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Semi-dynamical systems generated by autonomous Caputo fractional differential equations

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Abstract An autonomous Caputo fractional differential equation of order $\alpha \in (0, 1)$ in a finite dimensional space whose vector field satisfies a global Lipschitz condition is shown to generate a semi-dynamical system in the function space $\mathfrak{C}$ of continuous functions with the topology uniform convergence on compact subsets. This contrasts with a recent result of Cong & Tuan [3], which showed that such equations do not, in general, generate a dynamical system on the state space.

Keywords Caputo fractional differential equation, Existence and uniqueness solutions, Continuous dependence on the initial condition, Semi-dynamical systems, Volterra integral equations

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1 Introduction

The asymptotic behaviour of Caputo fractional differential equations in $\mathbb{R}^d$ has attracted much attention in the literature in recent years. It has often been asked if such equations generate an autonomous (or nonautonomous, if appropriate) dynamical system, since that would allow the theory of attractors

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to applied to them. In 2017 Cong & Tuan [3] showed that such equations do not generate a dynamical system on $\mathbb{R}^d$, except in special cases.

In this note, we observe that an "autonomous" Caputo fractional differential equation (Caputo FDE), i.e., with a time independent vector field, is formulated as an integral equation similar to a Volterra integral equation, but with an integrably singular rather than continuous kernel. This opens the door to Miller and Sell's formulation of Volterra integral equations as autonomous semi-dynamical systems, see Miller & Sell [10] and Sell [12, Chapter XI], and enables us to determine an autonomous semi-dynamical system representation of autonomous Caputo FDEs on the function space $\mathfrak{C}$ of continuous functions $f : \mathbb{R}^+ \to \mathbb{R}^d$ with the topology uniform convergence on compact subsets.

Consider an autonomous Caputo fractional differential equation of order $\alpha \in (0, 1)$ in $\mathbb{R}^d$ of the following form

$$^{C}D^{\alpha}_{0+}x(t) = g(x(t)), \qquad \text{for } t \in [0, T], \tag{1}$$

where $g : \mathbb{R}^d \to \mathbb{R}^d$ is globally Lipschitz continuous. We represent the solution of the Caputo FDE (1) with initial condition $x(0) = x_0$ by the integral equation

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} g(x(s))ds, \tag{2}$$

where $\Gamma(\alpha) := \int_0^\infty t^{\alpha-1} \exp(-t)dt$ is the Gamma function.

Define

$$a(t, s) := \frac{1}{\Gamma(\alpha)} (t - s)^{\alpha-1}, \qquad 0 \le s < t.$$

Then the integral equation (2) is a special case of the (singular) Volterra integral equation

$$x(t) = f(t) + \int_0^t a(t, s)g(x(s))ds \tag{3}$$

where $f : \mathbb{R}^+ \to \mathbb{R}^d$ is a continuous function. In the case of a Caputo FDE (1) $f(t) \equiv f(0) = x_0$.

As preparation, we first establish the existence and uniqueness of solutions of the integral equation (2) on any bounded time interval $0, T]$ for each $f \in \mathfrak{C}$, where $\mathfrak{C}$ is the space of continuous functions $f : \mathbb{R}^+ \to \mathbb{R}^d$ equipped with the topology uniform convergence on compact subsets, and then show their continuity in the initial data and the fractional exponent. For this we use the contraction mapping principle on the space $C([0, T], \mathbb{R}^d)$ with a norm weighted by an appropriate Mittag-Leffler function. The results assume that the vector field g satisfies a global Lipschitz condition, but as in Sell [12] we establish the semi-group property for a larger class of admissible vector fields, which are assumed to satisfy these preparatory results globally in time.

The extension to "nonautonomous" Caputo fractional differential equations and skew-product flows is sketched in the final section.

2 Preliminaries

The existence (local) and uniqueness and continuity in $f \in C([0,T],\mathbb{R}^d)$ of solutions of (3) are given in Miller [9] and Sell [12] provided that $a(t,s)$ is continuous at $s=t$. In our case $a(t,s)$ is integrably singular, but we can adapt the proof in Doan et al [5], which is for Itô stochastic versions of Caputo FDE to give the global existence and uniqueness of solutions; see also [3].

2.1 Global existence and uniqueness solutions

The global existence and uniqueness solutions of (1) and of the more general integral equation will be established when the vector field g satisfies the global Lipschitz condition:

(H1) There exists $L > 0$ such that for all $x, y \in \mathbb{R}^d$, $t \in [0,\infty)$

$$\|g(x) - g(y)\| \leq L\|x - y\|.$$

The proof follows by a contraction mapping argument, which gives only local existence if the usual supremum norm on continuous functions is used. Unlike ODES, these local solutions cannot be patched together to provide a global solution for Caputo FDE. This problem can be seen from the fact that two different trajectories of a Caputo FDE can intersect, see [3, Section 6]. To overcome this difficulty, we introduce a suitable Bielecki weighted norm on the space of continuous functions of the following form

$$\|x\|_\gamma := \sup_{t\in[0,T]} \frac{\|x(t)\|}{E_\alpha(\gamma t^\alpha)} \qquad \text{for all } x \in C([0,T],\mathbb{R}^d),$$

where $\gamma > 0$ is a suitable constant and the weight function is the Mittag-Leffler function $E_\alpha(\cdot)$ defined as follows:

$$E_\alpha(t) := \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + 1)} \quad \text{for all } t \in \mathbb{R}.$$

This approach is inherited from Doan et al [5] in which a weighted norm was introduced to prove the existence of solutions for stochastic fractional differential equations.

Theorem 1 *Assume that the vector field g satisfies the global Lipschitz condition in Assumption H1. Then for any $T > 0$ and for each $f \in C([0,T],\mathbb{R}^d)$ the integral equation* (3) *has a unique solution $x(t,f)$ on the interval* $[0,T]$.

Proof Since the proof is standard we just show the contraction property and how the weighted norm is used. Let $x, y, f \in C([0,T],\mathbb{R}^d)$ with

$$(\mathfrak{T}x)(t) = f(t) + \int_0^t a(t,s)g(x(s))\, ds, \quad (\mathfrak{T}y)(t) = f(t) + \int_0^t a(t,s)g(y(s))ds,$$

for each $t \in [0,T]$. Then

$$\begin{aligned}\|(\mathfrak{T}x)(t) - (\mathfrak{T}y)(t)\| &\leq \int_0^t a(t,s)\,\|g(x(s)) - g(y(s))\|\,ds \\ &\leq L\int_0^t a(t,s)\,\|x(s) - y(s)\|\,ds.\end{aligned}$$

By definition of $\|\cdot\|_\gamma$,

$$\|(\mathfrak{T}x)(t) - (\mathfrak{T}y)(t)\| \leq L\int_0^t a(t,s)E_\alpha(\gamma s^\alpha)\,ds\,\|x-y\|_\gamma. \tag{4}$$

Since $E_\alpha(\gamma t^\alpha)$ is a solution of the linear fractional differential equation ${}^C D^\alpha_{0+}x(t) = \gamma x(t)$ it follows that

$$E_\alpha(\gamma t^\alpha) = 1 + \gamma\int_0^t a(t,s)E_\alpha(\gamma s^\alpha)ds,$$

which together with (4) implies that

$$\frac{\|(\mathfrak{T}x)(t) - (\mathfrak{T}y)(t)\|}{E_\alpha(\gamma t^\alpha)} \leq \frac{L}{\gamma}\|x-y\|_\gamma.$$

Hence, by choosing $\gamma > L$ the operator $\mathfrak{T}$ is a contraction on $(C([0,T],\mathbb{R}^d), \|\cdot\|_\gamma)$ and its unique fixed point gives the unique solution of (1). The proof is complete.

2.2 Continuous dependence of the solution on the input function

We can also show the continuous dependence of solutions on the input function f, but we do not need the weighted norm for this. Instead, we will use the following version of Gronwall's lemma from Diethelm [4, Lemma 6.19].

Lemma 1 *Let α, μ, ν, $T \in \mathbb{R}^+$ and let $\Delta : [0,T] \to \mathbb{R}$ be a continuous function satisfying the inequality*

$$|\Delta(t)| \leq \mu + \frac{\nu}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|\Delta(s)|\,ds, \qquad t \in [0,T].$$

Then

$$|\Delta(t)| \leq \mu E_\alpha(\nu t^\alpha), \qquad t \in [0,T].$$

Theorem 2 *Assume that the vector field g satisfies the global Lipschitz condition in Assumption H1. Then for any $T > 0$ and for each $f \in C([0,T],\mathbb{R}^d)$ the unique solution $x(t,f)$ of the integral equation (3) depends continuously on f in the supremum norm.*

Proof Let $x_f, y_h \in \in C([0,T],\mathbb{R}^d)$ be the unique solutions of (3) corresponding to the inputs $f,\ h \in C([0,T],\mathbb{R}^d)$. Then,

$$x_f(t) - y_h(t) = f(t) - h(t) + \int_0^t a(t,s)(g(x(s)) - g(y(s)))ds.$$

Thus,

$$\|x_f(t) - y_h(t)\| \le \|f(t) - h(t)\| + L\int_0^t a(t,s)\|x(s) - y(s)\|\, ds.$$

The fractional Gronwall Lemma 1 then gives

$$\|x_f(t) - y_h(t)\| \le \|f(t) - h(t)\|\, E_\alpha(Lt^\alpha), \qquad 0 \le t \le T,$$

so, in the supremum norm on $C([0,T],\mathbb{R}^d)$,

$$\|x_f - y_h\|_\infty \le \|f - h\|_\infty \sup_{0\le t\le T} E_\alpha(Lt^\alpha) \le \|f - h\|_\infty\, E_\alpha(LT^\alpha).$$

The proof is complete.

2.3 Continuous dependence of the solution on the fractional exponent

We can also show the continuous dependence of solutions on fractional exponent α and the main ingredient is to use the suitable weighted norm as in Theorem 1. To make the statement concretely, we need to write the kernel function $a(t,s)$ depending on the fractional exponent, i.e. for any $\alpha \in (0,1)$

$$a_\alpha(t,s) := \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}, \qquad 0 \le s < t.$$

Then, thanks to Theorem 1 for any $T > 0$ and any $f \in C([0,T],\mathbb{R}^d)$ the equation

$$x(t) = f(t) + \int_0^t a_\alpha(t,s)g(x(s))ds \tag{5}$$

has a unique solution on $[0,T]$ denoted by $x_\alpha(t,f)$.

Theorem 3 *Assume that the vector field g satisfies the global Lipschitz condition in Assumption H1. Then for any $T > 0$ and for each $f \in C([0,T],\mathbb{R}^d)$ the unique solution $x_\alpha(t,f)$ of the integral equation* (5) *depends continuously on α in the supremum norm.*

Proof Choose and fix an arbitrary $\alpha \in (0,1)$. Now, let $\beta \in [\frac{\alpha}{2}, 1]$ be arbitrary. By definition of the solution of (5), we have

$$x_\alpha(t,f) - x_\beta(t,f) = \int_0^t a_\alpha(t,s)g(x_\alpha(s,f))\, ds - \int_0^t a_\beta(t,s)g(x_\beta(s,f))\, ds.$$

Then, by (H1) we have

$$\begin{aligned}\|x_\alpha(t,f)-x_\beta(t,f)\| &\leq \left\|\int_0^t a_\alpha(t,s)\left(g(x_\alpha(s,f))-g(x_\beta(s,f))\right)\,ds\right\|\\ &\quad+\left\|\int_0^t \left(a_\alpha(t,s)-a_\beta(t,s)\right)g(x_\beta(s,f))\,ds\right\|\\ &\leq L\int_0^t a_\alpha(t,s)\|x_\alpha(s,f)-x_\beta(s,f)\|\,ds\\ &\quad+M_\beta\int_0^t |a_\alpha(t,s)-a_\beta(t,s)|\,ds,\end{aligned}$$

where $M_\beta := \max_{0\leq s\leq T}\|g(x_\beta(s,f))\|$. Now, let $\gamma > 0$ be arbitrary and let $C([0,T],\mathbb{R}^d)$ be endowed with the weight norm $\|\cdot\|_\gamma$ defined as

$$\|x\|_\gamma := \sup_{t\in[0,T]}\frac{\|x(t)\|}{E_\alpha(\gamma t^\alpha)}\qquad \text{for all } x\in C([0,T],\mathbb{R}^d).$$

Then,

$$\begin{aligned}\frac{\|x_\alpha(t,f)-x_\beta(t,f)\|}{E_\alpha(\gamma t^\alpha)} &\leq L\int_0^t \frac{a_\alpha(t,s)E_\alpha(\gamma s^\alpha)}{E_\alpha(\gamma t^\alpha)}\,ds\,\|x_\alpha(\cdot,f)-x_\beta(\cdot,f)\|_\gamma\\ &\quad+M_\beta\int_0^t |a_\alpha(t,s)-a_\beta(t,s)|\,ds\\ &\leq \frac{L}{\gamma}\|x_\alpha(\cdot,f)-x_\beta(\cdot,f)\|_\gamma\\ &\quad+M_\beta\left|\int_0^T\frac{1}{(T-s)^{1-\alpha}}\,ds-\int_0^T\frac{1}{(T-s)^{1-\beta}}\,ds\right|.\end{aligned}$$

Consequently, for a fixed choice of γ satisfying that $\gamma > L$ we arrive at

$$\|x_\alpha(\cdot,f)-x_\beta(\cdot,f)\|_\gamma \leq \frac{M_\beta}{1-\frac{L}{\gamma}}\left|\int_0^T\frac{1}{(T-s)^{1-\alpha}}\,ds-\int_0^T\frac{1}{(T-s)^{1-\beta}}\,ds\right|.$$

Obviously,

$$\lim_{\beta\to\alpha}\left|\int_0^T\frac{1}{(T-s)^{1-\alpha}}\,ds-\int_0^T\frac{1}{(T-s)^{1-\beta}}\,ds\right| = 0$$

and the norm $\|\cdot\|_\gamma$ is equivalent to the sup norm $\|\cdot\|_\infty$. Then, to conclude the proof it is sufficient to show that $\sup_{\beta\in[\frac{\alpha}{2},1]} M_\beta < \infty$. By definition of M_β and (H1) we have

$$M_\beta \leq \|g(0)\| + L\sup_{0\leq t\leq T}\|x_\beta(t,f)\|. \tag{6}$$

On the other hand, from the fact that

$$x_\beta(t,f) = f(t) + \int_0^t a_\beta(t,s) g(x_\beta(s,f))\, ds$$

we derive that

$$\|x_\beta(t,f)\| \le \|f\|_\infty + \|g(0)\| \int_0^t a_\beta(t,s)\, ds + L \int_0^t a_\beta(t,s) \|x_\beta(s,f)\|\, ds.$$

A direct computation yields that $\int_0^t a_\beta(t,s)\, ds = \frac{t^\beta}{\beta\Gamma(\beta)}$. Hence,

$$\|x_\beta(t,f)\| \le \|f\|_\infty + \frac{\|g(0)\| T^\beta}{\beta\Gamma(\beta)} + L \int_0^t a_\beta(t,s) \|x_\beta(s,f)\|\, ds.$$

Then, by virtue of Lemma 1 we have

$$\|x_\beta(t,f)\| \le \left(\|f\|_\infty + \frac{\|g(0)\| T^\beta}{\beta\Gamma(\beta)} \right) E_\beta(LT^\beta),$$

which together with (6) implies that

$$\sup_{\beta \in [\frac{\alpha}{2},1]} M_\beta \le \|g(0)\| + L \left(\|f\|_\infty + \frac{\|g(0)\| 2T}{\alpha \min_{\beta \in [\frac{\alpha}{2},1]} \Gamma(\beta)} \right) \max_{\beta \in [\frac{\alpha}{2},1]} E_\beta(LT^\beta) < \infty.$$

The proof is complete.

3 Semi-group formulation

Let $\mathfrak{C}$ be the Banach space of continuous functions $f : \mathbb{R}^+ \to \mathbb{R}^d$ with the topology uniform convergence on compact subsets. This topology is induced by the metric

$$\rho(f,h) := \sum_{n=1}^\infty \frac{1}{2^n} \rho_n(f,h),$$

where

$$\rho_n(f,h) := \frac{\sup_{t\in[0,n]} \|f(t) - h(t)\|}{1 + \sup_{t\in[0,n]} \|f(t) - h(t)\|}.$$

We follow Chapter XI, pages 178-179, in Sell [12] closely, simplifying it to this "autonomous" case, and show that the singular Volterra integral equation (3) generates an autonomous semi-dynamical system on the space $\mathfrak{C}$.

Given $f \in \mathfrak{C}$ define the operator $T_\tau : \mathfrak{C} \to \mathfrak{C}$ for each $\tau \in \mathbb{R}^+$ by

$$(T_\tau f)(\theta) = f(\tau + \theta) + \int_0^\tau a(\tau + \theta, s) g(x_f(s))\, ds, \qquad \theta \in \mathbb{R}^+, \tag{7}$$

where x_f is a solution of the singular Volterra integral equation (3) for this f, i.e.,

$$x_f(t) = f(t) + \int_0^t a(t,s) g(x_f(s))\, ds.$$

Theorem 4 *Suppose that the vector field g is globally Lipschitz continuous. The integral equation* (3) *generalisation of the autonomous Caputo fractional differential equation* (1) *generates a semi-group of continuous operators* $\{T_\tau, \tau \in \mathbb{R}^+\}$ *on the space* $\mathfrak{C}$.

Proof We first show that $T_\tau : \mathcal{C} \to \mathcal{C}$ is continuous. Let $f, h \in \mathcal{C}$. Then, by (7)

$$\|T_\tau f(\theta) - T_\tau g(\theta)\| \leq \|f(\tau+\theta) - h(\tau+\theta)\| \\ +L \sup_{s\in[0,\tau]} \|x_f(s) - x_h(s)\| \int_0^\tau a(\tau+\theta, s)\, ds,$$

where L is the Lipschitz constant of g. A direct computation yields that

$$\int_0^\tau a(\tau+\theta, s)\, ds = \frac{1}{\Gamma(\alpha)} \int_0^\tau (\tau+\theta-s)^{\alpha-1}\, ds = \frac{1}{\alpha\,\Gamma(\alpha)} \left((\tau+\theta)^\alpha - \theta^\alpha\right).$$

Now, choose and fix $k \in \mathbb{N}$ with $k \geq \tau$. Then,

$$\sup_{\theta\in[0,n]} \|T_\tau f(\theta) - T_\tau g(\theta)\| \leq \sup_{t\in[0,k+n]} \|f(t)-g(t)\| + \frac{L(k+n)^\alpha}{\alpha\Gamma(\alpha)} \sup_{s\in[0,\tau]} \|x_f(s) - x_h(s)\|.$$

Using inequality $\frac{x}{1+x} \leq \frac{y}{1+y} + z$ provided that x, y, z are non-negative and $x \leq y + z$ yields that

$$\rho_n(T_\tau f, T_\tau g) \leq \rho_{n+k}(f, g) + \frac{L(k+n)^\alpha}{\alpha\Gamma(\alpha)} \sup_{s\in[0,\tau]} \|x_f(s) - x_h(s)\|.$$

Thus,

$$\rho(T_\tau f, T_\tau g) \leq 2^k \rho(f, g) + \frac{Lc}{\alpha\Gamma(\alpha)} \sup_{s\in[0,\tau]} \|x_f(s) - x_h(s)\|,$$

where $c := \sum_{n=1}^\infty \frac{(k+n)^\alpha}{2^n}$. By virtue of Theorem 2, $\sup_{s\in[0,\tau]} \|x_f(s) - x_h(s)\| \to 0$ as $\rho(f, g) \to 0$. Consequently, T_τ is continuous.

To complete the proof, we show that $\{T_\tau : \tau \in \mathbb{R}^+\}$ forms a semi-group. Note that

$$x_f(t) = f(t) + \int_0^t a(t, s) g(x_f(s))\, ds.$$

Then

$$\begin{aligned} x_f(t+\tau) &= f(t+\tau) + \int_0^{t+\tau} a(t+\tau, s) g(x_f(s))\, ds \\ &= f(t+\tau) + \left(\int_0^\tau + \int_\tau^{t+\tau}\right) a(t+\tau, s) g(x_f(s))\, ds \\ &= (T_\tau f)(t) + \int_\tau^{t+\tau} a(t+\tau, s) g(x_f(s))\, ds \\ &= (T_\tau f)(t) + \int_0^t a(t+\tau, r+\tau) g(x_f(r+\tau))\, dr, \qquad (r = s - \tau), \\ &= (T_\tau f)(t) + \int_0^t a(t, r) g(x_f(r+\tau))\, dr. \end{aligned}$$

Hence by the existence and uniqueness of solutions $x_f(t+\tau) = \psi(t)$, where $\psi(t)$ is a solution of

$$\psi(t) = (T_\tau f)(t) + \int_0^t a(t,s)g(\psi(s))\,ds.$$

We also have

$$\begin{aligned}
(T_\sigma(T_\tau f)))(\theta) &= (T_\tau f)(\sigma+\theta) + \int_0^\sigma a(\sigma+\theta,s)g(\psi(s))\,ds \\
&= f(\tau+\sigma+\theta) + \int_0^\tau a(\tau+\sigma+\theta,s)g(x_f(s))\,ds \\
&\qquad + \int_0^\sigma a(\sigma+\theta,s)g(\psi(s))\,ds \\
&= f(\tau+\sigma+\theta) + \int_0^\tau a(\tau+\sigma+\theta,s)g(x_f(s))\,ds \\
&\qquad + \int_\tau^{\tau+\sigma} a(\sigma+\theta,r-\tau)g(\psi(r-\tau))\,dr, \qquad (r = s+\tau).
\end{aligned}$$

Since $a(\sigma+\theta, r-\tau) = a(\tau+\sigma+\theta, r)$ and $\psi(r-\tau) = x_f(r)$ it follows that

$$\begin{aligned}
(T_\sigma(T_\tau f)))(\theta) = f(\tau+\sigma+\theta) &+ \int_0^\tau a(\tau+\sigma+\theta,s)g(x_f(s))\,ds \\
&+ \int_\tau^{\tau+\sigma} a(\tau+\sigma+\theta,r)g(x_f(r))\,dr.
\end{aligned}$$

This gives

$$(T_\sigma(T_\tau f))(\theta) = f(\tau+\sigma+\theta) + \int_0^{\tau+\sigma} a(\tau+\sigma+\theta,s)g(x_f(s))\,ds$$

On the other hand from the definition of the operator as in (7)

$$(T_{\sigma+\tau} f)(\theta) = f(\tau+\sigma+\theta) + \int_0^{\tau+\sigma} a(\tau+\sigma+\theta,s)g(x_f(s))\,ds$$

This means that

$$(T_{\sigma+\tau} f)(\theta) = (T_\sigma(T_\tau f))(\theta), \qquad \forall \tau,\theta,\sigma \geq 0,\ f \in \mathfrak{C},$$

that is

$$T_{\sigma+\tau} f = T_\sigma(T_\tau) f, \qquad \forall \tau,\sigma \geq 0,\ f \in \mathfrak{C}.$$

Remark 1 As in [12], we say that a vector field g in the integral equation (3) is *admissible* if it has a globally defined unique solution for each $f \in \mathfrak{C}$ with continuity in initial data. This holds if g is globally Lipschitz continuous as above, but weaker assumptions are also possible. The theorem above also holds for such admissible vector fields.

Remark 2 The above results also hold for autonomous Caputo fractional differential equations with a substantial time derivative, i.e., of the form

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{-\beta(t-s)} g(x(s))\, ds,$$

where $\beta > 0$. This can be seen by replacing $a(t,s)$ by

$$\tilde{a}(t,s) := \frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} e^{-\beta(t-s)}, \qquad 0 \leq s < t.$$

Note that $0 < \tilde{a}(t,s) \leq a(t,s)$.

Remark 3 (i) The theory of autonomous semi-dynamical systems [7] can be applied to the Caputo semi-group defined above. The solution $x(t,x_0)$ of the autonomous Caputo FDE (1) corresponds to a constant function $f_0(t) \equiv x_0$ and

$$x(t,x_0) \equiv (T_t f_0)(0).$$

Thus, when the semi-group $\{T_\tau, \tau \in \mathbb{R}^+\}$ has an attractor $\mathfrak{A} \subset \mathfrak{C}$, then an omega limit point $x \in \mathbb{R}^d$ of trajectories of the Caputo FDE satisfies $x = f(0)$ for some function $f \in \mathfrak{A}$.

(ii) In particular, if $g(x^*) = 0$, then $f^* \in \mathfrak{A}$ for the constant function $f^*(t) \equiv x^*$, i.e., x^* is a steady state solution of the system. But there may be functions $f^* \in \mathfrak{A}$ that are not constant functions, so the strict inclusion, $\Omega \subsetneq \mathfrak{A}(0)$ usually holds, where Ω is the union of all the above omega limits points.

(iii) It is important to note that the discussion in (ii) cannot be extended to periodic functions due to the non-existence of non-trivial periodic solutions of (1), see e.g. [2,6].

(iv) It is also interesting to consider the structure of the attractor $\mathfrak{A}$ when the fractional system is stable and attractive. We refer the readers to [1,8] for further discussion on the theory of stability and attractivity for fractional differential equations.

4 Non-autonomous Caputo FDE: skew-product flow

The above result can be extended to the nonautonomous case with a time dependent vector field $g(t,x)$. Then, again following Sell [12], we can show that a non-autonomous Caputo fractional differential equation generates a skew-product flow. We just sketch the details here.

In particular, we now use the integral equation

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s, x(s))\, ds. \tag{8}$$

We define the shift mappings

$$g_\tau(\cdot, x) = g(\tau + \cdot, x)$$

and (to match Sell's notation in [12])

$$a_\tau(\cdot, \cdot) = a(\tau + \cdot, \tau + \cdot),$$

Then, following Sell [12], we define

$$(T_\tau(f,g))(\theta) = f(\tau+\theta) + \int_0^\tau a_\tau(t+\theta, s) g_\tau(\varphi(s))\, ds,$$

so in our case we have in fact

$$(T_\tau(f,g))(\theta) = f(\tau+\theta) + \int_0^\tau a(t+\theta, s) g_\tau(\varphi(s))\, ds, \qquad \theta \in \mathbb{R}^+.$$

(Since $a_\tau(t,s) = a(t,s)$ is our case, it need not be considered as an independent variable here).

In the autonomous case a and g were fixed functions, so they appeared just parameters in the operators T_τ. Now, both f and g can vary in time, so they are the independent variables that determine the operators T_τ.

Let $\mathfrak{G}$ be an appropriate space of admissible functions $g : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^d$, see Sell [12] for some examples of such spaces. We can introduce a semi-group $\theta_\tau : \mathfrak{G} \to \mathfrak{G}$ defined by the shift $\theta_\tau g := g_\tau$ as our "driving system". Then we obtain a *skew-product flow*

$$\Pi : \mathbb{R}^+ \times \mathfrak{C} \times \mathfrak{G} \to \mathfrak{C} \times \mathfrak{G}$$

defined by

$$\Pi(\tau, f, g) := (T_\tau(f,g), g_\tau).$$

The proof is similar to that above with a bit more complicated notation. It is exactly as in Sell [12], pages 178-179. Essentially, here the operator $T_\tau(f,g)$: $\mathfrak{C} \times \mathfrak{G} \to \mathfrak{C}$ for each $\tau \in \mathbb{R}^+$ satisfies a cocycle property with respect to the driving system θ, [7].

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References

1. R. Agarwal, S. Hristova and D. O'Regan, Lyapunov functions and stability of Caputo fractional differential equations with delays, *To appear in Differential Equations and Dynamical Systems, https://doi.org/10.1007/s12591-018-0434-6.*
2. I. Area, J. Losada and J.J. Nieto, On quasi-periodicity properties of fractional integrals and fractional derivatives of periodic functions, *Integral Transforms Spec. Funct.* **27** (2016), no. 1, 1–16.
3. N.D. Cong and H.T. Tuan, Generation of nonlocal fractional dynamical systems by fractional differential equations, *Journal of Integral Equations and Applications* **29** (2017), 585–608.
4. K. Diethelm, *The Analysis of Fractional Differential Equations*, Springer Lecture Notes in Mathematics, vol. 2004, Springer, Heidelberg, 2010.
5. Doan Thai Son, Thi Huong Phan, Peter E. Kloeden and The Tuan Hoang, Asymptotic separation between solutions of Caputo fractional stochastic differential equations, *Stochastic Analysis and Applications* **39** (2018), 654–664.
6. E. Kaslik, S. Sivasundaram, Non-existence of periodic solutions in fractional-order dynamical systems and a remarkable difference between integer and fractional-order derivatives of periodic functions. *Nonlinear Anal. Real World Appl.* **13** (2012), no. 3, 1489–1497.
7. P. E. Kloeden and M. Rasmussen, *Nonautonomous dynamical systems*, American Mathematical Society, Providence (2011).
8. J. Losada, J.J. Nieto and E. Pourhadi, On the attractivity of solutions for a class of multi-term fractional functional differential equations *J. Comput. Appl. Math.* **312** (2017), 2–12.
9. R.K. Miller, *Nonlinear Volterra Integral Equations*, W.A. Benjamin, Menlo Park, 1971.
10. R.K. Miller and G.R. Sell, *Volterra Integral Equations and Topological Dynamics*, Memoir Amer. Math. Soc. vol. 102, 1970.
11. R.K. Miller and G.R. Sell, Existence, uniqueness and continuity of solutions of integral equations. An Addendum, *Annali di Matematica Pura ed Applicata*, **87** (1970), 281–286.
12. G.R. Sell. *Topological Dynamics and Ordinary Differential Equations*, Van Nostrand Reinhold Mathematical Studies, London, 1971.