CHAPTER 11 MODULES OVER ARTINIAN RINGS

In this chapter we consider those rings which are simplest, as measured by the complexity of the category of modules: the rings of finite representation type or, more generally (?), the right pure-semisimple rings.

A ring is right pure-semisimple iff every module over it is a direct sum of indecomposable submodules. These are precisely the rings whose every right module is totally transcendental (§1). To put this another way: a ring is right pure-semisimple iff the lattice P of pp-types has the ascending chain condition - so right pure-semisimplicity is the "with quantifiers" version of the right noetherian condition. The first section contains various equivalents to right pure-semisimplicity, as well as "local" versions of some results (i.e., applying to arbitrary theories closed under product, rather than just to T^*). I also include a proof of the fact that a right pure-semisimple ring is right artinian and that each of its indecomposable modules is of finite length.

A ring is of finite representation type if it is right pure-semisimple and if there are, up to isomorphism, only finitely many indecomposable modules. It is not known whether or not a right pure-semisimple ring must be of finite representation type. If the ring is an artin algebra, then the concepts are equivalent (though I don't prove that here): the artin algebras include the algebras which are finite-dimensional over a base field. We see (§2) that an artin algebra is of finite representation type if it has only finitely many indecomposable finitely generated modules (in fact, this is true of any right artinian ring). It is also seen that a countable ring is of finite representation type iff it has, up to isomorphism, fewer than $2^{\aleph_0}$ countable modules.

This chapter is concerned with finitely generated modules over right artinian rings. Such modules need not be pure-injective: yet we wish to use pp-types and the associated techniques, without having to step beyond the realm of finitely generated modules. In the third section, a theory of hulls is developed for finitely generated modules over right artinian rings. It is shown that if a is an element of such a module M, then there is a minimal direct summand of M containing a; this summand is unique up to a-isomorphism and depends only on the pp-type of a. We are then able to use these finitely generated analogues of hulls, more or less as we used the hulls of §4.1. Although we do not use the fact, it is nice to know that the two notions of hull fit together in the sense that the "finitely generated" hull of an element purely embeds in the hull of the element, and the one module is indecomposable iff the other is.

The main result of the fourth section is that a ring is of finite representation type iff every module over it has finite Morley rank. This second condition is equivalent to the lattice of pp-types having finite length. Thus finite representation type may be viewed as the "with quantifiers" version of the right artinian property. Our proof of the result is self-contained for artin algebras but, for general right artinian rings, we have to quote a result of Auslander which says, in effect, that if R is a ring of finite representation type then every irreducible type is neg-isolated. Indeed, this condition on irreducible types characterises the rings of finite representation type.

There is a supplementary section on the "pathologies" one encounters if no restriction is placed on the modules under consideration.

Throughout the chapter, I have made some attempt to distinguish between what is true for any totally transcendental theory closed under products and what is at least less local than this (though the correct setting for some of the results eludes me).

11.1 Pure-semisimple rings

The ring R is said to be **right pure-semisimple**, **rt. pss** for short, if every right R-module is a direct sum of indecomposable submodules. For instance, semisimple artinian rings are right pure-semisimple; for other examples, see below. As is suggested by the terminology, these generalise semisimple artinian rings, which may be seen as the quantifier-free version of right pure-semisimple rings (the modules over a semisimple artinian ring have complete elimination of quantifiers, so all embeddings are pure).

It turns out that, over a right pure-semisimple ring, every indecomposable module is finitely generated. The same class of rings is defined by the requirement that there be a cardinal κ such that every module is a direct sum of submodules each of cardinality no more than κ. It is not enough to require that there be only a set of indecomposable modules, as is shown by any regular ring which is not actually semisimple artinian.

Proposition 11.1 *Let T be a theory of modules, not necessarily complete, which is closed under products. Suppose that every model of T is a direct sum of indecomposable submodules. Then T is totally transcendental.*

Proof [Pr84; 2.1] Set $T_1=\mathrm{Th}(\oplus\{M_{T'}: T'$ is a complete extension of T and $M_{T'}$ is any chosen model of $T'\})$. Thus T_1 is the join, in the sense of §2.6, of the various T' and is itself a complete extension of T. So, if M is a model of T then M purely embeds in a model of T_1. Thus we reduce to the case where T is complete.

So suppose that T is complete. Recall that a module is totally transcendental iff it is Σ-pure-injective (3.2). I show that every module is t.t. by establishing first that every pure-injective module is Σ-pure-injective.

So let N be pure-injective. By assumption, one has a decomposition $N=\oplus_\Lambda N_\lambda$ where each N_λ is indecomposable and, being a direct summand of N, is also pure-injective.

Then $\mathrm{pi}(N^{(\aleph_0)})=\mathrm{pi}(\oplus_\Lambda N_\lambda{}^{(\aleph_0)})=\mathrm{pi}(\oplus_\Lambda \oplus_{i\in\omega} N_{\lambda,i})$ say, where for each $i\in\omega$ one has $N_{\lambda,i}\simeq N_\lambda$. By assumption, $\mathrm{pi}(N^{(\aleph_0)})$ has a representation $\oplus_\Sigma M_\sigma$ (say), where again the M_σ are indecomposable pure-injectives. Now, by 4.A14 the two decompositions $\mathrm{pi}(\oplus_{\Lambda\times\omega} N_{\lambda,i})$ and $\oplus_\Sigma M_\sigma=\oplus_\Sigma \overline{M}$ are essentially the same, in the sense that there is a bijection $f:\Lambda\times\omega\longrightarrow\Sigma$ such that $N_{\lambda,i}\simeq M_{f(\lambda,i)}$ for each λ, i.

Thus:
$N^{(\aleph_0)}\simeq(\oplus_\Lambda N_\lambda)^{(\aleph_0)}\simeq\oplus_{\Lambda\times i} N_{\lambda,i}\simeq\oplus_{\Lambda\times i} M_{f(\lambda,i)}\simeq\oplus_\Sigma M_\sigma=\mathrm{pi}(N^{(\aleph_0)})$. That is, $N^{(\aleph_0)}$ is pure-injective and so N is Σ-pure-injective, as desired.

Thus every module purely embeds in a t.t. module (namely its pure-injective hull). Hence every module is t.t. (3.7); as required. □

Applying this to T^*, we obtain the following characterisation of right pure-semisimple rings (the direction "$\Leftarrow$" by 3.14).

Theorem 11.2 *The ring R is right pure-semisimple iff every right R-module is a totally transcendental.* □

So we obtain the following list of equivalents.

Corollary 11.3 *The following conditions on the ring R are equivalent:*
(*i*) *R is right pure-semisimple;*
(*ii*) *every module is pure-injective;*
(*iii*) *every direct sum of pure-injective modules is pure-injective;*
(*iv*) *every pure-injective module is Σ-pure-injective.* □

I now go on to derive further equivalent characterisations of and information about such rings. I should say at this point that I am not going to try to carefully assign credits for these results, since they evolved over some time around the early/mid 70's and in a number of papers: see [Pr84] for references. I do, however, take the opportunity to add the following to the list of references in [Pr84]: [Aus71], [Gri70a], [KS75], [She77].

The next theorem replaces indecomposables by modules of bounded size: but the result is the same. The injective case is the Faith-Walker theorem [FW67; 1.1].

Theroem 11.4 [Gar80; Lemma 4] *Suppose that* $T = T^{\aleph_0}$ *is a theory of modules. Then the following conditions are equivalent:*

(*i*) T *is totally transcendental;*

(*ii*) *there is a cardinal* κ *such that every model of* T *is a direct sum of submodules each of cardinality no more than* κ;

(*iii*) *there is a cardinal* κ *such that every model of* T *purely embeds in a direct sum of modules each of which may be purely embedded in* $\tilde{M}$ *and is of cardinality no more than* κ.

Proof As in the first result, we may assume that T is actually complete. Already we have (i)$\Rightarrow$(ii) by **3.14** - the existence of κ is obvious since each summand may be taken to be the hull of a single element. Also (ii)$\Rightarrow$(iii) is trivial.

(iii)$\Rightarrow$(i) Since T is closed under products, it will be enough to show that T is superstable (**3.3**). This will be done by directly verifying the defining condition (**3.A** and comment following that). So let A be any subset of the monster model and let $N(A)$ be a copy of the hull of A. Let us assume, for convenience, that κ is infinite.

By assumption, there is a pure embedding of $N(A)$ into $N'' = \bigoplus_\Lambda N_\lambda$ where, for each $\lambda \in \Lambda$, one has that N_λ is a direct summand of $\tilde{M}$ and $|N_\lambda| \leqslant \kappa$. Since T is closed under products it follows that N'' also purely embeds into $\tilde{M}$. In particular, $N(A)$ has the same pp-type whether regarded (algebraically) as a pure submodule of N'' or as sitting inside $\tilde{M}$. So we may as well suppose that we are working inside the monster model.

Each element $a \in A$ is contained in a finite sub-sum of N''. Hence A is contained in a direct summand, $N' = \bigoplus_{\Lambda'} N_\lambda$, of N'' with $|\Lambda'| \leqslant |A| + \aleph_0$ and hence with $|N'| \leqslant \kappa(|A| + \aleph_0)$.

Since A is contained in the pure submodule N' of $\tilde{M}$, one has that its pp-type is the same, whether measured in $N(A)$, $\tilde{M}$ or N'. So the morphism $f: N(A) \longrightarrow N'$, which is the composition of the pure embedding of $N(A)$ in N with the canonical projection from N'' to N', preserves the pp-type of A. By **4.14** it follows that f preserves the pp-type of $N(A)$. In particular, f is an embedding of $N(A)$ into N'. Hence $|N(A)| \leqslant |N'|$, so $N(A)$ has cardinality bounded by $\kappa|A|$.

We show that T is superstable, by counting types over A. Set $\tilde{M} = N(A) \oplus M$ for some M. If p is a 1-type over A then take any realisation $c = (a_0, b) \in N(A) \oplus M$ of it.

Suppose also that the 1-type q, over A, is realised by $c' = (a_0, b') \in N(A) \oplus M$ where $\mathrm{tp}(b) = \mathrm{tp}(b')$. It is shown that q must equal p (so a bound on $|S_1(A)|$ will be obtained). Let $\varphi(v, \bar{a})$ be pp with $\bar{a}$ in A. Then the following assertions are equivalent: $\varphi(v, \bar{a}) \in p(v)$; $\varphi(c, \bar{a})$ holds; $\varphi(a_0, \bar{a}) \wedge \varphi(b, \bar{0})$ holds (on projecting); $\varphi(a_0, \bar{a}) \wedge \varphi(b', \bar{0})$ holds (by assumption); $\varphi(c', \bar{a})$ holds (on adding); $\varphi(v, \bar{a}) \in q(\bar{v})$. Thus $p^+ = q^+$ and so (**2.17**) $p = q$, as desired.

There are at most $|N(A)| \leqslant \kappa(|A| + \aleph_0)$ choices for a_0 and at most $|S_1(0)| \leqslant 2^{|T|}$ choices for $\mathrm{tp}(b)$. Hence $|S_1(A)| \leqslant \kappa(|A| + \aleph_0).2^{|T|}$. In particular, if $|A| \geqslant \kappa.2^{|T|}$ (a constant) then $|S_1(A)| \leqslant |A|$ - so T is superstable, as required. ▫

Corollary 11.5 *Suppose that* T *is a complete theory of modules such that there exists a cardinal* κ *with* $|N(A)| \leqslant \kappa|A|$ *for every subset* A *of the monster model. Then* T *is superstable.*

Proof This was shown in the last part of the proof of **11.4**. ▫

Corollary 11.6 *The following conditions on the ring* R *are equivalent:*

(*i*) R *is right pure-semisimple;*

(*ii*) *there exists a cardinal* κ *such that every module is (or even, purely embeds in,) a direct sum of modules, each of cardinality bounded by* κ;

(*iii*) *every module is totally transcendental;*
(*iv*) T^* *is totally transcendental;*
(*v*) *the lattice* $\mathrm{P}(R)$ *of pp-1-types has the ascending chain condition.*

Proof The equivalence of (iv) and (v) is immediate from **3.1**(c); that of (iii) and (iv) is clear by **3.7**. Setting $T=T^*$ in **11.4** yields (ii)$\Leftrightarrow$(iv). The equivalence of (i) and (iii) is **11.2**. □

The equivalence of (i) and (v) in **11.6** says that right pure-semisimplicity is the pp-version of the right noetherian condition. It will turn out that finite representation type is the corresponding strengthening of the right artinian condition.

Setting $T=T^*$ in **11.5** and applying **11.6** yields the next result.

Corollary 11.7 *The ring* R *is right pure-semisimple iff there exists* κ *such that for every* A $(\subseteq \tilde{M} \models T^*)$ *one has* $|N(A)| \leqslant \kappa |A|$. □

The corollary above is not "purely algebraic", since it explicitly refers to the pp-type of A; but there is the following stronger result.

Corollary 11.8 *The ring* R *is right pure-semisimple iff there is a cardinal* κ *such that for every module* M *one has* $|\bar{M}| \leqslant \kappa |M|$.

Proof The direction "$\Rightarrow$" is clear by what has been shown already. So suppose that the cardinality restriction is satisfied: we verify that the corresponding condition of **11.7** also holds. Given a subset A of the monster model of T^*, there exists a pure submodule, B, of $\tilde{M}$ which contains A and has cardinality no more than $|A|.|T|$ (add in witnesses for every pp formula with parameters in $A=A_0$; this gives A_1; repeat; ... and set $B=\bigcup_\omega A_i$). Then $|N(A)| \leqslant |\bar{B}| \leqslant \kappa|T|.|A|$ - so **11.7** does apply. □

One knows ([Sab70a; Cor2]) that, in any case, $|\bar{M}| \leqslant |M|^{(|R|+\aleph_0)}$. If R is regular one has complete elimination of quantifiers for T^* (**16.16**) and so, using **16.B**, a special case of **11.7** is the following.

Corollary 11.9 *Suppose that* R *is regular and non-artinian. Then for every cardinal* κ *there exists a module* M *such that* $|E(M)| > \kappa |M|$. □

The condition of right pure-semisimplicity is very strong. The next result details some of its consequences.

Theorem 11.10 *Suppose that* R *is right pure-semisimple. Then:*
(*a*) R *is right artinian;*
(*b*) *there are, up to isomorphism, at most* $|R|+\aleph_0$ *indecomposable modules;*
(*c*) *every indecomposable module is finitely generated.*

Proof (c) Let N be indecomposable and choose a non-zero element a of N. Let p be its pp-type in N and let φ be a pp formula which generates p modulo T^* (φ exists since T^* is t.t.). Then (cf. **8.4**) there is some finitely generated submodule, M, of N in which a lies and in which the pp-type of a is p.

Since M is, by **11.6**, pure-injective it has, as a direct summand, some copy N' of the hull of a. Since M is finitely generated, so is its direct summand N'. But N' is isomorphic to N: hence N is finitely generated (it follows easily that, in fact, $N=M$).

(b) This is immediate from (c) (or use that every indecomposable pure-injective is the hull of a pp-1-type and, since T^* is t.t., there are at most $|R|+\aleph_0$ of these).

(a) The original proof is [Ch60; 4.4]. I give one slightly closer to that in [Fai76].

Notice first that R is right noetherian: the lattice of right ideals embeds in the lattice of pp-1-types which, by **11.6**, has acc.

Let N be the **nilradical** of R (the sum of all the nilpotent ideals). Then the module $E(R/N)$ is finitely generated. For, by assumption, $E(R/N) = \oplus_I E_i$ for suitable indecomposable injectives E_i. Since R/N is finitely generated it is contained in some finite sub-sum which, by (c), is itself finitely generated. So $E(R/N)$ is also finitely generated, being a direct summand of this last injective module.

Now let E' be the injective hull of R/N as an R/N-module. Since E', as an R-module, is an essential extension of R/N it follows (see §1.1) that $R/N \leqslant E' \leqslant E(R/N)$ (as R-modules). Therefore, since E' is a submodule of a finitely generated R-module and since R is right noetherian, E' is itself finitely generated as an R-module (hence as an R/N-module).

Now, by Goldie's Theorem (see [St75; §2.2]), E' has the structure of a semisimple artinian ring of fractions of R/N. Let $c \in R/N$ be a regular element. Then, inside E', there is an inverse c^{-1} for c and one has $R/N \leqslant c^{-1}(R/N) \leqslant c^{-2}(R/N) \leqslant \dots$ (note $1 = c^{-1}.c \in c^{-1}(R/N)$). Since E' has acc on submodules, there is some integer n and some $b \in R/N$ such that $c^{-(n+1)} = c^{-n}b$. Hence $c^{-1} = b \in R/N$: that is, c already is invertible in R/N.

Thus, every regular element of R/N has an inverse in R/N. Hence R/N is its own ring of fractions. So, by Goldie's Theorem, R/N is a semisimple artinian ring.

Furthermore, since R is right noetherian there is an integer k with $N^k = 0$. Each factor N^i/N^{i+1} (an R/N-module) is finitely generated as an R-module, hence as an R/N-module. Therefore R is a finitely generated right R/N-module. Since the latter is artinian, it follows that R is right artinian. ▫

The property of being semisimple artinian is a two-sided one: but the corresponding question for pure-semisimplicity is open.

Open question 1 If R is right pure-semisimple is R left pure-semisimple?

A positive answer to this would have further consequences; for right and left pure-semisimplicity together imply finite representation type (see §2). A weaker (possibly - see **[Sim77a]**) question, suggested by **11.10**(a) is the following one.

Open question 2 If R is right pure-semisimple, is R necessarily left artinian?

Of course a positive answer to the first question entails an affirmative answer to the second. At least for hereditary rings (**[Sim81]**), a positive answer to the second question would imply a positive answer for the first: indeed, Simson reduces the question for hereditary rings to that for rings of the form $\begin{pmatrix} F & M \\ 0 & G \end{pmatrix}$ where F and G are division rings and M is a bimodule. Furthermore, Simson showed (**[Sim77a]**) that if every right pure-semisimple and left artinian ring is also left pure-semisimple, then *every* right pure-semisimple ring is left artinian (this follows by **8.A**).

Note the model-theoretic reformulation of the first open question.

Open question 1′ If every right R-module is totally transcendental, does it follow that every left R-module is totally transcendental?

In connection with the result of Simson mentioned above, one may show the following (perhaps it can be established more easily, but the proof does illustrate how one might use the results of §8.3). If one could show the result below with the cardinality restriction replaced by one on the dimensions, then it would, by **[Sim81**; 3.3**]**, imply that every right pure-semsimple hereditary ring is of finite representation type (the proof could hardly be described as delicate, but any extension of it would seem to need consideration of the geometry involved).

<u>Proposition 11.11</u> [Pr83; 1.18] *Let* R *be the matrix ring* $\begin{pmatrix} F & M \\ 0 & G \end{pmatrix}$, *where* F *and* G *are infinite division rings and* M *is an* (F,G)*-bimodule. Suppose that* $|F|<|G|$. *Then* R *is not right pure-semisimple.*

Proof It will be enough to produce a sequence of pp formulas, with $\ldots\varphi_n \to \varphi_{n-1} \to \ldots \to \varphi_1 \to \varphi_0$ (in every module) and none of the implications reversible. By the results of §8.3, it is equivalent to produce a sequence of matrices $A_n = \begin{pmatrix} \bar{r}_n \\ S_n \end{pmatrix}$ such that, for no matrices E,H is $\begin{pmatrix} 1 & \\ & E \\ 0 & \end{pmatrix} . A_{n+1} = A_n H$ and where A_{n+1} is formed from A_n by adding columns.

Therefore, the kind of equation that we want to avoid is $\begin{pmatrix} 1 & \bar{c} \\ 0 & D \end{pmatrix} . \begin{pmatrix} \bar{r}_{n+1} \\ S_{n+1} \end{pmatrix} = \begin{pmatrix} \bar{r}_n \\ S_n \end{pmatrix} . H$, which re-arranges to $\begin{pmatrix} \bar{r}_{n+1} \\ 0 \end{pmatrix} = \begin{pmatrix} \bar{c} & \bar{r}_n \\ D & S_n \end{pmatrix} . \begin{pmatrix} -S_{n+1} \\ H \end{pmatrix} \ldots (*)$.

Let us assume inductively that A_n has coefficients from M: so c and D may be assumed to have entries from F, and H to have entries from G. Also assume, inductively, that S_n is diagonal with non-zero entries on the diagonal: so if X is a matrix with entries in G then $S_n X = 0$ implies $X = 0$. Choose any non-zero element a of $\begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}$ and set $S_{n+1} = \begin{pmatrix} S_n & 0 \\ 0 & a \end{pmatrix}$.

Let $\mathcal{K}$ be the set of those matrices H with entries in G such that there exists a solution D to $DS_{n+1} = S_n H$. Note that, for $H \in \mathcal{K}$, the map $H \mapsto D$ is 1-1 (by assumption on S_n and choice of a). So, since D has entries in F, it follows that $|\mathcal{K}| \leqslant |F|$.

Now, we wish to choose $\bar{r}_{n+1} = (\bar{r}_n, x)$ such that the equation $(*)$ has no solution. That is, given H, choose $\bar{r}_{n+1}$ not equal to $-cS_{n+1} + \bar{r}_n H$. There are only $|F|$ possibilites for H and, for each of these, no more than $|F|$ possibilities for c so, since $|G| > |F| \geqslant \aleph_0$, a suitable choice of r_{n+1} may be made, as required. □

11.2 Pure-semisimple rings and rings of finite representation type

The ring R is said to be of **finite representation type**, FRT, if it is right pure-semisimple and if there are, up to isomorphism, only finitely many indecomposable modules. Thus, every R-module has the form $N_1^{(\kappa_1)} \oplus \ldots \oplus N_t^{(\kappa_t)}$, where $N_1, \ldots, N_t$ represent the finitely many isomorphism types of indecomposables and $\kappa_1, \ldots, \kappa_t$ are cardinals.

It is known that a ring is both right and left pure-semisimple iff it is of finite representation type (see §8.4). In particular, finite representation type is a two-sided property. The following question is, however, open and is, by the result just quoted, equivalent to the Open Question 1 of §1.

<u>Open question 1</u> If R is right pure-semisimple is R necessarily of finite representation type?

The answer is known to be affirmative for certain types of ring, and this will be discussed below.

Example 1

(a) Any semisimple artinian ring is of finite representation type.

(b) One kind of semisimple artinian ring is the group ring, $K[G]$, of a finite group G over a field K, where the characteristic of the field does not divide the order of the group. Even if the characteristic, p, of the field does divide the order of the group, it may be that $K[G]$ is of finite representation type. Specifically, $K[G]$ is of finite representation type iff each Sylow p-subgroup (maximal p-subgroup) is cyclic (see [CR62; 64.1]). Thus, for example, $F_2(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is not of finite representation type but $F_2(S_3)$ is, where F_2 denotes the field with two elements and S_3 is the symmetric group on three symbols (exercise: exhibit infinitely many finitely generated indecomposables over the first [See [CR62; 64.3]]).

Example 2 The ring $\mathbb{Z}_4$ is of finite representation type. Indeed, every $\mathbb{Z}_4$-module has the form $\mathbb{Z}_2{}^{(\kappa)} \oplus \mathbb{Z}_4{}^{(\lambda)}$ for suitable κ, λ.

Example 3 The ring $\begin{pmatrix} K & K \\ 0 & K \end{pmatrix}$ of upper-triangular 2x2 matrices over the field K is of finite representation type: it is just the path algebra of the Dynkin quiver A_2 (see §13.2). I justify this statement by describing all the modules.

Let e_{11}, e_{12}, e_{22} be the usual matrix units in R. Let M be any R-module. Then, as a K-vectorspace, M decomposes as $M.1 = Me_{11} \oplus Me_{22}$. Consider the annihilator, A, in Me_{11} of the element e_{12} - so $A = \{m \in M : m = me_{11} \text{ and } me_{12} = 0\}$. It is trivial to check that A is a submodule of M. It is not much more difficult to see that A is injective and is a direct sum of copies of the unique indecomposable (indeed simple) injective module $S = \begin{pmatrix} K & K \\ 0 & 0 \end{pmatrix} / \begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix}$.

Thus every R-module decomposes as a direct sum of copies of S and a module, M' say, in which the K-linear map $-\times e_{12} : M'e_{11} \longrightarrow M'e_{22}$ is monic with image $M'e_{12} = M'e_{11}e_{12}$. Such a module M' may usefully be thought of as a pair $(V = M'e_{22}, W = M'e_{11})$ of K-vectorspaces together with a specified embedding $W \longrightarrow V$ (and every such pair arises in this way from an R-module).

It is easy to show that direct-sum decompositions of R-modules without injective direct summands are equivalent to direct-sum decompositions (in the obvious sense) of the corresponding pairs of vectorspaces (cf. §3.A). And it is not difficult to see that every pair (V, W) is a direct sum of copies of just two indecomposable pairs: $(K, 0)$ and (K, K).

The indecomposable $(K, 0)$ corresponds to the simple projective module

$$P_1 = \begin{pmatrix} 0 & 0 \\ 0 & K \end{pmatrix} \simeq \begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix}$$

and (K, K) corresponds to the indecomposable projective

$$P_2 = \begin{pmatrix} K & K \\ 0 & 0 \end{pmatrix}$$

(exercise).

Thus, every R-module has the form $S^{(\kappa)} \oplus P_1{}^{(\lambda)} \oplus P_2{}^{(\mu)}$ for suitable cardinals κ, λ, μ. In particular, R is of finite representation type.

Exercise 1 Show that the ring $\begin{pmatrix} K & K & K \\ 0 & K & 0 \\ 0 & 0 & K \end{pmatrix}$ is of finite representation type, but that $\begin{pmatrix} K & K & K & K & K \\ 0 & K & 0 & 0 & 0 \\ 0 & 0 & K & 0 & 0 \\ 0 & 0 & 0 & K & 0 \\ 0 & 0 & 0 & 0 & K \end{pmatrix}$ is not. [Hint: c.f. Ex 13.2/1, **17.5**.]

Exercise 2 [Bau75; Thm 3] (R countable) Show that every R-module is $\aleph_0$-categorical iff R is a finite ring of finite representation type (cf. §4.C).

In order for a right artinian ring to be of finite representation type, it is enough that there be, up to isomorphism, only finitely many indecomposable finitely generated modules ([Tac73; §9]). This is false for arbitrary rings (cf. Ex 16.2/3)). I include a proof of this fact for the (important) special case of Artin algebras, after giving some related results (all from [Pr84]).

<u>Proposition 11.12</u> [Pr84; 2.5] *Suppose that there is a totally transcendental module C such that every finitely generated module purely embeds in C. Then R is right pure-semisimple.*

Proof Let M be any non-zero module and take a non-zero element a of M. Let p be the pp-type of a in M. Set A_0 to be the submodule generated by a and let p_0 be the pp-type of a in A_0 (so $p_0 \le p$).

Choose, if possible, a sequence $A_0 < A_1 < \ldots < M$ of finitely generated submodules of M such that, if p_i is the pp-type of a in A_i, then $p_0 \subset p_1 \subset \ldots \subset p$.

By hypothesis there is, for each i, a pure embedding $f_i : A_i \longrightarrow C$; by purity $\mathrm{pp}^C(f_i a_i) = p_i$. Since C is t.t., eventually $p_i = p_{i+1}$ - contradiction.

So there is some finitely generated submodule, A, of M with $\mathrm{pp}^A(a) = \mathrm{pp}^M(a)$ (since R is not assumed to be noetherian, **8.4** cannot be applied to conclude that p is finitely generated). Since A purely embeds in C, it is t.t. (but p being finitely generated in A need not imply that p is finitely generated in M, so we continue...). In particular A is pure-injective. So there is a copy $N(a)$ of the hull of a, which is a direct summand of A.

It is claimed that $N(a)$ is actually a direct summand of M; but this is immediate from **4.14**, since $\mathrm{pp}^{N(a)}(a) = \mathrm{pp}^M(a)$. Now we use an argument already seen in **3.14**. Let $\mathcal{F}$ be a family $\{N_i\}_i$ of finitely generated direct summands of M, such that the sum of the family is direct and pure in M, and which is maximal such (clearly Zorn's Lemma applies to give existence). Since $\oplus_i N_i$ can be embedded in some power of C - still a t.t. module - this direct sum is itself t.t. so, in particular, is pure-injective. Therefore $M = (\oplus_i N_i) \oplus N'$ say. If N' were non-zero then what was shown first would give a non-zero finitely generated direct summand of N' - contradicting maximality of $\mathcal{F}$.

Thus every module is a direct sum of finitely generated submodules. So by **11.6** (with $\kappa = |R| + \aleph_0$) one concludes that R is indeed right pure-semisimple. ▫

<u>Corollary 11.13</u> [Pr84; 2.6] *The ring R is right pure-semisimple iff every direct sum of finitely generated modules is totally transcendental.*

Proof $\Rightarrow$ This is by **11.6**.

$\Leftarrow$ This is by **11.12**. ▫

<u>Corollary 11.14</u> [Pr84; 2.7] *Suppose that every finitely generated R-module is totally transcendental and that there are only finitely many indecomposable finitely generated modules. Then R is of finite representation type.*

Proof Let C_1 be the direct sum of one copy of each of the finitely many indecomposable finitely generated modules. Since the sum is finite, the hypothesis implies (3.5) that C_1 is itself totally transcendental. Hence $C = C_1^{(\aleph_0)}$ is t.t. (3.4).

But every finitely generated module, being t.t., is a direct sum of indecomposable submodules (3.14) and so is isomorphic to a direct summand of C. Therefore the result follows by 11.12 and the definition of finite representation type. ◽

A ring is said to be an **Artin algebra** if its centre is an artinian ring and if the ring is finitely generated as a module (on the right, equally on the left) over its centre. The main examples are finite-dimensional algebras over fields. Observe that an Artin algebra must be both right and left artinian.

Lemma 11.15 *Suppose that R is an Artin algebra. Then every finitely generated R-module is totally transcendental of finite Morley rank (so is in particular pure-injective) with pp-rank bounded by its length as a module over the centre $C(R)$ of R.*

Proof Let M_R be finitely generated. Since R is finitely generated over $C(R)$, $M_{C(R)}$ is finitely generated. This implies, since $C(R)$ is of finite length, that $M_{C(R)}$ is of finite length. By 2.1, every pp-definable subgroup of M_R is a $C(R)$-module. Hence the length of M as a $C(R)$-module is a (finite) bound on the maximum length of a chain of pp-definable subgroups of M_R. ◽

Lemma 11.16 *Suppose that R is a finite-dimensional algebra over the algebraically closed field K and let M be an indecomposable finite-dimensional R-module. Then the pp-rank (and hence the Morley rank) of M is equal to the K-vectorspace dimension of M.*

Proof By 4.53, if $\varphi(M) > \psi(M)$ is a minimal pair of pp-definable subgroups of M then the quotient $\varphi(M)/\psi(M)$ is a 1-dimensional vectorspace over $\mathrm{End}M/J\mathrm{End}M$. Since M is finite-dimensional over K, this division ring is a finite extension of K so, since K is algebraically closed, it equals K and so $\varphi(M)/\psi(M)$ is a 1-dimensional vectorspace over K. Thus the result follows (the pp-rank equals the Morley rank since K is infinite!). ◽

Example 4 If K is not algebraically closed, then the conclusion of 11.16 may fail. Take K to be the real field and consider the simple module $K[X]/\langle X^2+1\rangle$. This is two-dimensional over K but is 1-dimensional over its endomorphism ring (the field of complex numbers). Therefore it has no proper non-trivial pp-definable subgroups, so it has pp-rank 1 but K-dimension 2.

Corollary 11.17 *Let R be an Artin algebra. Suppose that there are, up to isomorphism, only finitely many indecomposable finitely generated R-modules. Then R is of finite representation type.*

Proof This is immediate from 11.14 and 11.15. ◽

In fact, it is enough to assume just that R is right artinian, rather than an Artin algebra, in order to obtain the conclusion of 11.17, but the proof for this general case is rather different [Tac73; §9].

Exercise 3 [War78; 1.2] Show that a commutative noetherian ring over which every countably generated module is a direct sum of indecomposables is an artinian principal ideal domain – these are just the commutative rings of finite representation type.

The countable rings of finite representation type may be characterised in terms of the number of countable modules.

Theorem 11.18 [BM82] *Suppose that* R *is a countable ring. Then the following conditions are equivalent*:
(i) R *is of finite representation type*;
(ii) *there are only countably many countable modules up to isomorphism*;
(iii) *there are fewer than* $2^{\aleph_0}$ *countable modules up to isomorphism*;
$(ii)^o$-$(iii)^o$: *as* (ii)-(iii), *but for left modules.*

Proof Since R is not assumed to be right artinian, an appeal to **11.17** would be unjustified: so there really is something to prove.

As stated above, (i) is a right/left-symmetric condition (see **8.24**); (i)⇒(ii) is clear; trivially (ii)⇒(iii). So the implication (iii)⇒(i) remains.

First, it will be shown that every finitely generated module is totally transcendental. Assuming that this is not the case, consider some non-t.t. module A generated by $a_1,\ldots,a_n$ (say). Since A has an infinite descending chain of pp-definable subgroups (by **3.1**), it must be that the number of 1-types modulo $\mathrm{Th}(A)$ over A is $2^{\aleph_0}$ (compare proof of **3.1**(c)(i)⇒(iii)′). Since every element of A may be expressed as a term, using only $a_1,\ldots,a_n$ as parameters, one has $S_1(A)=S_1(a_1,\ldots,a_n)$. So there are $2^{\aleph_0}$ 1-types over $\{a_1,\ldots,a_n\}$. It follows (exercise) that there are $2^{\aleph_0}$ $(n+1)$-types modulo $\mathrm{Th}(A)$ over 0. But every $(n+1)$-type over 0 may be realised in a countable model of $\mathrm{Th}(A)$. So there must be $2^{\aleph_0}$ countable models to hold all these realisations - contrary to hypothesis.

Thus every finitely generated module is totally transcendental. Were there infinitely many non-isomorphic indecomposable finitely generated modules, $\{N_i : i\in\omega\}$ say, then one would have $2^{\aleph_0}$ countable modules: the $\bigoplus\{N_i : i\in I\}$, for $I\subseteq\omega$, provide $2^{\aleph_0}$ non-isomorphic (by **4.A14**) countable modules -contradiction.

The conclusion now follows by **11.14**. □

11.3 Finite hulls over artinian rings

It has been seen that if the ring R is right pure-semisimple or if it is an Artin algebra, then every finitely generated module is pure-injective. Therefore, when dealing only with finitely generated modules over such rings, we have the full machinery of hulls available. However, a finitely generated module over a right artinian ring need not be pure-injective (see Ex 14.2/1).

One may ask over what rings the finitely generated modules are pure-injective (and hence, if R is countable, totally transcendental)? In the two cases above where one has this property one also has existence of almost split sequences (see §13.1) on one side: is there a connection between finitely generated modules being pure-injective and existence of almost split sequences? (and, therefore, a connection with whether right pure-semisimple implies finite representation type).

What, then, are we to use in the general artinian case, since hulls are not available if one restricts to finitely generated modules? Is there still a connection between irreducible types and indecomposable modules? The material of §8.2 relates to this but, to get the best results (and the relativisation goes through almost completely), one must restrict to modules of finite length. No doubt, at least some of this section could be carried over to finitely presented modules over left perfect rings (cf. **[Sab71b]**).

Therefore I show first that if R is right artinian then, within the context of finitely generated modules, there are finitely generated analogues of hulls.

Throughout this section, R is assumed to be right artinian.

I begin with two useful, well-known, results about modules of finite length.

Proposition 11.19 (Fitting's Lemma) *If* M *is a module of finite length and if* f *is an endomorphism of* M, *then* $M=\mathrm{im}\,f^k\oplus\ker f^k$ *for some* $k\in\omega$.

Proof (outline) Use the (obvious) fact that f is monic iff it is epi iff it is an automorphism. We have $\operatorname{im} f^k = \operatorname{im} f^{k+1} = \dots$ and $\ker f^k = \ker f^{k+1} = \dots$, where k is the length of M. So, if $a \in M$, then $f^k a = f^{2k} b$ for some $b \in M$: thus $a = f^k b + (a - f^k b) \in \operatorname{im} f^k + \ker f^k$. Also, f^k is epi on $\operatorname{im} f^k$ (since the latter equals $\operatorname{im} f^{2k}$); so $\operatorname{im} f^k \cap \ker f^k = 0$. ▫

<u>Proposition 11.20</u> (Harada-Sai Lemma) [HaSa71] *Suppose that the modules $M_0, \dots, M_{n-1}$ are indecomposable and all of length bounded by $b \in \omega$. For each $i = 0, \dots, n-2$, let $f_i : M_i \longrightarrow M_{i+1}$ be a non-isomorphism. Suppose that the composition $f_{n-2} \dots f_1 f_0$ is non-zero. Then $n < 2^b$.*

Proof (outline) The following statement is proved by induction on k:
$l(\operatorname{im} f_{2^k-2} \dots f_1 f_0) \leqslant b - k$, where "$l$" denotes the length of a module. For $k \leqslant 1$ this is clear, (since f_0 is a non-isomorphism, and $l(M_0) \leqslant b$).

Suppose that the statement is true for the particular value k. Set $f = f_{2^k-2} \dots f_1 f_0$ and $h = f_{2^{k+1}-2} \dots f_{2^k}$. If either of these has image of length strictly less than $b-k$ then $l(\operatorname{im} f_{2^{k+1}-2} \dots f_0) = l(\operatorname{im} h f_{2^k-1} f) \leqslant b - (k+1)$, as required. So suppose otherwise and set $g = f_{2^k-1}$. Then suppose, for a contradiction, that $l(\operatorname{im} hgf) = b - k$.

By the induction hypothesis applied to f, we see that $\operatorname{im} f \cap \ker hg = 0$ $(*)$. By additivity of length, we have that $l(\operatorname{im} f) = b - k$, $l(\ker hg) = l(M_{2^k-1}) - l(\operatorname{im} hg)$ and $l(\operatorname{im} hg) = b - k$. Combining these equations, we see from $(*)$ that M_{2^k-1} is the direct sum of $\operatorname{im} f$ and $\ker hg$. But this module is indecomposable, so $\ker hg$ must be zero, and g is monic.

Similarly, one shows that g is epi. But that contradicts g being a non-isomorphism. ▫

Let $p \in \mathrm{P}_n{}^{\mathrm{f}}$ be a finitely generated pp-n-type. There is (**8.4**) a finitely generated module M containing a realisation $\bar{a}$ of p. Since R is right artinian, M has finite length; so there is a direct summand, $H(\bar{a})$, of M which contains $\bar{a}$ and which is minimal such. I show that $H(\bar{a})$ is unique up to $\bar{a}$-isomorphism: in fact, $H(\bar{a})$ is determined by p (as is $N(p)$).

So suppose that $\bar{b}$ in the finitely presented (so finite length) module M' realises p. Choose $H(\bar{b})$ in M' in the same way that $H(\bar{a})$ was chosen. Since $H(-)$ is a direct summand, one has $\mathrm{pp}^{H(\bar{a})}(\bar{a}) = p = \mathrm{pp}^{H(\bar{b})}(\bar{b})$. Therefore, by **8.5**, there are morphisms $f : H(\bar{a}) \longrightarrow H(\bar{b})$ and $g : H(\bar{b}) \longrightarrow H(\bar{a})$ with $f\bar{a} = \bar{b}$ and $g\bar{b} = \bar{a}$. In particular, gf fixes $\bar{a}$.

By Fitting's Lemma (**11.19**) there is $k \in \omega$ such that $H(\bar{a}) = \operatorname{im}(gf)^k \oplus \ker(gf)^k$. Since $\bar{a}$ is in $\operatorname{im}(gf)^k$ and since this module is a direct summand of $H(\bar{a})$, so of M, minimality of $H(\bar{a})$ gives $H(\bar{a}) = \operatorname{im}(gf)^k$. Therefore g must be epi: also $\ker(gf)^k = 0$ – so f must be monic. Then, by the symmetry of the situation, one concludes that f and g are mutually inverse. This establishes the following result.

<u>Proposition 11.21</u> [Pr83; §1] *(R right artinian) Let p be a finitely generated pp-n-type, and suppose that $\bar{a}$ in M and $\bar{b}$ in M' are realisations of p in finitely generated modules. Let $H(\bar{a})$ be a minimal direct summand of M containing $\bar{a}$ and let $H(\bar{b})$ be a minimal direct summand of M' containing $\bar{b}$. Then there is an isomorphism between $H(\bar{a})$ and $H(\bar{b})$ taking $\bar{a}$ to $\bar{b}$.* ▫

The module $H(\bar{a})$ may be called the **finite hull** or, if no confusion should arise, simply the **hull** of $\bar{a}$. If $p = \mathrm{pp}^{H(\bar{a})}(\bar{a})$ then $H(p) = H(\bar{a})$ is the (**finite**) **hull** of p.

Another question which arises is that of the relationship between $H(p)$ and $N(p)$. It should be fairly clear that p is irreducible iff $H(p)$ is indecomposable (use **8.7**); so $N(p)$ is indecomposable iff $H(p)$ is indecomposable. Nevertherless, this does not immediately relate the two modules $H(p)$ and $N(p)$ (although, for our purposes, **8.7** is all that is needed). An obvious question is: "What is the pure-injective hull of $H(p)$? Is it $N(p)$?". Equivalently: "Is the pure-injective hull of $H(p)$ indecomposable?". The next result answers this question (affirmatively).

Proposition 11.22 [Pr83; 2.12] *(R right artinian) Let p be any finitely generated n-type. Then the pure-injective hull of $H(p)$ is $N(p)$.*

Proof Let $\bar{a}$ realise p: it will be enough, by **4.6**, to show that if $\bar{b}$ is in $H(\bar{a})$ then $\mathrm{pp}(\bar{b}/\bar{a})$ is a maximal pp-type over $\bar{a}$ (where the over-theory may be taken to be that of $H(\bar{a})$).

Notice that $\mathrm{pp}(\bar{b}/\bar{a})$ is finitely generated. For, by **8.4**, $\mathrm{pp}^{H(\bar{a})}(\bar{b}\frown\bar{a})$ is finitely generated - say by the pp formula $\varphi(\bar{v},\bar{w})$. Then the formula $\varphi(\bar{v},\bar{a})$ generates $\mathrm{pp}(\bar{b}/\bar{a})$. So if $\mathrm{pp}(\bar{b}/\bar{a})$ were not maximal there would be a finitely generated pp-type over $\bar{a}$ strictly containing it. Then there would be a finitely generated module, M, containing $\bar{a}$ in such a way that $\mathrm{pp}^M(\bar{a})=p$, and containing a tuple $\bar{c}$ in M with $\mathrm{pp}(\bar{c}/\bar{a})\supset\mathrm{pp}(\bar{b}/\bar{a})$.

By **8.5** there would be morphisms $f:H(\bar{a})\longrightarrow M$ and $g:M\longrightarrow H(\bar{a})$, the first taking $\bar{a}\frown\bar{b}$ to $\bar{a}\frown\bar{c}$, the second fixing $\bar{a}$. Consider the endomorphism gf of $H(\bar{a})$. This morphism fixes $\bar{a}$ but is strictly pp-type increasing on $\bar{b}$. From the fact that $\bar{a}$ is fixed, we deduce quickly that gf is an isomorphism (an application of Fitting's Lemma, just as before **11.21**) and so cannot strictly increase the pp-type of $\bar{b}$ - contradiction.

Thus the pp-type of $H(\bar{a})$ over $\bar{a}$ is maximal. So, by **4.6** and **4.14**, $H(\bar{a})$ is purely embedded into $N(\bar{a})$, as required. ◻

Corollary 11.23 *(R right artinian) Let p be any finitely generated (pp-)type. If $f:H(\bar{a})\longrightarrow M$ is such that $\mathrm{pp}(f\bar{a})=p$ then f is a pure embedding.*

Proof This follows by **11.22** and **4.14**. ◻

Exercise 1 One may give a more algebraic proof of the fact that if $H(p)$ is indecomposable then so is its pure-injective hull.

Suppose that R is right noetherian such that every finitely generated module is a direct sum of indecomposable submodules, each with local endomorphism ring (e.g., suppose that R is right artinian or a principal ideal domain; also see [Bra79; 9.2]). Let M_R be finitely generated and indecomposable; then $\overline{M}$ is indecomposable.

[Let M be generated by $\bar{b}$ and suppose that $\overline{M}$ has a non-trivial decomposition as $N_1\oplus N_2$. Choose non-zero elements $a_i\in N_i$ and pp formulas φ_i linking a_i to $\bar{b}$ $(i=1,2)$. Then there is $M'=M_1\oplus M_2$ say, a finitely generated submodule of $\overline{M}$ with $a_i\in M_i$ and with M' containing $\bar{b}$ together with witnesses for the quantifiers in the $\varphi_i(a_i,\bar{b})$. Clearly M is pure in M' (it is even pure in $\overline{M}$) and so, since M'/M is finitely presented, M is a direct summand of M' (Exercise 2.3/1). By locality of endomorphism rings, one has the exchange property (see, e.g., [Fai76; 18.17]): so (decompose then recompose), $M'=M\oplus M''\oplus M_2$ say. On projecting $\varphi_2(a_2,\bar{b})$ to M_2 one obtains a contradiction to $\neg\varphi_2(a_2,\bar{0})$.]

Exercise 2 Give yet another proof of **11.22** using **8.4** and **8.7**.

Corollary 11.24 [Pr83; 2.13] *Suppose that R is right artinian and let M be finitely generated.*
(a) M is indecomposable iff $\overline{M}$ is indecomposable.
(b) M is a direct sum of n indecomposable modules iff the same is true of $\overline{M}$. ◻

Corollary 11.25 *(R right artinian) Let p be a finitely generated pp-type. Suppose that $H(p)$ is a direct sum of k indecomposable submodules. Then the algebraic weight of p is k (cf. §6.4).* ◻

Corollary 11.26 [Pr83; 2.14] *If R is right artinian then every finitely generated pp-type over 0 has finite weight.* ◻

One does need the type to be over 0 in **11.26** (see after **6.27**); and some condition on the ring is necessary - consider $R=M=\mathbb{Z}$, where the pp-type of the element $1_{\mathbb{Z}}$ has weight $\aleph_0$. The next result shows that if R is a right artinian ring then the correspondence $N \mapsto \overline{N}$ between indecomposable finitely generated modules and their pure-injective hulls, is 1-1 (modulo isomorphism).

<u>Proposition 11.27</u> *Suppose that R is right artinian and that N and N' are indecomposable finitely generated R-modules. If $\overline{N}\simeq\overline{N'}$ then $N\simeq N'$.*

Proof Take non-zero elements a, a' in N, N' respectively. Suppose that their pp-types are respectively generated (modulo T^*) by the pp formulas φ, φ'. We may consider N and N' as both purely embedded in the same copy $\overline{N}=\overline{N'}$ of the pure-injective hull. By **11.24**(a), $\overline{N}$ is indecomposable so, by **4.11**, there is a pp formula $\psi(v,w)$ such that $\overline{N}\models \psi(a,a')\wedge\neg\psi(a,0)$.

Since N is pure in $\overline{N}$, it satisfies $\exists w\,(\psi(a,w)\wedge\varphi'(w))$: say $b\in N$ witnesses "w". Since $\overline{N}$, so N, satisfies $\neg\psi(a,0)$, b is non-zero. Also, since b satisifes $\varphi'(w)$ and since this formula generates the pp-type of a', there is (**8.5**) a morphism $f: N' \longrightarrow N$ taking a' to b.

If f is an isomorphism, then we finish. Otherwise, repeat the argument with a' and b in place of a and a'. Since N and N' are of finite length, eventually we find that they are isomorphic or we reach a contradiction. ◘

The next lemma, which is immediate if R is an Artin algebra, will be useful later.

<u>Lemma 11.28</u> *Suppose that R is right artinian. Let M be indecomposable and finitely generated. Then the sub-poset $\{p\in \mathrm{P}_n{}^{\mathrm{f}} : H(p)\simeq M\}$ of $\mathrm{P}_n{}^{\mathrm{f}}$, consisting of those pp-n-types realised in M, has the acc - even has finite length.*

Proof Any strict chain in this subset of $\mathrm{P}_n{}^{\mathrm{f}}$ induces, by **8.5**, a sequence of non-isomorphisms from M to itself. But if the length of M is k then (**11.20**) any such chain has length bounded by 2^k, as required. ◘

One should beware that **11.28** does not say that M has the dcc on pp-definable subgroups (for then M would be t.t. - in contradiction with Ex14.2/1): not every pp-definable subgroup of M need have the form Sa where $S=\mathrm{End}M$ and $a\in M$.

Exercise 3 (cf. Exercise 2.3/3) Let M be a finitely presented module (over any ring) and let S be its endomorphism ring.

(i) If $a\in M$ and if φ is a pp formula equivalent to the pp-type of a in M, then $\varphi(M)=Sa$.

(ii) Every finitely generated S-submodule of M is a pp-definable subgroup of M_R.

(iii) If ${}_SM$ is noetherian then the S-submodules of M are precisely the pp-definable subgroups of M_R.

(iv) If M_R is weakly saturated and if every S-submodule of M is pp-definable, then ${}_SM$ is noetherian.

Exercise 4 [Pr83; 3.13] Prove the following slight strengthening of the Harada-Sai Lemma (**11.20**). Let $f_i: M_i \longrightarrow M_{i+1}$ $(i=0,\ldots,n-2)$ be a sequence of non-isomorphisms and let $\overline{a}_0$ be in M_0 such that, if we set $\overline{a}_1=f_0\overline{a}_0$, $\overline{a}_2=f_1\overline{a}_1$, ..., then M_i is the finite hull of $\overline{a}_i$. If each M_i has length no more than b then $n<2^b$.

"What makes a module indecomposable": this is a well known question of Auslander. On the basis of the results of this section, we can give one answer to this: a module (finitely generated over an artinian ring) is indecomposable iff, for every two non-zero elements a, b of it, there is a system of linear equations and a solution vector of the form $(a\ b\ \overline{x})$ but no solution vector of the form $(a\ 0\ \overline{y})$.

11.4 Finite Morley rank and finite representation type

It is shown in this section that the ring R has finite representation type iff every R-module has finite Morley rank. Essentially this is done by showing that both conditions are equivalent to the requirement that all irreducible types be isolated. In the important case of Artin algebras the proof given here is self-contained: for the general artinian case we need to quote a result of Auslander for one direction.

Actually, that every module having finite Morley rank implies that the ring is of finite representation type (**11.29**) already follows from **7.23** (or [**Zg84**; 8.12]) **5.13** and **5.18**.

Theorem 11.29 *If the Morley rank of the largest theory, T^*, of R-modules is finite, equal to n say, (that is, if the length of the lattice $\mathrm{P}_1(R)$ is n) then R has no more than n indecomposable modules up to isomorphism. In particular, R is of finite representation type.* □

Another result, from which **11.29** follows, is **9.4**.

The converse to **11.29** is easy if R is an Artin algebra.

Theorem 11.30 *Suppose that the ring R is such that every finitely generated module has its lattice of pp-definable subgroups of finite length. If R is of finite representation type then the Morley rank of T^*, and hence of every module, is finite.*

Proof Let $N_1, \ldots, N_k$ be the finitely generated indecomposable modules. The assumption on R implies that the Morley rank of $(N_1 \oplus \ldots \oplus N_k)^{(\kappa)}$ is finite for every cardinal κ. But every module is (since R is right pure-semisimple) a direct summand of such a sum of copies of $N_1 \oplus \ldots \oplus N_k$. Hence the result follows. □

Corollary 11.31 [**Pr84**; 3.9] *Let R be an Artin algebra. Then the following conditions are equivalent:*

(i) R is of finite representation type;

(ii) $\mathrm{P}_1(R)$ has finite length;

(iii) every module has finite Morley rank (and there is a uniform bound);

(iv) $\mathrm{MR}(T^*)$ is finite.

Proof Clearly (iii) and (iv) are equivalent by **5.21**; the equivalence of (ii) and (iv) is by **5.13** and **5.18**. Then, by **11.15**, **11.30** applies to give (i)$\Rightarrow$(iv). Finally **11.29** gives us (iv)$\Rightarrow$(i). □

Example 1 Let R be the path algebra of the quiver A_2 (see Ex 11.2/3). There are three indecomposables; two with pp-rank 1 and one with pp-rank 2. It is left as an exercise to show that the Morley rank of T^* is 4 (see **11.39** below).

Exercise 1 Let R be the ring of $n \times n$ matrices over some division ring. What is $\mathrm{MR}(T^*)$?

In fact **11.31** does not need the assumption that R is an Artin algebra: R being right artinian will do (note that all the conditions of **11.31** imply right pure-semisimplicity, so imply right artinian). But the general case involves more work and brings to light the importance of the property of irreducible (pp-)types being isolated. This property links up with elementary cogeneration (as has been seen in §9.4), with finite presentation of certain functors (§12.2) and hence ([**Aus74a**; 2.7]) with existence of almost split sequences.

The next result was proved first for totally transcendental theories [**Pr84**; 3.6]. Then the global case (i.e., R rt. pss) was generalised to the case where every finitely generated module is totally transcendental [**Pr83**]. Using the machinery of §3 one can now give a proof for arbitrary right artinian rings.

<u>Proposition 11.32</u> *Suppose that R is right artinian and let p be a finitely generated irreducible type. Then the following conditions are equivalent:*
(i) *p is isolated (as a pp-type, equivalently as a type);*
(ii) *if $\{M_\lambda\}_\lambda$ is a set of indecomposable finitely generated modules and if $H(p)$ purely embeds in $\prod_\lambda M_\lambda$ then $H(p)\simeq M_\lambda$ for some λ;*
(iii) *if $\{M_\lambda\}_\lambda$ is a set of indecomposable finitely generated modules and if $H(p)$ purely embeds in $\prod_\lambda M_\lambda$ then, for some λ, the morphism $H(p)\longrightarrow M_\lambda$ induced by the projection $\pi_\lambda:\prod_\mu M_\mu \longrightarrow M_\lambda$, is an isomorphism.*

Proof (i)⇒(iii) Let $\bar{a}$ in $H(\bar{a})=H(p)$ realise p. Inside the product $M=\prod M_\lambda$ one has $\bar{a}=(\bar{a}_\lambda)_\lambda$ for suitable $\bar{a}_\lambda$ in M_λ; set p_λ to be the pp-type of $\bar{a}_\lambda$ in M_λ. Since $H(\bar{a})$ purely embeds in M, one obtains $p=\bigcap_\lambda p_\lambda$. Since p is irreducible and isolated in $\mathrm{P^f}$ (**8.7**) it must be that $p=p_\lambda$ for some λ. Therefore, the projection π_λ preserves the pp-type of $\bar{a}$, hence (by **11.23**) is a pure embedding of $H(\bar{a})$ into M_λ. Since $M_\lambda/H(\bar{a})$ is finitely presented, it follows (Exercise 2.3/1) that $H(\bar{a})$ is a direct summand of M_λ But M_λ is indecomposable; so $\pi_\lambda:H(\bar{a})\longrightarrow M_\lambda$ is an isomorphism, as required.

(iii)⇒(ii) This is immediate.

(ii)⇒(i) Suppose, for a contradiction, that p is not isolated but that, nevertheless (ii) holds. Then there is a representation $p=\bigcap\{p_\lambda:\lambda\in\Lambda\}$, where the p_λ may be taken to be irreducible, finitely generated and with $p_\lambda>p$ for all λ (exercise: use the fact that if $q>p$ then there is a finitely generated pp-type p' with $q\geqslant p'>p$, and p' is a finite intersection of irreducible finitely generated pp-types).

Set $H_\lambda=H(\bar{a}_\lambda)$ where $\bar{a}_\lambda$ is some realisation of p_λ. Then, if $\bar{a}=(\bar{a}_\lambda)_\lambda\in\prod H_\lambda=H$ (say), one has that the pp-type of $\bar{a}$ in H is $\bigcap_\lambda p_\lambda=p$. So, by **8.5**, there is a morphism from $H(\bar{a})$ into H which, since it preserves the pp-type of $\bar{a}$, is a pure embedding (by **11.23**). Then from (ii) it follows that one has $H(\bar{a})\simeq H(\bar{a}_\lambda)$ for some λ.

If p_λ were not itself isolated then one could repeat the above argument with p_λ replacing p: and so on. Noting that $p<p_\lambda$ and that $H(p)=H(p_\lambda)$, we see that **11.28** guarantees the termination of this process after a finite number of steps. Therefore there is a finitely generated pp-type q with $H(q)=H(p)$ and with q isolated. Let $\bar{b}$ in $H(\bar{a})$ realise q.

By (i)⇒(iii) applied to q and H, some projection π_μ preserves the pp-type of $\bar{b}$. Hence (**11.23**) π_μ must preserve the pp-type of $\bar{a}$. That is, $p=p_\mu$ - contradiction (for p_μ is isolated), as required. ▫

<u>Corollary 11.33</u> *Suppose that R is right artinian. Let p and q be finitely generated irreducible pp-types with $H(p)\simeq H(q)$. Then p is isolated (with respect to a given T) iff q is isolated (with respect to T).*

Proof This follows from **11.32** since condition (ii) holds equally for p and q. Of course, if $H(p)$ is a t.t. module (e.g., if R is an Artin algebra) then this is immediate from the fact that the prime model realises exactly the isolated types.

Alternatively the result follows easily by **9.26** and **9.24** (say). ▫

Next, I give the totally transcendental version of **11.32** which overlaps with, but also diverges from that result.

<u>Proposition 11.34</u> [Pr84; 3.6] *Suppose that T is totally transcendental. Then the following conditions are equivalent for any irreducible type p over 0:*
(i) *p is isolated;*
(ii) *if $\{N_\lambda\}_\lambda$ is any set of indecomposable pure-injectives and if $N(p)$ is a direct summand of the product $\prod_\lambda N_\lambda$, then $N(p)\simeq N(p_\lambda)$ for some λ;*

(iii) *if* $\{N_\lambda\}_\lambda$ *is any set of indecomposable pure-injectives and if* $N(p)$ *is a direct summand of the product* $\prod_\lambda N_\lambda$ *then, for some* λ, *the canonical projection* $\pi_\lambda : \prod_\mu N_\mu \twoheadrightarrow N_\lambda$ *induces an isomorphism* $N(p) \simeq N_\lambda$.

Proof First note that none of the conditions (i), (ii), (iii) is changed by assuming that $T = T^{\aleph_0}$ (by **4.39** and (say) **9.26**).

(i)⇒(iii) and (iii)⇒(ii) are just as in the proof of **11.32**. A proof of (ii)⇒(i), very much like that in **11.32**, may be given using the t.t. condition in place of **11.23**.

Alternatively; let $M_0 = \bigoplus_\lambda N_\lambda^{(\kappa_\lambda)}$, with the N_λ being the hulls of all the isolated irreducible 1-types, be the prime model of T (**4.62**). By **9.33**, M_0 is an elementary cogenerator: so there is κ with $N(p)$ purely embedding into M_0^κ. Since $M_0^\kappa = (\bigoplus_\lambda N_\lambda^{(\kappa_\lambda)})^\kappa$ is pure in $\prod_\lambda N_\lambda^{\kappa_\lambda \times \kappa}$, one has the failure of (ii) if p is non-isolated (for then $N(p)$ cannot be isomorphic to any N_λ). ◻

The next result, needed for the converse of **11.29**, is due to Auslander.

<u>Theorem 11.35</u> [Aus76; 2.4] *Suppose that the ring* R *is of finite representation type. If* $\{N_\lambda\}_\lambda$ *is a set of indecomposable modules and if* N *is an indecomposable which purely embeds in the product* $\prod_\lambda N_\lambda$ *then, for some* λ, *one has* $N \simeq N_\lambda$. ◻

<u>Corollary 11.36</u> *If* R *is a ring of finite representation type then every irreducible type (in finitely many free variables) is isolated.*

Proof Since finite representation type implies right pure semisimplicity, hence every pp-type finitely generated, this follows by **11.35** and **11.32**. ◻

Of course, even if R has finite representation type one cannot expect all types in finitely many free variables to be isolated - for then T^* would be $\aleph_0$-categorical (assuming $|R| \leqslant \aleph_0$).

Example 2 Take R to be the ring $\mathbb{Q}$ of rationals; so $T^* = \mathrm{Th}(\mathbb{Q})$. Since T^* is not $\aleph_0$-categorical (for $\mathbb{Q} \equiv \mathbb{Q} \oplus \mathbb{Q}$) there must be non-isolated types (necessarily of weight $\geqslant 2$). Since there are only two 1-types (the type of the zero element, and that of any non-zero element), or by **11.36**, we must look to 2-types for non-isolation. In $\mathbb{Q} \oplus \mathbb{Q}$ let a be the element $(1,0)$ and let b be $(0,1)$ - so b is not a multiple of a. Then the type of the pair (a,b) is non-isolated (exercise).

The converse of **11.29** is then provided by **6.28**.

<u>Corollary 11.37</u> *If the ring* R *is of finite representation type then* T^* *has finite Morley rank.*

Proof Otherwise, **6.28** would imply that there was a non-isolated irreducible 1-type over 0, contradicting **11.36**. ◻

Summarising, we have the following.

<u>Theorem 11.38</u> [Pr84; 3.9] *The following conditions on the ring* R *are equivalent:*

(i) R *is of finite representation type;*

(ii) T^* *has finite Morley rank;*

(iii) *every* R*-module (equivalently every left* R*-module) has finite Morley rank;*

(iv) $\mathrm{P}_1(R)$ *has finite length;*

(v) *every irreducible type (in finitely many free variables) is isolated.*

If R *is right artinian, then a further equivalent is:*

(vi) $\mathcal{I}(T^*)$ *is finite.*

Proof The equivalence of (i) and (ii) is **11.29** and **11.37**. That (ii), (iii) and (iv) are equivalent follows as in **11.6**. By **11.36**, (i) implies (v).

Suppose that (v) holds. It follows by **6.28** that T^* has finite Morley rank - that is, (ii) holds. The last statement follows by **[Tac73**; §9**]** (cf. **11.17** above). □

Example 3 If R is not assumed to be right artinian, then $\mathcal{I}(T^*)$ being finite does not imply that R is of finite representation type. Consider Ex16.2/3.

Given a ring of finite representation type, the Morley rank of T^* is easily calculated if one knows the indecomposables. It is just the pp-rank of the module M which is a direct sum of one copy of each indecomposable (so, by **11.16**, if R is a K-algebra with K algebraically closed, it is just the K-dimension of M). This follows from the next result which, in turn, is an immediate consequence of **9.3**.

<u>Lemma 11.39</u> *Let M and N be modules of finite Morley rank. Then* $\mathrm{MR}(M \oplus N) = \mathrm{MR}(M) + \mathrm{MR}(N)$. □

Thus, for example, if R is the path algebra of the quiver E_6 then $\mathrm{MR}(T^*) = 156$.

Exercise 3 It's not difficult to classify the rings of finite representation type such that the Morley rank of T^* is very small.

(i) Show that if $\mathrm{MR}(T^*) \leqslant 2$ then R is semisimple artinian.

(ii) Show that if $\mathrm{MR}(T^*) = 3$ then either R is semisimple artinian or else $R/J \simeq J$ is a division ring (e.g. $R = \mathbb{Z}_4$).

(iii) Show that if $\mathrm{MR}(T^*) = 4$ then the only essentially new possibility is that R is of the form $\begin{pmatrix} D' & D \\ 0 & D \end{pmatrix}$ where D and D' are division rings and D has a (D', D)-bimodule structure (e.g. the path algebra of A_2). [Hint: show that $J^2 = 0$ - this is a useful first step in all three parts (and in the first two, pretty well the last step).]

Fix an integer d and consider the class $\mathcal{F}_d(K)$ of all d-dimensional algebras of finite representation type over the base field K. Any d-dimensional K-algebra can be described by d^3 structure constants, which tell how the elements of a chosen K-basis multiply together. In particular, any d-dimensional K-algebra with a chosen K-basis determines a point of affine K^{d^3}-space. Different bases of the same algebra are related by invertible $d \times d$ matrices over K - that is, by elements of $\mathrm{GL}_d(K)$. The action of $\mathrm{GL}_d(K)$ on K^d induces an action on the points of K^{d^3} which stabilises the set, X_d, of points corresponding to members of $\mathcal{F}_d(K)$. Under this action, the orbits are precisely the isomorphism classes of d-dimensional K-algebras of finite representation type. The question of whether there are, for each d and K, only finitely many algebras of finite representation type (up to isomorphism), was open.

Gabriel **[Gab75a]** showed that if K is algebraically closed then the set X_d is an open subset of the set Y_d of all members of K^{d^3} which correspond to algebras: there exist polynomials $f_1, \ldots, f_n \in K[X_i : 1 \leqslant i \leqslant d^3]$ such that a point $\overline{k}$ of Y_d^3 corresponds to an algebra of finite representation type iff not all of $f_1(\overline{k}), \ldots, f_n(\overline{k})$ are zero. Gabriel left open the question of whether there are such polynomials over the prime subfield.

Herrmann, Jensen and Lenzing in **[HJL81]** and **[JL82]** (also **[JL80]**) went some way towards answering this, by showing that the class $\mathcal{F}_d(K)$ for any field K is finitely axiomatisable so, using the elimination of quantifiers in the case that K is algebraically closed, they deduced that, for K algebraically closed, the set X_d is constructible over the prime subfield (that is, X_d is defined by a certain finite set of equations and inequations over the prime subfield). These papers contain a number of results on axiomatisability, effectivity, bounds on the number of indecomposables and the effect of extending the base field.

On the question of the finiteness of $\mathcal{F}_d(K)$, the further development of covering theory (see [BG82]), together with the fact that there is a bound, in terms of the dimension of the algebra, on the number of indecomposables (see [JL82: 3.6] and also [HJL81: 5.1]; alternatively, see [Bon82; §5] plus [BaBr81; §4]) shows that, outside of characteristic two (where there the map from an algebra to its Auslander-Reiten quiver need not be 1-1), there are indeed only finitely many d-dimensional K-algebras of finite representation type (there are, no doubt, a number of routes to this fact).

In any case, the multiplicative basis theorem [BGRS85] supercedes all this, at least for algebraically closed fields. The theorem says that, if R is a algebra of finite representation type over an algebraically closed field K then there is a K-basis of R in which the product of any two basis elements is either zero or a basis element. In particular, over such K, there are only finitely many orbits in $\mathcal{F}_d(K)$. For a discussion of the situation over non-algebraically closed fields, see [Gus85].

11.P "Pathologies"

The area that I touch on here is really rather large. So what I do is to direct the reader to some review papers and just mention some results which directly impinge on what is discussed elsewhere in the text. For a more balanced presentation, the reader should consult the review works that I mention.

In [Cor63], Corner showed that every countable reduced torsionfree ring is the endomorphism ring of some countable reduced torsionfree abelian group. He then gave examples of various pathologies which can arise in abelian groups (cf. Kaplansky's "Test Problems" [Kap54; p12]).

Corner's results were extended in the work of Brenner and Butler and Corner [BB65], [Bre67], [Cor69]. Brenner and Butler showed that every associative algebra over a field K can be realised as the algebra of those endomorphisms of a K-vectorspace which leave invariant a specified set of subspaces. Brenner improved this by showing that the number of subspaces may be taken to be five, provided the algebra is countably generated over K. Set theory began to make its appearance when Corner showed that it is enough to suppose that the number of generators of the algebra is less than the first strongly inaccessible cardinal. (In fact, Corner had already noted that a "proof" of Fuchs that there are arbitrarily large indecomposable torsionfree abelian groups failed at certain large cardinals.)

Since then, these results have been extended in many directions. One direction is, of course, to wild representation type (see Chpt.13). In another direction, one is concerned with realising algebras as endomorphism rings of members of various classes of modules. So this enterprise includes finding large indecomposable modules, finding large modules with no indecomposable direct summands, and so on. Set-theoretical techniques have turned out to be essential, and set-theoretical axioms beyond ZFC have to be invoked for some results.

The other source of related material is Shelah's solution to the Whitehead Problem ([She74], see [Ek76]).

The following are some survey papers: [CG85]; [Göb83]; [Göb84]; also see the introduction to [DuGö82].

The following sort of result, this one taken from [DuGö82] (also see [CG85]), is particularly striking. First note that a complete discrete rank 1 valuation domain does not have arbitrarily large indecomposable modules.

<u>Theorem</u> *Let R be a Dedekind domain, not a field. Let κ be an infinite cardinal. Then the following are equivalent:*

(i) R is not a complete discrete valuation domain;

(ii) there exists an indecomposable R-module of rank $\geqslant 2$

(iii) there exists an indecomposable R-module of rank $\geqslant \kappa$;

(iv) *there exists an R-module (not pure-injective!) of rank $\geqslant \kappa$ with no indecomposable summand;*

(v) *there exist R-modules of rank $\geqslant \kappa$ which do not satisfy Kaplansky's Test Problems;*

(vi) *if A is any cotorsion-free R-algebra then there exists an R-module with endomorphism algebra isomorphic to A.* ▫

There have been recent extensions of these results to arbitrary rings.

The above result can be used to answer in the negative a question of Kucera [Kuc87]. For it follows that there are arbitrarily large abelian groups with local endomorphism ring: since such a group M may be as large as one desires, the weight of M in $\mathrm{Th}(M^{\aleph_0})$ need not be 1.

It is shown in [DüGo85] (also see references in §15.1) that the class of all torsion theories of abelian groups (cf. §15.1) is a proper class: this contrasts with hereditary torsion theories, of which there can be at most 2^{λ} where $\lambda = 2^{|R|}$. Also see [DFS87].

[Hu83] contains related results on reflexive modules.

There are examples of the sort of pathology one obtains, even over tame rings, by working with arbitrary ("large") modules in [BrRi76].