ZEROS OF DIFFERENTIAL POLYNOMIALS OF MEROMORPHIC FUNCTIONS

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Abstract. Considering a transcendental meromorphic function f, a positive integer k and polynomials $Q_0, Q_1, \ldots, Q_k$. In this paper, we will prove that the frequency of distinct poles of f is governed by the frequency of zeros of the differential polynomial form $Q_0(f)Q_1(f') \ldots Q_k(f^{(k)})$ in f. We will also prove that the Nevanlinna defect of the differential polynomial form $Q_0(f)Q_1(f') \ldots Q_k(f^{(k)})$ in f satisfy

$$\sum_{a\in\mathbb{C}} \delta(a, Q_0(f)Q_1(f') \ldots Q_k(f^{(k)})) \leq 1$$

with suitable conditions on k and the degree of the polynomials.

Thus, our works are generalizations of a Mues's conjecture and Goldberg's conjecture to more general differential polynomials.

1. Introduction and main results

Let f be a transcendental meromorphic function, the Gol'dberg conjecture (see [3]) stated that the number of distinct poles of f is bounded by the number of zeros of the k-derivative $f^{(k)}$, where $k \geq 2$. In 1986, by a Wronskian method, Frank and Weissenborn [2] proved a part of the Gol'dberg conjecture where f has poles of multiplicity at most $k-1$. Another related result was established by Langley [7], who proved that if f is meromorphic function of finite order whose second derivative f'' has finite many zeros, then f has finite many poles. In 2013, by using the upper and lower estimates of the modification of the proximity function, Yamanoi [10] proved a generalization of the Gol'dberg conjecture, which states that for a transcendental meromorphic function f and $k \geq 2$ is a integer and let $\epsilon > 0$. Let $A \subset \mathbb{C}$ be a finite subset of complex number. Then, we have

$$(k-1)\overline{N}(r,f) + \sum_{a\in A} N_1\big(r, \frac{1}{f-a}\big) \leq N\big(r, \frac{1}{f^{(k)}}\big) + \epsilon T(r,f) \tag{1.1}$$

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as $r \to \infty$ possibly outside an exceptional set, where $N(r, f)$, $N_1\big(r, \frac{1}{f-a}\big)$,... will be defined in Section 2.

Let $a \in \mathbb{C}$ and let $\delta(a, f)$ be the Nevanlinna defect of function f. Then, the defect $\delta(a, f)$ is bounded in $[0, 1]$ and by the Nevanlinna Second Main Theorem, $\sum_{a\in\mathbb{C}} \delta(a, f) \leq 2$ for any meromorphic function f. For k is a positive integer, Mues [8] conjectured that the Nevanlinna defects of the k^{th} derivative of f satisfy

$$\sum_{a\in\mathbb{C}} \delta(a, f^{(k)}) \leq 1. \tag{1.2}$$

In the paper, Mues himself proved this conjecture for the case $k \geq 2$ and restricted to the class of meromorphic functions whose all of poles are simple. In 1990, Yang [11] and Ishizaki [5] obtained the upper bound for the sum in (1.2) is $\frac{2k+2}{2k+1}$. Then, Yang and Wang [12] proved that there exists a positive integer $K(f)$ such that the estimate (1.2) holds for $k \geq K(f)$. Wang [13] proved (1.2) holds for all $k \geq 0$ with at most four exceptions of k. Finally, Yamanoi [10] confirmed Mues conjecture without any additional hypotheses to meromorphic functions. It is known that the Gol'dberg's conjecture implies the Mues's conjecture.

In 2016, Jiang and Huang [6, Theorem 3] considered for differential monomials form $f^l(f^{(k)})^n$ where l, n, k are integers greater than 1. They obtained the upper bound for the sum of deficiencies of $f^l(f^{(k)})^n$ is $1 + \frac{1}{nk+n+l}$. However, this bound is not sharp.

Our aims in this paper are to give a generalization of the estimates (1.1) and (1.2) for the more general differential polynomials.

From now, let $k \geq 1$ be an integer and let $Q_i(z)$ be polynomials of degree q_i, $(i = 0, 1, 2, \dots, k)$ in $\mathbb{C}[z]$. We write

$$Q_i(z) = c_i \prod_{j=1}^{h_i} (z - \beta_{ij})^{q_{ij}}$$

with $c_i \in \mathbb{C}^*$ and $\sum_{j=1}^{h_i} q_{ij} = q_i$, for $i = 0, 1, 2, \dots, k$.

Set

$$\Phi := Q_0(f)Q_1(f') \dots Q_k(f^{(k)}) \tag{1.3}$$

and

$$q := q_0 + q_1 + \cdots + q_k.$$

Our result as following.

Theorem 1. *Let f be a transcendental meromorphic function in the complex plane. Let $k \geq 2$ be an integer, $\epsilon > 0$. Let $A \subset \mathbb{C}$ be a finite set of complex numbers. Then we have*

$$\sum_{i=0}^{k}(i-1)q_i\overline{N}(r,f) + q\sum_{a\in A} N_1\Big(r, \frac{1}{f-a}\Big) \leq N\Big(r, \frac{1}{\Phi}\Big) + \epsilon T(r,f),$$

for all $r > e$ outside a set $E \subset (e, \infty)$ of logarithmic density 0.

In the case that $Q_0(z) = Q_1(z) = \cdots = Q_{k-1}(z) = 1$ and $Q_k(z) = z$, we recover the result in [10] as a special case of our result.

Corollary 1. [10, Theorem 1.2] *Let f be a transcendental meromorphic function in the complex plane. Let $k \geq 2$ be an integer, $\epsilon > 0$. Let $A \subset \mathbb{C}$ be a finite set of complex numbers. Then we have*

$$(k-1)\overline{N}(r,f) + \sum_{a\in A} N_1\Big(r, \frac{1}{f-a}\Big) \leq N\Big(r, \frac{1}{f^{(k)}}\Big) + \epsilon T(r,f)$$

for all $r > e$ outside a set $E \subset (e, \infty)$ of logarithmic density 0.

Remark 1. The original Gol'dberg conjecture corresponds to the case $k = 2$, $Q_0(z) = Q_1(z) = \cdots = Q_{k-1}(z) = 1$, $Q_k(z) = z$ and $A = \emptyset$.

Theorem 2. *Let f be a meromorphic function, k be a positive integer. If one of the following conditions holds*

(a) $k \geq 2$ and there exists $\nu \in \{2, ..., k\}$ such that $Q_\nu(z)$ has a zero of order at least 2.
(b) $k \geq 1$ and $\Phi = Q_k(f^{(k)})$.
Then we have

$$\sum_{a\in\mathbb{C}} \delta(a, Q_0(f)Q_1(f')\ldots Q_k(f^{(k)})) \leq 1.$$

As a consequence, when we consider for $Q_0(z) = Q_1(z) = \cdots = Q_{k-1}(z) = 1$ and $Q_k(z) = z$, we can receive Mues conjecture as follow:

Corollary 2 (Mue Conjecture)**.** *Let f be a meromorphic function in the complex plane whose derivative f' is non-constant and $k \geq 1$ be an integer. Then we have*

$$\sum_{a\in\mathbb{C}} \delta(a, f^{(k)}) \leq 1.$$

When we consider for $Q_0(z) = z^l$, $Q_1(z) = \cdots = Q_{k-1}(z) = 1$ and $Q_k(z) = z^n$, where l, n, k are integers greater than 1, Theorem 2 implies [6, Theorem 3]. Moreover, we can improve their upper bound for the sum of deficiencies that $1 + \frac{1}{nk+n+l}$ to 1.

Corollary 3. *Let f be a transcendental meromorphic function in the complex plane, k, l, n be positive integers all at least 2. Then*

$$\sum_{a\in\mathbb{C}} \delta(a, f^l(f^{(k)})^n) \leq 1.$$

2. Preliminary on Nevanlinna's Theory

2.1. **Classical Nevanlinna Theory.** Let f be a meromorphic function on $\mathbb{C}$. In this paper, we assume readers are familiar with fundamental of some standard concepts in Nevanlinna Theory, in particular with the most usual of symbol $m(r, f), N(r, f)$, $\overline{N}(r, f)$, and $T(r, f)$,... (see [4, 9] for more detail). We define

$$N_1\Big(r, \frac{1}{f-a}\Big) = N\Big(r, \frac{1}{f-a}\Big) - \overline{N}\Big(r, \frac{1}{f-a}\Big).$$

We define the *Nevanlinna deficiency* by

$$\delta(a, f) = \liminf_{r\to\infty} \frac{m\big(r, \frac{1}{f-a}\big)}{T(r, f)} = 1 - \limsup_{r\to\infty} \frac{N\big(r, \frac{1}{f-a}\big)}{T(r, f)}.$$

The logarithmic derivative lemma can be stated as follows (see [9]).

Lemma 1 (Logarithmic Derivative Lemma)**.** *Let f be a non-constant meromorphic function on $\mathbb{C}$. Then*

$$m(r, \frac{f'}{f}) = o(T(r, f))$$

as $r \to \infty$ outside a subset of finite measure.

We state the first and second fundamental theorem in Nevanlinna theory (see e.g. [4], [9]):

Theorem 3 (The First Main Theorem)**.** *Let $f(z)$ be a meromorphic function and c is a finite complex number. Then*

$$T(r, \frac{1}{f-c}) = T(r, f) + O(1).$$

Theorem 4 (Second fundamental theorem)**.** *Let $a_1, \cdots, a_q$ be a set of distinct complex numbers. Let f be a non-constant meromorphic function on $\mathbb{C}$. Then, the inequality*

$$\sum_{j=1}^{q} m(r, \frac{1}{f-a_j}) \leq T(r,f) + \overline{N}(r,f) - N\Big(r, \frac{1}{f'}\Big) + o(T(r,f)),$$

holds for all r outside a set $E \subset (0, +\infty)$ with finite Lebesgue measure.

2.2. **Some results of Yamanoi.** In [10], Yamanoi improved more generalization of Nevanlinna theory. For convenient of readers, we would like to recall here.

We define the chordal distance between two points z and w in the complex plane by

$$[z,w] = \frac{|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}},$$

and

$$[z,\infty] = \frac{1}{\sqrt{1+|z|^2}}.$$

Let $\mathcal{R}_d$ be the set of all rational functions of degree less than or equal to d including the constant function which is identically equal to ∞. We define the modification of proximity function by

$$\bar{m}_{d,n}(r,f) = \sup_{(a_1,\ldots,a_n)\in(\mathcal{R}_d)^n} \int_0^{2\pi} \max_{1\leq j\leq n} \log \frac{1}{[f(re^{i\theta}), a_j(re^{i\theta})]} \frac{d\theta}{2\pi}.$$

A generalization of the first main theorem shows that $\bar{m}_{d,n}(r,f)$ is finite (see [10, Remark 2.3]).

For a meromorphic function f, we put

$$v(r,f,\theta) = \sup_{\tau\in[0,2\pi]} \Big(\sup_{t\in[\tau,\tau+\theta]} \log|f(re^{it})| - \inf_{t\in[\tau,\tau+\theta]} \log|f(re^{it})| \Big),$$

$$\lambda(r) = \min\Big\{1, \Big(\log^+ \frac{T(r,f)}{\log r}\Big)^{-1}\Big\}.$$

Lemma 2. [10, Proposition 3.1] *Let f be a transcendental meromorphic function in the complex plane. Let $\epsilon > 0$. Then we have*

$$v(r,f,\lambda(r)^{20}) \leq \epsilon T(r)$$

for all $r > e$ outside a set of logarithmic density zero.

Lemma 3. [10, Lemma 3.6] *Let f be a transcendental meromorphic function in the complex plane, and let k be a positive integer. Put*

$$u_k = (k+1)\log^+|f| + \log\frac{1}{|f^{(k)}|}.$$

Then given a positive integer n, we have

$$\int_0^{2\pi} u_k(re^{i\theta})\frac{d\theta}{2\pi} \leq \bar{m}_{k-1,n}(r,f) + (k-1)m(r,f) + v\Big(r,f,\frac{2\pi}{n}\Big) + v\Big(r,f^{(k)},\frac{2\pi}{n}\Big) + k\log(2\pi r) + 2kn\log 3$$

for all $r > 1$.

Lemma 4. [10, Theorem 1.4] *Let f be a transcendental meromorphic function on the complex plane. Let d and n be positive integers. Let $\epsilon > 0$. Let $B \subset \mathbb{C}\cup\{\infty\}$ be a finite set of points in the Riemann sphere and set $p = \#B$. Then we have*

$$\bar{m}_{d,n}(r,f) + \sum_{a\in B} N_1\Big(r,\frac{1}{f-a}\Big) \leq (2+\epsilon)T(r,f) + \frac{(p+n)^{17}}{\epsilon^4}T(r)^{4/5}(\log r)^{1/5}$$

for all $r > 0$ outside a set of finite linear measure $E_{f,d}$ which only depends on f and d.

3. Proof of Theorem 1

We first consider the following lemmas.

Lemma 5. *Let f be a transcendental meromorphic function in the complex plane. Let $\epsilon > 0$ be an arbitrary small positive constant. Let $\sigma : (e,\infty) \to \mathbb{N}_{>0}$ be a function such that*

$$\sigma(r) \sim \Big(\log^+\frac{T(r)}{\log r}\Big)^{20}.$$

Then, we have

$$v\Big(r,f,\frac{2\pi}{\sigma(r)}\Big) + v\Big(r,f^{(k+1)},\frac{2\pi}{\sigma(r)}\Big) + (k+1)\log(2\pi r) + 2(k+1)\sigma(r)\log 3 + o(T(r,f)) < \epsilon T(r,f), \tag{3.1}$$

for all $r > e$ outside a set of logarithmic density zero.

Proof. Applying Lemma 2, we have

$$v(r,f,\lambda(r)^{20}) < \frac{\epsilon}{21}T(r,f)$$

outside a set of logarithmic density zero. From the definition of function σ, we have

$$\frac{2\pi}{\sigma(r)} < 7\lambda(r)^{20}$$

for all r sufficiently large. Hence, we have

$$v\big(r, f, \frac{2\pi}{\sigma(r)}\big) < \frac{\epsilon}{3}T(r, f) \tag{3.2}$$

for all $r > e$ outside a set E_1 of logarithmic density zero.

From the Logarithmic Derivative Lemma, it is easy to see that

$$T(r, f^{(k+1)}) \leq (k+2)T(r, f) + o(T(r, f)). \tag{3.3}$$

Hence, by this estimate and again by Lemma 2, we have

$$v(r, f^{(k+1)}, \widehat{\lambda}(r)^{20}) < \frac{\epsilon}{42(k+2)}T(r, f^{(k+1)}) \leq \frac{\epsilon}{42}T(r, f),$$

for all $r > e$ outside a set of logarithmic density zero, where

$$\widehat{\lambda}(r) = \min\Big\{1, \Big(\log^+ \frac{T(r, f^{(k+1)})}{\log r}\Big)^{-1}\Big\}.$$

By the definition of $\lambda(r)$, $\widehat{\lambda}(r)$ and (3.3), we can see that $\lambda(r)^{20} < 2\widehat{\lambda}(r)^{20}$ for r sufficiently large. Hence, we have

$$v(r, f^{(k+1)}, \lambda(r)^{20}) < \frac{\epsilon}{21}T(r, f)$$

for all $r > e$ outside a set of logarithmic density zero. Therefore, we get

$$v\big(r, f^{(k+1)}, \frac{2\pi}{\sigma(r)}\big) < \frac{\epsilon}{3}T(r, f) \tag{3.4}$$

for all $r > e$ outside a set E_2 of logarithmic density zero.

On the other hand, since f is a transcendental meromorphic function, there exists a positive number $r_0 > e$ such that

$$(k+1)\log(2\pi r) + 2(k+1)\sigma(r)\log 3 + o(T(r, f)) < \frac{\epsilon}{3}T(r, f) \tag{3.5}$$

for all $r > r_0$. Put

$$E = [e, r_0] \cup E_1 \cup E_2.$$

Then E is a set of logarithmic density zero. Combining (3.2), (3.4) and (3.5), we deduce the inequality (3.1). □

Lemma 6. *Let k be a positive integer. Let f be a transcendental meromorphic function and Q_j be polynomials of one variable of degree q_i for $j = 0, 1, 2, \dots, k$. Let $q = \sum_{j=0}^{k} q_j$, and*

$$R_k = \log \frac{1}{|Q_0(f)Q_1(f') \dots Q_k(f^{(k)})|} + q(k+2)\log^+ |f|.$$

Then we have

$$\int_0^{2\pi} R_k(re^{i\theta})\frac{d\theta}{2\pi} \le q \int_0^{2\pi} u_{k+1}(re^{i\theta})\frac{d\theta}{2\pi} + o(T(r,f)).$$

Proof. Put $\Phi := Q_0(f)Q_1(f') \dots Q_k(f^{(k)})$. We have

$$\begin{aligned}\log \frac{1}{|\Phi|} &= q \log \frac{1}{|f^{(k+1)}|} + \log \frac{|f^{(k+1)}|^q}{|\Phi|} \\ &\le q \log \frac{1}{|f^{(k+1)}|} + \sum_{i=0}^{k}\sum_{j=1}^{h_i} q_{ij} \log \frac{|f^{(k+1)}|}{|f^{(i)} - \beta_{ij}|} + O(1) \\ &\le q \log \frac{1}{|f^{(k+1)}|} + \sum_{i=0}^{k}\sum_{j=1}^{h_i} q_{ij} \log^+ \frac{|f^{(k+1)}|}{|f^{(i)} - \beta_{ij}|} + O(1).\end{aligned}$$

From the Logarithmic Derivative Lemma

$$m\Big(r, \frac{f^{(k+1)}}{f^{(i)} - \beta_j}\Big) = o(T(r,f)),$$

for any $i = 0, ..., k$, we obtain

$$\begin{aligned}\int_0^{2\pi} R_k(re^{i\theta})\frac{d\theta}{2\pi} &\le q \int_0^{2\pi} u_{k+1}(re^{i\theta})\frac{d\theta}{2\pi} + \sum_{i=0}^{k}\sum_{j=1}^{h_i} q_{ij} m\Big(r, \frac{f^{(k+1)}}{f^{(i)} - \beta_j}\Big) + O(1) \\ &= q \int_0^{2\pi} u_{k+1}(re^{i\theta})\frac{d\theta}{2\pi} + o(T(r,f)).\end{aligned}$$

□

Recall the Jensen formula as follows (see [9]):

Lemma 7 (Jensen's Formula)**.** *Let $f \not\equiv 0$ be meromorphic function on $\bar{D}(r) = \{|z| \le r\}$, ($r < \infty$). Let $a_1, \dots, a_\mu$ denote the zeros of f in $\bar{D}(r)$, counting multiplicities, and let $b_1, \dots, b_\nu$ denote the poles of f in $\bar{D}(r)$, also counting multiplicities. Then if $f(0) \ne 0, \infty$, we have*

$$\log |f(0)| = \int_0^{2\pi} \log |f(re^{i\theta})| \frac{d\theta}{2\pi} - \sum_{i=1}^{\mu} \log \frac{r}{|a_i|} + \sum_{j=1}^{\nu} \log \frac{r}{|b_j|}.$$

Proof of Theorem 1. Put $\Phi := Q_0(f)Q_1(f')\dots Q_k(f^{(k)})$, and

$$R_k := q(k+2)\log^+|f| + \log\frac{1}{|\Phi|}.$$

Note that

$$N(r,\Phi) = \sum_{i=0}^{k} N(r, Q_i(f^{(i)})) = \sum_{i=0}^{k} q_i N(r, f^{(i)}) \\ = qN(r,f) + \sum_{i=0}^{k} iq_i\overline{N}(r,f). \tag{3.6}$$

Applying the Jensen's Formula to the meromorphic functions f and Φ, and using the First Main Theorem and the fact $\log x = \log^+ x - \log^+\frac{1}{x}$, and together with (3.6), we have

$$\begin{aligned}\int_0^{2\pi} R_k(re^{i\theta})\frac{d\theta}{2\pi} &= q(k+2)m\big(r,\frac{1}{f}\big) + q(k+2)\Big(N\big(r,\frac{1}{f}\big) - N\big(r,f\big)\Big) \\ &\quad + N\Big(r,\Phi\Big) - N\Big(r,\frac{1}{\Phi}\Big) + O(1) \\ &= q(k+2)\Big(m\big(r,\frac{1}{f}\big) + N\big(r,\frac{1}{f}\big)\Big) - q(k+1)N(r,f) \\ &\quad + \sum_{i=0}^{k} iq_i\overline{N}(r,f) - N\Big(r,\frac{1}{\Phi}\Big) + O(1) \\ &= q(k+2)T(r,f) - q(k+1)N(r,f) + \sum_{i=0}^{k} iq_i\overline{N}(r,f) \\ &\quad - N\Big(r,\frac{1}{\Phi}\Big) + O(1).\end{aligned} \tag{3.7}$$

On the other hand, by Lemma 3, Lemma 5 and Lemma 6, we have

$$\begin{aligned}\int_0^{2\pi} R_k(re^{i\theta})\frac{d\theta}{2\pi} &\le q\int_0^{2\pi} u_{k+1}(re^{i\theta})\frac{d\theta}{2\pi} + o(T(r,f))\\ &\le kq\, m(r,f) + q\,\bar{m}_{k,\sigma(r)}(r,f)\\ &\quad + q\Bigg(v\Big(r,f,\frac{2\pi}{\sigma(r)}\Big) + v\Big(r,f^{(k+1)},\frac{2\pi}{\sigma(r)}\Big)\\ &\quad + (k+1)\log(2\pi r) + 2(k+1)\sigma(r)\log 3\Bigg)\\ &\le kq\, m(r,f) + q\,\bar{m}_{k,\sigma(r)}(r,f) + \frac{\epsilon}{3}T(r,f) \qquad (3.8)\end{aligned}$$

for all $r > e$ and $\sigma : (e,\infty) \to \mathbb{N}_{>0}$ be a function as in Lemma 5.

Since f is a transcendental meromorphic function and by the definition of function σ, we have

$$\lim_{r\to\infty} \frac{(p+\sigma(r))^{17}T(r)^{4/5}(\log r)^{1/5}}{T(r)} = 0.$$

Hence, there exists a positive number r_1 such that

$$(p+\sigma(r))^{17}T(r)^{4/5}(\log r)^{1/5} < \frac{\epsilon^5}{3}T(r,f) \qquad (3.9)$$

for a given positive integer n and all $r > r_1$.

Now, let $A \subset \mathbb{C}$ be a finite set of complex numbers. Applying Lemma 4 to the case $B = A\cup\{\infty\}, d = k, p = \#A + 1$ and $n = \sigma(r)$, we obtain

$$\begin{aligned}&\bar{m}_{k,\sigma(r)}(r,f) + N_1(r,f) + \sum_{a\in A} N_1\Big(r,\frac{1}{f-a}\Big)\\ &\le \big(2 + \frac{\epsilon}{3q}\big)T(r,f) + \frac{(p+\sigma(r))^{17}}{q\epsilon^4}T(r)^{4/5}(\log r)^{1/5}\\ &\le \big(2 + \frac{2\epsilon}{3q}\big)T(r,f), \qquad (3.10)\end{aligned}$$

for all $r > 0$ outside a set of finite linear measure.

Combining (3.7), (3.8), and (3.10), we obtain

$$\sum_{i=0}^{k}(i-1)q_i\overline{N}(r,f) + q\sum_{a\in A} N_1\Big(r,\frac{1}{f-a}\Big) \le N\Big(r,\frac{1}{\Phi}\Big) + \epsilon T(r,f),$$

for all $r > e$ outside a set E of logarithmic density zero. This completes the proof of Theorem 1. □

4. Proof of Theorem 2

To prove the results, we need to prove the following lemmas.

Lemma 8. *Let f be a meromorphic function in the complex plane, k be a positive integer, Φ be defined as* (1.3). *We have*

$$T(r,\Phi) = O(T(r,f)), \quad \text{and} \quad o(T(r,\Phi)) = o(T(r,f)).$$

Proof. By characteristic function's properties, we have

$$\begin{aligned} T(r,\Phi) &\le \sum_{i=0}^{k} T(r, Q_i(f^{(i)})) + O(1) \\ &\le \sum_{i=0}^{k} q_i T(r, f^{(i)}) + o(T(r,f)) \\ &\le \sum_{i=0}^{k} q_i (i+1) T(r,f) + o(T(r,f)). \end{aligned}$$

Hence, we get

$$T(r,\Phi) = O(T(r,f)), \quad \text{and} \quad o(T(r,\Phi)) = o(T(r,f)).$$

The proof of Lemma 8 is completed. □

Proof of Theorem 2. We first consider the case that f is a rational function. Then Φ is a non-constant rational function. We have $\delta(a,\Phi) = 0$ for all $a \ne \Phi(\infty)$. Therefore, Theorem 2 holds when f is a rational function.

In the following, we assume that f is a transcendental meromorphic function. Let $a_1, a_2, \ldots, a_s$ be distinct complex numbers. Applying Theorem 4 for Φ and the complex numbers $a_1, a_2, \ldots, a_s$, we have

$$\begin{aligned} \sum_{i=1}^{s} m\Big(r, \frac{1}{\Phi - a_i}\Big) &\le T(r,\Phi) + \overline{N}(r,\Phi) - N\Big(r, \frac{1}{\Phi'}\Big) + o(T(r,\Phi)) \\ &= T(r,\Phi) + \overline{N}(r,f) - N\Big(r, \frac{1}{\Phi'}\Big) + o(T(r,f)) \qquad (4.1) \end{aligned}$$

for all r outside a set E of finite linear measure, where the second equality follows from Lemma 8 and the fact that $\overline{N}(r,\Phi) = \overline{N}(r,f)$.

If (a) holds, then there exists $\nu \in \{2, ..., k\}$ such that $Q_\nu(z)$ has at least one zero, for example $z = \beta_{\nu\eta}$, of order bigger than 1. Therefore, Φ' is divisible by $f^{(\nu)} - \beta_{\nu\eta}$.

If (b) holds, then Φ' is divisible by $f^{(k+1)}$.

In both cases, Φ' is divisible by $Q(f^{(i)})$, where $i \geq 2$ and

$$Q(f^{(i)}) := \begin{cases} f^{(\nu)} - \beta_{\nu\eta} & \text{if (a) holds,} \\ f^{(k+1)} & \text{if (b) holds.} \end{cases}$$

We have

$$N\Big(r, \frac{1}{\Phi'}\Big) \geq N\Big(r, \frac{1}{Q(f^{(i)})}\Big). \tag{4.2}$$

Applying Theorem 1 to differential polynomial $Q(f^{(i)})$ and the set $A = \emptyset$, we have

$$(i-1)\overline{N}(r, f) \leq N\Big(r, \frac{1}{Q(f^{(i)})}\Big) + \epsilon T(r, f),$$

for all $\epsilon > 0$ is an arbitrary small positive constant and for all $r > e$ outside a set E' of logarithmic density zero. Combining this inequality with (4.1), (4.2) and note that $i \geq 2$, we obtain, for all $r > e$ outside a set $E \cup E'$,

$$\sum_{i=1}^{s} m\Big(r, \frac{1}{\Phi - a_i}\Big) \leq T(r, \Phi) + \epsilon T(r, f) + o(T(r, f)).$$

Because $\epsilon > 0$ is arbitrary small constant, we get

$$\sum_{i=1}^{s} \delta(a_i, \Phi) \leq 1.$$

□

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