THE DENSITY OF IMAGES OF UNIT JACOBIAN DETERMINANT POLYNOMIAL MAPS OF $\mathbb{Z}^n$

NGUYEN VAN CHAU

ABSTRACT. Motivated by the Jacobian problem we establish the estimate

$$\#\{z \in F(\mathbb{Z}^n) : \max_{k=1,\dots,n} |z_k| \leq B\} = O(B^{n-1}), \text{ as } B \to +\infty,$$

for possible non-invertible maps $F = (F_1,\dots,F_n) \in \mathbb{Z}[X_1,\dots,X_n]^n$ with $\det DF \equiv 1$, where the implied constant depends on F.

1. INTRODUCTION

Let $n > 1$ be an integer and $X = (X_1,\dots,X_n)$ a n-tube of variables. For $F = (F_1,\dots,F_n) \in \mathbb{Z}[X]^n$, considered as a polynomial map from $\mathbb{C}^n$ into itself, let $JF := \det DF$ the Jacobian determinant of F. Our interest is with the asymptotic behavior of the quantity

$$\mathrm{N}_F(\mathbb{Z},B) := \#\{z \in F(\mathbb{Z}^n) : \max_{k=1,\dots,n} |z_k| \leq B\} \tag{1}$$

as $B \to +\infty$ for the case $JF \equiv 1$. The Jacobian conjecture (ref. [8, 1, 6]), posed by Ott-Heinrich Keller in [8] since 1939 and still open even for $n = 2$, says that if $F \in \mathbb{C}[X]^n$ with $JF \equiv c \neq 0$, then F is invertible. Following Van den Dries and McKenna [9], if this conjecture is false, then for $n \gg 1$ it has counterexamples $F \in \mathbb{Z}[X]^n$ with $JF \equiv 1$. This article is to establish a sharp upper bound of the quantity (1) for such possible counterexamples.

Theorem 1 (Main Theorem). *Suppose $F \in \mathbb{Z}[X]^n$ and $JF \equiv 1$. If F is not invertible, then*

$$\mathrm{N}_F(\mathbb{Z},B) = O(B^{n-1}),$$

where the implied constant depending on F.

The upper bound in the theorem is sharp in sense that if there is a non-invertible map $F \in \mathbb{Z}[X]^n$ with $JF \equiv 1$ and $F(0) = 0$, then $G(X,T) := (\frac{1}{T}F(TX),T) \in \mathbb{Z}[X,T]^{n+1}$, $JG \equiv 1$ and $\mathrm{N}_G(\mathbb{Z},B) \asymp B^n$. The asymptotic estimate follows from Theorem 1 and Schanuel's theorem [14], which asserts that $\#\{z \in \mathbb{Z}^n : \max_{k=1,\dots,n} |z_k| \leq B\} \asymp B^n$.

As an immediate consequence of Theorem 1, we obtain

Corollary 2. *Suppose $F \in \mathbb{Z}[X]^n$ and $JF \equiv 1$. The followings are equivalent:*

a) *F is invertible;*
b) *$\mathrm{N}_F(\mathbb{Z},B) \geq cB^{n-1+\varepsilon}$ as $B \to +\infty$ for some $\varepsilon > 0$ and $c > 0$.*

Institute of Mathematics, Vietnam Academy of Science and Technology, 18 Hoang Quoc Viet, 10307 Hanoi, Vietnam.

The author was partially supported by Vietnam National Foundation for Science and Technology Development (NAFOSTED) grant 101.04-20aa.bbbb, SIMON Foundation, Vietnam Institute for Advanced Study in Mathematics (VIASM) and CCCC.

In view of the well-known reduction results due to Yagzhev [16], Bass, Connel and Wright [1] and Druzkowski [5] and Corollary 1, to prove the Jacobian conjecture for all $n>1$ one may try to find a lower bound in (b) for all $n>1$ and all the cubic linear mappings $F\in\mathbb{Z}[X]^n$,

$$F_i(X)=X_i+(a_{i1}X_1+\cdots+a_{in}X_n)^3,\ a_{ij}\in\mathbb{Z},\ i,j=1,\dots,n,$$

satisfying $JF\equiv 1$.

The case $n=2$ of Theorem 1 can be easy reduced from a pioneer result in [10], which asserts that if $F\in\mathbb{Z}[X_1,X_2]^2$ with $JF\equiv 1$ is not invertible, then the numbers of integer points on the fibers $F_1=k$, $k\in\mathbb{Z}$, are uniformly bounded. In Section 3, instead of Theorem 1 we will prove a version for the case of $F\in \mathrm{O}_K[X]^n$, where $K=\mathbb{Q}$ or any imaginary quadratic number field and O_K is the ring of integral numbers of K (Theorem 7, Section 3). We will try to show the uniform boundedness of the numbers $\#(l\cap F(\mathrm{O}_K^n))$ for generic lines l in $\mathbb{C}^n$ (Lemma 4, Section 2). Our argument here is closed to those in [9, 10], which uses the special property of such fields K: $|r|\geq 1$ for all $0\neq r\in \mathrm{O}_K$.

In pursuing the Jacobian problem it would be useful to consider the following problem.

Question A: Let K be any number field of degree d over $\mathbb{Q}$ and $F\in K[X]^n$ with $JF\equiv 1$. Is

$$\mathrm{N}_F(K,B):=\#\{z\in F(K^n):\mathrm{Height}(z)\leq B\}=O(B^{(n-1)d})\qquad(2)$$

if F is not invertible?

In a contrast to (2), for any $f\in K[X]^n$ of geometric degree >1 the image set $f(K^n)$ is “thin” in K^n in sense of J-P. Serre ([15]), and hence,

$$\mathrm{N}_f(K,B)=O(B^{(n-\frac{1}{2})d}(\log B)^{\gamma}),\ 0<\lambda\leq 1,$$

by Cohen’s theorem on the density of thin sets (ref. [3, 15]). This bound seems to be optimum, for example, $\mathrm{N}_f(K,B)\asymp B^{(n-\frac{1}{2})d}$ for $f(X)=(X_1^2,X_2,\dots,X_n)$.

Uniform Bound Conjecture (see, for example, [2, 13]) asserts that for any number field K and any integer $g>1$ there is a constant $B(K,g)$ depending K and g such that any curve of genus g defined over K can have no more than $B(K,g)$ rational points. This conjecture implies a positive answer to Question A, except for when the preimages $F^{-1}(l)$ of generic lines l in $\mathbb{C}^n$ are irreducible curves of genus $g=0;1$ (Theorem 9, Section 3). Analogous to those in Corollary 2, a complete positive answer to Question A would allow to consider the Jacobian conjecture over number fields as a problem of finding lower bounds of $\mathrm{N}_F(K,B)$.

2. INTEGRAL POINTS ON PREIMAGES OF LINES

In sequels, $\overline{\mathbb{Q}}$ is the field of all algebraic numbers, K is any number field and O_K - the ring of integral numbers of K. Let us denote by $l(u,v)$ the complex line of direction $0\neq v\in\mathbb{C}^n$ passing through a point $u\in\mathbb{C}^n$, $l(u,v)=\{u+tv:t\in\mathbb{C}\}$.

Recall, for any dominant morphism $f:V\longrightarrow W$ of irreducible complex affine varieties V and W, the *bifurcation value set* E_f of f is the smallest closed algebraic subset of W such that the restriction $f:V\setminus f^{-1}(E_f)\longrightarrow W\setminus E_f$ defines a locally trivial smooth fibration. In the situation when $V=W=\mathbb{C}^n$ and $Jf\equiv c\neq 0$, either E_f is empty, i.e. f is invertible, or the bifurcation value set E_f coincides with the non-proper value set of F - the set of all values $a\in\mathbb{C}^n$ such that $f(x_k)\to a$ for some sequences $x_k\to\infty$. In the late case E_f is a hypersurface in $\mathbb{C}^n$ given by $H_f(X)=0$ for a non-constant $H_f\in\mathbb{C}[X]$ (ref. [7]). Let K_f denote the cone of tangents at infinity of E_f, i.e. $K_f=\{v\in\mathbb{C}^n:h_f(v)=0\}$, where h_f is the leading homogeneous of H_f.

Lemma 3. (Main Lemma in [12]) *Suppose $F \in \mathbb{C}[X]^n$ and $JF \equiv 1$, but F is not invertible. Then, there exists a non-constant polynomial $\sigma_F \in \mathbb{C}[U,V]$, $U = (U_1 \ldots, U_n)$ and $V = (V_1, \ldots, V_n)$, such that*

a) *$\sigma_F(\bar{u}, V) \not\equiv 0$ and $\sigma_F(U, \bar{v}) \not\equiv 0$ for $\bar{u} \notin E_F$ and $\bar{v} \notin K_F$, and*
b) *the preimages $F^{-1}(l)$, $l = l(u,v)$ with $u, v \in \mathbb{C}^n$ and $\sigma_F(u,v) \neq 0$, are irreducible affine curves of same genus g_F and number $n_F > 2$ of irreducible branches at infinity.*

Moreover, in the case $F \in K[X]^n$ for a number field K,

c) *$H_F, \sigma_F \in \overline{\mathbb{Q}}[X]$ and*
d) *for every line $l = l(u,v)$, $u, v \in K^n$ and $\sigma_F(u,v) \neq 0$, the preimage $F^{-1}(l)$ has at most finitely many integral points in K^n.*

From now on, we assume that K is $\mathbb{Q}$ or any imaginary quadratic number field. Such fields K can be characterized by the property: $|r| \leq 1$ for all $0 \neq r \in \mathrm{O}_K$. Let

$$\theta_K := \sup_{c \in \mathbb{C}} \#\{r \in \mathrm{O}_K : |c - r| \leq 1\},$$

This number θ_K is at most 7, since in any disk of radium 1 in $\mathbb{C}$ there is no more than seven distinct points s_i such that $|s_i - s_j| \geq 1$. For example, $\theta_{\mathbb{Q}} = 3$, $\theta_{\mathbb{Q}(\mathrm{i})} = 5$ and $\theta_{\mathbb{Q}(\mathrm{i}\sqrt{3})} = 7$.

The remains of this section is to prove the following.

Lemma 4. *In addition to Lemma 3, in the case $F \in \mathrm{O}_K[X]^n$ for $K = \mathbb{Q}$ or any imaginary quadratic number field,*

$$\begin{aligned} \#(l \cap F(\mathrm{O}_K^n)) &\leq \theta_K deg E_F \\ \#(F^{-1}(l) \cap \mathrm{O}_K^n) &\leq \theta_K \deg(F) \deg E_F \end{aligned}$$

for all $l = l(u,v)$, $u, v \in K^n$ with $\sigma_F(u,v) \neq 0$, where $\deg(F)$ is the geometric degree of F.

First, we present a variation of the earlier nice result due to Van den Dries in [9], which asserts that if $F \in \mathbb{Z}[X]^n$ and $JF \equiv 1$, but F is not invertible, then F maps the lattice $\mathbb{Z}^n$ into the narrow neighborhood $E_F + \{z \in \mathbb{C}^n : \max_{i=1,\ldots,n} |z_i| \leq 1\}$ of E_F.

Lemma 5. *Let $K = \mathbb{Q}$ or any imaginary quadratic number field. Suppose $F \in \mathrm{O}_K[X]^n$ and $a \in \mathrm{O}_K^n$ with $JF(a) = 1$. Let $0 < k \leq n$, $W := \{x \in \mathbb{C}^n : x_i = F_i(a), i = k+1, \ldots, n\}$, V denote the unique irreducible component of $F^{-1}(W)$ such that $a \in V$ and $f := F_{|V} : V \longrightarrow W$. Then, either*

a) *there is a point $c \in E_f$ such that $\max_{i=1,\ldots,k} |F_i(a) - c_i| \leq 1$, or*
b) *f is invertible, f^{-1} is a polynomial map of the variables X_is, $i = 1, \ldots, k$, with coefficients in O_K and $f : (V, V \cap \mathrm{O}_K^n) \longrightarrow (W, W \cap \mathrm{O}_K^n)$ is isomorphic.*

Proof. Our argument here is analogous to those in [9, 10], which is based on the implicit function theorem and the fact that power series with coefficients in O_K and with convergent radium > 1 must be polynomials.

Let $a \in \mathrm{O}_K^n$ be as in the statement and put

$$d := \inf_{c \in E_f} \max_{i=1,\ldots,k} |F_i(a) - c_i|.$$

Since E_f is closed, there is a point $c \in E_f$ such that $d = \max_{i=1,\ldots,k} |F_i(a) - c_i|$. So, we have to show only that "$d > 1$" $\Rightarrow (b)$.

Assume that $d > 1$. We may take an $\varepsilon > 0$ so that $d - \varepsilon > 1$. Replacing $F(X)$ by $F(X - a) - F(a)$, if necessary, without a loss of generality we may assume that $F \in \mathrm{O}_K[X]^n$, $F(0) = 0$ and $JF(0) = 1$. Then, by the implicit analytic function theorem there exists a unique local holomorphic inverse $G=(G_1,\dots,G_n)$ of F, defined on an open set U of $x\in\mathbb{C}^n$ with $|x_i|<r$ and $G_i\in \mathrm{O}_K[[X]]$, such that $G(0)=0$.

Let $B:=\{w\in W:|w_i|<d-\varepsilon, i=1,\dots,k\}$, which is a simply connected open subset of $W\setminus E_f$. Let $g=(g_1,\dots,g_n):=G:U\cap B\longrightarrow V$. Then, we have

$$g_i(X_1,\dots,X_k)=G(X_1,\dots,X_k,0,\dots,0),\ i=1,\dots,k,$$

which are power series in $\mathrm{O}_K[[X_1,\dots,X_k]]$.

Since $F:V\setminus f^{-1}(E_f)\longrightarrow W\setminus E_f$ is a unbranched smooth covering and $B\subset W\setminus E_f$, the holomorphic map g can be extended over B and determines a unique smooth lift of f with $g(0)=0$ and $f\circ g=Id_B$. Therefore, the power series g_i are absolutely convergence on the open box B. Since B contains the unit box $\{w\in W:|w_i|\le 1,\ i=1,\dots,k\}$, it follows that g_is are be polynomials of the variables X_i, $i=1,\dots,k$, and with coefficients in O_K. Thus, g is well defined over W and determines a unique inverse of f, as $f\circ g=Id_B$. In results, f gives an isomorphism of the pairs $(V,V\cap\mathrm{O}_K^n)$ and $(W,W\cap\mathrm{O}_K^n)$. □ □

Lemma 6. *Let $K=\mathbb{Q}$ or any imaginary quadratic number field. Let $F\in\mathrm{O}_K[X]^n$ with $JF\equiv 1$. Let $l:=l(u,v)$ be a line in $\mathbb{C}^n$, $u,v\in\mathrm{O}_K^n$. If $F^{-1}(l)\cap\mathrm{O}_K^n\neq\emptyset$ and $JF\equiv 1$ on $F^{-1}(l)$, then, either*

a) *$\#(l\cap F(\mathrm{O}_K^n))\le\theta_K\deg E_F$, or*
b) *$F^{-1}(l)$ has a component V isomorphic to $\mathbb{C}$ such that*

$$f:=F_{|V}:(V,V\cap\mathrm{O}_K^n)\longrightarrow(l,l\cap\mathrm{O}_K^n)$$

is isomorphic.

Proof. Given $l=l(u,v)$ with $u\in\mathrm{O}_K^n$ and $0\neq v\in\mathrm{O}_K^n$. We may assume that v is primitive, i.e. $\gcd(v_1,\dots,v_n)=1$. We need the following elementary fact:

Claim 1. *For any number field k and for any two non-zero primitive vectors $v,w\in\mathrm{O}_K^n$ there is a linear transform of O_k^n given by a matrix $T\in SL(n,\mathrm{O}_k)$, such that $T(v)=w$.*

(See, for example, in the proof of Theorem A' in [12]).

By this fact we can choose a matrix $T\in SL(n,\mathrm{O}_K)$ such that $T(v)=e$, $e:=(0,...,0,1)$. Replacing $F(X)$ by $T(F(X))$ and u by $T(u)$, we may assume that $l=l(u,e)$. Applying Lemma 5 to the case of $W=l$ and each of points $a\in F^{-1}(l)\cap\mathrm{O}_K^n$, either we obtain (b) or we have that

$$l\cap F(\mathrm{O}_K^n)\subset E_F+\{te:t\in\mathbb{C},\ |t|\le 1\}.$$

The late means that

$$l\cap F(\mathrm{O}_K^n)\subset\bigcup_{c\in l\cap E_F}(\{c+te:t\in\mathbb{C},\ |t|\le 1\}\cap\mathrm{O}_K^n)$$

that ensures $\#(l\cap F(\mathrm{O}_K^n))\le\theta_K\deg E_F$. □ □

Proof of Lemma 4. By Lemma 3 (b), for any given line $l=l(u,v)$, $u,v\in\mathrm{O}_K^n$ with $\sigma_F(u,v)\neq 0$, the preimage $F^{-1}(l)$ is an irreducible affine curves having more than two irreducible branches at infinity. In particular, $F^{-1}(l)$ has no any component diffeomorphic to $\mathbb{C}$. The desired upper bounds then follows from Lemma 6. □

3. Proof of Main Theorem

Instead of Theorem 1 we will prove the following version for when $K = \mathbb{Q}$ or any imaginary quadratic number field. For convenience, in sequels we denote

$$\mathrm{N}(\Omega, B) := \#\{z \in \Omega : \mathrm{Height}(z) \leq B\}$$

for subsets Ω of K^n.

Theorem 7. *Let $K = \mathbb{Q}$ or any imaginary quadratic number field. Suppose $F \in \mathrm{O}_K[X]^n$ and $JF \equiv 1$. If F is not invertible, then*

$$\mathrm{N}_F(\mathrm{O}_K, B) = O(B^{(n-1)d})$$

, where d is the degree of K over $\mathbb{Q}$.

Proof. Let F be as in the statement. Replacing $F(X)$ by $T(F(X))$ for some suitable linear transform $T \in \mathrm{GL}(n, \mathbb{Z})$, if necessary, without a loss of generality we may assume that the polynomial $H_F \in \overline{\mathbb{Q}}[X]$ defining the the bifurcation value set E_F satisfies

$$\deg_{X_i} H_F = \deg H_F,\ i = 1, \ldots, n. \tag{3}$$

Let $\sigma_F \in \overline{\mathbb{Q}}[U, V]$ be as in Lemma 3. We put

$$\begin{aligned}
&e_i := (e_{i1}, \ldots, e_{in}),\ e_{ij} = 0 \text{ for } j \neq i \text{ and } e_{ii} = 1;\\
&E_i := \{x \in \mathbb{C}^n : x_i = 0\};\\
&M_i := \{l(u, e_i) : u \in E_i, \sigma_F(u, e_i) \neq 0\}.
\end{aligned}$$

for $i = 1, \ldots, n$. By definitions the condition (3) ensures that $E_i \not\subset E_F$ and $e_i \notin K_F$. Therefore, by Lemma 3 (a) every polynomial $\sigma_F(U, e_i)$ does not identify zero on E_i. It follows that $(\mathbb{C}^n \setminus M_i)$s are closed algebraic proper subsets of $\mathbb{C}^n$.

Let S denote the intersection of all $(\mathbb{C}^n \setminus M_i)$, $i = 1, \ldots, n$. We will show that $S = \emptyset$. This allows us to cover $\mathbb{C}^n$ by the sets M_i,

$$\mathbb{C}^n = \bigcup_{i=1}^{n} M_i. \tag{4}$$

Indeed, if $S \neq \emptyset$, each of irreducible components of S is an affine variety of dimension at most $n-1$. Moreover, by definitions S contains all lines $l(z, e_i)$ passing through $z \in S$. So, the tangent space to S at each of smooth points of S must contain all n independent vectors e_i. This is impossible.

Now, by (4) the image set $F(\mathrm{O}_K^n)$ can be covered by the sets $M_i \cap F(\mathrm{O}_K^n)$, $i = 1, \ldots, n$. This allows us to estimate

$$\mathrm{N}(F(\mathrm{O}_K^n), B) \leq \sum_{i=1}^{n} \mathrm{N}(M_i \cap F(\mathrm{O}_K^n), B).$$

Hence, to get the desired upper bound of $\mathrm{N}(F(\mathrm{O}_K^n), B)$, it sufficients to establish

$$\mathrm{N}(M_i \cap F(\mathrm{O}_K^n), B) = O(B^{(n-1)d}).$$

To do it, observe that $l(u, e_i) \cap F(\mathrm{O}_K^n) \neq \emptyset$ if and only if $u \in E_i \cap \mathrm{O}_K^n$. So, by definitions we can represent

$$M_i \cap F(\mathrm{O}_K^n) = \bigcup_{\{u \in E_i \cap \mathrm{O}_K^n : \sigma_F(u, e_i) \neq 0\}} (l(u, e_i) \cap F(\mathrm{O}_K^n)).$$

Then, we have

$$\begin{aligned}
\mathrm{N}(M_i \cap F(\mathrm{O}_K^n), B) &= \mathrm{N}\Big(\bigcup_{\{u \in E_i \cap \mathrm{O}_K^n : \sigma_F(u,e_i) \neq 0\}} l(u,e_i) \cap F(\mathrm{O}_K^n), B\Big) \\
&\leq \theta_K \deg E_F \#\{u \in E_i \cap \mathrm{O}_K^n : \sigma_F(u,e_i) \neq 0, \max_k |u_k| \leq B\} \text{ (by Lemma 4)} \\
&\leq \theta_K \deg E_F \mathrm{N}(E_i \cap \mathrm{O}_K^n, B) \\
&\sim c\theta_K \deg E_F B^{(n-1)d} \text{ as } B \to +\infty \quad \text{(by Schanuel's theorem).}
\end{aligned}$$

Thus, we get

$$\mathrm{N}(M_i \cap F(\mathrm{O}_K^n), B) = O(B^{(n-1)d}).$$

□ □

***

Let us to conclude the paper by some discussions on the asymptotic behavior of the counting function $\mathrm{N}_F(K,B)$ for possible counterexamples $F \in K[X]^n$ to the Jacobian conjecture over an arbitrary number field K. Assume that $F \in K[X]^n$ is such a possible counterexample. Without a loss of generality we can assume $JF \equiv 1$.

i) Looking at the arguments in the proof of Theorem 7, the assumption that $K = \mathbb{Q}$ or any imaginary quadratic number field is used only to have the uniform bounds in Lemma 6. For any arbitrary number field K, the same arguments allows us to obtain the inequality

$$\mathrm{N}(F(K^n), B) \leq nc_F(B)\mathrm{N}(K^{n-1}, B), \tag{5}$$

where

$$c_F(B) := \sup_{l \in \Sigma_F} \mathrm{N}(l \cap F(K^n), B) \tag{6}$$

and

$$\Sigma_F := \{l = l(u,v) : u, v \in K^n,\ \sigma_F(u,v) \neq 0\}. \tag{7}$$

ii) Lemma 3 (b) says that the preimages $F^{-1}(l)$, $l \in \Sigma_F$, are irreducible affine curves having same genus g_F and same number $n_F > 2$ of irreducible branches at infinity. As one of most simple parts of the Jacobian conjecture, it is guessed

Conjecture 8. *There does not exist counterexamples $F \in \mathbb{C}[X]^n$ to the Jacobian conjecture such that $g_F = 0; 1$.*

In our knowledge this conjecture is still open, except the case of $n = 2$ and $g_F = 0$ (see [11]).

Finally, as mentioned in Section 1 we present here a conditionally positive answer to Question A.

Theorem 9. *Let K be any number field of degree d over $\mathbb{Q}$ and let $F \in K[X]^n$ with $JF \equiv 1$. Assuming Uniform Bound Conjecture and Conjecture 1, if F is not invertible, then*

$$\mathrm{N}_F(K,B) = O(B^{(n-1)d}).$$

Proof. Let $c_F(B)$ and Σ_F be as in (6-7). Let

$$C_F := \sup_{l\in\Sigma_F} \#(F^{-1}(l)\cap K^n) \leq +\infty.$$

Obviously, $c_F(B) \leq C_F$. Conjecture 1 ensures $g_F > 1$. Hence, by Uniform bound Conjecture $C_F \leq B(g_F, K)$ for a constant $B(g_F, K)$. So, from (5) and Schanuel's theorem we can deduce that

$$\mathrm{N}(F(K^n), B) \leq nB(g_F, K)\mathrm{N}(K^{n-1}, B) \asymp B^{(n-1)d}.$$

This means $\mathrm{N}_F(K, B) = O(B^{(n-1)d})$. □

REFERENCES

[1] H. Bass, E. Connell, D. Wright, *The Jacobian conjecture: reduction of degree and formal expansion of the inverse*, Bull. Amer. Math. Soc. (N.S.) **7** (1982), 287–330.

[2] L. Caporaso, J. Harris and M. Barry, *Uniformity of rational points*, J. Amer. Math. Soc. **10** (1997), no. 1, 1–35.

[3] S.D. Cohen, *The distribution of Galois groups and Hilbert's irreducibility theorem*, Proc. London Math. Soc. **43** (1981), 227–250.

[4] E. Connell, L. van den Dries, *Injective polynomial maps and the Jacobian Conjecture*, Journal of Pure and Applied Algebra **28** (1983), 235–239.

[5] L. M. Drużkowski, *An effective approach to Keller's Jacobian Conjecture*, Math. Ann. **264** (1983), 303–313.

[6] A. Van den Essen, Polynomial automorphisms and the Jacobian conjecture, Progress in Mathematics **190**, Birkhauser, Basel, 2000.

[7] Z. Jelonek, *The set of points at which a polynomial map is not proper*, Ann. Polon. Math. **LVIII.3** (1993), 259–266.

[8] O. Keller, *Ganze Cremona-Transformatione*, Monatsh. Math. Phys. **47** (1939), 299–306.

[9] K. McKenna, L. van den Dries, *Surjective polynomial maps, and a remark on the Jacobian problem*, Manuscripta Mathematica **67** (1) (1990), 1–15.

[10] Nguyen Van Chau, *Integer points on a curve and the plane Jacobian problem*, Ann. Polon. Math. **88** (2006), no. 1, 53–58.

[11] Nguyen Van Chau, *Jacobian pairs of two rational polynomials are automorphisms*, Vietnam Journal of Mathematics **42** (3), 401–406.

[12] Nguyen Van Chau, *Jacobian conjecture as a problem on integral points on affine curves*, Preprint, To appear.

[13] B. Poonen, *Uniform boundedness of rational points and preperiodic points*, preprint

[14] S. H. Schanuel, *Heights in number fields*, Bull. Soc. Math. France **107** (1979), 433–449.

[15] J-P. Serre, Lectures on the Mordell-Weil theorem, 3rd edition, Springer Fachmedien Wiesbaden, 1997.

[16] A.V. Yagzhev, *On Keller's problem* (in Russian). Sibirsk. Mat. Zh. **21** (1980), 141–150.

E-mail address: nvchau@math.ac.vn