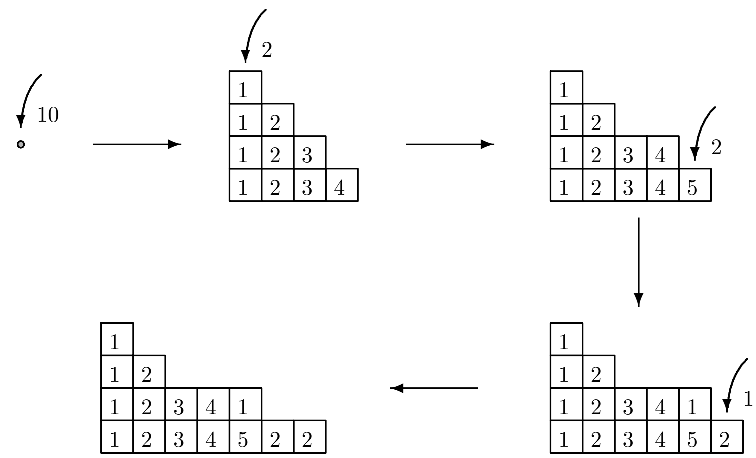


Fig. 7 A greatest chain from 0 to b. Each arrow $\rightarrow^k$ to a column i means that k grains are added to column i

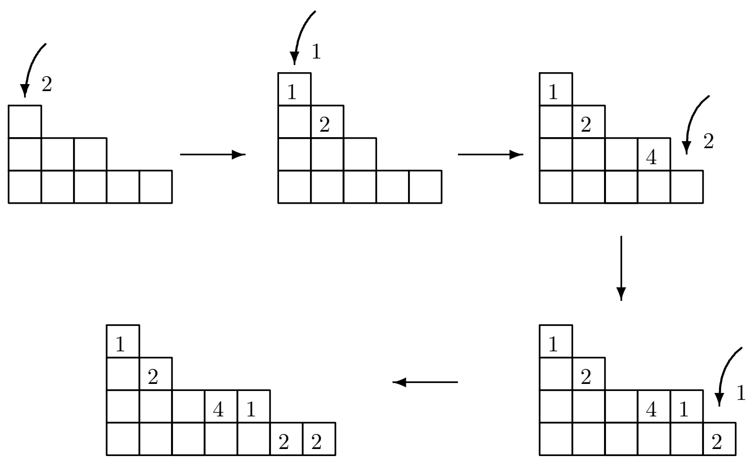
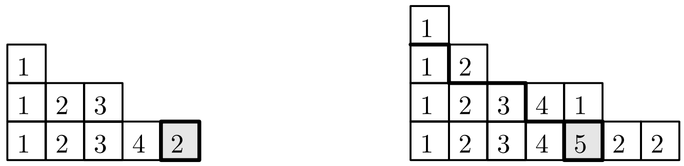


Fig. 8 A greatest chain from a to b. Each arrow $\rightarrow^k$ to a column i means that k grains are added to column i. The greatest length from a to b is 12

Furthermore, we remark that the avalanche chain of greatest length from (0) to a is unique. Indeed, from the proof of Theorem 3, we constate that the grain G at position (i, j) on the diagonal D_r for $r \geq 2$ (resp. Δ_1) has exactly $e_a(i, j)$ transitions if and only if G is added at the column $i_{r-1}+1$ (resp. 1) and then it slides diagonally and stops at position (i, j). So the diagonal D_{r-1} must be fulfilled,moreover the grain at position $(i+1, j-1)$ must be presented before the adding of the grain G. So by recurrence we claim that the avalanche chain of greatest length from (0) to a must be defined explicitly as in the proof of Theorem 3, hence it is unique.



(a) (b)

Fig. 9 (a):energy tableau of a; (b): energy tableau of b and $2 = e_a(5,1) \neq e_b(5,1) = 5$

However, there are many avalanche chains of greatest length from a to b. For instance, if we take $a = (2,2,1,1,1)$ and $b = (2,2,2,1,1,1)$. Then by Corollary 1 we have $l(a,b) = 2$. Moreover, we have two following avalanche chains of length 2:

$$a = (2,2,1,1,1) \longrightarrow (2,2,2,1,1) \longrightarrow (2,2,2,1,1,1) = b$$

$$\text{and } a = (2,2,1,1,1) \longrightarrow (2,2,1,1,1) \longrightarrow (2,2,2,1,1,1) = b.$$

3 Symmetric Sand Pile Model and unimodal sequences

Sand Pile Model was first introduced in the context of SOC phenomena. In order to bring it closer to real physical models, we consider the model such that grains can fall to the both sides (left and right). This generalized model is called two sided sand piles model [46] or symmetric sand piles model [19], and denoted by SSPM.

Definition 5 SSPM is a model defined by:

- Initial configuration: $(\underline{n})$.
- Local left vertical rule $\mathcal{L}$: $(\ldots, a_{i-1}, a_i, \ldots) \longrightarrow (\ldots, a_{i-1}+1, a_i-1, \ldots)$ if $a_{i-1} + 2 \leq a_i$.
- Local right vertical rule $\mathcal{R}$: $(\ldots, a_i, a_{i+1}, \ldots) \longrightarrow (\ldots, a_i - 1, a_{i+1}+1, \ldots)$ if $a_i \geq a_{i+1} + 2$.
- Global rule: we apply the $\mathcal{L}$ rule once , or the $\mathcal{R}$ rule once .

When studying this model, we formulate the characterization of reachable configurations and of fixed points. Note that a configuration is defined by its form and its position. If at the beginning, we have a pile of n sand grains at position 0, then a reachable configuration $a = (a_p, a_{p+1}, \ldots a_{p+k-1})$ can have grains in negative and positive positions. We call *position* of a the smallest index p such that $a_p > 0$, and we call the *form* of a the sequence $b = (b_1, \ldots, b_k)$ such that $b_i = a_{p+1-i}$ for all $1 \leq i \leq k$.

It is easy to see that a configuration can be represented by a unimodal sequence which is defined as follows.

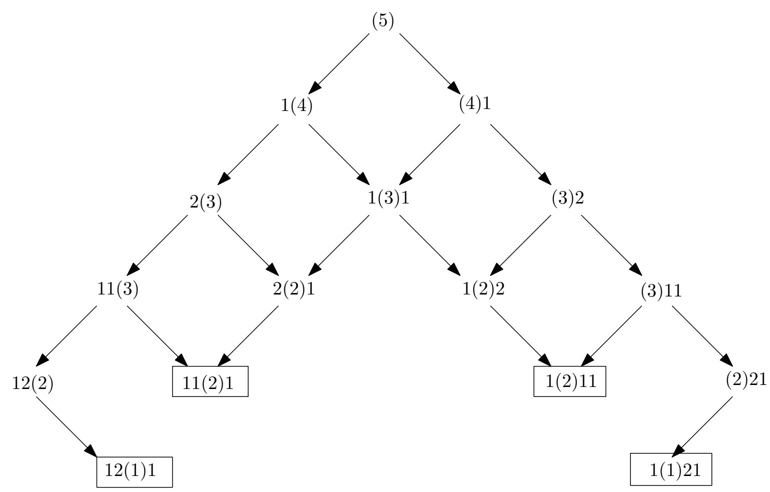


Fig. 10 Configuration space $SSPM(5)$

Definition 6 A *unimodal sequence* is a sequence of positive integers $(a_1, a_2, \ldots, a_k)$ such that there exists an index $1 \le i \le k$ satisfying the condition $a_1 \le a_2 \le \ldots a_{i_1} \le a_i \ge a_{i+1} \ge \ldots \ge a_{k-1} \ge a_k$. The quantities defined by

$$h(a) = \max\{a_i\}_{i=1}^{k} \quad \text{and} \quad w(a) = \sum_{i=1}^{k} a_i$$

are respectively called *the height* and *the weight* of a. We say also that a a unimodal sequence of $w(a)$.

Given an index $1 \le i \le k$, we denote

$$a_{<i} := (a_1, \ldots, a_{i-1}) \text{ and } a_{>i} := (a_{i+1}, \ldots, a_k),$$

$$a_{\le i} := (a_1, \ldots, a_{i-1}, a_i) \text{ and } a_{\ge i} := (a_i, a_{i+1}, \ldots, a_k),$$

and call them the *strict left sequence* and the *strict right sequence* of a by i, the *left sequence* and the *right sequence* of a by i, respectively.

We give a characterization for the form and the position of reachable configuration [46].

Theorem 4 *An integer sequence a is a configuration of $SSPM$ if and only if*

- *The form of a is a unimodal sequence which has a decomposition $a = (a_{<i}, a_{\ge i})$ where $a_{<i}$ and $a_{\ge i}$ are two partitions satisfying SPM condition.*
- *the position i satisfies:*
  - *if $i \ge 0$ then: $ia_i + \frac{i(i+1)}{2} + \sum_{j \ge i} a_j \le n$ if $a_{\ge i}$ begins with a slide step (subsequence of the form $(p, p_1, \ldots, q+1, q, q)$ with $p \ge q > 0$), or $ia_i + \frac{i(i-1)}{2} + \sum_{j \ge i} a_j \le n$ otherwise.*

– *if* $i < 0$ *then:* $-ia_{i-1} + \frac{i(i-1)}{2} + \sum_{j<i} a_j \leq n$ *if* $a_{<i}$ *begins with a slide step, or* $-ia_{i-1} + \frac{i(i+1)}{2} + \sum_{j<i} a_j \leq n$ *otherwise.*

Such a decomposition is called a $SSPM$ *decomposition*.
For the fixed point of $SSPM$, we give the following condition [46].

Theorem 5 *An integer sequence P is a fixed point of $SSPM(n)$ if P has an $SSPM$ decomposition at some position i such that:*

- $P_{<i}$ *and* $P_{\geq i}$ *are SPM fixed points and* $|P_i - P_{i-1}| \leq 1$,
- *the height* k *of* P *is either* $\lfloor \sqrt{n} \rfloor$ *or* $\lfloor \sqrt{n} \rfloor - 1$, *and*
- *the position* i *satisfies* $k + |i| \leq \lfloor \sqrt{2n} \rfloor$.

As consequence, the number of fixed point forms of $SSPM(n)$ is $\lceil \sqrt{n} \rceil$ [19].

4 Fixed points of Parallel Symmetric Sand Pile Model

The Parallel Symmetric Sand Pile Model is a variant of the Symmetric Sand Pile Model where we allow to apply at the same time all possible transitions [21, 43].

Definition 7 PSSPM is a model defined with the same initial configuration and local rule as SSPM, and with the following global rule.

- Global rule: we apply $\mathcal{L}$ and $\mathcal{R}$ in parallel on all possible columns. We apply at most once of the two rules on each column.

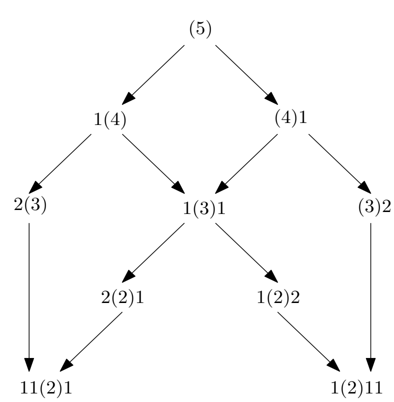


Fig. 11 Configuration space $PSSPM(5)$

4.1 Forms of fixed points

Remark that while the parallel SPM is deterministic, the parallel $SSPM$ is not because there may have columns from which grains can fall down on both sides. Recall that SPM has a unique fixed point which implies that PSPM have the same fixed point as SPM. But $SSPM$ can have more than one fixed points therefore $PSSPM$ may not have the same set of fixed points as SSPM. Actually, there exists fixed points of $SSPM$ which is not reachable in $PSSPM$ because its position is far from the position 0. Nevertheless, it is surprising that the set of forms of fixed points of $PSSPM$ is the same as that of SSPM. And this fact is the main result of this subsection [21].

Theorem 6 *The set of fixed point forms of $PSSPM(n)$ is equal to that of $SSPM(n)$. Consequently, there is $\lceil\sqrt{n}\rceil$ fixed point forms of $PSSPM(n)$.*

This theorem need a very long and technical proof but the idea is very constructive and can be presented as follows.

For a fixed point P of $SSPM(n)$, we construct a sequence of $PSSPM$ transitions to obtain a fixed point having the same form as P. Because we are interested in the form of P but not in its position, we can suppose that the center column of P is at position 0 (the center of a configuration a is the position i satisfying the condition that $|w(a_{<i})-w(a_{\geq i})|$ get the minimum value). In the constructed evolution, column 0 is always a highest one, so the choice of $PSSPM$ rules in each step is in fact the choice of the transition's direction at column 0.

- For a symmetric fixed point P, *i.e.* $(P_{<0})^{-1} = P_{>0}$, the evolution is an Alternating Procedure, described as follows: at odd steps, the rule R is applied at position 0, and at even steps, the rule L is applied at position 0. From (n) this procedure will converge to the symmetric fixed point P.
- For P not symmetric, we can suppose that the column 0 is the center of P, *i.e.*

$$d = |w(P_{>0}) - w(P_{<0})| = \min_i |w(P_{>i}) - w(P_{<i})|.$$

  Without loss of generality we may assume that $w(P_{>0}) - w(P_{<0}) > 0$. The evolution by $PSSPM$ rule is composed of three procedures:

  i) Pseudo-Alternating Procedure: a procedure from (n) to the configuration $Q = \big(1, 2, \ldots, d-1, (n-d^2), d, d-1, \ldots, 2, 1\big)$. Note that $w(Q_{>0}) - w(Q_{<0})$ is exactly d.
  ii) Alternating Procedure: a procedure from Q to the configuration R on which we could not apply any more the Alternating Procedure.
  iii) Deterministic procedure: a deterministic procedure from R to P, where at each of its step, on each position, only one rule can be applied.

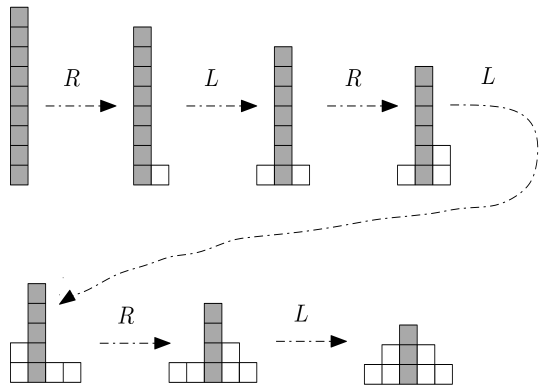


Fig. 12 6 first steps of Alternating Procedure from (9). The arrow together with the direction R or L (Right or Left) corresponding to the direction along which the transition at column 0 (dark column) is applied

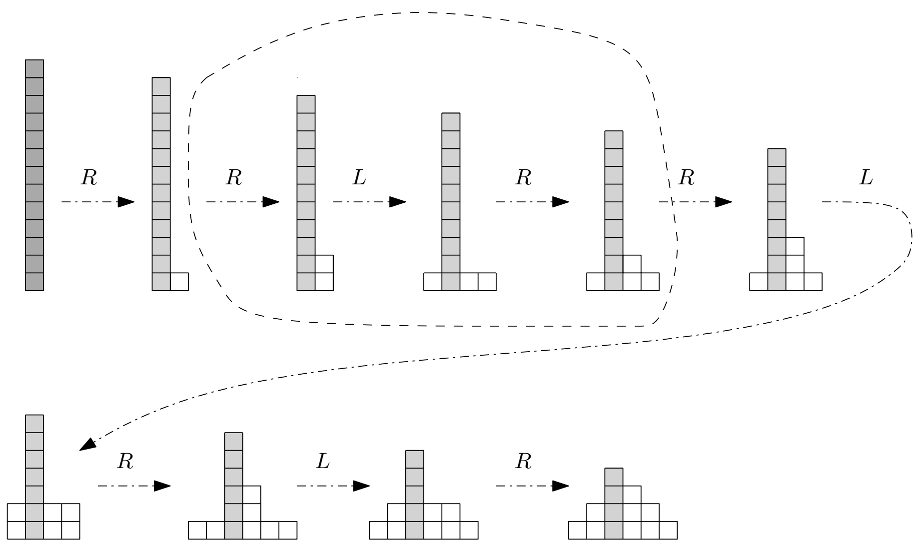


Fig. 13 9 first steps of Pseudo-Alternating Procedure on (13)

4.2 Positions of fixed points

We have already a characterization of the form of fixed points of $PSSPM$, but what about their positions?. Remark that for $SSPM$ one can obtained a fixed point at a position very far on the left (or on the right) when one applies as much as possible left transition (or right transition respectively). But for $PSSPM$ all transitions are applied at the same time at each step, so one can not apply as much left transition as he wants to. This explains why many fixed points of $SSPM$ very far on the left can not be reached by $PSSPM$. Nevertheless, we can prove that all fixed points of

$SSPM$ whose the position is between the leftmost and the rightmost fixed points of $PSSPM$ can be reached also by PSSPM. This is related to the continuity of fixed points of PSSPM. The notion of leftmost, rightmost and continuity will be explained clearly by using relation $\triangleleft$, a notion of closeness between configurations [43].

Definition 8 Let $\Delta(a,b)$ be the sequence of differences between configurations a and b, $\Delta_i(a,b) = a_i - b_i$.
We define a notion of similarity or closeness between configurations, denoted by the following relations:

$$\begin{aligned} a \triangleleft b &\iff \Delta(a,b) \in 0^*\bar{1}0^*10^* \\ a \overset{*}{\triangleleft} b &\iff \Delta(a,b) \in (0^*\bar{1}0^*10^*)^* \end{aligned}$$

where $\bar{1}$ is a minus one value. As a convention $\epsilon = 0^\omega$, so that $a = b$ implies $a \overset{*}{\triangleleft} b$.

Notation We use the symbols $\leq_{lex}$ to denote the lexicographic order over configurations. Note that $a \overset{*}{\triangleleft} b \Rightarrow a \leq_{lex} b$ and $a \triangleleft b \Rightarrow a <_{lex} b$.

Theorem 7 *[43]*

Let

$$\pi_0 <_{lex} \pi_1 <_{lex} \cdots <_{lex} \pi_{k-1} <_{lex} \pi_k$$

be the sequence of all fixed points of PSSPM(n) ordered lexicographically. Then this sequence has the following strong relation:

$$\pi_0 \triangleleft \pi_1 \triangleleft \cdots \triangleleft \pi_{k-1} \triangleleft \pi_k .$$

Moreover, for any fixed point π of SSPM(n) such that $\pi_0 \leq_{lex} \pi \leq_{lex} \pi_k$, there exists an index i, $0 \leq i \leq k$, such that $\pi_i = \pi$.

5 Signed Chip Firing Game and Symmetric Sandpile Model on the cycles

We explore the Sandpile Model and Chip Firing Game and an extension of these models on cycle graphs. These problems also have a strong relationship to the class of problems on cycles such as games of cards [12, 25, 30]. Furthermore, we are also interested in the signed versions of these models, *i.e.*, we allow the vertices to contain negative numbers of chips for CFG and the sandpiles to have negative heights for SPM. This also reflects deeply some natural phenomena: between sandpiles there may be holes (of negative heights), and besides the delivering chips from vertices containing many chips, it is dually possible receiving chips from vertices lacking (negative enough) chips [32]. We give the characterization of reachable configurations and of fixed points of each model. At the end, we give an explicit formula for the number of their fixed points [10].

5.1 SPM, CFG, $SSPM$ and SCFG on cycles: definitions and notations

Let C_n be a cycle graph of n vertices $\{1, 2, ..., n\}$ ($n \geq 3$). Each integer sequence $(a_1, a_2, \ldots, a_n)$ on vertices of C_n is called *circular distribution* and we say that vertex i contains a_i chips (note that a_i may be negative). We identify two circular distributions if they differ by a rotation of the cycle.

Definition 9 Let k be a non-negative integer. The *Sandpile model on C_n* of weight k (and its configuration space), denoted by $SPM(C_n, k)$, is described as follows:

(i) The initial configuration is $(k, 0, 0, \ldots, 0)$,
(ii)The evolution rule is the *right rule* as follows: a vertex gives one chip to its right neighbor vertex if it has at least 2 higher than this neighbor.

Definition 10 Let k be a non-negative integer. The *Symmetric Sandpile model on C_n* of weight k (and its configuration space), denoted by $SSPM(C_n, k)$, is described as follows:

(i) The initial configuration is $(k, 0, 0, \ldots, 0)$,
(ii)The evolution rule: addition to the right rule in $SPM(C_n, k)$, there is also the *left rule*, that means a vertex gives one chip to its left neighbor vertex if it has at least 2 higher than this left neighbor.

Definition 11 Let k be a non-negative integer. The *Chip Firing Game on C_n* of weight k (and its configuration space), denoted by $CFG(C_n, k)$, is described as follows:

(i) The initial configuration is $(k, 0, 0, \ldots, 0, -k)$,
(ii)The evolution rule is the *positive rule* as follows: a vertex containing at least 2 chips gives one chip to each of its two neighbors.

Definition 12 Let k be a non-negative integer. The *Signed Chip Firing Game on C_n* of weight k (and its configuration space), denoted by $SCFG(C_n, k)$, is described as follows:

(i) The initial configuration is $(k, 0, 0, \ldots, 0, -k)$.
(ii)The evolution rule: addition to the positive rule in $CFG(C_n, k)$, there is also the *negative rule*, that means a vertex containing at most -2 chips receives one chip from each of its two neighbors.

Notations.

- We define $SPM(C_n)$ the disjoint union of $SPM(C_n, k)$ for $k \geq 0$, and similarly for $SSPM(C_n)$, $CFG(C_n)$, $SCFG(C_n)$.
- Let a and b be two distributions of non-negative integers on C_n, we write $a \xrightarrow{(i,r)} b$ (resp. $a \xrightarrow{(i,l)} b$) if b is obtained from a by applying the rule at the vertex i on the right (resp. left); and $a \xrightarrow{(i,+)} b$ (resp. $a \xrightarrow{(i,-)} b$) if b is obtained from a by applying the positive rule (resp. negative rule) at the vertex i.

Remark. It is straightforward from the definitions that

- The configurations of $SPM(C_n)$ and $SSPM(C_n)$ are circular distributions of non-negative integers whereas the ones of $CFG(C_n)$ and $SCFG(C_n)$ are circular distributions of integers (may be negative),
- We have the two following inclusions

$$SPM(C_n, k) \subset SSPM(C_n, k) \text{ and } CFG(C_n, k) \subset SCFG(C_n, k).$$

Recall that two models are called isomorphic if there exists a bijection between their configuration spaces and this bijection preserves their evolution rule.

Now, let $a = (a_1, \ldots, a_n)$ be a circular distribution on C_n. We define

$$d(a) = (a_1 - a_2, \ldots, a_{n-1} - a_n, a_n - a_1).$$

It is straightforward that d is a well-defined map from the set of circular distributions on C_n to itself. Furthermore, we have the following result.

Proposition 3 *Under the map d two models $SPM(C_n, k)$ and $CFG(C_n, k)$ are isomorphic; and two models $SSPM(C_n, k)$ and $SCFG(C_n, k)$ are isomorphic.*

It is remarkable that although d is bijective from $SSPM(C_n, k)$ (resp. $SPM(C_n, k)$) to $SCFG(C_n, k)$ (resp. $CFG(C_n, k)$), it is not bijective from $SSPM(C_n)$ (resp. $SPM(C_n)$) to $SCFG(C_n)$ (resp. $CFG(C_n)$). Moreover, while $SSPM(C_n, k)$ and $SPM(C_n, k)$) are absolutely disjoint for different values k, $SCFG(C_n, k)$ and $CFG(C_n, k)$ may overlap each other, especially for values k differing by a multiple of n. Then a configuration of $SCFG(C_n)$ may correspond to many configurations of SSPM(Cn) whose weights differ by a multiple of n.

Next, we study a characterization for the configurations of the four models. Let $a = (a_1, a_2, \ldots)$ be a sequence of positive integers. A pair (a_i, a_{i+1}) is called a *cliff* (resp. *plateau*) of a at position i if $a_i - a_{i+1} \geq 2$ (resp. $a_i - a_{i+1} = 0$).

Theorem 8 *Let a be a circular distribution on C_n. Then a is a configuration of $SPM(C_n, k)$ if and only if there is a rotation vertices of C_n such that a (in the sequence form) is a configuration of $SPM(k)$ with the length at most n.*

Corollary 2 *Let $a = (a_1, a_2, \ldots)$ be a circular distribution. Then a is a configuration of $CFG(C_n, k)$ if and only if $d^{-1}(a)$ is a configuration of $SPM(C_n, k)$.*

Corollary 3 *The unique fixed point of $SPM(C_n, k)$ is of the form*

- $(p, p-1, \ldots, q, q, q-1, \ldots, 1, 0, \ldots)$ *if* $k \leq \frac{n(n-1)}{2}$, *where*

$$p = \left[\frac{3 + \sqrt{9 + 8k}}{2}\right] \textit{ and } q = k - \frac{p(p+1)}{2}.$$

- $(p, p-1, \ldots, q, q, q-1, \ldots, p-n+3, p-n+2)$ *if* $k \geq \frac{n(n-1)}{2} + 1$, *where*

$$p = \left[\frac{2k + (n-2)(n+1)}{2n}\right] \text{ and } q = k - \frac{(2p - n + 2)(n-1)}{2}.$$

Here $[x]$ is the largest integer not greater than x.

Next, we give a characterization for configurations of the $SSPM$s as well as $SCFG$s on C_n. To do this we first present the concept of 2-decomposable configurations on the cycle which is different a bit from the definition of LR-decomposition (left-right decomposition) on the line defined in [46].

Definition 13 Let $a = (a_1, a_2, \ldots, a_n)$ be a circular distribution on C_n, then a is called 2-decomposable at (i, j) with $1 \leq i \leq j \leq n$ if $(a_i, a_{i+1}, \ldots, a_j)$ and $(a_{i-1}, a_{i-2}, \ldots, a_1, a_n, \ldots, a_j{+}1)$ are SPM configurations. Furthermore, a is called 2-decomposable if there exist two indices $1 \leq i \leq j \leq n$ such that a is 2-decomposable at (i, j).

Theorem 9 *Let a be a circular distribution on C_n. Then a is a configuration of $SSPM(C_n)$ if and only if a is 2-decomposable.*

Corollary 4 *Let $a = (a_1, a_2, \ldots)$ be a circular distribution. Then a is a configuration of $SCFG(C_n, k)$ if and only if $d^{-1}(a)$ is 2-decomposable.*

5.2 Fixed points of $CFG(C_n)$ and $SCFG(C_n)$

Although we have a criterion for the configurations of $SCFG(C_n)$, it requires us to calculate their inverse images by d and then to check their 2-decomposability in $SSPM(C_n)$. In this section, we present a simple and direct characterization for the fixed points (not all their configurations) of $SCFG(C_n)$. Based on this characterization, we give an enumeration for these fixed points. We first classify the configurations of $CFG(C_n)$ and those of $SCFG(C_n)$ [10].

Proposition 4 *Let k, l be positive integers.*

(i) If $k \neq l \mod n$ then

$$CFG(C_n, k) \cap CFG(C_n, l) = \emptyset$$

and

$$SCFG(C_n, k) \cap SCFG(C_n, l) = \emptyset.$$

Consequently, the intersection of the set of fixed points of $SCFG(C_n, k)$ and that of $SCFG(C_n, l)$ is empty.

(ii)If $k = l \mod n$ and $k, l \geq (\frac{n+1}{2})^2$ then the set of fixed points of $CFG(C_n, k)$ (resp. $SCFG(C_n, k)$) is equal to that of $CFG(C_n, l)$ (resp. $SCFG(C_n, l)$).

As we remarked in the previous section that for large enough values of k in a residue class modulo n, although the set of fixed points of $SSPM(C_n, k)$ (resp.

$SPM(C_n, k)$) are disjoint, the heights of their columns differ up-to a constant. In other words, if $(a_1, \dots, a_n)$ is a fixed point of $SSPM(C_n, k)$ (resp. $SPM(C_n, k)$), then $(a_1{+}1, \dots, an{+}1)$ is a fixed point of $SSPM(C_n, k{+}n)$ (resp. $SPM(C_n, k{+}n)$). Hence their images by d in $SCFG(C_n, k)$ (resp. $CFG(Cn, k)$) and in $SCFG(C_n, k + n)$ (resp. $CFG(C_n, k{+}n)$) coincide. By Corollary 2.12, $CFG(C_n, k)$ has a unique fixed point whereas $SCFG(C_n, k)$ may have many fixed points. The set of fixed points of $SCFG(C_n)$ includes the fixed points of $SCFG(C_n, k)$ for small values of k and the n distinct residue classes of fixed points of $SCFG(C_n, k)$ for large values of k. For a small k, their fixed points can be found directly by taking the inverse images of d of 2-decomposable fixed points. We next characterize and enumerate the fixed points of $SCFG(C_n)$ for all $k \geq (\frac{n+1}{2})^2$.

For convenience, we denote by $FP(SCFG(C_n, k)$ the set of fixed points of $SCFG(C_n, k)$ and

$$FP(SCFG(C_n)) = \cap_{k \geq (\frac{n+1}{2})^2} FP(SCFG(C_n, k)).$$

Recall that each fixed point of $SCFG(C_n)$ is a circular distribution on C_n and its chips at vertices are $0, 1, -1$. By a rotation, first we can consider $FP(SCFG(C_n))$ as words on the alphabet $\{0, 1, \bar{1}\}$ where the letter $\bar{1}$ is understood as -1.

Theorem 10 *The set $FP(SCFG(C_n))$ is determined as follows*

- $FP(SCFG(C_3)) = \{(000); (10\bar{1}); (1\bar{1}0)\}$.
- $FP(SCFG(C_4)) = \{(0000); (1\bar{1}00); (10\bar{1}0); (100\bar{1}); (11\bar{1}\bar{1})\}$.
- $FP(SCFG(C_n))$*, with $n \geq 5$, consists of the words w on the alphabet* $\{0, 1, \bar{1}\}$ *satisfying the following properties:*
  - *w starts from* 1*;*
  - *in w, the number of occurrences of* 1 *is equal to that of* $\bar{1}$*;*
  - *w avoids the subsequences:* $\bar{1}1, 1001, \bar{1}00\bar{1}$ *and* 00000*;*
  - *If w has 4 occurrences of* 0 *then it must end by* 0 *and does not contain the sub-word* $1\bar{1}$.

Theorem 11 *The cardinality of $FP(SCFG(C_n))$ is*

- *3 if $n = 3$;*
- *5 if $n = 4$;*
- $\frac{(n-1)^2}{2}$ *if n is odd and $n \geq 5$;*
- $\frac{n(n-2)}{2}$ *if n is even and $n \geq 6$.*

6 Extension of Brylwaski's model

In the previous section, we presented many kinds of extensions of SPM model. For each kind of extension, it is natural to think about a similar extension of Brylawski's model. Recall that for the classical models, while the configuration space of SPM are clearly characterized with explicit criteria, we also have the exhaustive property of Brylawski's model: all integer partitions are reachable. Is we adapt the similar notion of slide transition in each extension of the SPM model, we have the corresponding extension of Brylawski's model. And this is very interesting that the exhaustive property remains for these extensions. It is also very surprising that the shortest and longest chains in extended models have the same length as in the classical model of Brylawski.

Let us recall here the definition of Brylawski's model.

Definition 14 Brylawski's model, with respect to a positive integer n, denoted by $L_B(n)$, is a model where configurations are partitions of n such that:

- Initial configuration: $(\underline{n})$.
- Local right vertical rule $\mathcal{R}$: $(\ldots, a_i, a_{i+1}, \ldots) \rightarrow (\ldots, a_i - 1, a_{i+1} + 1, \ldots)$ if $a_i \geq a_{i+1} + 2$.
- Local right horizontal rule $\mathcal{H}$: $(\ldots, p+1, p, \ldots, p, p-1, \ldots) \rightarrow (\ldots, p, p, \ldots, p, p, \ldots)$.
- Global rule: apply the $\mathcal{R}$ rule once, or the $\mathcal{H}$ rule once.

6.1 The symmetric Brylawski'model

Let us first investigate to the extended symmetric Brylawski's model (SB_L) [14].

Definition 15 The symmetric Brylawski's model, with respect to n, denoted by $SB_L(n)$, is a model defined by:

- Initial configuration: $(\underline{n})$.
- Local left vertical rule $\mathcal{L}$: $(\ldots, a_{i-1}, a_i, \ldots) \rightarrow (\ldots, a_{i-1} + 1, a_i - 1, \ldots)$ if $a_{i-1} + 2 \leq a_i$.
- Local right vertical rule $\mathcal{R}$: $(\ldots, a_i, a_{i+1}, \ldots) \rightarrow (\ldots, a_i - 1, a_{i+1} + 1, \ldots)$ if $a_i \geq a_{i+1} + 2$.
- Local left horizontal rule $\mathcal{L}_H$: $(\ldots, p+1, p, \ldots, p, p-1, \ldots) \rightarrow (\ldots, p, p, \ldots, p, p, \ldots)$.
- Local right horizontal rule $\mathcal{R}_H$: $(\ldots, p-1, p, \ldots, p, p+1, \ldots) \rightarrow (\ldots, p, p, \ldots, p, p, \ldots)$.
- Global rule: we apply the $\mathcal{L}$ rule once, or the $\mathcal{R}$ rule once.

The exhaustive property of SL_B is presented as follows.

Lemma 2 *The set of configuration forms of $SL_B(n)$ are the set of all unimodal sequences of weight n.*

We have the following straightforward result on fixed points.

Corollary 5 *$SL_B(n)$ has n fixed points of form $(1, \ldots, 1)$ where the first position can take value from $-n + 1$ to 0.*

And results on shortest and longest chains.

Proposition 5 *For $n \geq 4$, the shortest chains in $SL_B(n)$ have length $2n - 5$ and the longest chains in $SL_B(n)$ have the same length as that in $L_B(n)$ which is $\theta(n^{3/2})$.*

7 Infinite Extension

All the previous extensions have a common property: the total number of sand grains is unchanged. Now, we investigate an extended models where the total number of grains can be changed. Furthermore, we consider model where the first column has not a fix number but an infinite number of grains. It is natural to ask how one can construct the lattice $L_B(n+1)$ from the lattice $L_B(n)$. We construct a linear time algorithm which gives a translation from $L_B(n)$ to $L_B(n+1)$ and which reserves the lattice structure. From this algorithm, one can construct the lattice L_B with an arbitrary number of grains at the first position. And by the way, we can see that $L_B(\infty)$ is the limit of $L_B(n)$ where n goes to infinity. To do that, we consider the model with three transition rules: the vertical rule, the horizontal rule and the adding rule (adding one grain at the first position). It is obvious that all reachable configurations are still integer partitions, but what is interesting is that all integer partitions are reachable in this models [35].

7.1 Constructing $L_B(n+1)$ from $L_B(n)$

Before entering the core of algorithms, we need one more notation. If the k-tuple $a = (a_1, a_2, \ldots, a_k)$ is a partition, then the k-tuple $(a_1, a_2, \ldots, a_{i-1}, a_i + 1, a_{i+1}, \ldots, a_k)$ is denoted by $a^{\downarrow_i}$. In other words, $a^{\downarrow_i}$ is obtained from a by adding one grain on its i-th column. Notice that the k-tuple obtained this way is not necessarily a partition. If S is a set of partitions, then $S^{\downarrow_i}$ denotes the set $\{a^{\downarrow_i} | a \in S\}$. Finally, we denote by $Succ(a)$ the set of configurations directly reachable from a, *i.e.* the set $\{b \mid a \xrightarrow{i} b \text{ for some } i\}$.

Write $d_i(a) = a_i - a_{i+1}$ with the convention that $a_{k+1} = 0$. We say that a has a *cliff* at position i if $d_i(a) \geq 2$. If there exists an $\ell \geq i$ such that $d_j(a) = 0$ for all $i \leq j < \ell$ and $d_\ell(a) = 1$, then we say that a has a *slippery plateau* at i. Likewise, a has a *non-slippery plateau* at i if $d_j(a) = 0$ for all $i \leq j < \ell$ and it has a cliff at ℓ. The integer $\ell - i$ is called the *length* of the plateau at i. Note that in the special case $\ell = i$, the plateau is of length 0.

The set of elements of $L_B(n)$ that begin with a cliff, a slippery plateau of length ℓ and a non-slippery plateau of length ℓ are denoted by C, SP_ℓ, nSP_ℓ respectively.

Let $a = (a_1, a_2, \ldots, a_k)$ be a partition. It is clear that $a^{\downarrow_1}$ is again a partition. This define an embedding $\pi : L_B(n) \longrightarrow L_B(n)^{\downarrow_1} \subset L_B(n+1)$ which can be proved, by using infimum formula of $L_B(n)$ and $L_B(n+1)$, as a lattice map.

Lemma 3 *$L_B(n)^{\downarrow_1}$ is a sub-lattice of $L_B(n+1)$.*

Our next result characterizes the remaining elements of $L_B(n+1)$ that are not in $L_B(n)^{\downarrow_1}$.

Theorem 12 *For all $n \geq 1$, we have $L_B(n+1) = L_B(n)^{\downarrow_1} \sqcup_\ell SP_\ell^{\downarrow_{\ell+1}}$.*

Proof It is easy to check that each element in one of the sets $L_B(n)^{\downarrow_1}$ and $SP_\ell^{\downarrow_{\ell+1}}$ is an element of $L_B(n+1)$, and that these sets are disjoint. Now let us consider an element b of $L_B(n+1)$. If b begins with a cliff or a step then b is in $L_B(n)^{\downarrow_1}$. If b begins with a plateau of length $\ell+1, \ell \geq 2$, then b is in $SP_\ell^{\downarrow_{\ell+1}}$. □

Finally, we describe an algorithm to compute the successors of any given element of $L_B(n+1)$, thus giving a complete construction of $L_B(n+1)$ from $L_B(n)$.

Proposition 6 *Let x be an element of $L_B(n+1)$.*

1. *Suppose $x = a^{\downarrow_1} \in L_B(n)^{\downarrow_1}$.*
   **) If a is in C or nSP then $Succ(a^{\downarrow_1}) = Succ(a)^{\downarrow_1}$,*
   **) If a is in SP_l then $Succ(a^{\downarrow_1}) = Succ(a)^{\downarrow_1} \cup \{a^{\downarrow_{\ell+1}}\}$,*
2. *If $x = a^{\downarrow_{\ell+1}} \in SP_\ell^{\downarrow_{\ell+1}}$ for some $a \in SP_\ell$, then*
   **) If a has a cliff at $\ell+1$ or a non-slippery plateau at $\ell+1$, then $Succ(a^{\downarrow_{\ell+1}}) = Succ(a)^{\downarrow_{\ell+1}}$,*
   **) If a has a slippery plateau at $\ell+1$, let b such that $a \xrightarrow{\ell} b$ in $L_B(n)$, then $Succ(a^{\downarrow_{\ell+1}}) = (Succ(a) \setminus \{b\})^{\downarrow_{\ell+1}} \cup \{b^{\downarrow_\ell}\}$.* □

Proposition 6 makes it possible to write an algorithm to construct the lattice $L_B(n)$ in linear time (with respect to its size).

7.2 The infinite lattice $L_B(\infty)$

Imagine that (∞) is the initial configuration where the first column contains infinitely many grains and all the other columns contain no grains. Then the transitions V and H can be performed on (∞) just as if it is finite, and we call $L_B(\infty)$ the set of all the configurations reachable from (∞). A typical element a of $L_B(\infty)$ has the form $(\infty, a_2, a_3, \ldots, a_k)$. As in the previous section, we find that the dominance ordering on $L_B(\infty)$ (when the first component is ignored) is equivalent to the order induced by the dynamical model.

For any two elements $a = (\infty, a_2, \ldots, a_k)$ and $b = (\infty, b_2, \ldots, b_\ell)$ of $L_B(\infty)$, we define c by: $c_i = max(\sum_{j \geq i} a_j, \sum_{j \geq i} b_j) - \sum_{j > i} c_j$ for all i such that $2 \leq i \leq max(k, \ell)$. One can check that c is an element of $L_B(\infty)$, *i.e.* $c_1 = \infty$ and $c_i > c_{i+1}$ for all $i > 1$, and then $c = a \wedge b$. This implies that:

Theorem 13 *The set $L_B(\infty)$ is a lattice.* □

Now for any $n > 1$, there are two canonical embeddings of $L_B(n)$ in $L_B(\infty)$, defined by

$$\begin{array}{rccc} \pi : & L_B(n) & \longrightarrow & L_B(\infty) \\ & a = (a_1, a_2, \ldots, a_k) & \mapsto & \pi(a) = (\infty, a_2, \ldots, a_k), \end{array}$$

$$\begin{array}{rccc} \chi : & L_B(n) & \longrightarrow & L_B(\infty) \\ & a = (a_1, a_2, \ldots, a_k) & \mapsto & \chi(a) = (\infty, a_1, a_2, \ldots, a_k). \end{array}$$

The following result is straightforward:

Proposition 7 *Both π and χ are embedding of lattices.*

By using the embedding Π, one can consider $L_B(\infty)$ as the limit of $L_B(n)$ when n goes to infinity. By using the embedding χ, one can consider $L_B(\infty)$ as the disjoint

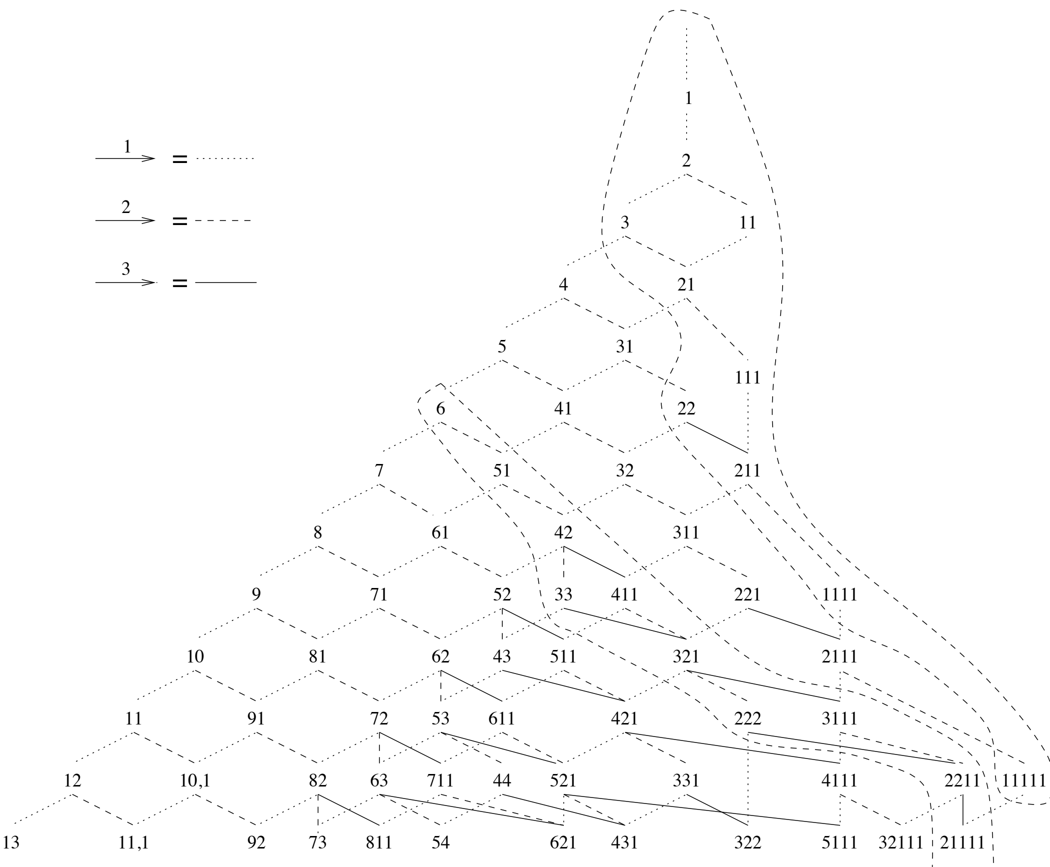


Fig. 14 The first elements and transitions of $L_B(\infty)$. As shown on this figure for $n = 6$, we have two ways to find parts of $L_B(\infty)$ isomorphic to $L_B(n)$ for any n.

union of $L_B(n)$ for all n, $L_B(\infty) = \bigsqcup_{n\geq 0} L_B(n)$.

7.3 The infinite binary tree $T_B(\infty)$

As shown in our procedure to construct $L_B(n+1)$ from $L_B(n)$, each element a of $L_B(n+1)$ is obtained from an element a' of $L_B(n)$ by adding of one grain: $a = a'^{\downarrow i}$ for some integer i. We will now represent this relation by a tree where $a \in L_B(n+1)$ is the son of $a' \in L_B(n)$ if and only if $a = a'^{\downarrow i}$ and we label with i the edge $a' \longrightarrow a$ in this tree. We denote this tree by $ST(\infty)$. The root of this tree is the empty partition $()$. We will show two ways to find the partitions of a given integer n in $T(\infty)$, which will make it possible to give an efficient and simple algorithm to compute them. Moreover, the recursive structure of this tree will allow us to obtain a recursive formula for the cardinal of $L_B(n)$ and some special classes of partitions.

From the construction of $L_B(n+1)$ from $L_B(n)$, it follows that the nodes of this tree are the elements of $\bigsqcup_{n\geq 0} L_B(n)$, and that each node a has at least one son, $a^{\downarrow 1}$, and one more if a begins with a slippery plateau of length l: the element $a^{\downarrow \ell+1}$. Therefore, $T(\infty)$ is a binary tree. We will call *left son* the first of two sons, and *right son* the other (if it exists). We call *the level n of the tree* the set of elements of depth n. The first levels of $T(\infty)$ are shown in Figure 14.
Like in the case of $L_B(\infty)$, there are two ways to find the elements of $L_B(n)$ in $ST(\infty)$. Based on the construction of $L_B(n+1)$ from $L_B(n)$ as given above, it is straightforward that:

Proposition 8 *The level n of $ST(\infty)$ is exactly the set of the elements of $L_B(n)$.* □

We will now give a recursive description of $T(\infty)$. We first define a certain kind of subtrees of $T(\infty)$. Then, we show how the whole structure of $T(\infty)$ can be described in terms of such subtrees.

Definition 16 We will call X_k *subtree* any subtree T of $T(\infty)$ which is rooted at an element $a = (\underbrace{i, \ldots, i}_{k}, a_{k+1}, \ldots)$ with $a_{k+1} \leq i-1$ and which is either the whole subtree of $T(\infty)$ rooted at a in the case a has only one son, or a and its left subtree otherwise. Moreover, we define X_0 as a simple node. □

The next proposition shows that all the X_k subtrees are isomorphic (see Figure 16).

Proposition 9 *An X_k subtree, with $k \geq 1$, is composed by a chain of $k+1$ nodes (the rightmost chain) whose edges are labeled $1, 2, \ldots, k$ and whose i-th node is the root of an X_{i-1} subtree for all i between 1 and $k+1$.*

This recursive structure and the above propositions allow us to give a compact representation of the tree by a chain (see Figure 17).

Theorem 14 *The tree $T(\infty)$ can be represented by the infinite chain defined as follows: the i-th node of this chain, $(\underbrace{1, \ldots, 1}_{i-1})$, is linked to the following node in the chain by an edge labeled with i and is the root of an X_{i-1} subtree.*

Moreover, we can prove a stronger property for each subtree in this chain:

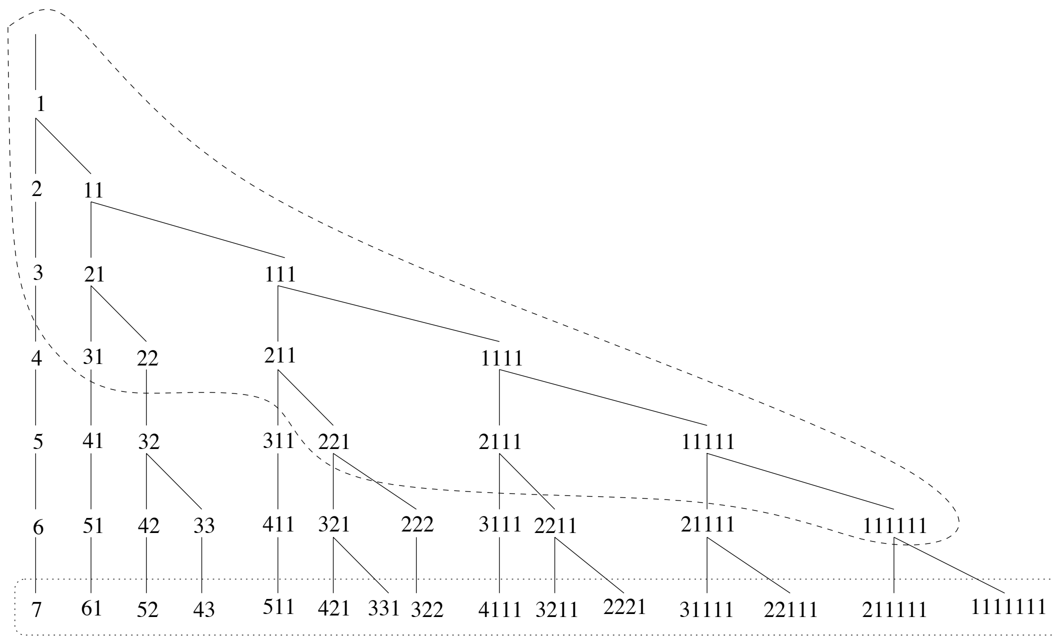


Fig. 15 Generating tree of partitions

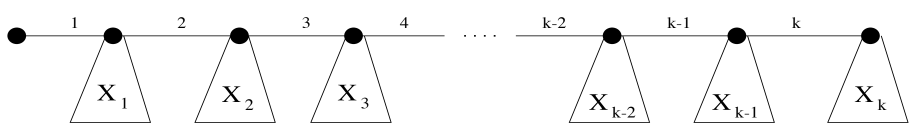


Fig. 16 Self-referencing structure of X_k subtrees

Corollary 6 *The X_k subtree of $T(\infty)$ with root $(1,\ldots,1)$ contains exactly the partitions of length k.*

We can now state our last result:

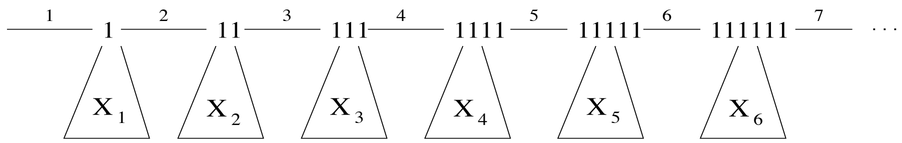


Fig. 17 Representation of T_P as a chain

Corollary 7 *Let $c(\ell, k)$ denote the number of paths in an X_k tree originating from the root and having length ℓ. We have:*

$$c(\ell, k) = \begin{cases} 1 & \text{if } \ell = 0 \text{ or } k = 1; \\ \sum_{i=1}^{inf(\ell,k)} c(\ell - i, i) & \text{otherwise.} \end{cases}$$

Moreover, $|L_B(n)| = c(n, n)$ and the number of partitions of n with length exactly k is $c(n - k, k)$. □

8 Conclusion

In this work, we seek characterization of reachable configurations and of stable configurations of many extensions of Sand Pile Model. In SPM and their extensions, a configuration is represented by an integer partition or by a unimodal sequence. In almost all cases, the initial configuration is just a 1-part partition. There are also models where the initial configuration is an arbitrary integer partition, some enumerations for this more general models were studied in [44], but questions about convergence time, the structure of configuration space, etc, remained open. It will be interesting to investigate similar problems for other classes of graphs such as trees or planar graphs.

The SPMs taken together when the number of grains is arbitrarily large form another lattice, called the infinite SPM. This yields an infinite lattice as well as an infinite tree on the set of all integer partitions. There are different ways to label the edges of this tree, each labeling gives rise to a generating function on the set of corresponding partitions. This approach is potentially useful in constructing partition identities. We provide some examples and discuss some questions around this point of view.

Acknowledgements This work was supported by the Vietnam National Foundation for Science ans Technology Development under the grant number NAFOSTED 101.99-2016.16 and by the Vietnam Institute for Advanced Study in Mathematics.

References

1. R. Anderson, L. Lovász, P. Shor, J. Spencer, E. Tardos, and S. Winograd. Disks, ball, and walls: analysis of a combinatorial game. *Amer. math. Monthly*, 96:481–493, 1989.
2. P. Bak. *How Nature Works: The Science of Self-Organized Criticality*. Springer, 1 edition, April 1999.
3. P. Bak, C. Tang, and K. Wiesenfeld. Self-organized criticality: An explanation of 1/f noise. *Phys. Rev. Lett.*, 59:381–384, 1987.
4. M. Baker and F. Shokrieh. Chip-firing games, potential theory on graphs, and spanning trees. *J. Combin. Theory Ser. A*, 120(1):164 – 182, 2013.
5. N. Biggs. Algebraic potential theory on graphs. *Bull. London math. Soc*, 29:641–682, 1997.
6. N. Biggs. Chip firing and the critical group on a graph. *J. Algebraic Combin.*, 9:25–42, 1999.
7. A. Björner and L. Lovász. Chip firing games on directed graphs. *J. Algebraic Combin.*, 1:305–328, 1992.
8. A. Björner, L. Lovász, and W. Shor. Chip-firing games on graphs. *European J. Combin.*, 12:283–291, 1991.
9. T. Brylawski. The lattice of interger partitions. *Discrete Math.*, 6:201–219, 1973.
10. R. Cori, T. H. D. Phan, and T. T. H. Tran. Signed chip firing games and symmetric sandpile models on the cycles. *RAIRO Inform. Théor. Appl.*, 47(2):133–146, 2013.
11. R. Cori and D. Rossin. On the sandpile group of dual graphs. *European J.Combin.*, 21:447–459, 2000.
12. J. Desel, E. Kindler, T. Vesper, and R. Walter. A simplified proof for the self-stabilizing protocol: a game of cards. *Inform. Proc. Lett.*, 54:327–328, 1995.
13. D. Dhar. Self-organized critical state of sandpile automaton models. *Phys. Rev. Lett.*, 64:1613–1616, 1990.
14. E. Duchi, R. Mantaci, T. H. D. Phan, and D. Rossin. Bidimensional sand pile and ice pile models. *Pure Math. Appl. (PU.M.A.)*, 17(1-2):71–96, 2006.
15. J. Durand-Lose. Grain sorting in the one dimensional sand pile model. *Complex Systems*, 10(3):195–206, 1996.
16. J. Durand-Lose. Parallel transient time of one-dimensional sand pile. *Theoret. Comput. Sci.*, 205(1-2):183–193, 1998.
17. K. Eriksson. No polynomial bound for the chip firing game on graphs. *Proc. Amer. Math. Soc.*, 112(4):1203–1205, 1991.
18. E. Formenti and B. Masson. On computing fixed points for generalized sandpiles. *International Journal of Unconventional Computing*, 2(1):13–25, 2006.
19. E. Formenti, B. Masson, and T. Pisokas. Advances in symmetric sandpiles. *Fundam. Inf.*, 76(1-2):91–112, 2007.
20. E. Formenti, K. Perrot, and E. Rémila. Computational complexity of the avalanche problem on one dimensional kadanoff sandpiles. *Proceedings of AUTOMATA'2014*, 8996(LNCS):21–30, 2014.
21. E. Formenti, T. V. Pham, Thi Ha Duong T. H. D. Phan, and T. T. H. Tran. Fixed-point forms of the parallel symmetric sandpile model. *Theoret. Comput. Sci.*, 533:1–14, 2014.
22. E. Goles and M. A. Kiwi. Games on line graphs and sand piles. *Theoret. Comput. Sci.*, 115:321–349, 1993.
23. E. Goles, M. Latapy, C. Magnien, M. Morvan, and H. D. Phan. Sandpile models and lattices: a comprehensive survey. *Theoret. Comput. Sci.*, 322(2):383–407, 2004.
24. E. Goles and M. Margenstern. Universality of chip firing game. *Theoret. Comput. Sci.*, 172:121–134, 1997.
25. E. Goles, M. Morvan, and H.D. Phan. Lattice structure and convergence of a game of cards. *Ann. of Combinatorics*, 6:327–335, 2002.
26. E. Goles, M. Morvan, and H.D. Phan. Sandpiles and order structure of integer partitions. *Discrete Appl. Math.*, 117:51–64, 2002.
27. E. Goles, M. Morvan, and H.D. Phan. The structure of linear chip firing game and related models. *Theoret. Comput. Sci.*, 270:827–841, 2002.

28. C. Greene and D. J. Kleiman. Longest chains in the lattice of integer partitions ordered by majorization. *European J.Combin.*, 7:1–10, 1986.
29. A. E. Holroyd, L. Levine, K. Mészáros, Y. Peres, J. Propp, and D. B. Wilson. Chip-firing and rotor-routing on directed graphs. In *In and out of equilibrium. 2*, volume 60 of *Progr. Probab.*, pages 331–364. Birkhäuser, Basel, 2008.
30. S.-T. Huang. Leader election in uniform rings. *ACM Trans. Programming Languages Systems*, 15(3):563–573, 1993.
31. L. P. Kadanoff, S. R. Nagel, L. Wu, and S. m. Zhou. Scaling and universality in avalanches. *Phys. Rev.*, A 39(12):6524–6537, Jun 1989.
32. R. Karmakar and S.S. Manna. Particle hole symmetry in a sandpile model. *J. Statistical Mechanics: Theory and Experiment*, 2005(01):L01002, 2005.
33. M. Latapy, R. Mataci, M. Morvan, and H.D. Phan. Structure of some sand piles model. *Theoret. Comput. Sci*, 262:525–556, 2001.
34. M. Latapy and H.D. Phan. The lattice structure of chip firing games. *Phys. D*, 115:69–82, 2001.
35. M. Latapy and T. H. D. Phan. The lattice of integer partitions and its infinite extension. *Discrete Math.*, 309(6):1357–1367, 2009.
36. M. H. Le, T. V. Pham, and T. H. D. Phan. A polynomial-time algorithm for reachability problem of a subclass of petri net and chip firing games. In *Computing and Communication Technologies, Research, Innovation, and Vision for the Future (RIVF), 2012 IEEE RIVF International Conference on*, pages 1–6. IEEE, 2012.
37. M. H. Le and T. H. D. Phan. Order structure and energy of conflicting chip firing game. *Acta Mathematica Vietnamica, year=2008*.
38. M. H. Le and T. H. D. Phan. Integer partitions in discrete dynamical models and ECO method. *Vietnam J. Math.*, 37(2-3):273–293, 2009.
39. Minh Ha Le and Thi Ha Duong Phan. Strict partitions and discrete dynamical systems. *Theoret. Comput. Sci.*
40. L. Levine. Sandpile groups and spanning trees of directed line graphs. *J. Combin. Theory Ser. A*, 118(2):350–364, 2011.
41. C. Magnien, H. D. Phan, and L. Vuillon. An extension of the model of chip firing game. *Discrete Math. Theoret. Comput. Sci.*, AA:229–244, 2001.
42. K. Perrot and T. V. Pham. Feedback arc set problem and np-hardness of minimum recurrent configuration problem of chip-firing game on directed graphs. *Ann. Comb.*, 19(2):373–396, 2015.
43. K. Perrot, T. V. Pham, and T. H. D. Phan. On the set of fixed points of the parallel symmetric sand pile model. *Automata 2011 - 17th International Workshop on Cellular Automata and Discrete Complex Systems, Discrete Math. Theor. Comput. Sci. Proc. AP*, pages 17–28, 2012.
44. K. Perrot and E. Rémila. Kadanoff sand pile model. avalanche structure and wave shape. *Theoret. Comput. Sci.*, 504:52–72, 2013.
45. T. V. Pham and T.H.D. Phan. Lattices generated by chip firing game models: Criteria and recognition algorithms. *European J. Combin.*, 34(5):812–832, 2013.
46. T. H. D. Phan. Two sided sand piles model and unimodal sequences. *RAIRO Inform. Théor. Appl.*, 42(3):631–646, 2008.
47. T. H. D. Phan and T. T. H. Tran. On the stability of sand piles model. *Theoret. Comput. Sci.*, 411(3):594–601, 2010.
48. Y. Le Borgne R. Cori. The riemann-roch theorem for graphs and the rank in complete graphs. *Electron. J. Combin.*, 23, 2016.
49. D. Rossin and Y. Le Borgne. On the identity of the sandpile group. *Discrete Math.*, 256, 3:756–790, 2002.
50. J. Spencer. Balancing vectors in the max norm. *Combinatorica*, 6:55–65, 1986.
51. R. P. Stanley. *Enumerative combinatorics. Vol. 2*. Cambridge University Press, Cambridge, 1999.
52. G. Tardos. Polynomial bound for a chip firing game on graphs. *SIAM J. Discrete Math.*, 1(3):397–398, 1988.