Finding Shortest θ-Gentle Paths on Polyhedral Terrains by the Method of Multiple Shooting

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Abstract

The problem of finding shortest θ-gentle paths can be stated as follows: given a triangulated terrain $\mathcal{T}$, two points $p, q \in \mathcal{T}$, and a parameter $\theta, (0 < \theta \leq \pi/2)$, finding shortest θ-gentle paths joining p and q on $\mathcal{T}$, which are shortest such that the slope of the paths at any point does not exceed θ. The special case $\theta = \pi/2$ was considered by M. Sharir and A. Schorr in *M. Sharir and A. Schorr, On shortest paths in polyhedral spaces. SIAM Journal on Computing, **15**(1), pp. 193–215 (1986).*

In this paper, the method of multiple shooting that consists of three factors is applied for approximately computing such shortest θ-gentle paths on triangulated terrains. These factors are terrain partition, straightness condition, and the update of shooting points. We also show that if the straightness condition is satisfied at all shooting points, then a locally shortest θ-gentle path is obtained. Otherwise, an approximately shortest θ-gentle path is obtained. Some advantages of the method of multiple shooting such as it does not rely on graph tools on the entire surface and the numbers of triangles in sequences of adjacent triangles are not too large (they do not exceed the number of faces of the terrain), are shown. The algorithm is implemented in C++ using CGAL and Open GL.

Keywords: approximate algorithm; method of multiple shooting; shortest path; shortest gentle path; straightest geodesic; terrain.

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1 Introduction

A variant of the shortest path problem is the shortest θ-gentle path problem (SGP problem for short): given two points p and q on a triangulated terrain $\mathcal{T}$, a path joining p and q on $\mathcal{T}$ is θ-gentle if the following slope constraint holds: the slope of the path at any point does not exceed a given positive number θ ($0 < \theta \leq \pi/2$), we need to find shortest θ-gentle paths joining p and q on $\mathcal{T}$. This problem can be seen as a generalization of traditional problem of finding shortest paths joining two points on a terrain when $\theta = \pi/2$, that is a well-studied problem in computational geometry.

Applications of slope-constrained shortest path problems appear in some fields. For example, when we design mobile robots for traversing on non-planar surfaces or we ski down a mountain and avoid a too steep path. In such cases, a path which is too steep should be replaced by a "zig-zag line" satisfying the slope constraint. In train transport, the railroads in Linz, Austria with the slope of 11.6% and the one in Tram 28 in Lisbon, Portugal which has a 14.5% grade are considered to be the steepest railroads in the world.

The case $\theta = \pi/2$ was considered by M. Sharir and A. Schorr [18]. Unfortunately, known star unfolding technique of Agarwal et al. [2], the modified Chen and Han's sequence tree [9] and the sequence of edges that the shortest path goes through [18] do not work for such a SGP problem ($0 < \theta \leq \pi/2$). Some special cases of the SGP problem have been investigated by Ahmed, Lubiw, and Maheshwari (see [3]). Notice that [4] shows the problem of minimizing the total number of bends in an SGP being NP-hard. To date, no exact algorithms have been given. There are several approaches for finding approximately such shortest θ-gentle paths. Most of these algorithms are based on graph tools. The algorithm of Ahmed, Lubiw, Maheshwari [3] models the problem as a graph whose nodes are Steiner points added along the edges of the triangulated terrain. Nöllenburg and Sautter [17] present an $(1 + \epsilon)$-approximation algorithm based on determining a new norm for finding approximately shortest θ-gentle paths on a sequence of adjacent triangles. The algorithms have not been implemented, and thus it is not clear how practical it is. Liu and Wong [13] propose an algorithm to solve approximately SGP problem using triangulated sequence trees. Unlike the Liu and Wong's method for computing shortest θ-gentle paths on terrain, which also generate a set of sequences of adjacent triangles of entire terrain, in this paper we will present a new approach, namely the method of multiple shooting, for computing shortest θ-gentle paths on sequences of adjacent triangles of sub-terrains with fewer number of triangles and implemente it on computer and therefore until now there are only two methods (i.e., Liu and Wong's method and our method of multiple shooting) having their implementations on computer. We also answer whether a shortest θ-gentle paths joining two given points exist or not.

Let us recall some previous works related to the method of multiple shooting. For solving ordinary differential equation (ODE) boundary value problems, a numerical method called multiple shooting method is introduced. A variant of multiple shooting method called the direct multiple shooting method is proposed by Bock and Plitt [8] to give an approximate solution of optimal control problems. Based on the idea of the multiple shooting method, multiple shooting approach in computational geometry has been posed in 2013 (see [6]) by An, Hai, and Hoai for finding approximately shortest paths joining two points p and q in a simple polygon. They expand the multiple shooting approach for solving approximately shortest path problem on convex polytopes in 3D without graph tools on the entire surface (see [11]). The method is also used successfully by An and Trang [7] to find shortest descending paths joining two points p and q on a triangulated terrain $\mathcal{T}$. It consists of three following factors:

(f1) Partition of the domain $\mathcal{T}$ (polygon, polytope) into subdomains (subpolygons, subpolytopes) is created by cutting slices between two points p and q. A subdomain is a sequence of adjacent triangles, that is deliberately created by two adjacent cutting slices. At each intersection between a slice and $\mathcal{T}$ we take a point, called a shooting point;

(f2) Consider a path on $\mathcal{T}$ joining two points p and q , formed by shooting points. Collinear / straightness condition is established at shooting points;

(f3) The algorithm enforces (f2) at all shooting points. Otherwise, an update of shooting points makes the paths joining two points p and q and formed by shooting points better.

In this paper, the factors (f1), (f2) and (f3), respectively are constructed in details in Sect. 4.2, Sect. 4.3, and Sect. 4.4, respectively. We show that if the straightness condition in (f2) is satisfied at all shooting points, then a locally shortest θ-gentle path is obtained (Prop. 6). Otherwise, an approximately shortest θ-gentle path is obtained (Theorem 1). In addition, all previous papers dealing with SGP problem [3, 13, 17] have not yet stated whether shortest θ-gentle paths joining two given points exist or not. The question is answered in Prop. 2 in Sect. 3.

The rest of the paper is organized as follows. Sect. 2 recalls preliminary notions. In Sect. 3 we present some properties of θ-gentle paths and shortest θ-gentle paths joining two points. Then we use the multiple shooting approach above to solve the SGP problem. (f1)-(f3) are specified and an algorithm based on (f1)-(f3) is introduced in Sect. 4 (Algorithm 1). The algorithm is implemented in C++ using CGAL and numerical results are given and visualized in Sect. 5 to describe how our method works (Examples 1-2). Some advantages of the multiple shooting approach such as it does not rely on graph tools on the entire surface and the numbers of triangles in sequences of adjacent triangles are not too large (they do not exceed the number of faces of the terrain), are shown in this section. Geometrical properties and their poofs for the correctness of the algorithm are arranged in Appendix 8.

2 Preliminaries

We recall some definitions and properties. For any points p, q in space, we denote $[p, q] := \{(1 - \lambda)p + \lambda q : 0 \leq \lambda \leq 1\}$, $(p, q) := \{(1 - \lambda)p + \lambda q : 0 < \lambda < 1\}$. For all $x \in (p, q)$, x is called an *interior point* of $[p, q]$.

A *terrain*, denoted by $\mathcal{T}$, is the graph of a continuous function $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ that assigns every point on a horizontal plane to an elevation. A terrain is usually represented by the Triangular Irregular Network (TIN) model consisting of a number of disjoint triangles (see [10]). Such a terrain is also called a *triangulated terrain*. A terrain is thus a polyhedral surface in $\mathbb{R}^3$ in which every vertical line intersects the surface at most once. Each face is a triangle having three vertices, connected by three edges. Each vertex is a point in a three-dimensional space. If an edge is located at the boundary of a triangulated terrain, it is owned by only one triangle.

Definition 1 ([5])**.** On a triangulated terrain $\mathcal{T}$, a *sequence of adjacent triangles*, denoted by $\mathcal{F}$, is defined by a list $(f_1, f_2, \ldots, f_m)$ of adjacent triangles $f_1, f_2, \ldots, f_m$ of $\mathcal{T}$, where f_i and f_{i+1} share a common edge e_i, for $i = 1, 2, \ldots, m - 1$.

Let p and q be two points on $\mathcal{F}$. If p is in the first triangle and q is in the final triangle, then $\mathcal{F}$ is called a sequence of adjacent triangles *joining* p *and* q.

Definition 2 ([15])**.** Let $[t_0, t_1]$ be a subset of $\mathbb{R}$. A *path* on the triangulated terrain $\mathcal{T}$ is a continuous map $\gamma : [t_0, t_1] \to \mathcal{T}$.

If $\gamma(t_0) = p, \gamma(t_1) = q$ (where $p, q \in \mathcal{T}$), then we say that γ *joins* p *and* q *on* $\mathcal{T}$. If $\gamma : [t_0, t_1] \to \mathcal{F}$ and $\gamma(t_0) = p, \gamma(t_1) = q$, (where $p, q \in \mathcal{F}$), then we say that γ is *a path joining* p *and* q *along the sequence of adjacent triangles* $\mathcal{F}$

We also call the image $\gamma([t_0, t_1])$ to be the path γ. In this paper, we only consider paths which are piecewise differentiable functions, i.e. diferentiable except for a finite number of points. The *slope of a segment* $[a, b]$ is the angle (in radian) between the line joining a, b and the horizontal plane, denoted by $\mathrm{sl}([a, b])$. Here, the angle between a line and a plane is defined by the angle between the line and its projection onto this plane. Consequently, the angle is in $[0, \pi/2]$. The *slope of a path* γ *at the point* $a \in \gamma$ is the angle (in radian) between the tangent to γ at p and the horizontal plane, denoted by $\mathrm{sl}_\gamma(a)$. We only consider the tangent which lies on triangles containing a. If such a tangent does not exist, the slope is defined to be the maximum value of the angles between one-sided tangents at a and the horizontal plane. Because angles between a line and a plane do not exceed $\pi/2$, $0 \le \mathrm{sl}_\gamma(a) \le \pi/2$. If γ is a polyline, namely it only includes segments γ_i, $i = 1, 2, \ldots, k$, then $\mathrm{sl}(\gamma) = \max\limits_{1 \le i \le k} \mathrm{sl}(\gamma_i)$.

We assume through out the paper that θ is the slope parameter ($0 < \theta \le \pi/2$) and p, q are two points on the triangulated terrain $\mathcal{T}$. Let γ be a path joining p and q on $\mathcal{T}$.

Definition 3 ([13])**.** The path γ is a θ*-gentle path* joining p and q on $\mathcal{T}$ if it satisfies the slope constraint at all its points (i.e., $\mathrm{sl}_\gamma(a) \le \theta$ for any $a \in \gamma$).

Definition 4 ([13])**.** The path γ is a *shortest* θ*-gentle path* joining p and q on $\mathcal{T}$, denoted by $\mathrm{SGP}_\mathcal{T}(p, q|\theta)$ or simply $\mathrm{SGP}_\mathcal{T}(p, q)$, if it is a θ-gentle path joining p and q on $\mathcal{T}$ and there is no other θ-gentle path γ' joining p and q on $\mathcal{T}$ such that $l(\gamma') < l(\gamma)$, where $l(.)$ denotes an arclength function.

A path joining p and q on $\mathcal{T}$ is called a *locally shortest* θ*-gentle path* if there exists a neighbor $U \subset \mathbb{R}^3$ of the path such that it is a shortest θ-gentle path joining p and q in $U \cap \mathcal{T}$.

The SGP problem is defined as follows: given a triangulated terrain $\mathcal{T}$, a source point p, a destination point q in $\mathcal{T}$, and a slope value θ, finding a shortest θ-gentle path joining p and q on $\mathcal{T}$.

3 Some Geometry Properties of the Shortest Gentle Paths

3.1 Reachability

In this section, we first recall the concept of reachability, which is given in [13] for the existence of θ-gentle paths. We then show when a vertex of a triangulated terrain is θ-unreachable.

Definition 5 ([13])**.** Given two distinct points a and a' on a terrain, a is said to be *reachable from* a' if and only if there exists a θ-gentle path from a' to a.

A point a is said to be θ*-reachable* if there exists another point a' of the terrain such that a is reachable from a'. Otherwise, a is said to be θ*-unreachable*.

By Lemma 5 [13], a point on a triangulated terrain $\mathcal{T}$ that is θ-unreachable, is a vertex of $\mathcal{T}$. To deal with the SPG problem, we thus only need to check whether each vertex (instead of all possible points on $\mathcal{T}$) is θ-unreachable or not. If the source point p and the destination point q are θ-reachable, then there is a θ-gentle path joining p and q on $\mathcal{T}$ (see Lemma 6 in [13]). This means that the existence of θ-gentle paths depends on the source point p and the destination point q. Therefore, if one of them, say p, is θ-unreachable, we deduce that p is a vertex. In this case, an approximate solution is obtained. We choose a point p' such that p' is not a vertex and the distance between p' and p is at most ε, where ε is a given positive number, then compute the θ-gentle path joining p' and q on the terrain.

A θ-unreachable vertex is also known as a *sharp* vertex defined in [3]. Additionally, in a triangle f, there are some vertices which are not θ-reachable from any points in f but can be θ-reachable from any point on another triangle. Such vertices are said to be *locally sharp in* f (see [3]).

To characterize a θ-unreachable vertex, we first introduce concepts of θ-cone. Take $a \in \mathbb{R}^3$, we construct a θ-*cone* whose vertex is a is as follows: let Δ be the line passing through a and being perpendicular to the horizontal plane. A θ-cone is formed by a set of lines passing through a and creating with Δ an angle not being exceed $\pi/2 - \theta$. Lines which create with Δ an angle equal to $\pi/2 - \theta$ are called generating lines of θ-cone. A θ-cone consists of two θ-*convex cones*, upper and lower ones are incident to the vertex a (see Fig. 1).

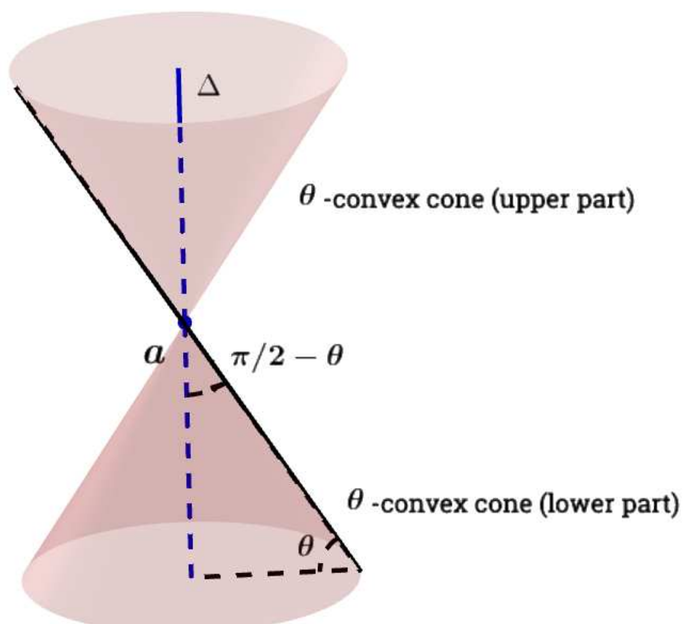

Figure 1: a θ-cone consists of two θ-convex cones

Proposition 1. *A vertex a of a triangulated terrain is θ-unreachable iff all its adjacent triangles completely lie in (i.e., in and not on the generating lines) one of the two θ-convex cones of the θ-cone whose vertex is a.*

The proof of Prop. 1 is given in the appendix (Sect. 8). Prop. 1 indicates that in order to check θ-unreachability of a vertex, we can construct a θ-cone having the vertex is a and then check whether all triangles adjacent to a lie completely in one of the two θ-convex cones of the θ-cone or not (see Fig. 2).

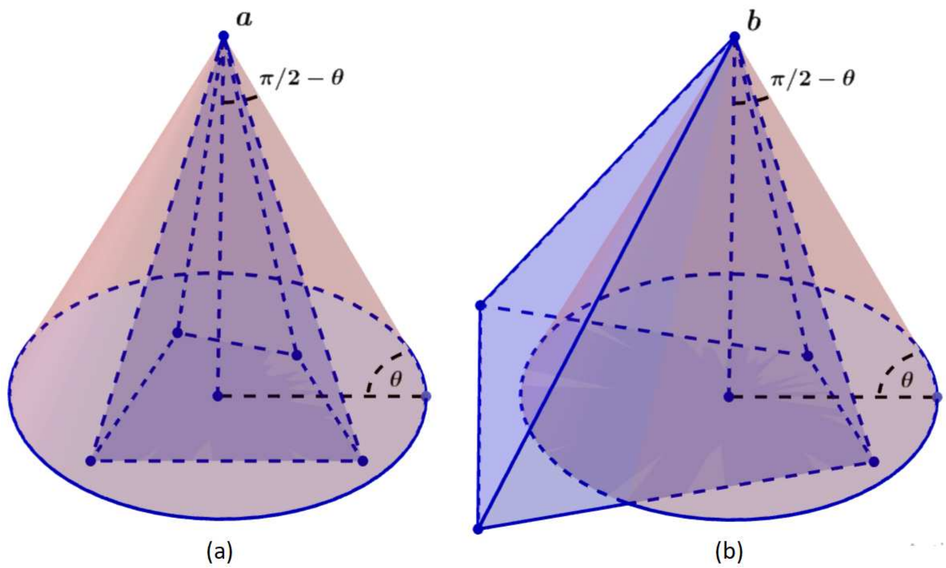

Figure 2: a) a is θ-unreachable b) b is θ-reachable

3.2 Steep and Flat Regions

We now present the concepts of steep and flat regions given in [3, 13, 17] for the existence of shortest θ-gentle paths joining two points on a triangle. The existence of shortest θ-gentle paths joining two points on a sequence of adjacent triangles of a triangulated terrain is then given (see Prop. 2).

Let a be a point of a triangle f on $\mathcal{T}$, and a slope parameter θ. The intersection between the θ-cone whose vertex is a and f forms a so-called *steep region* of a on f, denoted by $\mathrm{SR}_f(a)$. The steep region $\mathrm{SR}_f(a)$ can be $\{a\}$, a triangle, a polygon or entire triangle f (see Fig. 3).

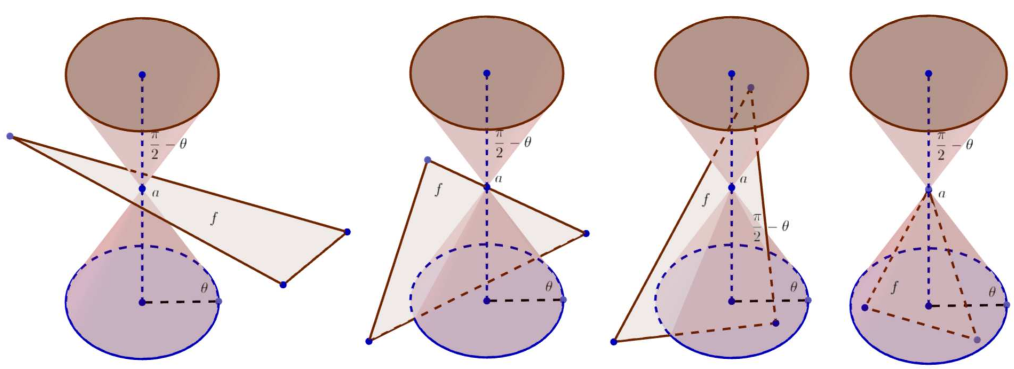

Figure 3: Steep regions in the triangle f.

Let a and b be θ-reachable points on the same triangle f of $\mathcal{T}$. If $b \notin \text{SR}_f(a)$, then $\text{sl}([a, b]) \leq \theta$ (i.e., $[a, b]$ is not too steep) and $\text{SGP}_f(a, b)$ is $[a, b]$. Otherwise, $\text{sl}([a, b]) > \theta$ (i.e., $[a, b]$ is too steep), we construct a polyline in $\text{SR}_f(a)$ starting from b to a such that the slope of segments of the polyline is equal to θ (see Fig. 4). Such a zig-zag line is called an *adjusted path* of $[a, b]$, denoted by $\text{adj}[a, b]$. Liu and Wong [13] shows that in such case, $\text{SGP}_f(a, b)$ is $\text{adj}[a, b]$ and its length is $\dfrac{|z(a) - z(b)|}{\sin\theta}$,

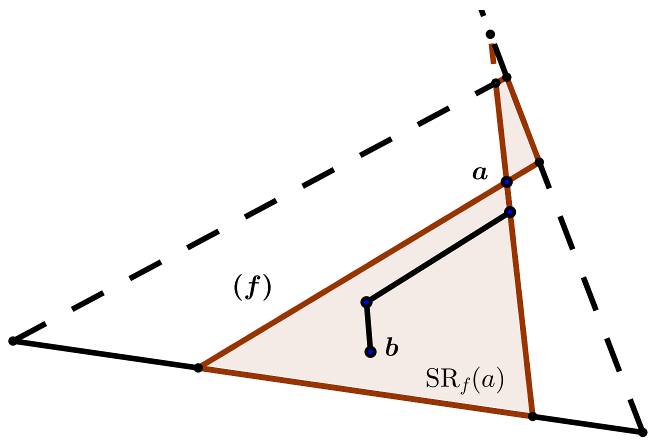

Figure 4: An adjusted path of $[a, b]$ when $[a, b]$ is too steep

where $z(a), z(b)$ are the z-coordinates of a, b. Hence, in both cases ($[a, b]$ is too steep or not), the length of $\text{SGP}_f(a, b)$ is given by:

$$l\left(\text{SGP}_f(a, b)\right) = \max\left\{\{l([a, b]), \frac{|z(a) - z(b)|}{\sin\theta}\right\}. \tag{1}$$

Moreover, Lemma 3 in [17] shows that the following defines a norm on $\mathbb{R}^3$:

$$\|a\|_s = \max\left\{\|a\|_2, \frac{|z(a)|}{\sin\theta}\right\}, \tag{2}$$

where a is any point in $\mathbb{R}^3$, $\|.\|_2$ is the Euclidean norm.

Combining (1) and (2) yields

$$\begin{aligned} \|a - b\|_s &= \max\left\{\|a - b\|_2, \frac{|z(a) - z(b)|}{\sin\theta}\right\} \\ &= \max\left\{l([a, b]), l(\text{adj}[a, b])\right\}. \end{aligned} \tag{3}$$

In the general case when a, b are two arbitrary points of a triangulated terrain $\mathcal{T}$, if $\text{SGP}_\mathcal{T}(a, b)$ exists, it is a polyline (see [3, 13]). We also obtain the triangle inequality for shortest θ-gentle paths: $l\left(\text{SGP}_\mathcal{T}(a, b)\right) \leq l\left(\text{SGP}_\mathcal{T}(a, c)\right) + l\left(\text{SGP}_\mathcal{T}(c, b)\right)$, where $a, b, c \in \mathcal{T}$ and suppose that $\text{SGP}_\mathcal{T}(a, b), \text{SGP}_\mathcal{T}(b, c), \text{SGP}_\mathcal{T}(c, a)$ exist. In the triangle inequality, $\mathcal{T}$ can be replaced by a triangle or a sequence of adjacent triangles.

Since a zig-zag line starting from b in $\text{SR}_f(a)$ in which the slope of segments of the zig-zag line is equal to θ, can be built by several ways, a number of adjusted paths of $[a, b]$ in a triangle whose the same length should be constructed (see Fig. 4). Hence, a shortest θ-gentle path joining two given points in a triangle can be determined in different ways. As a consequence, the shortest θ-gentle path on a triangle, a sequence of adjacent triangles and all the terrain may be not unique if it exists.

We assume $\Gamma_{\mathcal{F}}$ to be the set of all θ-gentle paths joining p and q on a sequence $\mathcal{F}$ of adjacent triangles and $\Gamma_{\mathcal{F}} \neq \emptyset$. Let $m = \inf\{l(\gamma) : \gamma \in \Gamma_{\mathcal{F}}\}$. The question is whether a shortest θ-gentle path joining p and q on $\mathcal{F}$ exists (i.e., whether or not there is a θ-gentle path γ_0 satisfying $l(\gamma_0) = m$). The answer is given in Prop. 2 below:

Proposition 2. *Given $p, q \in \mathcal{T}$, let $\mathcal{F}$ be a sequence of adjacent triangles joining p and q on $\mathcal{T}$, such that all vertices in $\mathcal{F}$ are θ-reachable from some point in $\mathcal{F}$. Let $\Gamma_{\mathcal{F}}$ denote the set of all θ-gentle paths joining p and q along $\mathcal{F}$ and $\Gamma_{\mathcal{F}} \neq \emptyset$. Then, there exists γ_0 in $\Gamma_{\mathcal{F}}$ such that $l(\gamma_0) \leq l(\gamma)$, for all $\gamma \in \mathcal{F}$. The path γ_0 is then a shortest θ-gentle path joining p and q along $\mathcal{F}$.*

The proof of Prop. 2 is given in the appendix (Sect. 8).

3.3 Pseudo Paths of Shortest θ-gentle Paths

Let $\mathcal{F}$ be a sequence of m adjacent triangles joining two points p and q on $\mathcal{T}$ and sharing common edges $e_1, e_2, \ldots, e_{m-1}$. We denote by x_i the intersection of $\text{SGP}_{\mathcal{F}}(p, q)$ and e_i, for $i = 1, 2, \ldots, m-1$. Then $\text{SGP}_{\mathcal{F}}(p, q)$ is the union of sub-paths $\text{SGP}_{f_i}(x_i, x_{i+1})$ in each triangle and these sub-paths are $[x_i, x_{i+1}]$ or the adjusted path of $[x_i, x_{i+1}]$, depending on the slope of $[x_i, x_{i+1}]$, $i = 0, 1 \ldots, m-1$, where $x_0 = p, x_m = q$ (see Fig. 5(a)).

Remark 1. Combining (1), (2), and (3) yields

$$l\left(\text{SGP}_{\mathcal{F}}(p, q)\right) = \|x_0 - x_1\|_s + \|x_1 - x_2\|_s + \ldots + \|x_{m-1} - x_m\|_s. \tag{4}$$

The polyline formed by consecutively connecting $x_0, x_1, \ldots, x_m$ is called a *pseudo path* of $\text{SGP}_{\mathcal{F}}(p, q)$, denoted by $\text{PSGP}_{\mathcal{F}}(p, q)$ (see Fig. 5(b)). i.e.,

$$\text{PSGP}_{\mathcal{F}}(p, q) = [x_0, x_1] \cup [x_1, x_2] \cup \ldots \cup [x_{m-1}, x_m].$$

We denote the right-hand side of Eq. (4) by $lp(\text{PSGP}_{\mathcal{F}}(p, q))$, then we can rewrite:

$$lp\left(\text{PSGP}_{\mathcal{F}}(p, q)\right) = l\left(\text{SGP}_{\mathcal{F}}(p, q)\right).$$

A shortest θ-gentle path joining p and q along a sequence of adjacent triangles can be reconstructed from a pseudo path joining p and q along the sequence of adjacent triangles. We can exactly find the length of shortest θ-gentle paths by their pseudo paths without constructing the orbits of these shortest θ-gentle paths. In each triangle, a sub-path of these shortest θ-gentle paths can be formed as the union of zig-zag lines being shown in Figure 4. We can also use an adjusting algorithm presented in [13] in oder to obtain shortest θ-gentle paths from a pseudo path in a triangle.

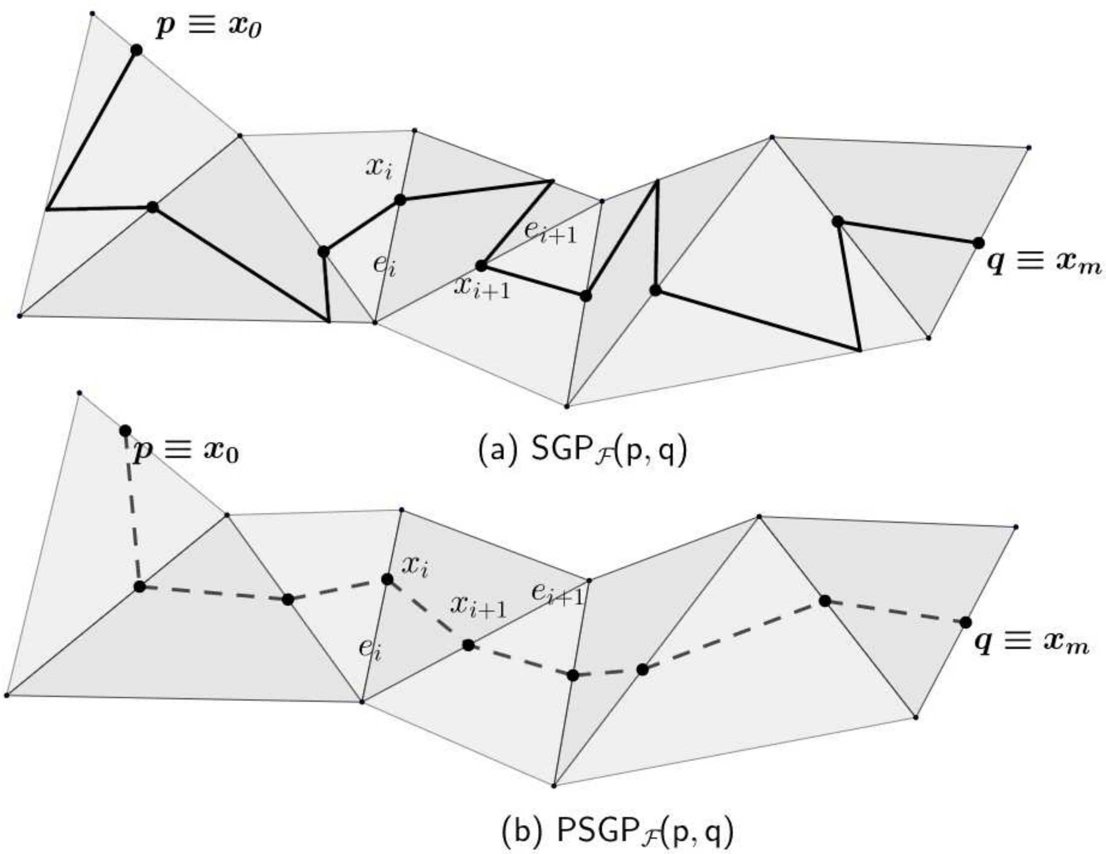

Figure 5: a) $\mathrm{SGP}_{\mathcal{F}}(p, q)$ b) Pseudo path $\mathrm{PSGP}_{\mathcal{F}}(p, q)$ of $\mathrm{SGP}_{\mathcal{F}}(p, q)$

4 The Method of Multiple Shooting (MMS) for Shortest Gentle Path Problem on Triangulated Terrains

Let $\Gamma_{\mathcal{T}}$ be the set of all θ-gentle paths joining p and q on $\mathcal{T}$ and $\Gamma_{\mathcal{T}} \neq \emptyset$ (if necessary, we can add assumptions as in Proposition 2 in order to get $\Gamma_{\mathcal{T}} \neq \emptyset$). A shortest θ-gentle path joining p and q on $\mathcal{T}$ is an element of $\Gamma_{\mathcal{T}}$, denoted by γ_0. Then

$$l(\gamma_0) = \inf_{\gamma \in \Gamma} \{l(\gamma)\}.$$

Definition 6. Let $p, q \in \mathcal{T}$ and let $\{\mathcal{F}^j\}$ be a set of sequences of adjacent triangles of $\mathcal{T}$ joining p and q, for $j > 0$. Assume that there exists a sequence of θ-gentle paths joining p to q on $\mathcal{T}$, denoted by $\{\gamma^j\}$ such that: $\gamma^j \subset \mathcal{F}^j$ and $l(\gamma^{j+1}) \leq l(\gamma^j)$. Then a θ-gentle path, denoted by γ, joining p to q such that

$$l(\gamma) = \inf_{j} \{l(\gamma^j)\},$$

is called an *approximate shortest θ-gentle path* joining p and q on $\mathcal{T}$.

Obviously, locally shortest θ-gentle paths joining p and q on $\mathcal{F} \subset \mathcal{T}$ are approximate shortest

θ-gentle paths joining p and q on $\mathcal{T}$. To see this, just choose $\gamma^j = \text{SGP}_{\mathcal{F}^j}(p,q)$, where $\mathcal{F}^j = \mathcal{F}, j = 1, 2, \ldots$

4.1 Three Factors of MMS

Based on the multiple shooting approach [6, 11], and the Polthier's straightest geodesic theory [16], an approximate algorithm to find a shortest θ-gentle path joining p and q on a convex terrain $\mathcal{T}$ is introduced. The multiple shooting approach consists of the following factors:

(f1) Partition of the terrain $\mathcal{T}$ into sub-terrains is created by cutting slices between two points a and b. The surface around a such sub-terrain is a sequence of adjacent triangles, that is deliberately created by two adjacent cutting slices. At each intersection between a slice and the terrain we take a point, called a shooting point;

(f2) Consider a path on $\mathcal{T}$ joining two points p and q, formed by shooting points. Straightness Condition is established at each shooting point;

(f3) The algorithm enforces (f2) at all shooting points. Otherwise, the update of shooting points makes paths joining two points p and q and formed by shooting points better, e.g, in non-increasing length.

We now describe these factors in details in the next subsections.

4.2 The Factor (f1): Partition

Given a triangulated terrain $\mathcal{T}$ with vertices v_i. Through v_i, we construct a plane $P(v_i)$ which parallels the horizontal plane, denoted by (Oxy). Let $\xi_i = \mathcal{T} \cap P(v_i)$. Then ξ_i is called a cutting slice of $\mathcal{T}$. Let $p, q \in \mathcal{T}$ such that $z(p) > z(q)$ and $v_1, v_2, \ldots, v_k$ be vertices of $\mathcal{T}$ between p, q in term of z -coordinate, i.e., $z(p) > z(v_i) > z(q), i = 1, 2, \ldots k$. From now on we assume that: $\{v_1, v_2, \ldots, v_k\}$ is non-empty, $v_0 := p, v_{k+1} := q$, $\xi_0 := \xi_p$ and $\xi_{k+1} := \xi_q$ (ξ_p and ξ_q are cutting slices of $\mathcal{T}$ through p and q, respectively). In conclusion, we divide $\mathcal{T}$ into suitable sub-terrains $\mathcal{T}_i$ by a set of cutting slices $\xi_0, \ldots, \xi_{k+1}$ satisfying:

$$\begin{aligned}
&\xi_i \parallel (Oxy), z(\xi_i) > z(\xi_{i+1}), \xi_i (i = 1, 2, \ldots, k) \text{ strictly separates } p \text{ and } q, \\
&\text{There is no vertex of } \mathcal{T} \text{ between planes } \xi_i \text{ and } \xi_{i+1}, \\
&\mathcal{T}_i \text{ is bounded by } \mathcal{T}, \text{ and the cutting slices } \xi_i \text{ and } \xi_{i+1}, i = 0, 1, .., k,
\end{aligned} \tag{5}$$

where $z(\xi_i)$ denotes the height of ξ_i, i.e., z-coordinate of an arbitrary point of ξ_i, for $i = 0, 1, \ldots, k$. Clearly, $\cup_{i=0}^{k} \mathcal{T}_i \subset \mathcal{T}$, int $\mathcal{T}_i \cap$ int $\mathcal{T}_j = \emptyset$, $\xi_i \cap \xi_j = \emptyset$, for $i \neq j$.

The following assumption will be needed throughout the paper. The terrain considered has the property that each of the cutting slices ξ_i obtained by partition (f1) is simply connected. Obviously, if the terrain is convex, then that condition is completely satisfied.

The surface of $\mathcal{T}_i$ which does not include the relative interior of ξ_i and ξ_{i+1} (i.e., $\mathcal{T}_i \setminus (\text{ri}\xi_i \cup \text{ri}\xi_{i+1})$) is a sequence of adjacent triangles which does not contain any vertex of $\mathcal{T}$, where riξ_i, riξ_{i+1} is the relative interior of ξ_i, ξ_{i+1}, respectively.

Take a set of initial shooting points $u_i \in \text{bd}\xi_i, (i = 1, 2, \dots, k)$, $u_0 = p, u_{k+1} = q$. By Proposition A.5 [11], there are not more than two proper ordered sequences of adjacent triangles around $\mathcal{T}_i$, containing u_i and u_{i+1}, where $u_{i+1} \in \text{bd}\xi_{i+1}$, u_i belongs to the first triangle of the sequence, u_{i+1} belongs to the final triangle of the sequence. We denote such a proper ordered sequence by $\mathcal{F}_i$. According the Proposition 5 [13], all θ-unreachable points are vertices of a triangulated terrain. Thus we only need to check whether each vertex (instead of all possible points on $\mathcal{T}$ or $\mathcal{F}_i$) is θ-unreachable or not.

Proposition 3. *Every vertex in $\mathcal{F}_i$ of $\mathcal{T}$ is θ-reachable from some points in $\mathcal{F}_i$.*

The proof of Prop. 3 is given in the appendix (Sect. 8). We know that there are maybe more than one shortest θ-gentle path joining two given points in $\mathcal{F}_i$, by Lemma 4 [3], there exists at least one shortest θ-gentle path that crosses the relative interior of each triangle of $\mathcal{F}_i$ at most once. By Prop. 3, we obtain a stronger result in such case.

Proposition 4. *For any shortest θ-gentle path γ joining u_i and u_{i+1} along the sequence of adjacent triangles $\mathcal{F}_i$, if γ crosses any triangle of $\mathcal{F}_i$ more than once, we can always find another shortest θ-gentle path joining u_i and u_{i+1} along $\mathcal{F}_i$, such that it crosses each triangle of $\mathcal{F}_i$ at most once and has the same length as γ.*

The proof of Prop. 4 is given in the appendix (Sect. 8).

4.3 The Factor (f2): Straightness Condition

As previously mentioned, p, q are two points on $\mathcal{T}$ such that $z(p) > z(q)$ and (5) holds. Throughout the paper, we use superscript indices for objects in j^{th}-iterative step such as $u_i^j, \mathcal{F}_i^j, \dots$.

With $j = 0$, u_i^0 are initial shooting points $(i = 1, 2, \dots, k)$. At a j^{th}-iterative step, we assume that there is a set of shooting points $u_i^j \in \text{bd}\xi_i$, for $i = 1, 2, \dots, k, (u_0^j \equiv p, u_{k+1}^j \equiv q)$. Particularly, u_i^j is a shooting point on the cutting polygon ξ_i and u_{i-1}^j, u_{i+1}^j are shooting points on the polygons ξ_{i-1}, ξ_{i+1}, respectively. With each $\mathcal{T}_i$, we take a sequence of adjacent triangles $\mathcal{F}_i$ and find $\text{SGP}_{\mathcal{F}_i}(u_i^j, u_{i+1}^j)$. We consider $\chi^j = \cup_{i=0}^k \text{PSGP}_{\mathcal{F}_i}(u_i^j, u_{i+1}^j)$, where $\text{PSGP}_{\mathcal{F}_i}(u_i^j, u_{i+1}^j)$ is the pseudo path of $\text{SGP}_{\mathcal{F}_i}(u_i^j, u_{i+1}^j)$, for $i = 0, 1, \dots, k$. There are two following cases at the point u_i^j:

Case 1 (Common edge case for the shooting point u_i^j): χ^j passes two triangles of $\mathcal{F}_{i-1}, \mathcal{F}_i$, which have a common edge $e_i \subset \text{bd}\xi_i$ and $u_i^j \in e_i$ (see Fig. 6(a)).

Case 2 (Non-common edge case for the shooting point u_i^j): χ^j passes two triangles of $\mathcal{F}_{i-1}, \mathcal{F}_i$, which share the vertex u_i^j and these triangles have not any common edges (see Fig. 6(b)).

Proposition 5. *We assume that $\chi^j = \cup_{i=0}^k \text{PSGP}_{\mathcal{F}_i}(u_i^j, u_{i+1}^j)$, where $\text{PSGP}_{\mathcal{F}_i}(u_i^j, u_{i+1}^j)$ is the pseudo path of $\text{SGP}_{\mathcal{F}_i}(u_i^j, u_{i+1}^j)$, for $i = 0, 1, \dots, k$. Let $x \in \chi^j$, there exists at least one $\text{SGP}_{\mathcal{F}_i}(u_i^j, u_{i+1}^j)$ with some index i such that x belongs to this path.*

The proof of Prop. 5 is given in the appendix (Sect. 8).

We take x_i^j such that:

$$x_i^j \in \text{PSGP}_{\mathcal{F}_i}(u_i^j, u_{i+1}^j), x_i^j \neq u_i^j, x_i^j \neq u_{i+1}^j, i = 0, 1, \dots k. \quad (6)$$

We need to construct a sequence of adjacent triangles $\mathcal{S}_i$ joining x_{i-1}^j and x_i^j, for $i = 1, 2 \dots k$ as follows:

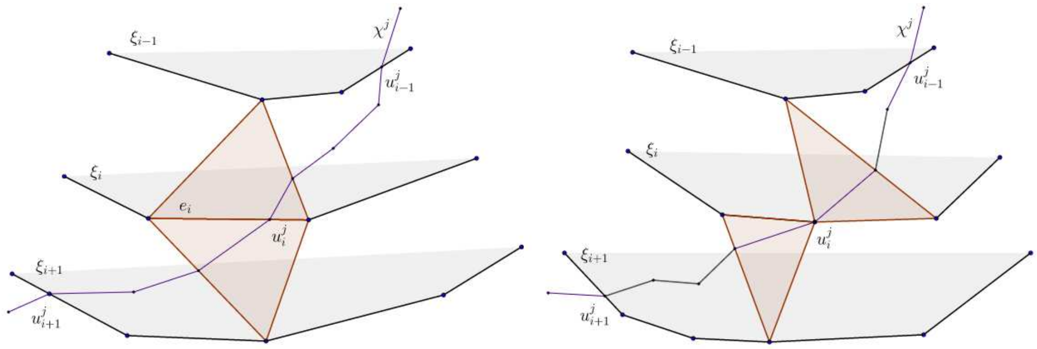

Figure 6: (a) χ^j goes through a interior point u_i^j of edge e_i, (b) χ^j goes through the vertex u_i^j of the polygon ξ_i.

(A1) If Common Edge Case for the shooting point u_i^j happens, $\mathcal{S}_i$ is the sequence of minimum number of triangles in $\mathcal{F}_{i-1}, \mathcal{F}_i$ such that $\mathcal{S}_i$ contains χ's sub-path joining x_{i-1}^j and x_i^j, where x_{i-1}^j is in the first triangle, and x_i^j is in the final triangle of $\mathcal{S}_i$.

(A2) If Non-common Edge Case for the shooting point u_i^j happens, we construct two adjacent sequences of adjacent triangles as follows:

They are sequences of minimum number of triangles in $\mathcal{F}_{i-1}$ and $\mathcal{F}_i$ such that the sequences of adjacent triangles contain χ's sub-path joining x_{i-1}^j and x_i^j, where x_{i-1}^j is in the first triangle, and x_i^j is in the final triangle of $\mathcal{S}_i$. Since χ^j goes through the vertex u_i^j of the polygon ξ_i, several triangles sharing the vertex u_i^j in sub-terrain $\mathcal{T}_{i-1}$ and $\mathcal{T}_i$ are added to the sequences of adjacent triangles. This adding ensures the adjacency of triangles in a sequence of adjacent triangles. Such two sequences of adjacent triangles are denoted by $\mathcal{S}_i^*$ and $\mathcal{S}_i^{**}$.

We now state the factor (f2): Straightness Condition

At each shooting point u_i^j, at j-step, with x_{i-1}^j, x_i^j determined by (6) we check the follow condition (called the Straightness Condition) to decide whether the algorithm stops or continues. The Straightness Condition of the factor (f2) is defined as follows:

(B1) If Common Edge Case for the shooting point u_i^j happens, then with $\mathcal{S}_i$ defined by (A1), the Straightness Condition is

$$lp\left(\text{PSGP}_{\mathcal{S}_i}(x_{i-1}^j, x_i^j)\right) = lp\left(\text{PSGP}_{\mathcal{F}_i}(x_{i-1}^j, u_i^j)\right) + lp\left(\text{PSGP}_{\mathcal{F}_i}(u_i^j, x_i^j)\right).$$

(B2) If Non-common Edge Case for the shooting point u_i^j happens, then with $\mathcal{S}_i^*, \mathcal{S}_i^{**}$ defined by (A2), the Straightness Condition is

$$\begin{aligned} lp\left(\text{PSGP}_{\mathcal{S}_i^*}(x_{i-1}^j, x_i^j)\right) &= lp\left(\text{PSGP}_{\mathcal{S}_i^{**}}(x_{i-1}^j, x_i^j)\right) \\ &= lp\left(\text{PSGP}_{\mathcal{F}_i}(x_{i-1}^j, u_i^j)\right) + lp\left(\text{PSGP}_{\mathcal{F}_i}(u_i^j, x_i^j)\right). \end{aligned}$$

(By Remark 2, a $\text{SGP}(p, q)$ may pass through a vertex of the polytope. Hence (B2) is included in the Straightness Condition).

The following proposition shows that if the straightness condition is satisfied at all shooting points, then a locally shortest θ-gentle path is obtained.

Proposition 6. *Let p, q be two points on $\mathcal{T}$ and (5) hold. Let τ be a path formed by shooting points $u_i, i = 0, 1, \dots k+1$, where $u_0 \equiv p, u_{k+1} \equiv q$. Denote the sub-path joining u_i and u_{i+1} of τ by $\tau(u_i, u_{i+1})$ and a proper ordered sequence of adjacent triangles around $\mathcal{T}_i$, containing u_i and u_{i+1} by $\mathcal{F}_i$, where u_i belongs to the first triangle of the sequence, u_{i+1} belongs to the final triangle of the sequence, $i = 0, 1, \dots, k$. Suppose that $\tau(u_i, u_{i+1})$ is a shortest θ-gentle path joining u_i and u_{i+1} along $\mathcal{F}_i$, $\tau(u_i, u_{i+1})$ and the traditional shortest path joining u_i and u_{i+1} along $\mathcal{F}_i$ does not coincide and the Straightness Condition* (B1) - (B2) *is satisfied at u_i, for all $i = 1, 2, \dots k$. Then τ is a shortest θ-gentle path joining p and q on the sequence of adjacent triangles $\bigcup_{i=0}^{k} \mathcal{F}_i$. Furthermore, if all intersections of τ and edges of $\bigcup_{i=0}^{k} \mathcal{F}_i$ are interior points, then τ is a locally shortest θ-gentle path.*

The proof of Prop. 6 is given in the appendix (Sect. 8).

4.4 The Factor (f3): Update of Shooting Points

Suppose that at j-iterative step, we have a set of shooting points $u_i^j, i = 0, 1, \dots, k+1$, where the source point $u_0^j \equiv p$ and the destination point $u_{k+1}^j \equiv q$.

We assume that the triple $(u_{i-1}^j, u_i^j, u_{i+1}^j) \in \text{bd}\ \xi_{i-1} \times \text{bd}\ \xi_i \times \text{bd}\ \xi_{i+1}$ does not satisfy the Straightness Condition (B1) - (B2) in Sect. 4.3. We update shooting points $u_i^j \in \text{bd}\ \xi_i$ to $u_i^{j+1} \in \text{bd}\ \xi_i$, $i = 1, 2 \dots k$ and $u_0^{j+1} = u_0^j = p, u_{k+1}^{j+1} = u_{k+1}^j \equiv q$, such that the length of the paths formed by shooting points is descending, i.e.,

$$l\left(\cup_{i=0}^{k}\text{SGP}_{\mathcal{F}_i^{j+1}}(u_i^{j+1}, u_{i+1}^{j+1})\right) < l\left(\cup_{i=0}^{k}\text{SGP}_{\mathcal{F}_i^{j}}(u_i^{j}, u_{i+1}^{j})\right).$$

The update depends on u_i^j and two triangles that the pseudo path $\chi^j = \cup_{i=0}^{k}\text{PSGP}_{\mathcal{F}_i^j}(u_i^j, u_{i+1}^j)$ passes through (Common Edge Case and Non-common Edge Case of Sect. 4.3).

Take x_{i-1}^j, x_i^j in the pseudo path χ^j satisfying (6), we construct the sequence of adjacent triangles $\mathcal{S}_i^j$ joining x_{i-1}^j and x_i^j according to (A1) and (A2) in Sect. 4.3. If the Straightness Condition in Sect. 4.3 is satisfied at all shooting points u_i^j, for all $i = 1, 2, \dots, k$, the algorithm stops. Otherwise, there exists a shooting point u_i^j such that it does not satisfy the Straightness Condition (B1) - (B2). Then we update shooting points u_i^j to u_i^{j+1} as follows:

(C1) If Common Edge Case for the shooting point u_i^j happens and u_i^j does not satisfy the Straightness Condition (B1), we then set u_i^{j+1} as the intersection of $\text{PSGP}_{\mathcal{S}_i^j}(x_{i-1}^j, x_i^j)$ with e_i^j, where e_i^j is the common edge of two triangles which is passed by χ^j (see Fig. 7(a)).

(C2) If Non-common Edge Case for the shooting point u_i^j happens and u_i^j does not satisfy the Straightness Condition (B2), we then set u_i^{j+1} as the intersection of $\text{PSGP}_{\mathcal{S}_i^j}(x_{i-1}^j, x_i^j)$ with e_i^j

or e'^j_i (see Fig. 7(b)), where there are two edges of ξ_i parallel to the horizontal plane and they have the common vertex u^j_i (these edges are denoted by e_i, e'_i) and $\mathcal{S}^j_i \in \{\mathcal{S}^{*j}_i, \mathcal{S}^{**j}_i\}$ such that

$$lp\left(\text{PSGP}_{\mathcal{S}^j_i}(x^j_{i-1}, x^j_i)\right) = \min\{lp\left(\text{PSGP}_{\mathcal{S}^{*j}_i}(x^j_{i-1}, x^j_i)\right), lp\left(\text{PSGP}_{\mathcal{S}^{**j}_i}(x^j_{i-1}, x^j_i)\right)\}.$$

By updating (C1)-(C2), we obtain:

$$lp\left(\text{PSGP}_{\mathcal{S}^j_i}(x^j_{i-1}, x^j_i)\right) < lp\left(\text{PSGP}_{\mathcal{F}^j_i}(x^j_{i-1}, u^j_i)\right) + lp\left(\text{PSGP}_{\mathcal{F}^j_i}(u^j_i, x^j_i)\right). \tag{7}$$

Indeed, if Common Edge Case for the shooting point u^j_i happens, from the triangle inequality for shortest θ-gentle paths, $lp(\text{PSGP}_{\mathcal{S}^j_i}(x^j_{i-1}, x^j_i)$ is the length of shortest θ-gentle paths joining x^j_{i-1} and x^j_i along $\mathcal{S}^j_i$, then $lp(\text{PSGP}_{\mathcal{S}^j_i}(x^j_{i-1}, x^j_i)$ is less than or equal to the sum of the length of θ-gentle paths joining x^j_{i-1} and u^j_i and the length of θ-gentle paths joining u^j_i and x^j_i along $\mathcal{F}^j_i$. On the other hand, if u^j_i does not satisfy the Straightness Condition (B1), the equal sign cannot occur. Therefore, we get (7).

Likewise, suppose that Non-common Edge Case for the shooting point u^j_i happens and u^j_i does not satisfy the Straightness Condition (B2). We obtain:

$$\begin{aligned}\min\{lp\left(\text{PSGP}_{\mathcal{S}^{*j}_i}(x^j_{i-1}, x^j_i)\right), lp\left(\text{PSGP}_{\mathcal{S}^{**j}_i}(x^j_{i-1}, x^j_i)\right)\}\\ < lp\left(\text{PSGP}_{\mathcal{F}^j_i}(x^j_{i-1}, u^j_i)\right) + lp\left(\text{PSGP}_{\mathcal{F}^j_i}(u^j_i, x^j_i)\right).\end{aligned}$$

Hence, we also get (7).

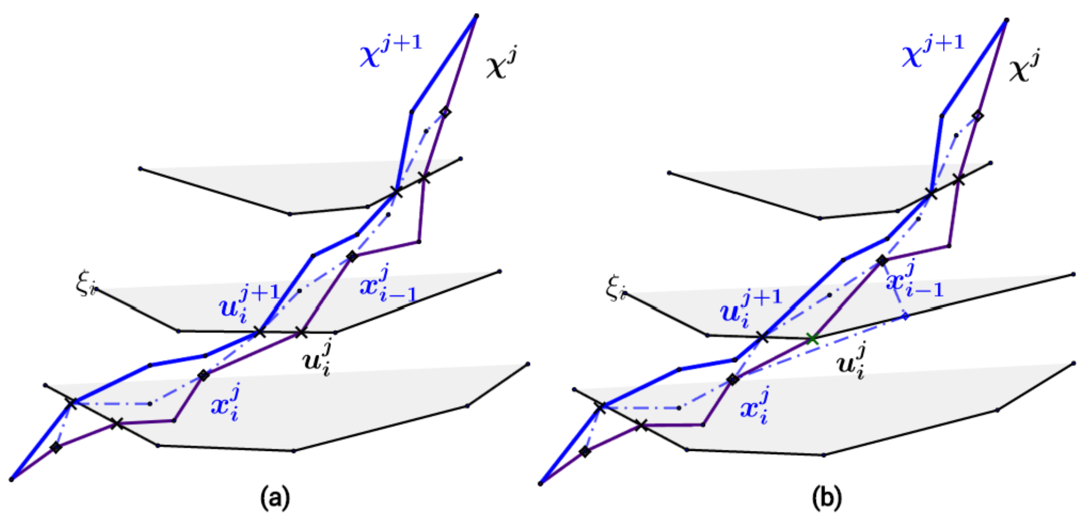

Figure 7: Updating shooting points u^j_i to u^{j+1}_i, where (a) illustrates Common Edge Case for u^j_i happens and (b) illustrates Non-common Edge Case for u^j_i happens. After updating from $\chi^j = \bigcup_{i=1}^k \text{PSGP}_{\mathcal{F}^j_i}(u^j_i, u^j_{i+1})$, we get $\chi^{j+1} = \bigcup_{i=1}^k \text{PSGP}_{\mathcal{F}^{j+1}_i}(u^{j+1}_i, u^{j+1}_{i+1})$

Proposition 7. *We assume τ_0 to be a shortest θ-gentle path joining p and q on $\mathcal{T}$. Let $u_i = \tau_0 \cap \mathrm{bd}\xi_i$, $i = 1, 2 \ldots k$, where ξ_i is determined by (5) in Sect. 4.3. Then the Straightness Condition* (B1) - (B2) *is satisfied at all u_i.*

The proof of Prop. 7 is given in the appendix (Sect. 8).

4.5 Construction of Sequence of Adjacent Triangles

We assume $\mathcal{F}_i^j$ to be a sequence of adjacent triangles joining u_i^j and u_{i+1}^j on the sub-terrain $\mathcal{T}_i$ at j-step. After updating shooting points, at $(j+1)$-step, we need to construct a sequence of adjacent triangles $\mathcal{F}_i^{j+1}$ joining u_i^{j+1} and u_{i+1}^{j+1} on $\mathcal{T}_i$, where u_i^{j+1} is in the first triangle and u_{i+1}^{j+1} is in the final triangle of $\mathcal{F}_i^{j+1}$.

We will construct $\mathcal{F}_i^{j+1}$ at $(j+1)$-step such that $\mathcal{F}_i^{j+1}$ includes minimum triangles number which contains $\mathrm{PSGP}_{\mathcal{F}_i^j}(u_i^j, u_{i+1}^j)$ and

a. If u_i^j and u_{i+1}^j satisfy the Straightness Condition, we then chose $\mathcal{F}_i^{j+1} \equiv \mathcal{F}_i^j$.

b. Otherwise, suppose that u_i^j (or u_{i+1}^j or both) is not satisfied the Straightness Condition, according to Sect. 4.4, u_i^j is updated to u_i^{j+1} at the next step. We need to add minimum triangles number in sub-terrain $\mathcal{T}_i$. The adding must ensure the adjacency of triangles in $\mathcal{F}_i^{j+1}$ and containing u_i^{j+1}, u_{i+1}^{j+1}.

4.6 Algorithm

For each iteration, we need to update shooting points to get a better path. Proposition 8 shows that the decreasing of length of χ^j is still satisfied as $j \to +\infty$. Then Procedure STRAIGHTNESS CONDITION AND UPDATE $(\chi^j, \chi^{j+1}, flag_i)$ of Algorithm 1 performs checking Straightness Condition and updating the shooting points u_i^j to u_i^{j+1}.

Proposition 8. *Procedure* STRAIGHTNESS CONDITION AND UPDATE *$(\chi^j, \chi^{j+1}, flag_i)$ gives*

$$l\left(\bigcup_{i=0}^{k} \mathrm{SGP}_{\mathcal{F}_i^{j+1}}(u_i^{j+1}, u_{i+1}^{j+1})\right) \leq l\left(\bigcup_{i=0}^{k} \mathrm{SGP}_{\mathcal{F}_i^{j}}(u_i^{j}, u_{i+1}^{j})\right).$$

If the Straightness Condition (B1) - (B2) *is not satisfied at some shooting point u_i^j, then the inequality above is strict.*

The proof of Prop. 8 is given in Sect. 8.

procedure STRAIGHTNESS CONDITION AND UPDATE($\chi^j, \chi^{j+1}, flag_i$)

Require: $flag_i = 0, (i = 1, 2, \ldots, k)$, $\chi^j = \cup_{i=0}^{k}\text{PSGP}_{\mathcal{F}_i^j}(u_i^j, u_{i+1}^j)$ ▷ $flag_i = 1$ *then Straighness Condition* (B1)-(B2) *holds*

Ensure: Determined if $flag_i = 0$ or $flag_i = 1, (i = 1, 2, \ldots, k)$ and $\chi^{j+1} = \cup_{i=0}^{k}\text{PSGP}_{\mathcal{F}_i^{j+1}}(u_i^{j+1}, u_{i+1}^{j+1})$ such that

$$l\left(\cup_{i=0}^{k}\text{SGP}_{\mathcal{F}_i^{j+1}}(u_i^{j+1}, u_{i+1}^{j+1})\right) \leq l\left(\cup_{i=0}^{k}\text{SGP}_{\mathcal{F}_i^{j}}(u_i^{j}, u_{i+1}^{j})\right).$$

 for $i = 1, 2, \ldots, k$ **do**

  Take x_i^j on $\text{PSGP}_{\mathcal{F}_i^j}(u_i^j, u_{i+1}^j)$ such that x_i^j does not coincide with u_i^j, u_{i+1}^j

  if Common Edge Case for u_i^j happens **then**

   Construct the sequence of adjacent triangles $\mathcal{S}_i^j$ due to (A1) in Sect. 4.3

   Find $\text{PSGP}_{\mathcal{S}_i^j}(x_{i-1}^j, x_i^j)$

   if $lp(\text{PSGP}_{\mathcal{S}_i^j}(x_{i-1}^j, x_i^j)) = lp(\text{PSGP}_{\mathcal{F}_i^j}(x_{i-1}^j, u_i^j)) + lp(\text{PSGP}_{\mathcal{F}_i^j}(u_i^j, x_i^j))$ **then**

   ▷ *check the Straightness Condition* (B1)

    Set $u_i^{j+1} = u_i^j$, $flag_i = 1$.

   else Set u_i^{j+1} to be the intersection of $\text{PSGP}_{\mathcal{S}_i^j}(x_{i-1}^j, x_i^j)$ and e_i^j ▷ *due to (C1)*

   end if

  else Construct two sequences of adjacent triangles $\mathcal{S}_i^{*j}$ and $\mathcal{S}_i^{**j}$ due to (A2) in Sect. 4.3

   if $lp(\text{PSGP}_{\mathcal{S}_i^{*j}}(x_{i-1}^j, x_i^j)) = lp(\text{PSGP}_{\mathcal{S}_i^{**j}}(x_{i-1}^j, x_i^j)) = lp(\text{PSGP}_{\mathcal{F}_i^j}(x_{i-1}^j, u_i^j)) + lp(\text{PSGP}_{\mathcal{F}_i^j}(u_i^j, x_i^j))$ **then** ▷ *check the Straightness Condition* (B2)

    Set $u_i^{j+1} = u_i^j$ and $flag_i = 1$

   else Find $\mathcal{S}_i^j \in \{\mathcal{S}_i^{*j}, \mathcal{S}_i^{**j}\}$ such that
$lp(\text{PSGP}_{\mathcal{S}_i^j}(x_{i-1}^j, x_i^j)) = \min\{lp(\text{PSGP}_{\mathcal{S}_i^{*j}}(x_{i-1}^j, x_i^j)), lp(\text{PSGP}_{\mathcal{S}_i^{**j}}(x_{i-1}^j, x_i^j))\}$

    Set u_i^{j+1} to be the intersection of $\text{PSGP}_{\mathcal{S}_i^j}(x_{i-1}^j, x_i^j)$ and e_i^j or $e_i^{'j}$ ▷ *due to (C2)*

   end if

  end if

 end for

end procedure

Algorithm 1 FINDING AN APPROXIMATELY SHORTEST θ-GENTLE PATH JOINING TWO GIVEN POINTS ON A TRIANGULATED TERRAIN

Require: A triangulated terrain $\mathcal{T}$, a number $0 < \theta \leq \pi/2$, and $p, q \in \mathcal{T}$.
Ensure: An approximate shortest θ-gentle path joining p and q on $\mathcal{T}$.

Divide $\mathcal{T}$ into $\mathcal{T}_i$ satisfying (5) by cutting slices $\xi_1, \ldots, \xi_k$, $(i = 1, \ldots, k)$. ▷ *partition*
Initialize $u_i^0 \in \xi_i$, $(i = 1, \ldots, k)$, construct $\mathcal{F}_i^0$, then find $\text{PSGP}_{\mathcal{F}_i^0}(u_i^0, u_{i+1}^0)$.
$j := 0$, $\chi^j := \cup_{i=0}^k \text{PSGP}_{\mathcal{F}_i^j}(u_i^j, u_{i+1}^j)$.
loop
  Call Procedure 1
  if $flag_i = 1, i = 1, \ldots, k$ **then** ▷ *the Straightness Condition holds*
    stop and **return** $\chi^j := \cup_{i=0}^k \text{PSGP}_{\mathcal{F}_i^j}(u_i^j, u_{i+1}^j)$.
    else Construct sequence of adjacent triangles $\mathcal{F}_i^{j+1}$, $(i = 0, \ldots, k)$. ▷ *it is shown in 4.4*
    Find $\text{PSGP}_{\mathcal{F}_i^{j+1}}(u_i^{j+1}, u_{i+1}^{j+1})$ on $\mathcal{T}_i$, $i = 1, \ldots, k$.
    $\chi^{j+1} := \cup_{i=0}^k \text{PSGP}_{\mathcal{F}_i^{j+1}}(u_i^{j+1}, u_{i+1}^{j+1})$.
  end if
  $j := j + 1$.
end loop
Adjust the pseudo path χ^j to get the θ-gentle path $\tau^j := \cup_{i=0}^k \text{SGP}_{\mathcal{F}_i^j}(u_i^j, u_{i+1}^j)$. ▷ *$\tau^j$ satisfies slope requirement and its length is computed by (4)*

Theorem 1. *The path given by Algorithm 1 is an approximate shortest θ-gentle path joining p and q on $\mathcal{T}$.*

The proof of Theorem 1 is given in the appendix (Sect. 8).

It is easily seen that if the Straightness Condition is satisfied at all shooting points, then by Theorem 1, an approximate shortest θ-gentle path is obtained. Otherwise, we update shooting points, then check if the Straightness Condition is satisfied at all shooting points or not. If the Straightness Condition is never satisfied, we get a sequence of θ-gentle paths joining two given points whose lengths are strictly decreasing. According to Theorem 1, an approximate shortest θ-gentle path is obtained, too.

5 Numerical Results

We implement Algorithm 1 in C++ code using CGAL, then compile and run the code on Ubuntu Linux platform Intel Core i5-7200U, CPU 568 2.50GHz. The numerical results are visualized by OpenGL.

Example 1. Consider triangulated terrains $\mathcal{T}$ which are polyhedral surfaces with 8 vertices and p, q as given in Table 1 and a slope parameter $\theta = \pi/12$.

In Sect. 8.3 of the Appendix, the brute-force search gives the exact shortest θ-gentle path τ_0 joining two points $p = 1$ and $q = 5$ in triangulated terrain $\mathcal{T}$ with $l(\tau_0) = 38.63046$ after solving 28 convex optimization problems corresponding to 28 sequences of adjacent triangles.

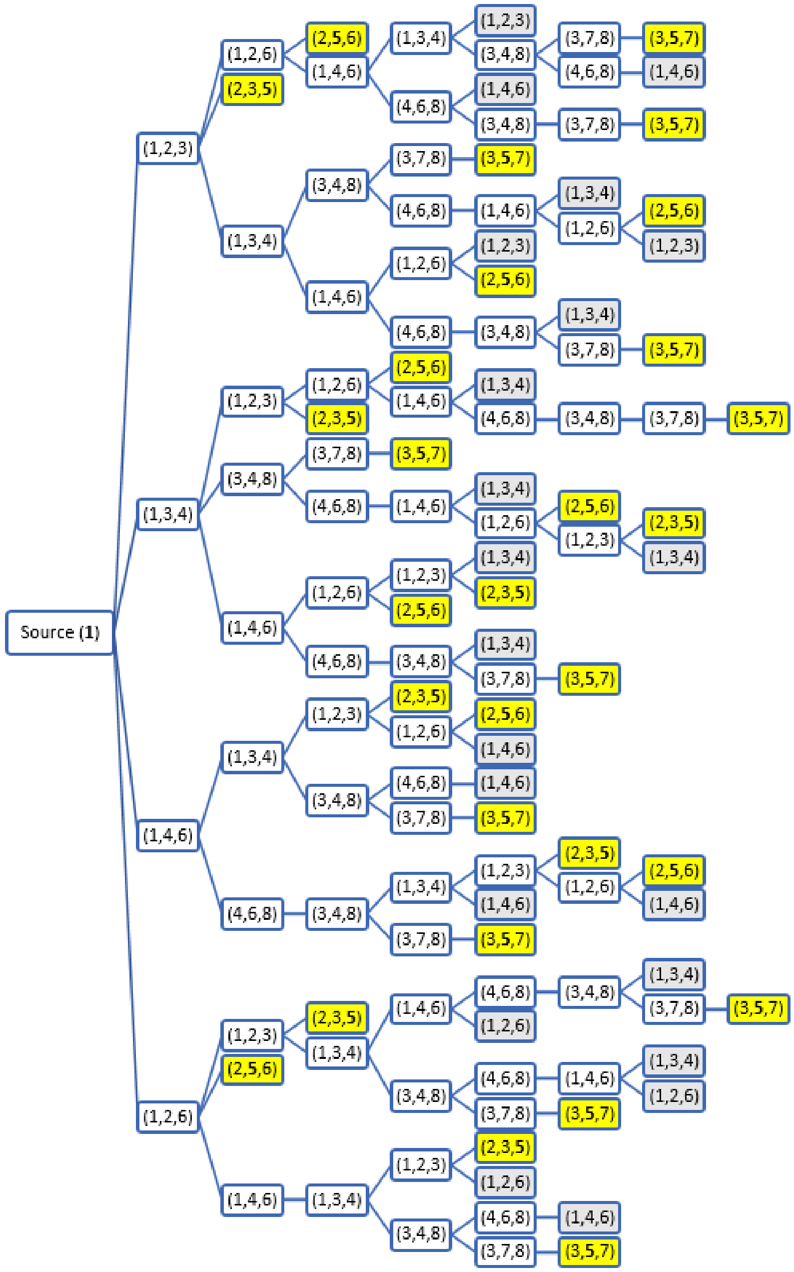

Figure 8: Finding the optimal solution of Example 1: the brute-force search gives a graph displaying all sequences of adjacent triangles joining the vertices $p = 1$ and $q = 5$, where each node in the tree of sequences is a triangled face (a, b, c) with three vertices a, b, c of the terrain. There are 28 sequences of adjacent triangles joining 1 and 5 (coloured in yellow) and the others are self-cutting sequences.

.

Table 1: Coordinates of vertices of $\mathcal{T}$, p and q

Vertex index	x-coordinate	y-coordinate	z-coordinate
$1 \equiv p$	20	20	8
2	20	35	6
3	35	12	4
4	6	1	2
$5 \equiv q$	50	40	0
6	20	60	0
7	40	0	0
8	0	0	0

Sect. 8.3 of the Appendix gives approximate shortest θ-gentle paths τ corresponding to the tolerances ϵ. Algorithm 1 gives approximate shortest θ-gentle paths with their length $l(\tau)$ after a number of iterations (Table 2).

Table 2: Comparison between algorithms for solving Example 1.

	Liu and Wong's algorithm				Algorithm 1 (method of multiple shooting)			
ϵ	Removed vertices	Num. of sub-prob.	Max. num. of variables	$l(\tau)$	Num. of iterations	Num. of sub-prob.	Max. num. of variables	$l(\tau)$
0.025	3	17	6	39.5958	14	7	3	39.5977
0.02412	2	13	5	39.5622	15	7	3	39.3924
0.00356	–	28	7	38.6304	22	7	3	38.7679

As can be seen in Table 2, if we choose a small enough tolerance, Liu and Wong's algorithm gives a shorter path. However, the number of variables in convex optimization problems occurring in their algorithm increases. In addition, if the tolerance is too small, the algorithm cannot find any vertex to be removed such that the distance requirement is satisfied. It then uses the brute-force search mentioned above to give the exact shortest θ-gentle path τ_0 with $l(\tau_0) = 38.6304$ after solving 28 sub-problems (i.e., convex optimization problems) corresponding to 28 sequences of adjacent triangles. By contrast, our Algorithm 1 establishes iterations. For each iterating step, the number of sub-problems is 7 and the maximum number of variables is small.

Example 2. Consider the terrains with 25 vertices, 50 vertices and 100 vertices (see Fig. 9), where $\theta = \pi/6$ and p, q are randomly selected on the terrains for each test (Table 3).

$\text{PSGP}_{\mathcal{F}_i^0}(u_i^0, u_{i+1}^0)$ is calculated by Agarwal et al.s algorithm [1]. At each shooting point u_i^0, we check the Straightness Condition (B1)-(B2) to decide whether the algorithm stops or continues by using the procedure Check the Straightness Condition (B1)-(B2) and Update Shooting Points (χ^0) in Sect. 4.6, where $\chi^0 = \cup_{i=0}^k \text{PSGP}_{\mathcal{F}_i^0}(u_i^j, u_{i+1}^0)$. The algorithm stops when either Straightness Condition B1-B2 holds at all shooting points or $\left| lp\left(\text{PSGP}_{\mathcal{F}_i^{j-1}}(u_i^{j-1}, u_{i+1}^{j-1})\right) - lp\left(\text{PSGP}_{\mathcal{F}_i^j}(u_i^j, u_{i+1}^j)\right)\right| < \delta$, since $\left\{ lp\left(\text{PSGP}_{\mathcal{F}_i^j}(u_i^j, u_{i+1}^j)\right)\right\}$ is convergent, where $\delta > 0$ is an accuracy constant.

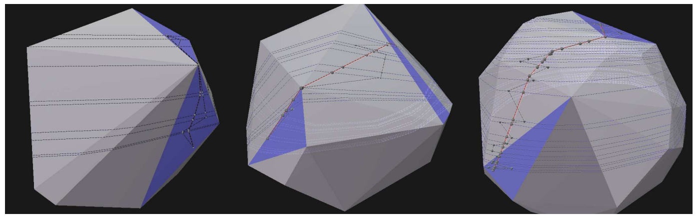

Figure 9: A visualization of the algorithm on terrains which are polyhedral surfaces with 25 vertices (left), 50 vertices (center), and 100 vertices (right).

Table 3: The lengths of paths joining p to q and the number of iterations of the convergence of Algorithm 1 for terrains with $\theta = \pi/6$

Number of vertices of $\mathcal{T}$	$p = (x(p), y(p), z(p))$ $q = (x(q), y(q), z(q))$	Number of slices	Number of iterations	Length of initial path	Length of final path
25	$p = (55.7376, 1.4176, 77.8723)$ $q = (85.3552, 11.6324, -0.78132)$	9	17	125.728	100.19
25	$p = (35.5585, -37.9061, 73.2294)$ $q = (63.7484, -43.3758, 0.954288)$	8	36	200.003	108.28
25	$p = (50.6517, -13.3067, 74.5624)$ $q = (-3.36289, -85.3131, -35.9806)$	14	193	323.407	173.246
50	$p = (-69.07, 46.1851, 3.25405)$ $q = (4.01055, 80.7934, -46.3964)$	10	196	162.297	106.799
50	$p = (11.2359, 52.67, 75.4174)$ $q = (94.3164, 1.24132, 21.458)$	12	93	150.218	125.661
50	$p = (-9.36114, 77.8444, 46.6735)$ $q = (5.60548, 74.6719, -60.5558)$	23	189	436.962	140.081
100	$p = (27.8305, 21.2531, 83.8312)$ $q = (63.4817, -69.2619, 5.09992)$	28	78	294.23	151.773
100	$p = (29.1146, -4.98809, 88.0256)$ $q = (42.4595, -86.3349, 17.7956)$	27	120	246.457	129.403
100	$p = (9.90211, -15.9185, 94.678)$ $q = (-9.8933, -91.5309, 14.4459)$	28	299	261.3	134.819

6 Concluding Remarks

In this paper, an approximate algorithm for computing an shortest θ-gentle path joining two points on the terrain based on the multiple shooting approach is already introduced. Ideas of the method of multiple shooting may be used for finding energy-minimizing paths on terrains [19]. It is also suitable for the problem of finding shortest paths on terrains where its face is weighted polygon.

7 Acknowledgments

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8 Appendix

8.1 Some Remarks

Remark 2. We known that there is no shortest path joining two given points on a polyhedron passing through a convex vertex v of the polyhedron unless v is the source or destination of the shortest path. Unlike, shortest θ-gentle paths joining two given points on a polyhedron (a terrain) can pass through vertices of the polyhedron (the terrain) as shown below.

Example 3. Consider a triangulated terrain $\mathcal{T}$ with 9 vertices $a(0.5, 0.5, 1), b(0, 4, 0), c(1, 0, 0), e(0, -1, 0), f(-1, 0, 0), d(2, 0, -5), g(0, 10, -5), h(0, -2, -5), i(-2, 0, 5)$. Take $\theta = \pi/3$, $p \equiv a, q(1, -1, -5)$.

We will show that there is a shortest θ-gentle path joining p and q on $\mathcal{T}$ passing through the (convex) vertex c. Indeed, along the sequence of adjacent triangles $\triangle pce, \triangle ecd, \triangle edh$, a pseudo path of a shortest θ-gentle path joining p and q is the polyline joining p, c and q. The length of this shortest θ-gentle path is ≈ 6.998248. Along the sequence of adjacent triangles $\triangle pcb, \triangle bcg, \triangle gcd, \triangle dch$, the length of shortest θ-gentle paths joining p and q is ≈ 6.998248. Similarly, for remaining sequences of adjacent triangles joining p and q on $\mathcal{T}$, we see that the lengths of shortest θ-gentle paths joining p and q along these sequences of adjacent triangles are greater than ≈ 6.998248. Hence, there is a shortest θ-gentle path joining p and q on $\mathcal{T}$ passing through the (convex) vertex c.

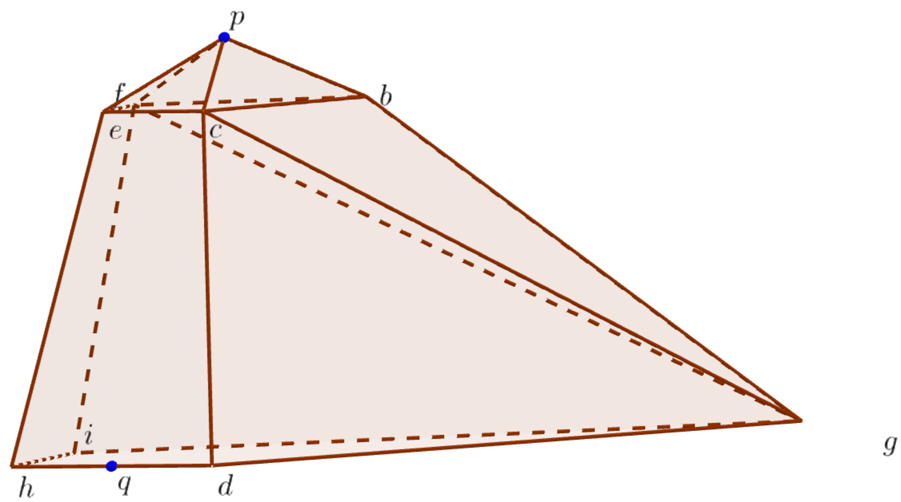

Figure 10: A shortest θ-gentle path joining p and q on $\mathcal{T}$ passing through the (convex) vertex c is the polyline joining p, c and q.

8.2 The Proofs of some Results

The proof of Proposition 1:

Proof. Suppose that the vertex a is θ-unreachable, we will show that all triangles adjacent to a lie completely in one of two θ-convex cones (see Fig. 2). On the contrary, indeed, if there is some triangle f adjacent to a that does not lie completely in one of two θ-convex cones, then f has an edge e adjacent to a that does not lie completely in one of two θ-convex cones. This mean that the angle between e and the horizontal plane does not exceed θ, e is then not too steep. Therefore, a is θ-reachable from any point in e. This conflicts with a is θ-unreachable. Now, we assume that all triangles adjacent to a lie completely in one of two θ-convex cones, we will prove that a is the θ-unreachable. On the contrary, indeed, if a is θ-reachable, then there is a θ-gentle path γ joining a and some point in some triangle f adjacent to a. Then the angle of the tangent of γ at a and the horizontal plane is greater than or equal to the slope of edges of f. Since all triangles adjacent to a lie completely in one of two θ-convex cones, the slope of edges of f is greater than θ. Thus, $\mathrm{sl}_\gamma(a) > \theta$. This conflicts with the fact that γ is a θ-gentle path. It completes the proof. □

The proof of Proposition 2:

Proof. Suppose that $\mathcal{F}$ has $k+1$ adjacent triangles. We denote $\mathbb{E} = \mathbb{R}^3$, $\mathcal{E} := e_1 \times e_2 \times \ldots \times e_k \in \mathbb{E}^k$, where e_i is the common edge of adjacent triangles. The set $\mathcal{E}$ is closed and bounded in $\mathbb{E}^k$ with product topology. Then $\mathcal{E}$ is compact. Let $x_i \in e_i, i = 1, 2, \ldots, k$, $x_0 \equiv p, x_{k+1} \equiv q$. Consider the function $\Phi : \mathbb{E}^k \longrightarrow \mathbb{R}, \Phi(x_1, x_2, \ldots, x_k) = \sum_{i=0}^{k} \|x_i - x_{i+1}\|_s$, where $\|.\|_s$ determines in the formula (2):

$$\|a\|_s = \max\left\{\|a\|_2, \frac{|z(a)|}{\sin\theta}\right\}$$

We will show that Φ is continuous by proving that for all sequences $(x_1^{(n)}, x_2^{(n)}, \ldots, x_k^{(n)})$ converging to $(x_1, x_2, \ldots, x_k)$ in $\mathbb{E}^k$, we have that $\Phi(x_1^{(n)}, x_2^{(n)}, \ldots, x_k^{(n)})$ converges to $\Phi(x_1, x_2, \ldots, x_k)$ in $\mathbb{R}$ or $|\Phi(x_1^{(n)}, x_2^{(n)}, \ldots, x_k^{(n)}) - \Phi(x_1, x_2, \ldots, x_k)| \to 0$ as $n \to \infty$.

$$\begin{aligned}\Phi(x_1^{(n)}, x_2^{(n)}, \ldots, x_k^{(n)}) &= \sum_{i=0}^{k} \|x_i^{(n)} - x_{i+1}^{(n)}\|_s \\ &\leq \sum_{i=0}^{k} (\|x_i^{(n)} - x_i\|_s + \|x_i - x_{i+1}\|_s + \|x_{i+1} - x_{i+1}^{(n)}\|_s)\end{aligned}$$

where $x_0^{(n)} \equiv p, x_{k+1}^{(n)} \equiv q$. Moreover, $\Phi(x_1, x_2, \ldots, x_k) = \sum_{i=0}^{k} \|x_i - x_{i+1}\|_s$, we conclude that:

$$|\Phi(x_1^{(n)}, x_2^{(n)}, \ldots, x_k^{(n)}) - \Phi(x_1, x_2, \ldots, x_k)| \leq \sum_{i=0}^{k} (\|x_i^{(n)} - x_i\|_s + \|x_{i+1} - x_{i+1}^{(n)}\|_s) \qquad (*)$$

Because $(x_1^{(n)}, x_2^{(n)}, \ldots, x_k^{(n)})$ converges to $(x_1, x_2, \ldots, x_k)$ in $\mathbb{E}^k$, $x_1^{(n)} \to x_1, x_2^{(n)} \to x_2, \ldots,$ $x_k^{(n)} \to x_k$ in $\mathbb{R}^3$ as $n \to \infty$. Hence, $\sum_{i=0}^{k} \|x_i^{(n)} - x_i\|_2 \to 0$ and $\sum_{i=0}^{k} \|x_{i+1} - x_{i+1}^{(n)}\|_2 \to 0$ as $n \to \infty$, where $\|.\|_2$ is the Euclidean norm in $\mathbb{R}^3$. Two norms in a finite dimensional space are equivalent, then $\|.\|_2$ and $\|.\|_s$ are equivalent in $\mathbb{R}^3$. Accordingly, the sequence $\{x_i^{(n)}\}$ converges with the norm $\|.\|_2$ iff $\{x_i^{(n)}\}$ converges with the norm $\|.\|_s$, for all $\{x_i^{(n)}\} \subset \mathbb{R}^3$. Consequently, $\sum_{i=0}^{k} \|x_i^{(n)} - x_i\|_s \to 0$ and $\sum_{i=0}^{k} \|x_{i+1} - x_{i+1}^{(n)}\|_s \to 0$ as $n \to \infty$. From (*), it follows that: $0 \leq |\Phi(x_1^{(n)}, x_2^{(n)}, \ldots, x_k^{(n)}) - \Phi(x_1, x_2, \ldots, x_k)| \to 0$ as $n \to \infty$. This implies that Φ is continuous.

Consequently, Φ is continuous and $\mathcal{E}$ is compact, there exists $(x_1^*, x_2^*, \ldots, x_k^*) \in \mathcal{E}$ such that $\Phi(x_1^*, x_2^*, \ldots, x_k^*) = \min\{\Phi(x_1, x_2, \ldots, x_k) : (x_1, x_2, \ldots, x_k) \in \mathcal{E}\}$. We connect the points $x_0, x_1^*, x_2^*,$ $\ldots, x_k^*, x_{k+1}$ by the polyline that includes the segments $[x_0, x_1^*]$, $[x_1^*, x_2^*], \ldots, [x_k^*, x_{k+1}]$. Since all vertices in $\mathcal{F}$ are θ-reachable from some point in $\mathcal{F}$, all points in $\mathcal{F}$ are θ-reachable from some point in $\mathcal{F}$. Therefore, we can adjust the polyline if necessary to get the θ-gentle path σ and its length is $l(\sigma) = \|x_0 - x_1^*\|_s + \|x_1^* - x_2^*\|_s + \ldots + \|x_k^* - x_{k+1}\|_s = \Phi(x_1^*, x_2^*, \ldots, x_k^*)$. We are now in position to prove that σ is a shortest θ-gentle path in $\Gamma_{\mathcal{F}}$ joining p and q along $\mathcal{F}$.

Suppose that γ is an arbitrary element in $\Gamma_{\mathcal{F}}$. It follows that γ is the θ-gentle path joining p, q along $\mathcal{F}$. Let the intersections between γ and the edges $e_1, e_2, \ldots, e_k$ alternately be $x_1, x_2, \ldots, x_k$. Because x_i and x_{i+1} are in the same triangles, the length of θ-sub gentle path joining x_i and x_{i+1} is greater than or equal to the length of $\text{SGP}_{f_{i+1}}(x_i, x_{i+1})$ that indeed is $\|x_i - x_{i+1}\|_s$. Thus, $l(\gamma) \geq \sum_{i=0}^{k} \|x_i - x_{i+1}\|_s \geq \Phi(x_1^*, x_2^*, \ldots, x_k^*) = l(\sigma)$ for all $\gamma \in \Gamma_{\mathcal{F}}$, where $x_0 \equiv p, x_{k+1} \equiv q$. Therefore, σ is a shortest θ-gentle path in $\Gamma_{\mathcal{F}}$ joining p and q along $\mathcal{F}$. □

The proof of Proposition 3:

Proof. For each the vertex $v \in \mathcal{F}_i$, there are two edges e, e' share the vertex v and they are edges of polygon ξ_i. As e (resp. e') is adjacent to only one triangle in $\mathcal{F}_i$, we call this triangle f (resp. f'). According to the construction sequence of adjacent triangles $\mathcal{F}_i$, f or f' belongs to $\mathcal{F}_i$. Because e and e' are parallel to the horizontal plane, they are not steep. The vertex v is θ-reachable from some point in the relative interior of e or e'. Hence, v is θ-reachable from some point in $\mathcal{F}_i$. □

The proof of Proposition 4:

Proof. Suppose that γ crosses a triangle f of $\mathcal{F}_i$ more than once, it suffices to show that we can replace the sub-path between the first and the last visit point in f by a shortcut which is not too steep, remains in triangle f, and its length does not exceed the original.

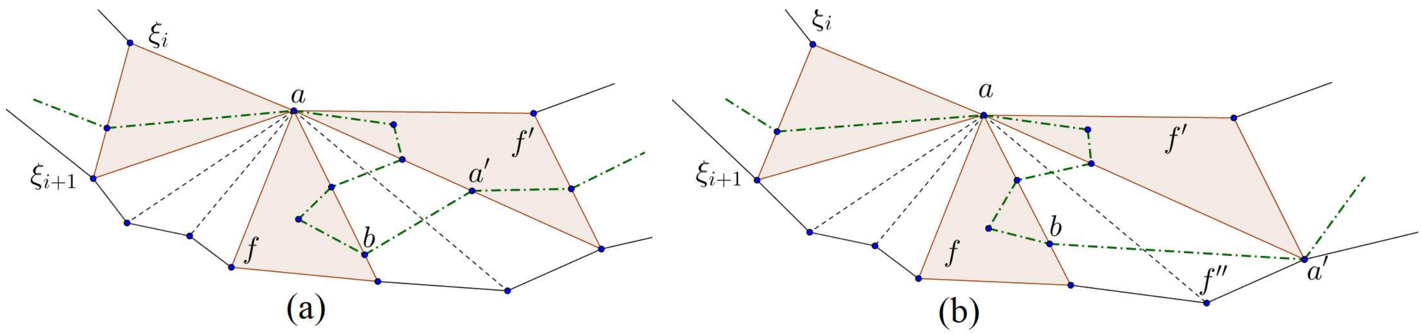

Figure 11: Illustration for the proof in Prop. 4

Let a and b be the first and last intersection points between γ and f via $\mathcal{F}_i$, respectively. Since γ is a shortest θ-gentle path joining u_i and u_{i+1} along $\mathcal{F}_i$, the sub-path of γ joining a and b must be a shortest θ-gentle path joining a and b, too.

If a and b are θ-reachable from all of points in f, we can replace the sub-path joining a and b by $[a, b]$ if this segment is not too steep or adj$([a, b])$ in the opposite, without changing the length. Hence, we get a path which passes through f only once and has the same length as γ.

If a is θ-unreachable from all points in f, a is then the vertex of sequence of adjacent triangles $\mathcal{F}_i$ (as all θ-unreachable points are vertices of the terrain). By Prop. 3, a is θ-reachable from some triangle f' in $\mathcal{F}_i$ ($f' \neq f$). Because b is the last intersection point between γ and f, γ has to come back f' again (see Fig. 11). The path γ cuts f' at another point $a' \neq a$.

If a' is θ-reachable from some point in f' (see Fig. 11a), we can replace the sub-path joining a and a' by $[a, a']$ or adj$[a, a']$ without changing the length. We then get a path which passes through f only once and has the same length as γ.

If a' is θ-unreachable from f', a' is a vertex of f'. According to Prop. 3, a' is θ-reachable from some point in $\mathcal{F}_i$. We deduce that a' is θ-reachable from points in the triangle f'' which is adjacent to aa' ($f'' \neq f'$, see Fig. 11b). We can replace the sub-path joining a and a' by the adjusted path adj$[a, a']$ without changing the length of the path, where adj$([a, a'])$ includes two parts: the one starting from a and lying on f' and the other lying on f''.

Summarizing, we get the θ-gentle path, which passes through f only once and has the same length as γ. □

The proof of Proposition 5:

Proof. If x is the intersection of χ^j and some edge of a triangle in $\mathcal{F}_i$ ($i = 0, 1, \ldots, k$), then x clearly belongs to shortest θ-gentle paths whose pseudo is χ^j.

Otherwise, x is an interior point of some line segment $[a, b]$ such that a, b are both the intersections between χ^j and the edges of a triangle f in $\mathcal{F}_i$ ($i = 0, 1, \ldots, k$). We consider two cases:

a) If $b \notin \mathrm{SR}_f(a)$ (where $\mathrm{SR}_f(a)$ is the steep region of a in f), then $[a,b]$ belongs to shortest θ-gentle paths whose pseudo is χ^j. The point x then belongs to these shortest θ-gentle paths.

b) If $b \in \mathrm{SR}_f(a)$, then $[a,b]$ is steep. We construct the adjusted paths adj$[a,x]$, adj$[x,b]$ and then adj$[a,x] \cup$ adj$[x,b]$ is θ-gentle and lies completely in f.

$$l(\text{adj}[a,x] \cup \text{adj}[x,b]) = \frac{|z(x) - z(a)|}{\sin\theta} + \frac{|z(b) - z(x)|}{\sin\theta} = \frac{|z(b) - z(a)|}{\sin\theta} = l(\text{adj}[a,b]).$$

Consequently, adj$[a,x] \cup$ adj$[x,b]$ is also a shortest θ-gentle path joining a and b in f. We construct a shortest θ-gentle path along $\mathcal{F}_i$ such that when it passes through f, its orbit is adj$[a,x] \cup$ adj$[x,b]$. Indeed, x belongs to it.

Therefore, the proof is completed. □

The proof of Propositon 6:

Proof. Suppose that χ is the pseudo path of τ, χ is then a polyline in which endpoints of its line segments lie on edges of the terrain or of bdξ_i. Let $\bigcup_{i=0}^{k} \mathcal{F}_i$ be the sequence of adjacent triangles with edges $e_1, e_2, \ldots, e_m$, where e_i is the common edge of adjacent triangles. We denote $\mathcal{E} := e_1 \times e_2 \times \ldots \times e_m \in \mathbb{E}^m$, where $\mathbb{E} = \mathbb{R}^3$.

Firstly, if all intersections of τ and edges of $\bigcup_{i=0}^{k} \mathcal{F}_i$ are interior points, we show that τ is a locally shortest θ-gentle path. We only need to prove the following claim at u_i, where u_i is a shooting point in an edge of $\mathrm{bd}\xi_i$.

Claim 1: " *If $\tau(p, u_i)$ and $\tau(u_i, q)$ respectively, are shortest θ-gentle paths joining p and u_i, u_i and q respectively, along corresponding sequences of adjacent triangles and the Straightness Condition* (B1) *is satisfied at u_i, then τ is a locally shortest θ-gentle*".

Before presenting Lemma 1 to prove Claim 1, we need following notations: Let $B(a,\delta)$ be a close sphere with the center a, radius δ and e is an edge with a direction vector $\overrightarrow{e}$. Two intersections of the boundary of $B(a,\delta)$ and the line containing e are $a \pm \delta \dfrac{\overrightarrow{e}}{\|e\|}$, denoted by $a \pm \delta$.

Lemma 1. *In $\mathbb{R}^3$, let f be a triangle with edges e, g, h and a (resp. u) be an interior point of g (resp. e). Suppose* $\operatorname{int}\mathrm{SR}_f(a) \neq \emptyset$*. Then for all x in (a,u), there exist positive numbers ϵ and δ such that: for all b in $(a-\delta, a+\delta)$, for all b' in $(u-\epsilon, u+\epsilon)$, we have:*

$$lp([b,b']) = lp([b,x]) + lp([x,b']). \tag{8}$$

The proof of Lemma 1:

Proof. Since definition of steep regions, $\mathrm{SR}_f(b)$ is a polygon having edges parallel or coincident with edges of $\mathrm{SR}_f(a)$. For each x in (a,u), we construct lines through x and parallel to two edges through a of $\mathrm{SR}_f(a)$. Let intersections of these lines and the line containing e (resp. g) be r and r' (resp. s and s'). Take $\epsilon > 0$ such that $[u-\epsilon, u+\epsilon] \subset [r, r'] \cap \mathrm{SR}_f(a)$. There exists $\delta > 0$ such that $[a-\delta, a+\delta] \subset [s, s'] \cap g$. Then for all b in $(a-\delta, a+\delta)$, $x \in \mathrm{SR}_f(a) \cap \mathrm{SR}_f(b)$ and $[u-\epsilon, u+\epsilon] \subset \mathrm{SR}_f(a) \cap \mathrm{SR}_f(b)$. Since any steep region is convex, $\mathrm{SR}_f(a)$ and $\mathrm{SR}_f(b)$ are convex. Therefore, $\mathrm{SR}_f(a) \cap \mathrm{SR}_f(b)$ is convex. Then the triangle given by three points $x, u-\epsilon, u+\epsilon$, is in $\mathrm{SR}_f(a) \cap \mathrm{SR}_f(b)$. Thus, for all b' in $(u-\epsilon, u+\epsilon)$, $x, b' \in \mathrm{SR}_f(b)$. By constructing $\mathrm{adj}[b,b']$ in $\mathrm{SR}_f(b)$ such that $\mathrm{adj}[b,b']$ is through x, we have $lp[b,b'] = lp[b,x] + lp[x,b']$. The proof is completed. □

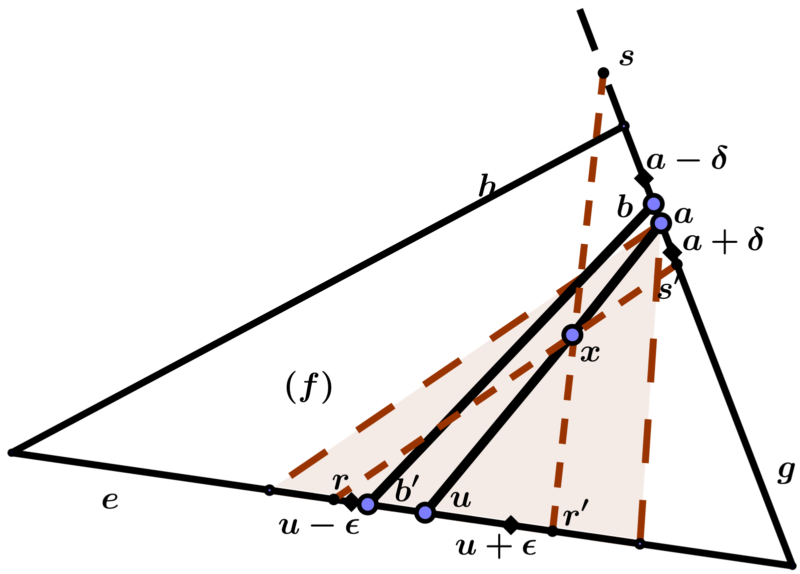

Figure 12: Illustration of the proof of Lemma 1

When a, u are not interior points, the neighbor at u (resp. a) has one of the following forms: $[u-\delta, u]$ or $[u, u+\delta]$ (resp. $[a-\delta, a]$ or $[a, a+\delta]$). The state similar to Lemma 1 also holds in these cases.

Proof of Claim 1. Return to the problem, in order to prove τ is a locally shortest θ-gentle path, we need to show that there exists a neighbor U of χ, $(U \subset \mathcal{E})$ such that the length of all θ-gentle paths on U joining p and q are longer than or equal to the length of τ.

Let f and f' be two triangles sharing the edge e_j ($f \in \mathcal{T}_{i-1}, f' \in \mathcal{T}_i$). Then e_{j-1}, e_{j+1} are the edges of f and f', respectively. Let a (resp. a') be intersection of $\tau(p, u_i)$ and e_{j-1} (resp. $\tau(u_i, q)$ and e_{j+1}). There are two cases happening at u_i as follows:

Case 1: $u_i \in \text{SR}_f(a) \cap \text{SR}_{f'}(a')$.

Let x, x' be two points in $[a, u_i]$ and $[u_i, a']$, respectively. According to Lemma 1, there exist positive numbers $\epsilon_1, \epsilon_2, \delta, \delta'$ satisfying the conclusion of the Lemma. Let $\epsilon := \min\{\epsilon_1, \epsilon_2\}$ and

$$U := e_1 \times e_2 \times \ldots \times (a-\delta, a+\delta) \times (u_i - \epsilon, u_i + \epsilon) \times (a' - \delta', a' + \delta') \times \ldots \times e_m.$$

Let τ^* be a θ- gentle path joining p and q in U. We will show that $l(\tau^*) \geq l(\tau)$. We denote $u_i^* = \tau^* \cap e_i, b = \tau^*(p, u_i^*) \cap e_{i-1}, b' = \tau^*(u_i^*, q) \cap e_{i+1}$. Then $u_i^* \in (u_i - \epsilon, u_i + \epsilon), b \in (a-\delta, a+\delta), b' \in (a' - \delta', a' + \delta')$. We "add" x and x' to the orbit of τ^* similarly to (8) as follows:

$$l(\tau^*) = l(\tau^*(p, b)) + lp([b, x]) + lp([x, u_i^*]) + lp([u_i^*, x']) + lp([x', b']) + l(\tau^*(b', q)) \quad (9)$$

Since Straightness Condition (B1) is satisfied at u_i, we conclude that

$$lp([x, u_i^*]) + lp([u_i^*, x']) \geq lp([x, u_i]) + lp([u_i, x']) \quad (10)$$

Combining (9) and (10) yields

$$l(\tau^*) \geq l(\tau^*(p, b)) + lp([b, x]) + lp([x, u_i]) + lp([u_i, x']) + lp([x', b']) + l(\tau^*(b', q)). \quad (11)$$

Because $\tau^*(p,b) \cup adj([b,x]) \cup adj([x,u_i])$ is a θ-gentle path joining p and u_i, we have

$$l(\tau^*(p,b)) + lp([b,x]) + lp([x,u_i]) \geq l(\tau(p,u_i)). \tag{12}$$

Similarly, $adj([u_i,x']) \cup adj([x',b']) \cup \tau^*(b',q)$ is a θ-gentle path joining u_i and q, we have

$$lp([u_i,x']) + lp([x',b']) + l(\tau^*(b',q)) \geq l(\tau(u_i,q)). \tag{13}$$

Combining (11), (12) and (13) yields $l(\tau^*) \geq l(\tau)$.

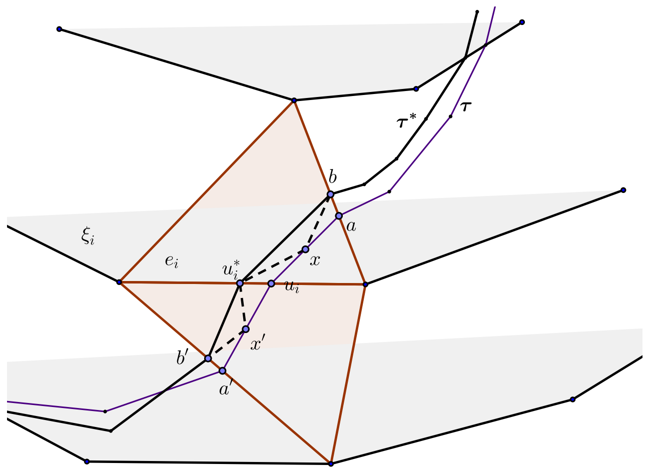

Figure 13: Illustration to the proof of Case 1

Case 2: $u_i \notin \mathrm{SR}_f(a) \cap \mathrm{SR}_{f'}(a')$.

Since for all $i = 0,1,\ldots,k$, $\tau(u_i,u_{i+1})$ and the traditional shortest path joining u_i and u_{i+1} along $\mathcal{F}_i$ does not coincide, there exists at least one segment of $\chi(p,u_i)$ and one of $\chi(u_i,q)$ such that the segments are steep. Let $[a,t]$ (resp. $[t',a']$) be a such segment of $\chi(p,u_i)$ (resp. $\chi(u_i,q)$). It may happen either $[a,t] \in f$ or $[t',a'] \in f'$. Let e_v (e_w, respectively) be the edge containing a (a', respectively). Then $t \in e_{v+1}$ and $t' \in e_{w-1}$.

Let $\epsilon > 0$, x, x' be two points in $[a,t]$ and $[t',a']$, respectively. According to Lemma 1, there exist $\delta > 0, \delta' > 0$ satisfying the conclusion of the Lemma.

$$U := e_1 \times e_2 \times \ldots \times (a-\delta, a+\delta) \times (t-\epsilon, t+\epsilon) \times \ldots \times (t'-\epsilon, t'+\epsilon) \times (a'-\delta', a'+\delta') \times \ldots \times e_m.$$

Let τ^* be a θ- gentle path joining p and q in U. We will show that $l(\tau^*) \geq l(\tau)$. We denote $u^* = \tau^* \cap e_i$, $b = \tau^*(p,u_i^*) \cap e_v$, $n = \tau^*(p,u_i^*) \cap e_{v+1}$, $b' = \tau^*(u_i^*,q) \cap e_w$, $n' = \tau^*(u_i^*,q) \cap e_{w-1}$. Then $n \in (t-\epsilon, t+\epsilon)$, $n' \in (t'-\epsilon, t'+\epsilon)$, $b \in (a-\delta, a+\delta)$, $b' \in (a'-\delta', a'+\delta')$. We “add” x and x' to the orbit of τ^* similarly to (8) as follows:

$$\begin{aligned} l(\tau^*) = l(\tau^*(p,b)) + lp([b,x]) &+ lp([x,n]) + l(\tau^*(n,u_i^*)) \\ &+ l(\tau^*(u_i^*,n') + lp([n',x']) + lp([x',b']) + l(\tau^*(b',q)). \end{aligned} \tag{14}$$

Since Straightness Condition (B1) is satisfied at u_i, we have

$$lp([x,n]) + l(\tau^*(n,u_i^*)) + l(\tau^*(u_i^*,n') + lp([n',x']) \geq l(\tau(x,u_i)) + l(\tau(u_i,x')). \tag{15}$$

Combining (14) and (15) yields

$$l(\tau^*) \geq l(\tau^*(p,b)) + lp([b,x]) + l(\tau(x,u_i)) + l(\tau(u_i,x')) + lp([x',b']) + l(\tau^*(b',q)). \quad (16)$$

Since $\tau^*(p,b) \cup adj([b,x]) \cup \tau(x,u_i)$ is a θ-gentle path joining p and u_i, we have

$$l(\tau^*(p,b)) + lp([b,x]) + l(\tau(x,u_i)) \geq l(\tau(p,u_i)). \quad (17)$$

Likewise, $\tau(u_i,x') \cup adj([x',b']) \cup \tau^*(b',q)$ is a θ-gentle path joining u_i and q, we have

$$l(\tau(u_i,x')) + lp([x',b']) + l(\tau^*(b',q)) \geq l(\tau(u_i,q)). \quad (18)$$

Combining (16), (17) and (18) yields $l(\tau^*) \geq l(\tau)$.

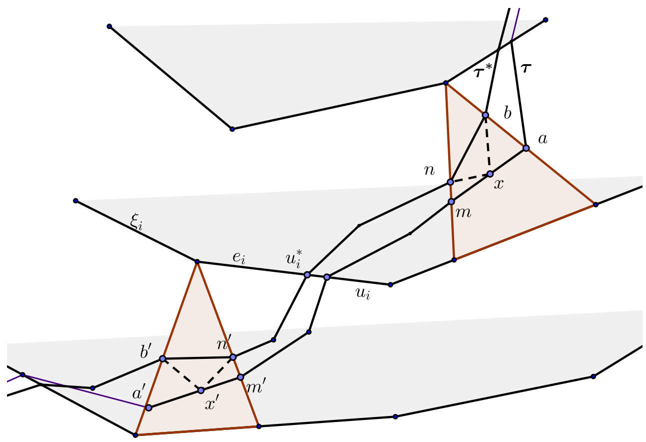

Figure 14: Illustration of the proof of Case 2

Summarily, τ is a locally shortest θ-gentle path and Claim 1 is proven. □

We see that Claim 1 is true in the case u_i is interior point of e_i. However, when all intersections of τ and edges of $\bigcup_{i=0}^{k} \mathcal{F}_i$ are arbitrary points (not interior points), a similar claim holds with the same proof. Therefore, the length of all shortest θ-gentle paths on $\bigcup_{i=0}^{k} \mathcal{F}_i \cap U$ (where U is a neighbor of $\mathcal{T}$) joining p and q are more than or equal to the length of τ. Accordingly, τ is a locally shortest θ-gentle path in $\bigcup_{i=0}^{k} \mathcal{F}_i$.

Assume that τ' is a θ-gentle path joining p and q on $\bigcup_{i=0}^{k} \mathcal{F}_i$. Let x_i be the intersection of τ' and e_i, for $i = 1, 2, \dots, m$. Then we can write $x_i = \lambda_i c_i + (1-\lambda_i) d_i$, where $\lambda_i \in [0,1]$ and c_i, d_i are endpoints of the edge e_i. Then $l(\tau') = \|p - x_1\|_s + \|x_1 - x_2\|_s + \ldots + \|x_{m-1} - x_m\|_s + \|x_m - q\|_s$, where $\|x_i - x_{i+1}\|_s$ is the length of adj$[x_i, x_{i+1}]$, calculated as follows:

$$\max\{\sqrt{((x_i)_x - (x_{i+1})_x)^2 + ((x_i)_y - (x_{i+1})_y)^2 + ((x_i)_z - (x_{i+1})_z)^2}, \frac{|(x_i)_z - (x_{i+1})_z|}{\sin\theta}\}.$$

The problem of finding a shortest θ-gentle path on the sequence of adjacent triangles $\bigcup_{i=0}^{k} \mathcal{F}_i$ becomes the problem of finding τ' such that $l(\tau')$ is minimum. This is the problem of minimizing the sum of $\|.\|_s$ norms as follows:

$$\begin{aligned} &\text{minimize}_{\lambda_i, i=1,2,\dots,m} && f(\lambda_1, \lambda_2, \dots, \lambda_m) \\ &\text{subject to} && \lambda_i \in [0,1], i = 1, 2, \dots, m \end{aligned} \quad \text{(P1)}$$

Since f is a convex function on $[0,1]^m$, (P1) is a convex optimal problem. We know that every locally solution of a convex optimal problem is global solution. Therefore, τ is also shortest θ-gentle path in $\bigcup_{i=0}^{k}\mathcal{F}_i$, because τ is a locally shortest θ-gentle path in $\bigcup_{i=0}^{k}\mathcal{F}_i$.

Summarizing, τ is a shortest θ-gentle path joining p and q on the sequence of adjacent triangles $\bigcup_{i=0}^{k}\mathcal{F}_i$. Furthermore, if all intersections of τ and edges of $\bigcup_{i=0}^{k}\mathcal{F}_i$ are interior points, then τ is a locally shortest θ-gentle path. This completes the proof. □

The proof of Proposition 7:

Proof. Let $u_0 \equiv p, u_{k+1} \equiv q$. We can write $\tau_0 = \cup_{i=0}^{k}\text{SGP}_{\mathcal{F}_i}(u_i, u_{i+1})$, where $\text{SGP}_{\mathcal{F}_i}(u_i, u_{i+1})$ is a shortest θ-gentle path joining u_i and u_{i+1} on $\mathcal{F}_i$ and $\mathcal{F}_i$ is a sequence of adjacent triangles joining u_i and u_{i+1} such that $\mathcal{F} = \bigcup_{i=0}^{k}\mathcal{F}_i$ is a sequence of adjacent triangles along τ_0. Denote the pseudo path of $\text{SGP}_{\mathcal{F}_i}(u_i, u_{i+1})$ by $\text{PSGP}_{\mathcal{F}_i}(u_i, u_{i+1})$. For $i = 0, 1, \ldots, k$, take x_i on $\text{PSGP}_{\mathcal{F}_i}(u_i, u_{i+1})$ such that x_i does not coincide with u_i and u_{i+1}. According to Prop. 5, there exists at least a shortest θ-gentle path $\text{SGP}_{\mathcal{F}_j}(u_i, u_{i+1})$ of $\text{PSGP}_{\mathcal{F}_j}(u_i, u_{i+1})$ such that x_i belongs to $\text{SGP}_{\mathcal{F}_j}(u_i, u_{i+1})$.

It's easy to prove that any sub-path of a shortest θ-gentle path is also a shortest θ-gentle path. For $i = 1, 2, \ldots, k$, denote the sub-path joining x_{i-1} and x_i of τ_0 by $\tau_0(x_{i-1}, x_i)$. Since τ_0 is a shortest θ-gentle path joining p and q on $\mathcal{T}$, $\tau_0(x_{i-1}, x_i)$ is a shortest θ-gentle path joining x_{i-1} and x_i on $\mathcal{T}$. If Common Edge Case for u_i happens, $\tau_0(x_{i-1}, x_i)$ is a shortest θ-gentle path along the sequence of adjacent triangles $\mathcal{F}_{i-1} \cup \mathcal{F}_i$. Therefore, (B1) of the Straightness Condition must happen at u_i.

If Non-common Edge Case for u_i happens, then according to the construction of sequences of adjacent triangles $\mathcal{S}_i^*$ and $\mathcal{S}_i^{**}$, $\tau_0(x_{i-1}, x_i)$ is the θ-gentle path joining x_{i-1} and x_i along $\mathcal{S}_i^*$ and $\mathcal{S}_i^{**}$. We will show that $\tau_0(x_{i-1}, x_i)$ is a shortest θ-gentle path joining x_{i-1} and x_i along $\mathcal{S}_i^*$ and $\mathcal{S}_i^{**}$. Since τ_0 is a shortest θ-gentle path joining p and q on entire $\mathcal{T}$, $\tau_0(x_{i-1}, x_i)$ is a shortest θ-gentle path joining x_{i-1} and x_i on entire $\mathcal{T}$. Suppose that there exists a θ-gentle path, denoted by γ, joining x_{i-1} and x_i along $\mathcal{S}_i^*$ or $\mathcal{S}_i^{**}$ such that $l(\gamma) < l(\tau_0(x_{i-1}, x_i))$. We can construct a θ-gentle path joining p and q on entire $\mathcal{T}$ from τ_0, denoted by τ_1, by only replacing $\tau_0(x_{i-1}, x_i)$ by γ. Therefore, $l(\tau_1) < l(\tau_0)$. This contradicts the fact that τ_0 is a shortest θ-gentle path joining p and q on entire $\mathcal{T}$. Hence, $\tau_0(x_{i-1}, x_i)$ is a shortest θ-gentle path joining x_{i-1} and x_i along $\mathcal{S}_i^*$ and $\mathcal{S}_i^{**}$. Therefore, (B2) of the Straightness Condition must happen at u_i. This completes the proof. □

Remark 3. With the constructions sequences of adjacent triangles $\mathcal{F}_i^j$, $\mathcal{F}_i^{j+1}$ and $\mathcal{S}_i^j$ in Sect. 4.5, and note that $u_i^{j+1} = u_i^{*j}$, we deduce that:

$$l\left(\text{SGP}_{\mathcal{S}_i^j}(x_{i-1}^j, x_i^j)\right) \le l\left(\text{SGP}_{\mathcal{F}_{i-1}^j}(x_{i-1}^j, u_i^{j+1})\right) + l\left(\text{SGP}_{\mathcal{F}_i^j}(u_i^j, x_{i+1}^{j+1})\right);$$

$$l\left(\text{SGP}_{\mathcal{F}_i^{j+1}}(u_i^{j+1}, u_{i+1}^{j+1})\right) \le l\left(\text{SGP}_{\mathcal{S}_{i-1}^j}(u_i^{j+1}, x_i^{j+1})\right) + l\left(\text{SGP}_{\mathcal{S}_i^j}(x_i^{j+1}, u_{i+1}^{j+1})\right).$$

The proof of Proposition 8:

Proof. Assume that Straightness Condition (B1) - (B2) is not satisfied at some shooting point u_i^j, $\tau^j = \cup_{i=0}^{k}\text{SGP}_{\mathcal{F}_i^j}(u_i^j, u_{i+1}^j)$. According to Prop. 5 and $x_i^j \in \text{PSGP}_{\mathcal{F}_i^j}(u_i^j, u_{i+1}^j)$, x_i^j belongs to $\text{SGP}_{\mathcal{F}_i^j}(u_i^j, u_{i+1}^j)$, for $j = 0, 1, \ldots, k$ (see Fig. 15). Hence, instead of proving the results on the length of shortest θ-gentle paths, we can move to proving the corresponding results about the sum of $\|.\|_s$ norms of their pseudo paths.

On the sub-terrain $\mathcal{T}_{i-1}$, we have:

$$lp\left(\text{PSGP}_{\mathcal{F}^j_{i-1}}(u^j_{i-1},u^j_i)\right) = lp\left(\text{PSGP}_{\mathcal{F}^j_{i-1}}(u^j_{i-1},x^j_{i-1})\right) + lp\left(\text{PSGP}_{\mathcal{F}^j_{i-1}}(x^j_{i-1},u^j_i)\right). \tag{19}$$

On the sub-terrain $\mathcal{T}_i$, we get:

$$lp\left(\text{PSGP}_{\mathcal{F}^j_{i}}(u^j_{i},u^j_{i+1})\right) = lp\left(\text{PSGP}_{\mathcal{F}^j_{i}}(u^j_{i},x^j_{i})\right) + lp\left(\text{PSGP}_{\mathcal{F}^j_{i}}(x^j_{i},u^j_{i+1})\right). \tag{20}$$

Combining Remark 3, with formulas (19) and (20), yields:

$$\begin{aligned}
l\left(\bigcup_{i=0}^{k}\text{SGP}_{\mathcal{F}^j_i}(u^j_i,u^j_{i+1})\right) &= \sum_{i=0}^{k} l\left(\text{SGP}_{\mathcal{F}^j_i}(u^j_i,u^j_{i+1})\right)\\
&= \sum_{i=0}^{k} lp\left(\text{PSGP}_{\mathcal{F}^j_i}(u^j_i,u^j_{i+1})\right)\\
&= lp\left(\text{PSGP}_{\mathcal{F}^j_0}(u^j_0,x^j_0)\right) + \sum_{i=1}^{k-1} lp\left(\text{PSGP}_{\mathcal{F}^j_{i-1}}(x^j_i,u^j_i)\cup\text{PSGP}_{\mathcal{F}^j_i}(u^j_i,x^j_{i+1})\right)\\
&\quad + lp\left(\text{PSGP}_{\mathcal{F}^j_{k+1}}(x^j_k,u^j_{k+1})\right)
\end{aligned}$$

Since the construction the sequence of adjacent triangles $\mathcal{S}^j_i$ from $\mathcal{F}^j_i$ by adding triangles to ensure the adjacency of triangles in $\mathcal{S}^j_i$, we see that:

$$lp\left(\text{PSGP}_{\mathcal{F}^j_{i-1}}(x^j_i,u^j_i)\cup\text{PSGP}_{\mathcal{F}^j_i}(u^j_i,x^j_{i+1})\right) \geq lp\left(\text{PSGP}_{\mathcal{S}^j_i}(x^j_i,x^j_{i+1})\right). \tag{21}$$

Updating shooting points u^{*j}_i yields:

$$lp\left(\text{PSGP}_{\mathcal{S}^j_i}(x^j_i,x^j_{i+1})\right) = lp\left(\text{PSGP}_{\mathcal{S}^j_i}(x^j_i,u^{*j}_i)\cup\text{PSGP}_{\mathcal{S}^j_i}(u^{*j}_i,x^j_{i+1})\right). \tag{22}$$

Combining (21) with (22) gives:

$$\begin{aligned}
&l\left(\bigcup_{i=0}^{k}\text{SGP}_{\mathcal{F}^j_i}(u^j_i,u^j_{i+1})\right)\\
&\geq lp\left(\text{PSGP}_{\mathcal{F}^j_0}(u^j_0,x^j_0)\right) + \sum_{i=1}^{k-1} lp\left(\text{PSGP}_{\mathcal{S}^j_i}(x^j_i,u^{*j}_i)+\text{PSGP}_{\mathcal{S}^j_i}(u^{*j}_i,x^j_{i+1})\right) + lp\left(\text{PSGP}_{\mathcal{F}^j_{k+1}}(x^j_k,u^j_{k+1})\right)\\
&= lp\left(\text{PSGP}_{\mathcal{F}^j_0}(u^{j+1}_0,x^j_0)\right) + \sum_{i=1}^{k-1} lp\left(\text{PSGP}_{\mathcal{S}^j_i}(x^j_i,u^{j+1}_i)+\text{PSGP}_{\mathcal{S}^j_i}(u^{j+1}_i,x^j_{i+1})\right) + lp\left(\text{PSGP}_{\mathcal{F}^j_{k+1}}(x^j_k,u^{j+1}_{k+1})\right)\\
&= lp\left(\text{PSGP}_{\mathcal{F}^j_0}(u^{j+1}_0,x^j_0)\right) + lp\left(\text{PSGP}_{\mathcal{S}^j_0}(x^j_0,u^{j+1}_1)\right) + lp\left(\text{PSGP}_{\mathcal{S}^j_0}(u^{j+1}_1,x^j_1)\right) + lp\left(\text{PSGP}_{\mathcal{S}^j_1}(x^j_1,u^{j+1}_2)\right)\\
&+ \ldots + lp\left(\text{PSGP}_{\mathcal{S}^j_k}(u^{j+1}_k,x^j_k)\right) + lp\left(\text{PSGP}_{\mathcal{F}^j_{k+1}}(x^j_k,u^{j+1}_{k+1})\right).
\end{aligned}$$

For $i = 0, 1, \ldots, k$, $lp\left(\mathrm{PSGP}_{\mathcal{S}_i^j}(u_i^{j+1}, x_i^j)\right) + lp\left(\mathrm{PSGP}_{\mathcal{S}_{i+1}^j}(x_i^j, u_{i+1}^{j+1})\right) \geq lp\left(\mathrm{SGP}_{\mathcal{F}_i^j}(u_i^j, u_{i+1}^j)\right)$, we obtain:

$$
\begin{aligned}
& l\left(\bigcup_{i=0}^{k} \mathrm{SGP}_{\mathcal{F}_i^j}(u_i^j, u_{i+1}^j)\right) \\
& \geq lp\left(\mathrm{PSGP}_{\mathcal{F}_0^{j+1}}(u_0^{j+1}, u_1^{j+1})\right) + lp\left(\mathrm{PSGP}_{\mathcal{F}_1^{j+1}}(u_1^{j+1}, u_2^{j+1})\right) + \ldots + lp\left(\mathrm{PSGP}_{\mathcal{F}_k^{j+1}}(u_k^{j+1}, u_{k+1}^{j+1})\right) \\
& = l\left(\bigcup_{i=0}^{k} \mathrm{SGP}_{\mathcal{F}_i^{j+1}}(u_i^{j+1}, u_{i+1}^{j+1})\right).
\end{aligned}
$$

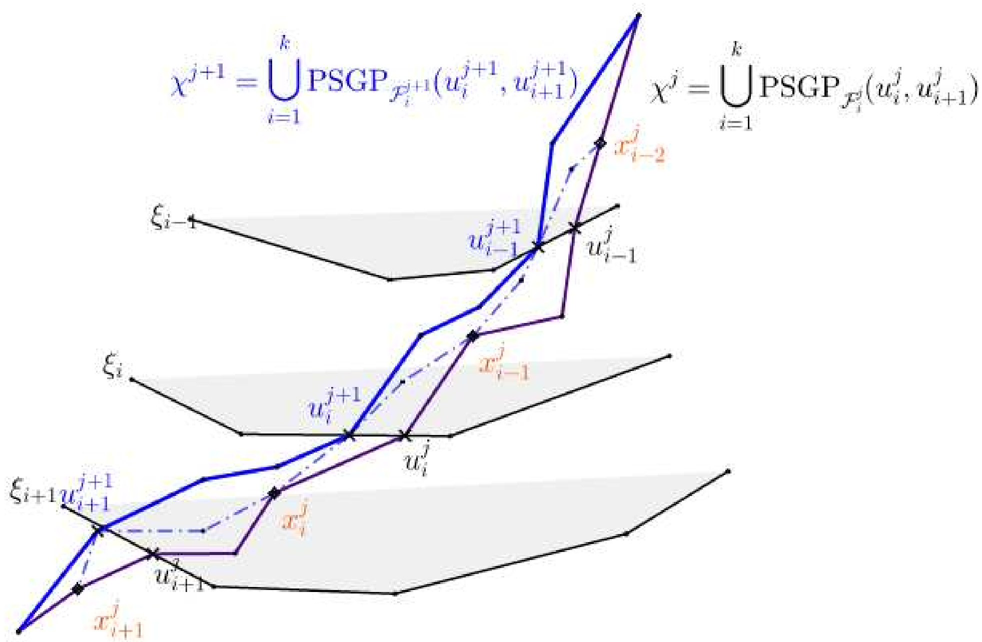

Figure 15: Update shooting points

If the Straightness Condition (B1) - (B2) does not hold at some shooting point u_i^j then the inequality in the formula (21) is strict. This completes the proof. □

The proof of Theorem 1:

Proof. At j-step, the algorithm 1 defines a family of paths $\tau^j = \bigcup_{i=0}^{k} \mathrm{SGP}_{\mathcal{F}_i^j}(u_i^j, u_{i+1}^j)$. For $i = 0, 1, \ldots, k$, take x_i^j on $\mathrm{PSGP}_{\mathcal{F}_i^j}(u_i^j, u_{i+1}^j)$ such that x_i^j does not coincide with u_i^j and u_{i+1}^j. There is a set of points $x_i^j, i = 0, 1, \ldots, k$ on the pseudo path $\chi^j = \cup_{i=0}^{k} \mathrm{PSGP}_{\mathcal{F}_i}(u_i^j, u_{i+1}^j)$.

Firstly, suppose that the Straightness Condition (B1) - (B2) in Sect. 4.3 does not hold at some shooting point u_i^j.

Case 1. If u_i^j does not satisfy (B1), then

$$lp\left(\mathrm{PSGP}_{\mathcal{S}_i^j}(x_{i-1}^j, x_i^j)\right) < lp\left(\mathrm{PSGP}_{\mathcal{F}_{i-1}^j}(x_{i-1}^j, u_i^j)\right) + lp\left(\mathrm{PSGP}_{\mathcal{F}_i^j}(u_i^j, x_{i+1}^j)\right)$$

where $\mathcal{S}_i^j$ is defined by (A1) in Sect. 4.3. Call the intersection of $\text{PSGP}_{\mathcal{S}_i^j}(x_{i-1}^j, x_i^j)$ with e_i^j is u_i^{*j}. In such case, u_i^j is updated to $u_i^{j+1} \equiv u_i^{*j}$.

Case 2. If u_i^j does not satisfy (B2), then

$$\min\{lp\left(\text{PSGP}_{\mathcal{S}_i^{*j}}(x_{i-1}^j, x_i^j)\right),\ lp\left(\text{PSGP}_{\mathcal{S}_i^{**j}}(x_{i-1}^j, x_i^j)\right)\} < lp\left(\text{PSGP}_{\mathcal{F}_{i-1}^j}(x_{i-1}^j, u_i^j)\right) + lp\left(\text{PSGP}_{\mathcal{F}_i^j}(u_i^j, x_{i+1}^j)\right)$$

where $\mathcal{S}_i^{*j}, \mathcal{S}_i^{**j}$ is defined in (A2) by Sect. 4.3. Let $\mathcal{S}_i^j \in \{\mathcal{S}_i^{*j}, \mathcal{S}_i^{**j}\}$ such that

$$lp\left(\text{PSGP}_{\mathcal{S}_i^j}(x_{i-1}^j, x_i^j)\right) = \min\{lp\left(\text{PSGP}_{\mathcal{S}_i^{*j}}(x_{i-1}^j, x_i^j)\right),\ lp\left(\text{PSGP}_{\mathcal{S}_i^{**j}}(x_{i-1}^j, x_i^j)\right)\}.$$

Call the intersection of $\text{PSGP}_{\mathcal{S}_i^j}(x_{i-1}^j, x_i^j)$ with e_i^j or e'^j_i is u_i^{*j}. In such case, u_i^j is updated to $u_i^{j+1} = u_i^{*j}$.

Hence, using the procedure finding the pseudo path $\text{PSGP}_{\mathcal{F}_i^{j+1}}(u_i^{j+1}, u_{i+1}^{j+1})$ of $\text{SGP}_{\mathcal{F}_i^{j+1}}(u_i^{j+1}, u_{i+1}^{j+1})$ constructed by Sect. 4.5, the length with $\|.\|_s$ norm of the pseudo path $\chi^{j+1} := \bigcup_{i=0}^k \text{PSGP}_{\mathcal{F}_i^{j+1}}(u_i^{j+1}, u_{i+1}^{j+1})$ joining p and q along the sequence of adjacent triangles $\cup_{i=0}^k \mathcal{F}_i^{j+1}$ is equal to the length of $\tau^{j+1} := \bigcup_{i=0}^k \text{SGP}_{\mathcal{F}_i^{j+1}}(u_i^{j+1}, u_{i+1}^{j+1})$.

We thus obtain a family of θ-gentle paths τ^j joining p and q along the sequence of adjacent triangles $\cup_{i=0}^k \mathcal{F}_i^{j+1}$. Applying Prop. 8 gives $l(\tau^{j+1}) < l(\tau^j)$. Therefore, the sequence of length of the family of paths τ^j is strictly reduced. The sequence $\{l(\tau^j)\}$ is then a convergent sequence. Denote $\sigma = \inf\{l(\tau^j), j \in \mathbb{N}\}$. Two norms $\|.\|_2$ and $\|.\|_s$ are equivalent in $\mathbb{R}^3$. Therefore, analysis similar to that in the proof of Proposition 1.4.11 in [15] shows that the sequence of corresponding pseudo paths χ^j has a subsequence χ^{j_n} that converges uniformly to some path χ. Denote τ is the θ-gentle path that gets by adjusting χ. Since $lp(\chi^{j_n}) = l(\tau^{j_n})$, we get $\lim_{n\to\infty} l(\tau^{j_n}) = l(\tau)$. Combining $\{\tau^{j_n}\} \subset \{\tau^j, j \in \mathbb{N}\}$ with the formula defining σ, we have $\sigma \le \lim_{n\to\infty} l(\tau^{j_n}) = l(\tau)$, i.e. $\sigma \le l(\tau)$. Since $\chi^{j_n} \rightrightarrows \chi$, we get $lp(\chi) \le \liminf_{n\to\infty} lp(\chi^{j_n})$. Otherwise, $lp(\chi^{j_n}) = l(\tau^{j_n})$, $lp(\chi) = l(\tau)$. Then $l(\tau) \le \sigma$. In conclusion, $l(\tau) = \sigma$. According to Definition 2, τ is an approximate shortest θ-gentle path. Secondly, suppose that at n_0^{th}-step, the Straightness Condition (B1) - (B2) in Sect. 4.3 holds at all shooting points. We receive a sequence of paths $\{(\tau^j), j \in \mathbb{N}\}$, where $\tau^j = \bigcup_{i=0}^k \text{SGP}_{\mathcal{F}_i^j}(u_i^j, u_{i+1}^j)$, for $j = 1, 2, \ldots, n_0$ and from n_0 onwards, all τ^j coincide with τ^{n_0}. As the sequence of length of the family of paths τ^j is not increasing, we get $\lim_{j\to\infty} l(\tau^j) = l(\tau^{n_0})$. According to Definition 2, τ^{n_0} is an approximate shortest θ-gentle path. This completes the proof. □

8.3 Solving Example 1

a) Finding the optimal solution of the problem given in Example 1:

We use the brute-force search to find all possible candidates for the solution and checking whether each candidate satisfies problem's statement. The method executes as follows: starting from vertex 1, we find all sequences of adjacent triangles (not self-cutting) joining the vertices 1 and 5. A tree of the sequences as seen in Fig 8 is constructed by 28 such sequences of adjacent triangles in which a shortest θ-gentle τ_0 joining p and q in the triangulated terrain $\mathcal{T}$ has the length $l(\tau_0) = 38.63046$.

We now solve the sub-problem for finding shortest θ-gentle paths on each sequence of adjacent triangles. Assume that τ is a θ-gentle path joining p and q on $\mathcal{F}$, where $\mathcal{F}$ is a sequence of adjacent triangles with common edges of adjacent triangles are consecutively $e_1, e_2, \ldots, e_m$. Let x_i be the intersection of τ and e_i, for $i = 1, 2, \ldots, m$. Then we can write $x_i = \lambda_i c_i + (1 - \lambda_i) d_i$, where $\lambda_i \in [0, 1]$ and c_i, d_i are endpoints of the edge e_i. Then $l(\tau) = \|p - x_1\|_s + \|x_1 - x_2\|_s + \ldots + \|x_{m-1} - x_m\|_s + \|x_m - q\|_s$, where $\|x_i - x_{i+1}\|_s$ is the length of $\mathrm{adj}[x_i, x_{i+1}]$, calculated as follows:

$$\max\{\sqrt{((x_i)_x - (x_{i+1})_x)^2 + ((x_i)_y - (x_{i+1})_y)^2 + ((x_i)_z - (x_{i+1})_z)^2}, \frac{|(x_i)_z - (x_{i+1})_z|}{\sin\theta}\}.$$

To find a shortest θ-gentle path on the sequence of adjacent triangles $\mathcal{F}$, we find τ such that $l(\tau)$ is minimum. This becomes the convex optimization problem (P1). Since the object function of (P1) is non-smooth, the difficultly of solving this problem increases when the number of variables rise.

b) Using Liu and Wong's algorithm for finding an approximate solution of the problem given in Example 1:

Firstly, Liu and Wong [13] present an algorithm called Surface Simplification to simplify surface $\mathcal{T}$ to surface $\tilde{\mathcal{T}}$ such that any mapped path found by their algorithm "Path Mapping" satisfies the distance requirement $\Delta(\mathcal{T}, \tilde{\mathcal{T}}) < 1 + \epsilon$ and the slope requirement. They check whether a vertex of $\mathcal{T}$ is removed or not using the condition $\Delta(\mathcal{T}, \tilde{\mathcal{T}}) < 1 + \epsilon$. Table 2 indicates that number of vertices to be removed depends on the tolerance ϵ.

Then, a simpler terrain $\tilde{\mathcal{T}}$ is generated. Next, they construct a tree of sequences of adjacent triangles connecting p and q and find all shortest θ-gentle paths joining two points p and q along these sequences of adjacent triangles. In such a case, they have to solve a number of sub-problems in oder to gain a shortest θ-gentle paths $\tilde{\tau}$ joining two corresponding points $\tilde{p}$ and $\tilde{q}$ on $\tilde{\mathcal{T}}$. Next, a path on $\mathcal{T}$ having the same shadow as $\tilde{\tau}$ is constructed. After adjusting this path to satisfying the slope requirement, they obtain an approximation θ-gentle path joining two points p and q, say τ. Table 2 indicates such paths τ corresponding to tolerances ϵ. If ϵ is too small, no vertex of $\mathcal{T}$ is removed and therefore there is no $\tilde{\mathcal{T}}$. Then the brute-force search mentioned in a) is used to find the optimal solution.