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ON THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO TIME-FRACTIONAL ELLIPTIC EQUATIONS DRIVEN BY A MULTIPLICATIVE WHITE NOISE

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Abstract. This paper is devoted to study of time-fractional elliptic equations driven by a multiplicative noise. By combining the eigenfunction expansion method for symmetry elliptic operators, the variation of constant formula for strong solutions to scalar stochastic fractional differential equations, Ito's formula and establishing a new weighted norm associated with a Lyapunov–Perron operator defined from this representation of solutions, we show the asymptotic behaviour of solutions to these systems in the mean square sense. As a consequence, we also prove existence, uniqueness and the convergence rate of their solutions.

1. Introduction.

Calculus (derivative and integral) is an ideal tool to describe evolutionary processes. Typically, each evolutionary process is represented by a system of differential equations. By studying (qualitative or quantitative) solutions of equations, one can know the current state as well as predict the past or future posture of the process. However, common phenomena in life are history dependent. For these phenomena, extrapolating its posture at a future time from the past depends on both local observation and the whole past. Moreover, dependence in general is not the same at all times. Fractional calculus (fractional derivative and fractional integral) is one of the theories that come up to meet those requirements.

Fractional differential equations (equations contains fractional derivatives) are of great interest in the last four decades due to its application in describing real-world problems, such as in signal processing, in financial mathematics, in biotechnology, in image processing, in control theory, and in mathematical psychology where fractional-order systems may be used to model the behaviour of human beings, specifically, the way in which a person reacts to external influences depends on the experience he or she has made in the past.

The time-fractional diffusion equations have been introduced in Physics by Nigmatullin [17] to describe super slow diffusion process in a porous medium with the structure type of fractal geometry (for example the Koch's tree). From the

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probabilistic point of view, Metzler and Klafter [15] have pointed out that a time-fractional diffusion equation generates a non-Markovian diffusion process with a long memory. Roman and Alemany [19] have considered continuous-time random walks on fractals and observed that the average probability density of random walks on fractals obeys a diffusion equation with a fractional time derivative asymptotically. Moreover, a time-fractional diffusion equation also is used to model a relaxation phenomena in complex viscoelastic materials, see, e.g., [10].

The existence of solutions to time-fractional partial differential equations has been studied by many authors. In [8], using Fourier transform, the authors have built a fundamental solution for elliptic equations with smooth coefficient. By Galerkin method and the Yoshida approximation sequence, in [21] Zacher has proposed a way to prove existence of certain weak solutions to abstract evolutionary integro-differential equations in Hilbert spaces. Using the operator theory in functional analysis and the eigenfunction expansion method for symmetry elliptic operators, in [20] Sakamoto and Yamamoto have proved the existence and uniqueness of the weak solution for a fractional diffusion-wave equation. Recently, using a De Giorgi–Nash type estimation, in [1] and [22], the authors established the existence and Holder continuity of weak solutions for fractional parabolic equations. By proposing a definition of the Caputo derivative on a finite interval in fractional Sobolev spaces, Gorenflo, Luchko and Yamamoto [12] have investigated solutions (in the distribution sense) to time-fractional diffusion equations from the operator theoretic viewpoint.

In contrast to existence theory of solutions to deterministic fractional partial differential equations, there are very few researches on stochastic fractional partial differential equations. Using the integration by parts, Ito's formula and the Parseval's identity, an L_2-theory for stochastic time-fractional partial differential equations is presented in [5] by Chen and co-authors. By a choosing a framework for infinite dimensional stochastic integration, in [3] Baeumer, Geissert and Kovacs have showed the unique mild solution to a class of semi-linear Volterra stochastic evolution equations is mean-p Holder continuous. In [4], Chen, Hu and Nualart have studied nonlinear stochastic time-fractional slow and fast diffusion equations. They have proven the non-negativity of the fundamental solution, existence and uniqueness of solutions together with the moment bounds of these solutions. In some cases, they have obtained the sample path regularity of the solutions. Based on monotonicity techniques, in [16] the authors have developed a method to solve (stochastic) evolution equations on Gelfand triples with time-fractional derivative.

To our knowledge, until now, almost no research on the asymptotic behavior of solutions to stochastic fractional partial differential equations has been published. Motivated by this fact, this paper is devoted to study the stability in the mean square sense for time-fractional elliptic equations driven by a multiplicative white noise. The paper is organised as follows. In Section 2, we recall a framework of stochastic fractional differential equations. In Section 3, we first introduce a definition of mild solution. Then we prove a theorem on existence and uniqueness and the main result of the paper on the asymptotic behavior of solution (Theorem 3.4).

2. **Fractional calculus and stochastic fractional differential equations.** We briefly recall an abstract framework of fractional calculus and stochastic fractional differential equations.

Let $\alpha \in (0,1]$, $[0,b] \subset \mathbb{R}$ and $x : [0,b] \to \mathbb{R}$ be a measurable function such that $\int_0^b |x(\tau)|\, d\tau < \infty$. The *Riemann–Liouville integral operator of order* α is defined by

$$(I^\alpha_{0+}x)(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} x(\tau)\, d\tau,$$

where $\Gamma(\cdot)$ is the Gamma function. The *Riemann–Liouville fractional derivative* ${}^{RL}D^\alpha_{0+}x$ of x on $[0,b]$ is defined by

$${}^{RL}D^\alpha_{0+}x(t) := (DI^{1-\alpha}_{0+}x)(t), \quad \text{for almost } t \in [0,b],$$

where $D = \frac{d}{dt}$ is the usual derivative. The *Caputo fractional derivative* of x on $[0,b]$ is defined by

$${}^{C}D^\alpha_{0+}x(t) = {}^{RL}D^\alpha_{0+}(x(t) - x(0)) \quad \text{for almost } t \in [0,b].$$

The Caputo fractional derivative of a d-dimensional vector function

$$x(t) = (x_1(t), \ldots, x_d(t))^T$$

is defined component-wise as

$$({}^{C}D^\alpha_{0+}x)(t) := ({}^{C}D^\alpha_{0+}x_1(t), \ldots, {}^{C}D^\alpha_{0+}x_d(t))^T.$$

Let $A \in \mathbb{R}^{d\times d}$ and $f : [0,\infty) \to \mathbb{R}^d$ is a continuous vector-valued function. As showed in [18, p. 140], the equation with the fractional order $\alpha \in (0,1)$

$$\begin{aligned} {}^{C}D^\alpha_{0+}x(t) &= Ax(t) + f(t), \quad t > 0,\\ x(0) &= x_0 \in \mathbb{R}^d, \end{aligned}$$

has a unique solution x on $[0,\infty)$ which has the presentation

$$x(t) = E_\alpha(t^\alpha A)x_0 + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}((t-\tau)^\alpha A) f(\tau) d\tau, \quad t \geq 0,$$

where $E_{\alpha,\beta} : \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}$ is the Mittag-Leffler function defined by

$$E_{\alpha,\beta}(A) := \sum_{k=0}^{\infty} \frac{A^k}{\Gamma(\alpha k + \beta)}$$

and

$$E_\alpha(A) := E_{\alpha,1}(A).$$

For more details on Mittag-Leffler functions, we refer the reader to the monographs [18, 11].

Next, we discuss a fractional stochastic differential equation of order $\alpha \in (\frac{1}{2}, 1)$ in the following form

$${}^{C}D^\alpha_{0+}X(t) = AX(t) + b(t, X(t)) + \sigma(t, X(t)) \frac{dW_t}{dt}, \quad t > 0, \tag{1}$$

where W_t is a standard scalar Brownian motion on an underlying complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} := \{\mathcal{F}_t\}_{t\in[0,\infty)}, \mathbb{P})$ and $b, \sigma : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d$ are measurable functions satisfying the following conditions.

(H1) There exists $L > 0$ such that for all $x, y \in \mathbb{R}^d$, $t \in [0,\infty)$

$$\|b(t,x) - b(t,y)\| + \|\sigma(t,x) - \sigma(t,y)\| \leq L\|x - y\|.$$

(H2) $\sigma(\cdot,0)$ is essentially bounded, i.e.,

$$\|\sigma(\cdot,0)\|_\infty := \text{ess sup}_{\tau\in[0,\infty)}\|\sigma(\tau,0)\| < \infty$$

almost sure and $b(\cdot,0)$ is L^2-locally integrable, i.e., for any $T>0$

$$\int_0^T \|b(\tau,0)\|^2\, d\tau < \infty.$$

For each $t\in[0,\infty)$, let $\mathfrak{X}_t := \mathbb{L}^2(\Omega,\mathcal{F}_t,\mathbb{P})$ be the space of all mean square integrable functions $f:\Omega\to\mathbb{R}^d$ with $\|f\|_{\text{ms}} := \sqrt{\mathbb{E}\|f\|^2}$. A process $\xi:[0,\infty)\to\mathbb{L}^2(\Omega,\mathcal{F},\mathbb{P})$ is said to be $\mathbb{F}$-adapted if $\xi(t)\in\mathfrak{X}_t$ for all $t\geq 0$. We now restate the notion of classical solution to (1), see e.g., [2, p. 209] and [7].

Definition 2.1 (Classical solution of stochastic time-fractional differential equation)**.** For each $\eta\in\mathfrak{X}_0$, an $\mathbb{F}$-adapted process X is called a solution of (1) with the initial condition $X(0)=\eta$ if for every $t\in[0,\infty)$ it satisfies

$$X(t) = \eta + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}(AX(\tau)+b(\tau,X(\tau)))d\tau + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\sigma(\tau,X(\tau))dW_\tau. \tag{2}$$

It was proved in [7] that for any $\eta\in\mathfrak{X}_0$, there exists a unique solution $\phi(t,\eta)$ of (2). The following result gives a special presentation of $\phi(t,\eta)$.

Theorem 2.2 (A variation of constant formula for stochastic time-fractional differential equation)**.** *Let $\eta\in\mathfrak{X}_0$ arbitrary. Then the classical solution $\phi(t,\eta)$ to (1) with the initial condition $\phi(0,\eta)=\eta$ has the form*

$$\phi(t,\eta) = E_\alpha(t^\alpha A)\eta + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}((t-\tau)^\alpha A)b(\tau,\phi(\tau,\eta))\,d\tau + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}((t-\tau)^\alpha A)\sigma(\phi(\tau,\eta))\,dW_\tau,\quad t\geq 0.$$

Proof. See [2, Theorem 2.3]. □

As an application of the preceding theorem, we obtain an explicit representation of the solution to stochastic linear inhomogeneous fractional differential equations.

Corollary 1. *Consider the system (1) with the initial data $X(0)=\eta$. Assume that the coefficient functions b and σ only depend on the time variable t. Then the explicit solution to the problem*

$$^{C}D^\alpha_{0+}X(t) = AX(t) + b(t) + \sigma(t)\,\frac{dW_t}{dt},\quad t>0,$$
$$X(0) = \eta,$$

as

$$\phi(t,\eta) = E_\alpha(t^\alpha A)\eta + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}((t-\tau)^\alpha A) b(\tau)\, d\tau + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}((t-\tau)^\alpha A)\sigma(\tau)\, dW_\tau, \quad t \geq 0.$$

Remark 1. In this paper, we consider stochastic fractional differential equations driven by a scalar Brownian motion. The solution of these equations is defined as square integrable processes. Thus, from the Ito's formula, we see that it makes sense when the fractional order of these equations belongs to the interval $(\frac{1}{2}, 1)$.

3. **Asymptotic behavior in the mean square sense of solutions to time-fractional stochastic elliptic equations driven by a multiplicative white noise.** Let U be a bounded domain in $\mathbb{R}^d$ with the boundary $\partial U \in C^1$. We consider the time-fractional stochastic elliptic equation of the order $\alpha \in (1/2, 1)$

$$\frac{\partial^\alpha u(t,x)}{\partial t^\alpha} = \sum_{i,j=1}^d \partial_{x_i}(a_{ij}(x)\partial_{x_j} u(t,x)) + c(x)u(t,x) + \beta u(t,x) + \gamma u(t,x)\frac{dW_t}{dt}, \tag{3}$$

where

(a1) β, γ are arbitrary coefficients, $u(t,x) \in \mathbb{R}$ with $t \in \mathbb{R}_+$, $x \in \bar{U}$;
(a2) $a_{ij} \in C^1(\bar{U})$, $a_{ij} = a_{ji}$ for all $1 \leq i,j \leq d$ and there exists $\theta > 0$ such that $\sum_{i,j=1}^d a_{ij}(x)\xi_i\xi_j \geq \theta \|\xi\|^2$ for all $x \in \bar{U}$, $\xi \in \mathbb{R}^d$;
(a3) $c \in C(\bar{U})$, $c(x) \leq 0$ for all $x \in \bar{U}$;
(a4) $(W_t)_{t\in[0,\infty)}$ is a standard scalar Brownian motion on an underlying complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} := \{\mathcal{F}_t\}_{t\in[0,\infty)}, \mathbb{P})$.

Assume that the initial condition

$$u(0,\cdot) = f \in L^2(U) \tag{4}$$

is $\mathcal{F}_0$-measurable and the Dirichlet condition

$$u(t,x) = 0, \quad t \geq 0,\ x \in \partial U. \tag{5}$$

Let $\{e_j\}_{j=1}^\infty$ be an orthonormal basis of $L^2(U)$ which are eigenvectors of the elliptic operator $\mathcal{L}$ defined by

$$\mathcal{L}u = -\left(\sum_{i,j=1}^d \partial_{x_i}(a_{ij}\partial_{x_j} u) + c(x)u\right)$$

with respect to eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \ldots$, $\lambda_n \to \infty$ as $n \to \infty$ (see, e.g., [9, p. 335]). Suppose that $\{u(t,\cdot)\}_{t\geq 0}$ is a solution to the system (3), (4) and (5) and denote $y_j := \langle u, e_j\rangle_{L^2(U)}$, $1 \leq j < \infty$. Then y_j satisfies the equation

$$^C D^\alpha_{0+} y_j(t) = (-\lambda_j + \beta)y_j(t) + \gamma y_j(t) dW_t, \quad t > 0$$

with the initial condition $y_j(0) = \langle f, e_j\rangle_{L^2(U)}$. By virtue of Corollary 1, we have

$$y_j(t) = E_\alpha(-(\lambda_j - \beta)t^\alpha)f_j + \gamma \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(\lambda_j-\beta)(t-s)^\alpha) y_j(s) dw_s,\ t \geq 0.$$

This suggests us to establish a definition of solution as below. On the $L^2(U)$, we introduce two families of operators $\{S(t)\}_{t\geq 0}$ and $\{R(t)\}_{t>0}$ defined by

$$S(t)v := \sum_{j=1}^{\infty} E_\alpha((-\lambda_j+\beta)t^\alpha)\langle v, e_j\rangle_{L^2(U)} e_j, \quad t \geq 0,\ v \in L^2(U), \tag{6}$$

and

$$R(t)v := \sum_{j=1}^{\infty} t^{\alpha-1} E_{\alpha,\alpha}((-\lambda_j+\beta)t^\alpha)\langle v, e_j\rangle_{L^2(U)} e_j, \quad t > 0,\ v \in L^2(U). \tag{7}$$

Definition 3.1 (Mild solution of fractional stochastic elliptic equation)**.** Let $T > 0$ be arbitrary. A $L^2(U)$-valued process $\{\mathbf{u}(t)\}_{t\in[0,T]}$ is called a mild solution of the problem (3), (4), (5) on the interval $[0, T]$ if

$$\mathbf{u}(t) = S(t)f + \gamma \int_0^t R(t-s)\mathbf{u}(s)dW_s, \quad t \in [0,T],$$

where the integral $\int_0^t R(t-s)\mathbf{u}(s)dW_s$ is defined by

$$\Big\langle \int_0^t R(t-s)\mathbf{u}(s)dW_s, x\Big\rangle_{L^2(U)} = \int_0^t \langle R(t-s)\mathbf{u}(s), x\rangle_{L^2(U)} dW_s$$

for all $x \in L^2(U)$, $t \geq 0$.

This definition is a stochastic version of the deterministic case motivated by the variation of constant formula (see, e.g., [13, Definition 2.1]).

Now denote by $\mathbb{H}_T$ the space of all $L^2(U)$-valued processes $\{\mathbf{u}(t)\}_{t\in[0,T]}$ which are predictable and satisfy

$$\sup_{t\in[0,T]} \mathbb{E}\|\mathbf{u}(t)\|^2_{L^2(U)} < \infty.$$

It is obvious that $\mathbb{H}_T$ is a Banach space with the norm

$$\|\mathbf{u}\|_{\mathbb{H}_T} := \sqrt{\sup_{t\in[0,T]} \mathbb{E}\|\mathbf{u}(t)\|^2_{L^2(U)}}.$$

We collect some properties of families of operators $\{S(t)\}_{t\geq 0}$ and $\{R(t)\}_{t>0}$ in the lemma below.

Lemma 3.2. *Let $\{S(t)\}_{t\geq 0}$ and $\{R(t)\}_{t>0}$ are families of operators defined as in (6) and (7), respectively. Suppose that $\beta < \lambda_1$. Then, the following statements hold.*

(i) For any $v \in L^2(U)$ and $T > 0$, we have $S(\cdot)v \in C([0,T]; L^2(U))$. Furthermore,

$$\|S(t)v\|^2_{L^2(U)} \leq \sup_{t\geq 0} E_\alpha((-\lambda_1+\beta)t^\alpha)^2 \|v\|^2_{L^2(U)}, \quad \forall t \geq 0.$$

(ii) For any $T > 0$ and let $u \in \mathbb{H}_T$. We have $\int_0^\cdot R(\cdot - s)u(s)dW_s \in C([0,T]; \mathbb{H}_T)$. On the other hand,

$$\mathbb{E}\Big\|\int_0^t R(t-s)u(s)dW_s\Big\|^2_{L^2(U)} \leq \int_0^\infty s^{2\alpha-2} E_{\alpha,\alpha}((-\lambda_1+\beta)s^\alpha)^2 ds \sup_{t\in[0,T]} \|u(t)\|^2_{ms}$$

for any $t \in [0,T]$.

Proof. (i) For $v \in L^2(U)$, it is obvious that

$$\begin{aligned}\|S(t)v\|^2_{L^2(U)} &= \sum_{j=1}^{\infty} \langle S(t)v, e_j\rangle^2_{L^2(U)} \\ &= \sum_{j=1}^{\infty} E_\alpha((-\lambda_j+\beta)t^\alpha)^2 \langle v, e_j\rangle^2_{L^2(U)} \\ &\le \sup_{t\ge 0} E_\alpha((-\lambda_1+\beta)t^\alpha)^2 \|v\|^2_{L^2(U)}.\end{aligned}$$

Moreover, the series $\sum_{j=1}^{\infty} E_\alpha((-\lambda_j+\beta)t^\alpha)^2\langle v, e_j\rangle^2_{L^2(U)}$ is uniformly convergent on $[0,T]$. Thus, the series $\sum_{j=1}^{\infty} E_\alpha((-\lambda_j+\beta)t^\alpha)\langle v, e_j\rangle_{L^2(U)} e_j$ is also uniformly convergent on this interval which together with the fact that the function $E_\alpha((-\lambda_j+\beta)t^\alpha)$ is continuous implies that $S(\cdot)v \in C([0,T]; L^2(U))$.
(ii) For any $T > 0$ and $u \in \mathbb{H}_T$, we see that

$$\begin{aligned}\left\|\int_0^t R(t-s)\mathbf{u}(s)dW_s\right\|^2_{L^2(U)} &= \sum_{j=1}^{\infty}\left\langle \int_0^t R(t-s)\mathbf{u}(s)dW_s, e_j\right\rangle^2_{L^2(U)} \\ = \sum_{j=1}^{\infty}\left(\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}((-\lambda_j+\beta)(t-s)^\alpha)\langle \mathbf{u}(s), e_j\rangle_{L^2(U)} dW_s\right)^2.\end{aligned}$$

Hence,

$$\begin{aligned}&\mathbb{E}\left\|\int_0^t R(t-s)\mathbf{u}(s)dW_s\right\|^2_{L^2(U)} \\ &\quad= \sum_{j=1}^{\infty}\mathbb{E}\Big(\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}((-\lambda_j+\beta)(t-s)^\alpha)\langle \mathbf{u}(s), e_j\rangle_{L^2(U)} dW_s\Big)^2 \\ &\quad= \sum_{j=1}^{\infty}\int_0^t (t-s)^{2\alpha-2}E_{\alpha,\alpha}((-\lambda_j+\beta)(t-s)^\alpha)^2\mathbb{E}\langle \mathbf{u}(s), e_j\rangle^2_{L^2(U)} ds \\ &\quad\le \int_0^t (t-s)^{2\alpha-2}E_{\alpha,\alpha}((-\lambda_1+\beta)(t-s)^\alpha)^2\mathbb{E}\|\mathbf{u}(s)\|^2_{L^2(U)} ds \\ &\quad\le \int_0^\infty s^{2\alpha-2}E_{\alpha,\alpha}((-\lambda_1+\beta)s^\alpha)^2 ds \sup_{t\in[0,T]}\mathbb{E}\|\mathbf{u}(t)\|^2_{L^2(U)}\end{aligned}$$

for all $t \in [0,T]$, which implies that $\int_0^{\cdot} R(\cdot - s)\mathbf{u}(s)dW_s$ is bounded in $\mathbb{H}_T$. By the same arguments as in the proof of (i), we have $\int_0^{\cdot} R(\cdot - s)u(s)dW_s \in C([0,T];\mathbb{H}_T)$. The proof is complete. □

The following theorem shows a result on existence and uniqueness of mild solution to the problem (3) with conditions (4), (5).

Theorem 3.3 (Existence and uniqueness of mild solution to time-fractional stochastic elliptic equation)**.** *Suppose that $\beta < \lambda_1$. The system (3), (4), (5) has a unique mild solution in $\mathbb{H}_T$.*

Proof. On the space $\mathbb{H}_T$ we establish an operator $\mathcal{T}_f$ by

$$\mathcal{T}_f\mathbf{u}(t) = S(t)f + \gamma\int_0^t R(t-s)\mathbf{u}(s)dW_s, \quad t \in (0,T],$$

and $\mathcal{T}_f\mathbf{u}(0) = f$. First, we prove that this operator is well-defined. Indeed, by Lemma 3.2, we obtain for every $t \geq 0$

$$\|S(t)f\|^2_{L^2(U)} \leq \sup_{t\geq 0} E_\alpha((-\lambda_1+\beta)t^\alpha)^2\|f\|^2_{L^2(U)}, \tag{8}$$

and

$$\mathbb{E}\left\|\int_0^t R(t-s)\mathbf{u}(s)dW_s\right\|^2_{L^2(U)} \leq \int_0^\infty s^{2\alpha-2}E_{\alpha,\alpha}((-\lambda_1+\beta)s^\alpha)^2 ds \sup_{t\in[0,T]}\mathbb{E}\|\mathbf{u}(t)\|^2_{L^2(U)},$$

which together with (8) implies that

$$\|\mathcal{T}_f\mathbf{u}\|_{\mathbb{H}_T} = \sup_{t\in[0,T]}\mathbb{E}\|\mathcal{T}_f\mathbf{u}(t)\|^2_{L^2(U)} < \infty.$$

Note that for any $\rho > 0$, the norm $\|\cdot\|_{\mathbb{H}_T}$ and the norm $\|\cdot\|_{\mathbb{H}_T,w}$ defined by $\|\mathbf{u}\|_{\mathbb{H}_T,w} := \sqrt{\sup_{t\in[0,T]}\exp(-\rho t)\mathbb{E}\|\mathbf{u}(t)\|^2_{L^2(U)}}$ are equivalent. Next, we show that the operator $\mathcal{T}_f$ is contractive on $\mathbb{H}_T$ with respect to the norm $\|\cdot\|_{\mathbb{H}_T,w}$. For any $\mathbf{u},\mathbf{v} \in \mathbb{H}_T$, $t \in [0,T]$, we obtain the estimates

$$\begin{aligned}
&\exp(-\rho t)\mathbb{E}\|\mathcal{T}_f\mathbf{u}(t) - \mathcal{T}_f\mathbf{v}(t)\|^2_{L^2(U)}\\
&\leq \gamma^2\int_0^t \exp(-\rho(t-s))(t-s)^{2\alpha-2}E_{\alpha,\alpha}((-\lambda_1+\beta)(t-s)^\alpha)^2\\
&\qquad \exp(-\rho s)\mathbb{E}\|\mathbf{u}(s)-\mathbf{v}(s)\|^2_{L^2(U)}ds\\
&\leq \gamma^2\sup_{t\geq 0}E_{\alpha,\alpha}((-\lambda_1+\beta)t^\alpha)^2\int_0^t\exp(-\rho s)s^{2\alpha-2}ds\\
&\qquad \sup_{t\in[0,T]}\exp(-\rho t)\mathbb{E}\|\mathbf{u}(t)-\mathbf{v}(t)\|^2_{L^2(U)}\\
&\leq \frac{\gamma^2\sup_{t\geq 0}E_{\alpha,\alpha}((-\lambda_1+\beta)t^\alpha)^2\Gamma(2\alpha-1)}{\rho^{2\alpha-1}}\\
&\qquad \sup_{t\in[0,T]}\exp(-\rho t)\mathbb{E}\|\mathbf{u}(t)-\mathbf{v}(t)\|^2_{L^2(U)}\\
&= \frac{\gamma^2\sup_{t\geq 0}E_{\alpha,\alpha}((-\lambda_1+\beta)t^\alpha)^2\Gamma(2\alpha-1)}{\rho^{2\alpha-1}}\|\mathbf{u}-\mathbf{v}\|^2_{\mathbb{H}_T,w}.
\end{aligned} \tag{9}$$

Due to the estimate (9), we obtain

$$\|\mathcal{T}_f\mathbf{u}-\mathcal{T}_f\mathbf{v}\|^2_{\mathbb{H}_T,w} \leq \frac{\gamma^2\sup_{t\geq 0}E_{\alpha,\alpha}((-\lambda_1+\beta)t^\alpha)^2\Gamma(2\alpha-1)}{\rho^{2\alpha-1}}\|\mathbf{u}-\mathbf{v}\|^2_{\mathbb{H}_T,w}.$$

Thus, for $\rho > 0$ large enough, for example,

$$\frac{\gamma^2\sup_{t\geq 0}E_{\alpha,\alpha}((-\lambda_1+\beta)t^\alpha)^2\Gamma(2\alpha-1)}{\rho^{2\alpha-1}} < 1,$$

then $\mathcal{T}_f$ is contractive in $\mathbb{H}_T$. The proof is complete. □

Remark 2. The proof of Theorem 3.3 is true with any $T > 0$ arbitrarily. Thus, the problem (3), (4), (5) has the unique global mild solution on the $[0,\infty)$.

By combining the eigenfunction expansion method for symmetric elliptic operators and stability of solutions to stochastic fractional differential equations, we show the asymptotic behavior in the mean square sense of solutions to the time-fractional stochastic elliptic equation (3) with the data (4) and (5) in the following result.

Theorem 3.4 (Asymptotic behavior in the mean square sense of the mild solution to time-fractional stochastic elliptic equation)**.** *Consider the system* (3)*,* (4) *and* (5)*:*

$$\begin{aligned}\frac{\partial^\alpha u(t,x)}{\partial t^\alpha} &= \sum_{i,j=1}^d \partial_{x_i}(a_{ij}(x)\partial_{x_j}u(t,x)) + c(x)u(t,x)\\ &\quad + \beta u(t,x) + \gamma u(t,x)\frac{dW_t}{dt},\\ u(0,\cdot) &= f \in L^2(U),\\ u(t,x) &= 0, \quad t \geq 0,\ x \in \partial U.\end{aligned}$$

Assume that $\beta < \lambda_1$*. Then, it has a unique global mild solution* $\mathbf{u}$ *on* $[0,\infty)$*. Moreover, the following statements hold.*

(i) For

$$\gamma^2 \int_0^\infty s^{2\alpha-2} E_{\alpha,\alpha}(-(\lambda_1-\beta)s^\alpha)^2 ds < 1 \tag{10}$$

and any $\delta \in (0,1)$*, we have*

$$\sup_{t\geq 0} t^\delta \mathbb{E}\|\mathbf{u}(t)\|^2_{L^2(U)} < \infty. \tag{11}$$

(ii) For

$$\gamma^2 \int_0^\infty s^{2\alpha-2} E_{\alpha,\alpha}(-(\lambda_1-\beta)s^\alpha)^2 ds > 1, \tag{12}$$

there exists f *such that the solution* $\mathbf{u}$ *with the initial condition* $\mathbf{u}(0) = f$ *satisfies*

$$\lim_{t\to\infty} \mathbb{E}\|\mathbf{u}(t)\|^2_{L^2(U)} \to 0.$$

Remark 3. The equation (3) can be thought as a stochastic perturbed model of the time-fractional elliptic equation

$$\frac{\partial^\alpha u(t,x)}{\partial t^\alpha} = \sum_{i,j=1}^d \partial_{x_i}(a_{ij}(x)\partial_{x_j}u(t,x)) + c(x)u(t,x), \quad t > 0.$$

Theorem 3.4(i) shows that under small perturbations, for example $\beta < \lambda_1$ and γ satisfies (10), the asymptotic stability (in the mean square sense) of (3) is guaranteed. However, if the noise is large (e.g., the condition (12) is satisfied), its stability will be broken.

Remark 4. For $\alpha \in (\frac{1}{2}, 1)$, $\lambda < 0$, from [6, Lemma 3] or [4, Lemma 5.1 (a)], there has a positive constant C which only depends on α, λ such that

$$t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha) \leq \frac{C}{t^{\alpha+1}},\ \forall t \geq 1.$$

On the other hand, from the increasing monotonicity of the function $E_{\alpha,\alpha}(\cdot)$ on $(-\infty, 0]$ (see, e.g., [11, Lemma 4.25, p. 86]), we see that

$$E_{\alpha,\alpha}(\lambda t^\alpha) \leq 1,\ \forall t \geq 0.$$

Thus,

$$\begin{aligned}\int_0^\infty s^{2\alpha-2}E_{\alpha,\alpha}(\lambda s^\alpha)^2 ds &= \int_0^1 s^{2\alpha-2}E_{\alpha,\alpha}(\lambda s^\alpha)^2 ds + \int_1^\infty s^{2\alpha-2}E_{\alpha,\alpha}(\lambda s^\alpha)^2 ds\\ &\le \int_0^1 s^{2\alpha-2} ds + C^2\int_1^\infty \frac{1}{t^{2\alpha+2}}dt\\ &= \frac{1}{2\alpha-1}+\frac{C^2}{2\alpha+1}.\end{aligned}$$

Proof of Theorem 3.4. From Remark 2, we see that the problem (3), (4) and (5) has a unique mild solution on $[0,\infty)$. Denote this solution by $\{\mathbf{u}(t)\}_{t\ge 0}$ and $y_j := \langle \mathbf{u}, e_j\rangle_{L^2(U)}$, $1\le j<\infty$.

(i) Let $\mathbb{H}$ be the space of all $L^2(U)$-valued processes $\{\xi(t)\}_{t\in[0,\infty)}$ which are predictable and satisfy

$$\sup_{t\in[0,\infty)} \mathbb{E}\|\xi(t)\|^2_{L^2(U)} < \infty.$$

This is a Banach space with the norm

$$\sup_{t\in[0,\infty)} \sqrt{\mathbb{E}\|\xi(t)\|^2_{L^2(U)}}.$$

We establish a functional on $\mathbb{H}$ as below. For any $\xi\in\mathbb{H}$, we define

$$\|\xi\|_w := \sup_{t\in[0,\infty)} \sqrt{\alpha(t)}\|\xi(t)\|_{ms},$$

where

$$\alpha(t)=\begin{cases} T^\delta, & t\in[0,T],\\ t^\delta, & t\ge T,\end{cases}$$

T is a positive constant and chosen later. We denote $\mathbb{H}_w := \{\xi\in\mathbb{H} : \|\xi\|_w<\infty\}$. It is obvious that $(\mathbb{H},\|\cdot\|_w)$ is also a Banach space. Under the assumption (10), by the same arguments as in the proof of Theorem 3.3, we see that the mild solution $\mathbf{u}$ of the system (3)–(5) is bounded on $[0,\infty)$ in the mean square sense. To complete the proof of this part, we will show that the mild solution $\mathbf{u}\in\mathbb{H}_w$.

For any $\xi\in\mathbb{H}_w$, we establish

$$\mathcal{T}_f\xi(t) := S(t)f + \gamma\int_0^t R(t-s)\xi(s)dW_s,\quad t\in(0,T].$$

This operator is contractive with respect to the norm $\|\cdot\|_w$. Indeed, for any $\xi,\hat{\xi}\in\mathbb{H}_w$, we have

$$\begin{aligned}\|\mathcal{T}_f\xi(t)-\mathcal{T}_f\hat{\xi}(t)\|^2_{ms} &= \left\|\int_0^t R(t-s)\xi(s)dW_s\right\|_{ms}\\ &\le \int_0^t (t-s)^{2\alpha-2}E_{\alpha,\alpha}(a(t-s)^\alpha)^2\|\xi(s)-\hat{\xi}(s)\|^2_{ms}ds,\quad \forall t\ge 0,\end{aligned}$$

where $a = -\lambda_1 + \beta$. We first consider $t \in [0, T]$. In this case,

$$\begin{aligned}\alpha(t)\|\mathcal{T}_f\xi(t) - \mathcal{T}_f\hat{\xi}(t)\|_{ms}^2 &\le T^\delta\gamma^2 \int_0^t (t-\tau)^{2\alpha-2}E_{\alpha,\alpha}(a(t-\tau)^\alpha)^2\|\xi(\tau) - \hat{\xi}(\tau)\|_{ms}^2 d\tau \\ &\le \gamma^2 \int_0^t (t-\tau)^{2\alpha-2}E_{\alpha,\alpha}(a(t-\tau)^\alpha)^2\alpha(\tau)\|\xi(\tau) - \hat{\xi}(\tau)\|_{ms}^2 d\tau \\ &\le \gamma^2 \int_0^\infty s^{2\alpha-2}E_{\alpha,\alpha}(as^\alpha)^2 ds\|\xi - \hat{\xi}\|_w^2. \end{aligned} \tag{13}$$

Next, for $t > T$, then

$$\begin{aligned}\alpha(t)\|\mathcal{T}_f\xi(t) - \mathcal{T}_f\hat{\xi}(t)\|_{ms}^2 &\le t^\delta\gamma^2 \int_0^t (t-\tau)^{2\alpha-2}E_{\alpha,\alpha}(a(t-\tau)^\alpha)^2\|\xi(\tau) - \hat{\xi}(\tau)\|_{ms}^2 d\tau \\ &\le t^\delta\gamma^2 \int_0^t (t-\tau)^{2\alpha-2}E_{\alpha,\alpha}(a(t-\tau)^\alpha)^2\tau^{-\delta}d\tau\|\xi - \hat{\xi}\|_w^2. \end{aligned} \tag{14}$$

Note that on the interval $[0, t/2]$,

$$\begin{aligned} t^\delta \int_0^{t/2} &(t-\tau)^{2\alpha-2}E_{\alpha,\alpha}(a(t-\tau)^\alpha)^2\tau^{-\delta}\, d\tau \\ &\le t^\delta \int_0^{t/2} \frac{C}{(t-\tau)^{2+2\alpha}}\tau^{-\delta}d\tau \\ &\le \frac{Ct^\delta}{(t/2)^{2\alpha+2}}\int_0^{t/2}\tau^{-\delta}d\tau \\ &\le \frac{C2^{2\alpha+\delta+1}}{(1-\delta)t^{2\alpha+1}} \\ &\le \frac{C2^{2\alpha+\delta+1}}{(1-\delta)T^{2\alpha+1}}, \end{aligned} \tag{15}$$

on the interval $[t/2, t-M]$,

$$\begin{aligned} t^\delta \int_{t/2}^{t-M} &(t-\tau)^{2\alpha-2}E_{\alpha,\alpha}(a(t-\tau)^\alpha)^2\tau^{-\delta}d\tau \\ &\le \frac{t^\delta}{(t/2)^\delta}\int_{t/2}^{t-M}\frac{C}{(t-\tau)^{2\alpha+2}}d\tau \\ &\le \frac{C2^\delta}{(2\alpha+1)M^{2\alpha+1}}, \end{aligned} \tag{16}$$

and on $[t-M, t]$,

$$\begin{aligned} t^\delta \int_{t-M}^{t} &(t-\tau)^{2\alpha-2}E_{\alpha,\alpha}(a(t-\tau)^\alpha)^2\tau^{-\delta}d\tau \\ &\le \frac{t^\delta}{(t-M)^\delta}\int_{t-M}^{t}(t-\tau)^{2\alpha-2}E_{\alpha,\alpha}(a(t-\tau)^\alpha)^2 d\tau \\ &\le \frac{t^\delta}{(t-M)^\delta}\int_0^\infty s^{2\alpha-2}E_{\alpha,\alpha}(as^\alpha)^2 ds. \end{aligned} \tag{17}$$

From (15), (16), (17) and (10), we choose $M > 0$ and $T > 2M$ such that for any $t > T$

$$\frac{\gamma^2 C 2^{2\alpha+\delta+1}}{(1-\delta)T^{2\alpha+1}} + \frac{C\gamma^2 2^\delta}{(2\alpha+1)M^{2\alpha+1}} + \frac{\gamma^2 t^\delta}{(t-M)^\delta}\int_0^\infty s^{2\alpha-2}E_{\alpha,\alpha}(as^\alpha)^2 ds < 1.$$

This combines with (13) and (14) showing that $\mathcal{T}_f$ is contractive on the space $(\mathbb{H}_w, \|\cdot\|_w)$. On the other hand, it is easy to see that $\mathcal{T}_f$ is bounded in this space. Hence, by Banach fixed point theorem, there exists a unique fixed point ξ^* in $(\mathbb{H}_w, \|\cdot\|_w)$ which is also a mild solution to the origin system. Due to the fact that the system has a unique mild solution in $\mathbb{H}$, it implies the solution $\mathbf{u} = \xi^* \in \mathbb{H}_w$ and the estimate (11) is proved.

(ii) Without loss of generality, choose the initial data f such that $y_j(0) = \langle f, e_j\rangle_{L^2(U)} \in \mathfrak{X}_0 \setminus \{0\}$. To complete the proof of this part, we will prove that

$$\|y_j(t)\|_{ms} \to 0 \tag{18}$$

as $t \to \infty$. Indeed, suppose that (18) is not true, that is,

$$\lim_{t\to\infty} \mathbb{E}y_j(t)^2 = 0.$$

Put $h(t) = \mathbb{E}y_j(t)^2$, $t \geq 0$. It is worth noting that $h(t) > 0$ for all $t \geq 0$. By this fact there exists an increasing monotone consequence $\{t_k\}_{k=1}^\infty$ with $0 < t_1 < t_2 < ... < t_k \to \infty$ and

$$0 < h(t_k) = \min_{s\in[0,t_k]} h(s), \quad k = 1, 2, \ldots$$

Thus, for any $k = 1, 2, \ldots$, we have

$$\begin{aligned}
h(t_k) &= E_\alpha((-\lambda_j+\beta)t_k^\alpha)^2\mathbb{E}y_j(0)^2 \\
&\qquad + \gamma^2\int_0^{t_k}(t_k-s)^{2\alpha-2}E_{\alpha,\alpha}((-\lambda_j+\beta)(t_k-s)^\alpha)^2 h(s)ds \\
&\geq E_\alpha((-\lambda_j+\beta)t_k^\alpha)^2\mathbb{E}y_j(0)^2 \\
&\qquad + \gamma^2\int_0^{t_k}(t_k-s)^{2\alpha-2}E_{\alpha,\alpha}((-\lambda_j+\beta)(t_k-s)^\alpha)^2 ds\, h(t_k) \\
&\geq E_\alpha((-\lambda_j+\beta)t_k^\alpha)^2\mathbb{E}y_j(0)^2 + \gamma^2\int_0^{t_k} s^{2\alpha-2}E_{\alpha,\alpha}((-\lambda_j+\beta)s^\alpha)^2 ds\, h(t_k) \\
&\geq \gamma^2\int_0^{t_k} s^{2\alpha-2}E_{\alpha,\alpha}((-\lambda_j+\beta)s^\alpha)^2 ds\, h(t_k).
\end{aligned}$$

This together with (12) that

$$\gamma^2\int_0^{t_k} s^{2\alpha-2}E_{\alpha,\alpha}((-\lambda_j+\beta)s^\alpha)^2 ds > 1$$

for k large enough leads to

$$h(t_k) > h(t_k),$$

a contradiction. The proof is complete. □

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