ON THE CONDITIONAL PLURISUBHARMONIC ENVELOPES OF BOUNDED FUNCTIONS

HOANG-SON DO AND GIANG LE

Abstract. In this paper, we extend some recent results of Guedj-Lu-Zeriahi [GLZ19] about psh envelopes of bounded functions on bounded domains in $\mathbb{C}^n$. We also present a result on the regularity of psh envelopes.

1. Introduction

In [GLZ19], Guedj-Lu-Zeriahi studied quasi-plurisubharmonic envelopes on compact Kähler manifolds and plurisubharmonic envelopes on domains of $\mathbb{C}^n$. By using and extending an approximation process due to Berman [Ber19], they show that the (quasi-)plurisubharmonic envelope of a viscosity super-solution is a pluripotential super-solution of a given complex Monge-Ampère equation. Our goal is to extend Guedj-Lu-Zeriahi's results for *conditional plurisubharmonic envelopes* on domains of $\mathbb{C}^n$.

Let $\Omega \subset \mathbb{C}^n$ be a bounded domain. Denote by $\mathcal{M}$ the set of Borel measures μ on Ω satisfying $\mu = (dd^c\varphi)^n$ for some bounded plurisubharmonic function φ in Ω. If $\mu \in \mathcal{M}$ and u is a bounded function in Ω then we define

$$P(u, \mu, \Omega) := (\sup\{v \in PSH(\Omega) \cap L^\infty(\Omega) : v \leq u, (dd^c v)^n \geq \mu\})^*.$$

By [Kol98], we have $fd\lambda \in \mathcal{M}$ for every $f \in L^p(\Omega), p > 1$, where λ is the Lebesgue measure in $\mathbb{C}^n$. If $f \in L^p(\Omega), p > 1$, then we also denote $P(u, f, \Omega) := P(u, fd\lambda, \Omega)$. The first main result of this paper is the following:

Theorem 1.1. *Assume that $\Omega \subset \mathbb{C}^n$ is a bounded pseudoconvex domain. Suppose that $f \in L^p(\Omega)(p > 1)$ and $g \in C(\Omega)$ are non-negative functions. If u is a bounded viscosity subsolution to the equation*

$$(dd^c w)^n = gd\lambda, \tag{1}$$

on Ω then $(dd^c P(u, f, \Omega))^n \leq \max\{f, g\}d\lambda$ in the pluripotential sense in Ω.

Corollary 1.2. *Assume that $\Omega \subset \mathbb{C}^n$ is a bounded pseudoconvex domain and and $0 \leq f, g \in L^p(\Omega), p > 1$. Suppose that u is a continuous plurisubharmonic on Ω such that $(dd^c u)^n = gd\lambda$ in the pluripotential sense. Then*

$$(dd^c P(u, f, \Omega))^n \leq \max\{f, g\}d\lambda.$$

In this paper, we also study the continuity of $P(u, f, \Omega)$ when u is continuous. Our second main result is the following:

Theorem 1.3. *Assume that Ω is a smooth strictly pseudoconvex domain. If $0 \leq f \in L^p(\Omega), p > 1$, and $u \in C(\overline{\Omega})$ then $P(u, f, \Omega) \in C(\overline{\Omega})$.*

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Corollary 1.4. *Assume that $\Omega \subset \mathbb{C}^n$ is a smooth strictly pseudoconvex domain and $U \subset \Omega$ is a hyperconvex domain. Then, for every $E \Subset U$, for each $0 \leq f \in L^p(\Omega), p > 1$, if $P(-\chi_E, f, U)$ is continuous then $P(-\chi_E, f, \Omega)$ is continuous.*

2. Some general properties

In this section, we give some properties of $P(u, \mu, \Omega)$, mainly about the convergence and stability. Some of them have been proved in [GLZ19] for the case $\mu = 0$.

Proposition 2.1. *Let u be a bounded function on Ω and $\mu \in \mathcal{M}$. Denote*

$$T = \{v \in PSH(\Omega) \cap L^\infty(\Omega) : v \leq u \text{ quasi everywhere, } (dd^c v)^n \geq \mu\}.$$

Then $P(u, \mu, \Omega) \in T$. Moreover, $P(u, \mu, \Omega) = \sup\{v : v \in T\}$.

Here $v \leq u$ *quasi everywhere* means that there exists a pluripolar set N such that $v \leq u$ on $\Omega \setminus N$.

Proof. Since negligible sets are pluripolar [BT82], we have $P(u, \mu, \Omega) = \sup_{v \in S} v$ quasi everywhere, where

$$S = \{v \in PSH(\Omega) \cap L^\infty(\Omega) : v \leq u, (dd^c v)^n \geq \mu\}.$$

Hence, $P(u, f, \Omega) \leq u$ quasi everywhere.

By Choquet lemma, there exists a sequence of functions $u_j \in S$ such that $P(u, f, \Omega) = (\sup_j u_j)^*$. Note that if $v, w \in PSH(\Omega) \cap L^\infty(\Omega)$ and $(dd^c v)^n, (dd^c w)^n \geq \mu$ then $(dd^c \max\{v, w\})^n \geq \mu$. Hence $(dd^c (\max_{j \leq k} u_j))^n \geq \mu$ for every k. Letting $k \to \infty$ and using [BT82, Theorem 2.6](see also [Kli91, Theorem 3.6.1]), we get $(dd^c P(u, f, \Omega))^n \geq \mu$. Then $P(u, f, \Omega) \in T$.

Now, let v be an arbitrary element of T. Then there exists $\varphi \in PSH^-(\Omega)$ such that $\{v > u\} \subset \{\varphi = -\infty\}$. Denote $M = \sup |u - v|$. We have

$$v_\epsilon := v + \max\{\epsilon\varphi, -M\} \in S,$$

for every $\epsilon > 0$. Letting $\epsilon \searrow 0$, we obtain

$$v = (\lim_{\epsilon \to 0^+} v_\epsilon)^* \leq P(u, \mu, \Omega).$$

Thus $P(u, \mu, \Omega) = \sup\{v : v \in T\}$. □

Corollary 2.2. *Let u be a bounded function on Ω and $\mu \in \mathcal{M}$. Then*

$$P(u, \mu, \Omega) = P(P(u, 0, \Omega), \mu, \Omega).$$

Proposition 2.3. *Let u be a bounded function on Ω and $\mu \in \mathcal{M}$. If Ω_j is an increasing sequence of relative compact domains in Ω such that $\cup_j \Omega_j = \Omega$ then $P(u, \mu, \Omega_j)$ decreases to $P(u, \mu, \Omega)$.*

Proof. By the definition, we have

$$P(u, \mu, \Omega) \leq P(u, \mu, \Omega_{j+1}) \leq P(u, \mu, \Omega_j),$$

on Ω_j for every j. Denote $v = \lim_{j \to \infty} P(u, \mu, \Omega_j)$. Then v is a bounded plurisubharmonic function on Ω satisfying

$$P(u, \mu, \Omega) \leq v, \tag{2}$$

and

$$(dd^c v)^n \geq \mu. \tag{3}$$

It follows from Proposition 2.1, that $P(u,\mu,\Omega_j)\leq u$ quasi everywhere on Ω_j. Then $v\leq u$ quasi everywhere on Ω. Hence, by the last assertion of Proposition 2.1 and by (3), we get

$$v\leq P(u,\mu,\Omega). \tag{4}$$

Combining (2) and (4), we obtain $v=P(u,\mu,\Omega)$. Thus $P(u,\mu,\Omega_j)$ decreases to $P(u,\mu,\Omega)$ as $j\to\infty$. □

Proposition 2.4. *Let $u, u_j (j\in\mathbb{Z}^+)$ be bounded functions on Ω and $\mu\in\mathcal{M}$. Then the following statements hold:*

(i) *If u_j decreases to u as $j\to\infty$ then $P(u_j,\mu,\Omega)$ decreases to $P(u,\mu,\Omega)$.*

(ii) *Assume that u_j is continuous for every j. If u_j increases to u as $j\to\infty$ then $P(u_j,\mu,\Omega)$ increases to $P(u,\mu,\Omega)$ almost everywhere.*

Proof. (i) By the definition, we have

$$P(u,\mu,\Omega)\leq P(u_{j+1},\mu,\Omega)\leq P(u_j,\mu,\Omega),$$

for every j. Then

$$v:=\lim_{j\to\infty}P(u_j,\mu,\Omega)\geq P(u,\mu,\Omega). \tag{5}$$

Since $(dd^cP(u_j,\mu,\Omega))^n\geq\mu$ for every j, we also have

$$(dd^cv)^n\geq\mu. \tag{6}$$

It follows from Proposition 2.1 that $P(u_j,\mu,\Omega)\leq u_j$ quasi everywhere on Ω_j. Letting $j\to\infty$, we get $v\leq u$ quasi everywhere on Ω. Hence, by the last assertion of Proposition 2.1 and by (3), we have

$$v\leq P(u,\mu,\Omega). \tag{7}$$

Combining (5) and (7), we obtain $v=P(u,\mu,\Omega)$. Thus $P(u_j,\mu,\Omega)$ decreases to $P(u,\mu,\Omega)$ as $j\to\infty$.

(ii) By the defintion, we have

$$P(u,\mu,\Omega)\geq P(u_{j+1},\mu,\Omega)\geq P(u_j,\mu,\Omega),$$

for every j. Then

$$v:=(\lim_{j\to\infty}P(u_j,\mu,\Omega))^*\leq P(u,\mu,\Omega). \tag{8}$$

We will show that $v=\sup_{w\in T}w$, where

$$T=\{w\in PSH(\Omega)\cap L^\infty(\Omega): w\leq u \text{ quasi everywhere}, (dd^cw)^n\geq\mu\}.$$

Since $(dd^cP(u_j,\mu,\Omega))^n\geq\mu$ for every j, we have

$$(dd^cv)^n\geq\mu. \tag{9}$$

Combining (8) and (9) and using Proposition 2.1, we get that

$$v\in T. \tag{10}$$

Let $\varphi\in T$. Since $\varphi-u\leq 0$ and $u_j-u\nearrow 0$, we have $\max\{\varphi-u_j,0\}$ decreases to 0. Denote by $\hat\varphi$ the upper semicontinuous extension of φ to $\overline{\Omega}$, i.e.,

$$\hat\varphi(\xi):=\lim_{r\to0^+}\sup_{B(\xi,r)\cap\Omega}\varphi,\ \forall\xi\in\partial\Omega.$$

Then $\max\{\hat{\varphi} - u_j, 0\}$ decreases to 0 on $\overline{\Omega}$ as $j \to \infty$. It follows from Dini's theorem that $\max\{\varphi - u_j, 0\}$ converges uniformly on $\overline{\Omega}$ to 0. Hence, for every $\epsilon > 0$, there exists j such that

$$\varphi - \epsilon \leq u_j.$$

Then

$$\varphi - \epsilon \leq P(u_j, \mu, \Omega) \leq v.$$

Since φ and ϵ are arbitrary, we get

$$v \geq \sup_{w \in T} w. \tag{11}$$

Combining (10) and (11), we have

$$v = \sup\nolimits_{w \in T} w.$$

Hence, by Proposition 2.1, we obtain $v = P(u, \mu, \Omega)$. Thus, $P(u_j, \mu, \Omega)$ increases to $P(u, \mu, \Omega)$ almost everywhere. □

Proposition 2.5. *Let u be a bounded function on Ω and $0 \leq f, g \in L^p(\Omega)$ for some $p > 1$. Then, there exists a uniform constant $C > 0$ such that*

$$|P(u, f, \Omega) - P(u, g, \Omega)| \leq C(\|f - g\|_{L^p(\Omega)})^{1/n}.$$

Proof. Let D be a smooth strictly pseudoconvex domain in $\mathbb{C}^n$ such that $\Omega \Subset D$. Then, by [Kol98, Corollary 3.1.3], there exists $\phi \in PSH(D) \cap C(\overline{D})$ such that $(dd^c\phi)^n = \chi_\Omega |f - g|$ and $\phi|_{\partial D} = 0$. By Proposition 2.1, we have

$$P(u, f, \Omega) + \phi|_\Omega \leq P(u, g, \Omega),$$

and

$$P(u, g, \Omega) + \phi|_\Omega \leq P(u, f, \Omega).$$

Then

$$\sup_\Omega |P(u, f, \Omega) - P(u, g, \Omega)| \leq \sup_\Omega |\phi|. \tag{12}$$

Using [GKZ08, Theorem 1.1] for $\phi/(\|f - g\|_{L^p(\Omega)})^{1/n}$ and 0, with $\gamma = 0$, we have

$$\sup_D \frac{|\phi|}{(\|f - g\|_{L^p(\Omega)})^{1/n}} \leq C, \tag{13}$$

where $C > 0$ is a uniform constant.

Combining (12), (13), we get

$$|P(u, f, \Omega) - P(u, g, \Omega)| \leq C(\|f - g\|_{L^p(\Omega)})^{1/n}.$$

□

Proposition 2.6. *Let $\Omega \subset \mathbb{C}^n$ be a bounded hyperconvex domain. Assume that $u \in USC(\Omega) \cap L^\infty(\Omega)$, $v \in LSC(\Omega) \cap L^\infty(\Omega)$, $\mu \in \mathcal{M}$ and $W \Subset \Omega$. Denote $M = \sup_W |u - v|$. Then*

$$Cap(\{|P(u, \mu, W) - P(v, \mu, W)| \geq M\epsilon\}, \Omega) \leq \frac{2(n!)^2}{\epsilon^n} Cap(\{|u - v| \geq \epsilon\} \cap \overline{W}, \Omega),$$

for every $\epsilon > 0$.

Here $Cap(E,\Omega)$ is the relative capacity defined by Bedford-Taylor [BT82] as follows:

$$Cap(E,\Omega)=\sup\{\int_E (dd^c v)^n : v\in PSH(\Omega,[0,1])\}. \tag{14}$$

Proof. Denote $E_1=\{u-v\geq\epsilon\}\cap\overline{W}$, $E_2=\{u-v<-\epsilon\}\cap W$ and $\delta=Cap(\{|u-v|\geq\epsilon\}\cap\overline{W},\Omega)$. Then

$$Cap(E_j,\Omega)\leq\delta,\quad j=1,2.$$

Since E_1 is compact and E_2 is open, we have

$$Cap(E_j,\Omega)=Cap^*(E_j,\Omega),\quad j=1,2.$$

Denote $E=E_1\cup E_2$. We have

$$Cap^*(E,\Omega)\leq Cap^*(E_1,\Omega)+Cap^*(E_2,\Omega)\leq 2\delta. \tag{15}$$

Let $h_E=\sup\{h\in PSH^-(\Omega): h|_E\leq -1\}$. It follows from [BT82, Proposition 6.5] that

$$\int_\Omega (dd^c h_E^*)^n = Cap^*(E,\Omega)\leq 2\delta. \tag{16}$$

By using [Xin96, Lemma 1] for h_E^* and 0, we get

$$\int_\Omega (-h_E^*)^n (dd^c h)^n \leq (n!)^2\int_\Omega (dd^c h_E^*)^n\leq 2(n!)\delta,$$

for all $h\in PSH(\Omega,[0,1])$. Hence

$$Cap(\{h_E^*<-\epsilon\},\Omega)\leq\frac{2(n!)^2\delta}{\epsilon^n}. \tag{17}$$

Note that, by [BT82], $h_E^*=h_E$ quasi everywhere. Then, by the definition of h_E, we have

$$u+Mh_E^*\leq v \text{ and } v+Mh_E^*\leq u,$$

quasi everywhere in W. Hence

$$P(u,\mu,W)+Mh_E^*\leq P(v,\mu,W) \text{ and } P(v,\mu,W)+Mh_E^*\leq P(u,\mu,W).$$

Then

$$Cap(\{|P(u,\mu,W)-P(v,\mu,W)|\geq M\epsilon\},\Omega)\leq Cap(\{h_E^*<-\epsilon\},\Omega)\leq\frac{2(n!)^2\delta}{\epsilon^n}.$$

□

3. Proof of the main theorems

3.1. **Proof of Theorem 1.1.** We use the same method as in the proof of [GLZ19, Theorem 3.9.]. First, we prove a special case of Theorem 1.1.

Proposition 3.1. *Assume that $\Omega \subset \mathbb{C}^n$ is a bounded smooth strictly pseudoconvex domain and $0 \leq f, g \in C(\overline{\Omega})$. If $u \in C(\overline{\Omega})$ is a viscosity subsolution to the equation*

$$(dd^c w)^n = g d\lambda, \tag{18}$$

on Ω then $(dd^c P(u, f, \Omega))^n \leq \max\{f, g\} d\lambda$ in the pluripotential sense in Ω.

Proof. By [BT76], for every $j \in \mathbb{Z}^+$, there exists $u_j \in PSH(\Omega) \cap C(\overline{\Omega})$ such that

$$(dd^c u_j)^n = \max\{e^{j(u_j - u)} g, f\} d\lambda, \tag{19}$$

in the pluripotential sense in Ω and $u_j = u$ on $\partial\Omega$.

By [EGZ11], u_j satisfies (19) in the viscosity sense. Applying the viscosity comparison principle [EGZ11, DDT19] to the equation $(dd^c w)^n = \max\{e^{j(w-u)} g, f\} d\lambda$, we get $u_j \leq u$ and $u_j \leq u_{j+1}$ for every $j \in \mathbb{Z}^+$. Denote $v := (\lim_{j\to\infty} u_j)^*$. We have

$$\max\{f, g\} d\lambda \geq (dd^c u_j)^n \xrightarrow{weak} (dd^c v)^n \geq f d\lambda,$$

and

$$v \leq P(u, f, \Omega) \leq u.$$

It remains to show that $v \geq P(u, f, \Omega)$. For every $j \in \mathbb{Z}^+$, $\epsilon > 0$ and $h \in PSH(\Omega, [-1, 0))$, we denote

$$E(j, \epsilon, h) = \{z \in \Omega : u_j < P(u, f, \Omega) - \epsilon + \epsilon h\}.$$

By Proposition 2.1, we have $(dd^c P(u, f, \Omega))^n \geq f d\lambda$. Then

$$\int_{E(j,\epsilon,h)} (f d\lambda + \epsilon^n (dd^c h)^n) \leq \int_{E(j,\epsilon,h)} (dd^c(P(u, f, \Omega) + \epsilon dd^c h))^n. \tag{20}$$

By the Bedford-Taylor comparison principle, we have

$$\int_{E(j,\epsilon,h)} (dd^c(P(u, f, \Omega) + \epsilon dd^c h))^n \leq \int_{E(j,\epsilon,h)} (dd^c u_j)^n. \tag{21}$$

Since $u_j - u \leq -\epsilon$ in $E(j, \epsilon, h)$, we get

$$\int_{E(j,\epsilon,h)} (dd^c u_j)^n = \int_{E(j,\epsilon,h)} \max\{e^{j(u_j-u)} g, f\} d\lambda \leq \int_{E(j,\epsilon,h)} \max\{e^{-j\epsilon} g, f\} d\lambda. \tag{22}$$

Combining (20), (21) and (22), we obtain

$$\epsilon^n \int_{E(j,\epsilon,h)} (dd^c h)^n \leq \int_{E(j,\epsilon,h)} (\max\{e^{-j\epsilon} g, f\} - f) d\lambda.$$

Then

$$\epsilon^n \int_{\{u_j \leq P(u,f,\Omega) - 2\epsilon\}} (dd^c h)^n \leq \int_\Omega (\max\{e^{-j\epsilon} g, f\} - f) d\lambda.$$

Since $h \in PSH(\Omega, [-1, 0))$ is arbitrary, it implies that

$$Cap(\{u_j \leq P(u,f,\Omega) - 2\epsilon\}, \Omega) \leq \frac{1}{\epsilon^n} \int\limits_{\Omega} (\max\{e^{-j\epsilon} g, f\} - f) d\lambda,$$

where $Cap(.,\Omega)$ is the relative capacity of Bedford-Taylor (see (14)). Letting $j \to \infty$, we get

$$Cap(\{v \leq P(u,f,\Omega) - 2\epsilon\}, \Omega) \leq \lim_{j\to\infty} Cap(\{u_j \leq P(u,f,\Omega) - 2\epsilon\}, \Omega) = 0.$$

Then $v \geq P(u,f,\Omega) - 2\epsilon$. Since $\epsilon > 0$ is arbitrary, we obtain $v \geq P(u,f,\Omega)$. □

Proposition 3.2. *Assume that $\Omega \subset \mathbb{C}^n$ is a bounded pseudoconvex domain and $W \Subset \Omega$ is a smooth strictly pseudoconvex domain. Suppose that $0 \leq f, g \in C(\Omega)$. If u is a bounded viscosity subsolution to the equation*

$$(dd^c w)^n = g d\lambda, \tag{23}$$

on Ω then $(dd^c P(u,f,W))^n \leq \max\{f,g\} d\lambda$ in the pluripotential sense in W.

Proof. Let $A > \sup_\Omega |u|$ and denote $B = \inf\{d(x,y) : x \in W, y \in \partial\Omega\} > 0$. For every $m > m_0 := \frac{2A}{B}$, we consider

$$u_m(z) := \inf\{u(z+\xi) + m|\xi| : |\xi| < \frac{m_0 B}{m}\},$$

for $z \in \overline{W}$. Then (u_m) is an increasing sequence of continuous functions in $\overline{W}$ satisfying $\lim_{m\to\infty} u_m = u$ and

$$(dd^c u_m)^n \leq g_m d\lambda,$$

in W in the viscosity sense, where $g_m(z) = \sup\{g(z+\xi) : |\xi| < \dfrac{m_0 B}{m}\}$. By Proposition 3.1, we have

$$(dd^c P(u_m, f, W))^n \leq \max\{f, g_m\} d\lambda,$$

in the pluripotential sense in W for all $m > m_0$.

Letting $m \to \infty$ and using Proposition 2.4, we obtain

$$(dd^c P(u,f,W))^n \leq \max\{f,g\} d\lambda.$$

□

Proof of Theorem 1.1. Since Ω is pseudoconvex, there exists an increasing sequence of smooth strictly pseudoconvex domains $\Omega_j \Subset \Omega$ such that $\cup_j \Omega_j = \Omega$ (see, for example, [Hor73, Theorem 2.6.11]). Let f_k be a sequence of continuous functions in $\mathbb{C}^n$ such that f_k converges to f in L^p as $k \to \infty$. By Proposition 3.2, we have

$$(dd^c P(u, f_k, \Omega_j))^n \leq \max\{f_k, g\} d\lambda,$$

in the pluripotential sense in Ω_j for all $j, k \in \mathbb{Z}^+$. Letting $j \to \infty$ and using Proposition 2.3, we get

$$(dd^c P(u, f_k, \Omega))^n \leq \max\{f_k, g\} d\lambda,$$

in the pluripotential sense in Ω for all $j, k \in \mathbb{Z}^+$. Moreover, it follows from Proposition 2.5 that $P(u, f_k, \Omega)$ converges uniformly to $P(u,f,\Omega)$ as $k \to \infty$. Thus

$$(dd^c P(u,f,\Omega))^n = \lim_{k\to\infty} (dd^c P(u,f_k,\Omega))^n \leq \lim_{k\to\infty} \max\{f_k, g\} d\lambda = \max\{f,g\} d\lambda.$$

The proof is completed. □

3.2. **Proof of Theorem 1.3.** We proceed through some lemmas.

Lemma 3.3. *Assume that Ω is a smooth strictly pseudoconvex domain and $u, f \in C^\infty(\overline{\Omega})$ with $f \geq 0$. Then, there exists $C > 0$ such that, for every $\delta > 0$, if $\Omega_\delta := \{z \in \Omega : d(z, \partial\Omega) > \delta\} \neq \emptyset$ then*

$$P(u, f, \Omega_\delta) \leq P(u, f, \Omega) + C\delta,$$

on Ω_δ.

Proof. Since Ω is a smooth strictly pseudoconvex, there exists $\rho \in C^\infty(\overline{\Omega}) \cap PSH(\Omega)$ such that $\rho|_{\partial\Omega} 0$, $\inf_\Omega \det(\rho_{\alpha\overline{\beta}}) > 0$ and

$$-C_1 d(z, \partial\Omega) \leq \rho(z) \leq -C_2 d(z, \partial\Omega),$$

for every $z \in \Omega$, where $0 < C_2 < C_1$.

Let $M \gg 1$ such that $(M\rho + u)$ is plurisubharmonic in Ω and $(dd^c(M\rho + u))^n \geq f d\lambda$.

For every $0 < \delta < 1$, if $\Omega_\delta \neq \emptyset$ then we define

$$v_\delta = \begin{cases} M\rho + u & \text{on} \quad \Omega \setminus \Omega_\delta, \\ \max\{M\rho + u, P(u, f, \Omega_\delta) - 2MC_1\delta\} & \text{on} \quad \Omega_\delta. \end{cases}$$

Then $v_\delta \in PSH(\Omega) \cap L^\infty(\Omega)$, $v_\delta \leq u$ and $(dd^c v_\delta)^n \geq f d\lambda$. Hence

$$v \leq P(u, f, \Omega). \tag{24}$$

Moreover, by the definition of v_δ, we have

$$v_\delta|_{\Omega_\delta} \geq P(u, f, \Omega_\delta) - 2MC_1\delta. \tag{25}$$

By (24) and (25), we obtain

$$P(u, f, \Omega_\delta) \leq P(u, f, \Omega) + 2MC_1\delta,$$

on Ω_δ.

The proof is completed. □

Lemma 3.4. *Let $u \in PSH(\Omega) \cap L^\infty(\Omega)$ and $0 \leq f \in L^p(\Omega), p > 1$. Suppose that $\phi : \Omega \to W$ is a biholomorphic mapping. Then*

$$P(u, f, \Omega) \circ \phi^{-1} = P(u \circ \phi, (f \circ \phi).|J_\phi|^2, \phi^{-1}(\Omega)).$$

Proof. The proof is straightforward from the definitions of $P(u, f, \Omega)$ and $P(u \circ \phi, (f \circ \phi).|J_\phi|^2, \phi^{-1}(\Omega))$. □

Lemma 3.5. *Assume that Ω is a smooth strictly pseudoconvex domain and $u, f \in C^\infty(\overline{\Omega})$ with $f \geq 0$. Then $P(u, f, \Omega)$ is Lipschitz.*

Proof. Since Ω is bounded and smooth, there exists a constant $A > 0$ such that, for every $z_0, z \in \Omega$,

$$|z - z_0| \leq A \inf\{length(\gamma) : \gamma \in C^1([0, 1], \Omega), \gamma(0) = z_0, \gamma(1) = z\}.$$

Hence, $P(u, f, \Omega)$ is Lipschitz iff

$$\sup_{z_0 \in \Omega} \limsup_{z \to z_0} \frac{|P(u, f, \Omega)(z) - P(u, f, \Omega)(z_0)|}{|z - z_0|} < \infty. \tag{26}$$

Let $a, b \in \Omega, a \neq b$, such that $\delta := |a-b| \leq \dfrac{1}{2}\min\{d(a,\partial\Omega), d(b,\partial\Omega)\}$. Since $a - b + \Omega_\delta \subset \Omega$, we have

$$P(u,f,\Omega)(a) \leq P(u,f,b-a+\Omega_\delta)(a) \tag{27}$$

By using Lemma 3.4 for $\phi(z) = z + b - a$, we have

$$P(u,f,b-a+\Omega_\delta)(a) = P(u(z+b-a), f(z+b-a), \Omega_\delta)(b). \tag{28}$$

Since $u, f \in C^1(\overline{\Omega})$, there exists $C_1 > 0$ such that

$$|u(z) - u(z')| \leq C_1|z - z'|,$$

and

$$|f(z) - f(z')| \leq C_1|z - z'|,$$

for every $z, z' \in \Omega$. Hence

$$P(u(z+b-a), f(z+b-a), \Omega_\delta)(b) \leq C_1\delta + P(u, (f - C_1\delta)_+, \Omega_\delta)(b). \tag{29}$$

By Proposition 2.5, there exists $C_2 > 0$ that does not depend on δ such that

$$P(u, (f - C_1\delta)_+, \Omega_\delta) \leq C_2\delta + P(u, f, \Omega_\delta). \tag{30}$$

Moreover, it follows from Lemma 3.3 that

$$P(u, f, \Omega_\delta) \leq C_3\delta + P(u, f, \Omega), \tag{31}$$

where $C_3 > 0$ that does not depend on δ. By combining (27), (28), (29), (30) and (31), we obtain

$$P(u,f,\Omega)(a) \leq P(u,f,\Omega)(b) + C\delta,$$

where $C > 0$ that does not depend on δ. Similarly, we have

$$P(u,f,\Omega)(b) \leq P(u,f,\Omega)(a) + C\delta.$$

Then

$$|P(u,f,\Omega)(a) - P(u,f,\Omega)(b)| \leq C\delta = C|a-b|.$$

Thus, for every $a \in \Omega$,

$$\limsup_{b\to a} \frac{|P(u,f,\Omega)(b) - P(u,f,\Omega)(a)|}{|b-a|} \leq C,$$

and we get (26). □

Proof of Theorem 1.3. Let $u_j, f_j \in C^\infty(\overline{\Omega})$ such that $f_j \geq 0$, u_j converges uniformly to u and f_j converges in $L^p(\Omega)$ to f. By Lemma 3.5, we have $P(u_k, f_j, \Omega)$ is continuous for every $j, k \in \mathbb{Z}^+$. Moreover, since u_j converges uniformly to u, we have $P(u_k, f_j, \Omega)$ converges uniformly to $P(u, f_j, \Omega)$ as $k \to \infty$. Hence $P(u, f_j, \Omega)$ is continuous for every j. Since f_j converges in $L^p(\Omega)$ to f, it follows from Proposition 2.5 that $P(u, f_j, \Omega)$ converges uniformly to $P(u, f, \Omega)$ as $j \to \infty$. Then $P(u, f, \Omega)$ is continuous.

The proof is completed. □

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Institute of Mathematics, Vietnam Academy of Science and Technology, 18 Hoang Quoc Viet, Hanoi, Vietnam

Email address: hoangson.do.vn@gmail.com, dhson@math.ac.vn

Department of Mathematics, Hanoi National University of Education, 136-Xuan Thuy, Cau Giay, Hanoi, VietNam

Email address: legiang@hnue.edu.vn, legiang01@yahoo.com