IET Research Journals

Brief Paper

ISSN 1751-8644
doi: 0000000000
www.ietdl.org

Global attractivity and asymptotic stability of mixed-order fractional systems

H. T. Tuan[1], *H. Trinh*[2,*]

[1] *Institute of Mathematics, Vietnam Academy of Science and Technology, 18 Hoang Quoc Viet, 10307 Hanoi, Vietnam*
[2] *School of Engineering, Deakin University, Geelong, VIC 3217, Australia*
* *E-mail: hieu.trinh@deakin.edu.au*

Abstract: This study investigates the asymptotic properties of mixed-order fractional systems. By using the variation of constants formula, properties of real Mittag-Leffler functions, and Banach fixed-point theorem, the authors first propose an explicit criterion guaranteeing global attractivity for a class of mixed-order linear fractional systems. The criterion is easy to check requiring the system's matrix to be strictly diagonally dominant and elements on the main diagonal of this matrix to be negative (this is called NSDD). The authors then show the asymptotic stability of the trivial solution to a nonlinear mixed-order fractional system linearized along its equilibrium point such that its linear part is NSDD. Two numerical example is given to illustrate the effectiveness of the results over existing ones in the literature.

1 Introduction

Fractional calculus, i.e. calculus of integrals and derivatives of arbitrary orders, is a fascinating field of mathematics borne out of the traditional definitions of calculus integral and derivative operators. There are sufficient studies to show that fractional-order differential equations are suitable for capturing many complex phenomena in science and engineering, particularly when describing memory and hereditary properties of dynamical processes (for examples, see [1]-[5]). One of the most important topics in the qualitative theory of fractional-order differential equations is the study of the asymptotic behaviors of their solutions. This topic has received significant research attention in the literature (for examples, see [8]-[15] and the references therein).

In this paper we discuss on mixed-order fractional systems. They arise, for example, the Basset equation (it describes the forces that occur when a spherical particle sinks in a incompressible viscous fluid, see [16]), the Bagley–Torvik equation (this equation performs the motion of a rigid plate immersed in a Newtonian fluid, see [17]) and a general FitzHugh-Nagumo neuronal model which expresses a biological neuron's spiking behavior, see [18]. Recently, linear mixed-order fractional systems are used to formulate a model of national economies in a study of commonwealth countries which cannot be simply divided into clear groups of independent and dependent variables, see [19].

In contrast to systems having a fractional derivative term, the research on asymptotic behavior of solutions to mixed-order fractional systems is still in its early phase. So far, only some results on these systems are available. By using Laplace transform, in [20, Theorem 1], the authors proposed a criterion based on a characteristic equation to test global attractivity of mixed-order linear fractional systems. However, the criterion is not suitable for dealing with systems of high dimensions and irrational fractional orders. Mixed-order linear fractional systems with rational fractional derivatives were discussed in [21, Theorem 4]. By exploiting the monotonic and the asymptotic properties of constant delay systems by virtue of the positivity, and comparing trajectory of time-varying delay systems with that of constant delay systems, [22, Theorem 2] proved that asymptotic stability of mixed-order positive fractional systems is not sensitive to the magnitude of delays. The existence and uniqueness of solutions to general mixed-order fractional systems was considered in [23, Theorem 2.3]. The authors also derived a criterion based on the spectrum of the matrix of coefficients to test global attractivity of the trivial solution to mixed-order fractional upper-triangle systems. In [25], the authors derived a sufficient condition for the stability of the trivial solutions to mixed-order fractional systems, see [25, Theorem 4]. However, there are limitations in the works mentioned above. In particular, it is not easy to check the attractivity criteria presented in [20, 21] because they lead to finding solutions to fractional-order characteristic polynomials, a task which is very complicated for systems with high dimension, and impossible for systems with irrational fractional derivatives. Whereas, the result [25, Theorem 4] is elegant but it is too hard to compute the contractive coefficient γ in this theorem (see [25, condition (14), pp. 6]).

From the above analysis we find that studying asymptotic properties of fixed-order fractional systems is a complicated task. In our view, it is still a long way from being able to find a general and efficient criterion that characterizes their stability even in the linear case. Our aim is to propose simple and explicit theorems proving asymptotic stability of the trivial solution to a special class of systems with multi fractional orders.

Let the linear systems on the interval $[0, \infty)$

$$^{C}D^{\alpha_i}_{0+}x_i(t) = \sum_{j=1}^{d} a_{ij}x_j(t), \qquad i = 1, \dots, d, \tag{1}$$

where $\alpha_i \in (0, 1]$, coefficients $a_{ij} \in \mathbb{R}$, $i, j = 1, \dots, d$, and the Caputo differential operator of order α_i is defined by

$$^{C}D^{\alpha_i}_{0+}x(t) = J^{1-\alpha_i}Dx(t)$$

with the classical derivative D and the Riemann-Liouville integral operator

$$J^{1-\alpha_i}x(t) = \frac{1}{\Gamma(1-\alpha_i)}\int_0^t (t-\tau)^{-\alpha_i}x(\tau)\,d\tau,$$

see e.g. [2].

The first our contribution is to show that if the matrix of coefficients $(a_{ij})_{1\le i,j\le d}$ is strictly diagonally dominant and the entries on its main diagonal are negative then this system is globally attractive. To do this we focus on elements on the main diagonal. Due to their role in determining asymptotic behavior of the solutions, we consider (1) as a diagonal system and the ones out-off the main diagonal as perturbed terms. Then applying the variation of constants formula and properties of real Mittag-Leffler functions, we calculate clearly the contractive coefficient of a Lyapunov–Perron type operator associated with the system which guarantees its stability.

Next, by the linearized method, we also obtain a result on asymptotic stability (in the Lyapunov sense) of the trivial solution to a nonlinear mixed-order fractional system linearized along its equilibrium point

$$ {}^{C}D^{\alpha_i}_{0+}x_i(t) = \sum_{j=1}^{d} a_{ij}x_j(t) + f_i(x(t)), \qquad i = 1, \dots, d, \quad (2) $$

where the matrix $(a_{ij})_{1\le i,j\le d}$ satisfies two conditions as mentioned above, and the nonlinear function $f = (f_1, \dots, f_d) : \mathbb{R}^d \to \mathbb{R}^d$ describes a small perturbation satisfying:

(i) f is continuous;
(ii) $f(0) = 0$;
(iii) f is Lipschitz continuous in a neighborhood of the origin and

$$ \lim_{r\to 0} \ell_f(r) = 0, $$

in which

$$ \ell_f(r) := \sup_{x,y\in B_{\mathbb{R}^d}(0,r)} \frac{\|f(x)-f(y)\|}{\|x-y\|} \quad (3) $$

with $\|\cdot\|$ is a norm in $\mathbb{R}^d$ and $B_{\mathbb{R}^d} := \{x \in \mathbb{R}^d : \|x\| \le r\}$.

Notation. We denote by $\mathbb{N}$ the set of all natural numbers, $\mathbb{R}$ the set of all real numbers, $\mathbb{R}_{\ge 0}$ the set of nonnegative real numbers and $\mathbb{C}$ the set of complex numbers. Let $\mathbb{R}^d$ be the d-dimensional Euclidean space endowed with a norm $\|\cdot\|$. Due to the fact that every norm in $\mathbb{R}^d$ are equivalent, hence without loss of generality, in this paper we only use the max norm, that is, for any $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ we mean $\|x\| = \max\{|x_1|, \dots, |x_d|\}$. The set $C([0,\infty), \mathbb{R}^d)$ is the space of continuous functions from $[0,\infty)$ to $\mathbb{R}^d$, and $\left(C_\infty(\mathbb{R}^d), \|\cdot\|_\infty\right) \subset C([0,\infty), \mathbb{R}^d)$ is the space of all continuous functions $\xi : [0,\infty) \to \mathbb{R}^d$ which are bounded on $[0,\infty)$, i.e.,

$$ \|\xi\|_\infty := \sup_{t\ge 0} \|\xi(t)\| < \infty. $$

It is well known that $\left(C_\infty(\mathbb{R}^d), \|\cdot\|_\infty\right)$ is a Banach space.

2 Main result

Consider the system (1). For convenience, we use the notation

$$ {}^{C}D^{\hat{\alpha}}_{0+}x(t) := \begin{pmatrix} {}^{C}D^{\alpha_1}_{0+} & & \\ & \ddots & \\ & & {}^{C}D^{\alpha_d}_{0+} \end{pmatrix} x(t) = \begin{pmatrix} {}^{C}D^{\alpha_1}_{0+}x_1(t) \\ \vdots \\ {}^{C}D^{\alpha_d}_{0+}x_d(t) \end{pmatrix} $$

and $A = (a_{i,j})_{1\le i,j\le d} \in \mathbb{R}^{d\times d}$. Then this system is rewritten in the form

$$ {}^{C}D^{\hat{\alpha}}_{0+}x(t) = Ax(t). \quad (4) $$

Assume that the matrix A satisfies two conditions:

(C1) A is a strictly diagonally dominant matrix, i.e. $|a_{ii}| > \sum_{1\le j\le d, j\ne i} |a_{ij}|$ for $1 \le i \le d$; and

(C2) the entries on the main diagonal of this matrix are negative, i.e. $a_{ii} < 0$ for $1 \le i \le d$.

Our main result is to prove global attractivity of (4), i.e. showing that its every non-trivial solution converge to the origin at the infinity. This is stated in the following theorem.

Theorem 1 (Global attractivity of mixed-order linear fractional systems)**.** *Consider the system* (4) *with the matrix A satisfies the conditions (C1) and (C2). Then for any $x^0 \in \mathbb{R}^d$, the solution $\varphi(\cdot, x^0)$ of* (4) *which starts from x^0 converges to the origin.*

To prove this theorem we need some estimates concerning Mittag-Leffler functions. For any $\beta, \gamma > 0$, a function $E_{\beta,\gamma}(\cdot) : \mathbb{C} \to \mathbb{C}$ defined by

$$ E_{\beta,\gamma}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta k + \gamma)}, \quad z \in \mathbb{C}, $$

with $\Gamma(\cdot)$ is the Gamma function, is called the Mittag-Leffler function, see e.g. [26, Chapters 3–4].

Lemma 1. *For an arbitrary integer $p \ge 1$, $\beta \in (0,1)$ and γ is an arbitrary complex number, the following statement holds. For $z \in \{\lambda \in \mathbb{C} : \alpha\pi/2 < |arg(\lambda)| \le \pi\}$, we have*

$$ E_{\beta,\gamma}(z) = -\sum_{k=1}^{p} \frac{z^{-k}}{\Gamma(\gamma - \beta k)} + \mathcal{O}(|z|^{-1-p}), $$

when $|z| \to \infty$.

Proof: See [4, Theorem 1.4, pp. 33]. □

Lemma 2. *Let $0 < \beta \le 1$ and $\lambda > 0$. Then the following statements hold:*

(i) The function $E_\beta(-\lambda t^\beta)$ is strictly decreasing on $[0,\infty)$ and

$$ \lim_{t\to\infty} E_\beta(-\lambda t^\beta) = 0. $$

(ii)

$$ \int_0^t \tau^{\beta-1} E_{\beta,\beta}(-\lambda\tau^\beta)\, d\tau = t^\beta E_{\beta,\beta+1}(-\lambda t^\beta) $$

for all $t > 0$.
(iii)

$$ \int_0^\infty \tau^{\beta-1} E_{\beta,\beta}(-\lambda\tau^\beta)\, d\tau = \frac{1}{\lambda}. $$

Proof: (i) The proof is obtained directly from the completely monotonic property of the Mittag-Leffler function, see e.g. [26, Proposition 3.23, pp. 47] and Lemma 1.
(ii) See [4, Formula (1.99), pp. 24].
(iii) From (ii), we obtain

$$ 0 \le \int_0^t \tau^{\beta-1} E_{\beta,\beta}(-\lambda\tau^\beta)\, d\tau = t^\beta E_{\beta,\beta+1}(-\lambda t^\beta), \quad \forall t > 0. $$

On the other hand by virtue of Lemma 1, for $p = 1$, we have

$$ \begin{aligned} E_{\beta,\beta+1}(z) &= \frac{1}{z\Gamma(1)} + \mathcal{O}(\frac{1}{|z|^2}) \\ &= \frac{1}{z} + \mathcal{O}(\frac{1}{|z|^2}) \end{aligned} $$

as $z \in \{\lambda \in \mathbb{C} : \alpha\pi/2 < |\arg(\lambda)| \le \pi\}$ and $|z| \to \infty$. Let $z = -\lambda t^\beta$ $(\lambda > 0)$, then

$$ E_{\beta,\beta+1}(-\lambda t^\beta) = \frac{1}{\lambda t^\beta} + \mathcal{O}(\frac{1}{\lambda^2 t^{2\beta}}) $$

as $t \to \infty$. Hence,

$$ \lim_{t\to\infty} t^\beta E_{\beta,\beta+1}(-\lambda t^\beta) = \frac{1}{\lambda}. $$

This implies that

$$\int_0^\infty \tau^{\beta-1} E_{\beta,\beta}(-\lambda\tau^\beta)\, d\tau = \lim_{t\to\infty} t^\beta E_{\beta,\beta+1}(-\lambda t^\beta) = \frac{1}{\lambda},$$

which completes the proof. □

Proof of Theorem 1*:* Due to the existence and uniqueness of solutions to the system (4), if $x^0 = 0$, we have $\varphi(t, 0) = 0$ for all $t \geq 0$. Hence, to complete the proof of this theorem we only study non-trivial solutions. For any $x^0 \in \mathbb{R}^d \setminus \{0\}$, let

$$\varepsilon := \frac{\|x^0\|}{1 - \max_{1\leq i\leq d} \frac{\sum_{1\leq j\leq d, j\neq i} |a_{ij}|}{|a_{ii}|}}. \tag{5}$$

In the space $C([0,\mathbb{R}), \mathbb{R}^d)$, we denote the ball with the radius ε centered at the origin by $B_{C_\infty}(0,\varepsilon)$, i.e.

$$B_{C_\infty}(0,\varepsilon) := \{\xi \in C([0,\infty), \mathbb{R}^d) : \|\xi\|_\infty \leq \varepsilon\}.$$

Next, we establish a Lyapunov–Perron type operator on $B_{C_\infty}(0,\varepsilon)$ as follows. For any $\xi \in B_{C_\infty}(0,\varepsilon)$ let

$$(\mathcal{T}_{x^0}\xi)(t) = ((\mathcal{T}_{x^0}\xi)_1(t), \ldots, (\mathcal{T}_{x^0}\xi)_d(t)), \quad t \geq 0,$$

where for $i \in \{1, \ldots, d\}$

$$(\mathcal{T}_{x^0}\xi)_i(t) := E_{\alpha_i}(a_{ii}t^{\alpha_i})x_i^0 + \int_0^t (t-\tau)^{\alpha_i-1} E_{\alpha_i,\alpha_i}(a_{ii}(t-\tau)^{\alpha_i}) \sum_{1\leq j\leq d, j\neq i} a_{ij}\xi_j(\tau)\, d\tau.$$

First, we show that $\mathcal{T}_{x^0}(B_{C_\infty}(0,\varepsilon)) \subset B_{C_\infty}(0,\varepsilon)$. Indeed, for any $t > 0$, we obtain the following estimates

$$\begin{aligned}
|(\mathcal{T}_{x^0}\xi)_i(t)| &\leq E_{\alpha_i}(a_{ii}t^{\alpha_i})|x_i^0| + \int_0^t (t-\tau)^{\alpha_i-1} E_{\alpha_i,\alpha_i}(a_{ii}(t-\tau)^{\alpha_i}) \sum_{1\leq j\leq d, j\neq i} |a_{ij}||\xi_j(\tau)|\, d\tau \\
&\leq \|x^0\| + \|\xi\|_\infty \int_0^t \tau^{\alpha_i-1} E_{\alpha_i,\alpha_i}(a_{ii}\tau^{\alpha_i})\, d\tau \sum_{1\leq j\leq d, j\neq i} |a_{ij}| \\
&\leq \|x^0\| + \|\xi\|_\infty \int_0^\infty \tau^{\alpha_i-1} E_{\alpha_i,\alpha_i}(a_{ii}\tau^{\alpha_i})\, d\tau \sum_{1\leq j\leq d, j\neq i} |a_{ij}|
\end{aligned}$$

This together (5) and Lemma 2(iii) implies

$$\begin{aligned}
(\mathcal{T}_{x^0}\xi)_i(t) &\leq \varepsilon \left(1 - \max_{1\leq i\leq d} \frac{\sum_{1\leq j\leq d, j\neq i} |a_{ij}|}{|a_{ii}|}\right) \\
&\quad + \max_{1\leq i\leq d} \frac{\sum_{1\leq j\leq d, j\neq i} |a_{ij}|}{|a_{ii}|}\varepsilon \\
&= \varepsilon, \quad \forall i \in \{1, \cdots, d\}.
\end{aligned}$$

Hence, $\|\mathcal{T}_{x^0}\xi\|_\infty \leq \varepsilon$ for all $\xi \in B_{C_\infty}(0,\varepsilon)$. Furthermore, it is easy to show that this operator is contractive with the coefficient of contraction as $\max_{1\leq i\leq d} \frac{\sum_{1\leq j\leq d, j\neq i} |a_{ij}|}{|a_{ii}|}$. From Banach fixed point theorem, $\mathcal{T}_{x^0}$ has a unique fixed point ξ^* in $B_{C_\infty}(0,\varepsilon)$. It is worth noting that from [23, Theorem 2.3], the system (4) with the initial condition $x(0) = x^0$ has a unique solution. Moreover, by applying the variation of constants formula as in [24, Lemma 3.1] for each component of (4), this solution must be a fixed point of the operator $\mathcal{T}_{x^0}$. Thus for any $x^0 \in \mathbb{R}^d \setminus \{0\}$, the system (4) has a unique solution as the fixed point ξ^*. This solution belongs to the ball $B_{C_\infty}(0,\varepsilon)$. Finally, we show that $\lim_{t\to\infty} \xi^*(t) = 0$. Let $a = \limsup_{t\to\infty} \|\xi^*(t)\|$. Assume that $a > 0$. Let $T_0 > 0$ is a constant such that

$$\|\xi^*(t)\| \leq \frac{aM}{p}, \quad \forall t \geq T_0, \tag{6}$$

where $p := \max_{1\leq i\leq d} \frac{\sum_{1\leq j\leq d, j\neq i} |a_{ij}|}{|a_{ii}|} < 1$ and $p < M < 1$. From the presentation of ξ^*, for each $i \in \{1, \ldots, d\}$, we have

$$\begin{aligned}
\limsup_{t\to\infty} |\xi_i^*(t)| \leq\ & \limsup_{t\to\infty} E_{\alpha_i}(a_{ii}t^\alpha)|x_i^0| \\
&+ \limsup_{t\to\infty} \int_0^{T_0} (t-\tau)^{\alpha_i-1} E_{\alpha_i,\alpha_i}(a_{ii}(t-\tau)^{\alpha_i}) \sum_{1\leq j\leq d, j\neq i} |a_{ij}\xi_j^*(\tau)|\, d\tau \\
&+ \limsup_{t\to\infty} \int_{T_0}^t (t-\tau)^{\alpha_i-1} E_{\alpha_i,\alpha_i}(a_{ii}(t-\tau)^{\alpha_i}) \sum_{1\leq j\leq d, j\neq i} |a_{ij}\xi_j^*(\tau)|\, d\tau.
\end{aligned}$$

Due to Lemma 2,

$$\limsup_{t\to\infty} E_{\alpha_i}(a_{ii}t^\alpha)|x_i^0| = 0$$

and

$$\begin{aligned}
\limsup_{t\to\infty} & \int_0^{T_0} (t-\tau)^{\alpha_i-1} E_{\alpha_i,\alpha_i}(a_{ii}(t-\tau)^{\alpha_i}) \sum_{1\leq j\leq d, j\neq i} |a_{ij}\xi_j^*(\tau)|\, d\tau \\
&\leq \limsup_{t\to\infty} \int_{t-T_0}^t \tau^{\alpha_i-1} E_{\alpha_i,\alpha_i}(a_{ii}\tau^{\alpha_i})\, d\tau \sum_{1\leq j\leq d, j\neq i} |a_{ij}| \max_{t\in[0,T_0]} \|\xi^*(t)\| \\
&= 0.
\end{aligned}$$

Moreover, from Lemma 2(iii) and (6), we have

$$\begin{aligned}
\limsup_{t\to\infty} & \int_{T_0}^t (t-\tau)^{\alpha_i-1} E_{\alpha_i,\alpha_i}(a_{ii}(t-\tau)^{\alpha_i}) \sum_{1\leq j\leq d, j\neq i} |a_{ij}\xi_j^*(\tau)|\, d\tau \\
&\leq \int_0^\infty t^{\alpha_i} E_{\alpha_i,\alpha_i}(a_{ii}t^{\alpha_i})\, dt\, \frac{aM}{p} \sum_{1\leq j\leq d, j\neq i} |a_{ij}| \\
&= aM.
\end{aligned}$$

Hence,

$$0 < a = \limsup_{t\to\infty} \|\xi^*(t)\| \leq aM < a,$$

a contradiction. This implies that

$$\limsup_{t\to\infty} \|\xi^*(t)\| = a = 0.$$

The proof is complete. □

Remark 1. *Unlike the approach considered in [20, Theorem 1] and [21, Theorem 4], our method does not use any characteristic polynomial equation to analyze asymmetrical behavior of solutions. Our result is independent of fractional orders of the system and depends only on its matrix of coefficients A. It is clear that our condition is very easy and computationally simple to check, as well as it can deal with mix-order fractional systems of high dimensions and irrational fractional derivatives.*

Remark 2. *Consider the system* (4). *For any non-singular matrix* $T \in \mathbb{R}^{d\times d}$, *in general, we have* ${}^{C}D^{\hat{\alpha}}_{0+}Tx(t) \neq T\,{}^{C}D^{\hat{\alpha}}_{0+}x(t)$. *Hence, the matrix transformation technique (which is used in the commensurate case* $\alpha_1 = \alpha_2 = \cdots = \alpha_d$, *see [9, Section 3.1]) does not hold for mixed-order fractional systems (the non-commensurate case). In our opinion, the qualitative theory of mixed-order fractional systems is still in its infancy. In particular, the question about the relationship between stability of systems and the spectrum of the linearization of the "vector fields" is still open.*

Example 1. *Consider the mixed order fractional system on the interval* $(0,\infty)$

$$\begin{cases} {}^{C}D^{\alpha_1}_{0+}x_1(t) = -2x_1(t) + x_2(t), \\ {}^{C}D^{\alpha_2}_{0+}x_2(t) = -2x_1(t) - 3x_2(t), \end{cases} \tag{7}$$

where $\alpha_1, \alpha_2 \in (0, 1]$ *are rational number and* $\alpha_1 \leq \alpha_2$. *For this system, we can easily obtain matrix A, where* $A = \begin{pmatrix} -2 & 1 \\ -2 & -3 \end{pmatrix}$. *It is easy to see that matrix A satisfies all the conditions stated in Theorem* 1, *and hence we immediately conclude that the system* (7) *is globally attractive.*

In the following discussion, we compare our result to existing ones in the literature. Suppose that $\alpha_1 = \frac{p_1}{q_1}$ *and* $\alpha_2 = \frac{p_2}{q_2}$ *with* $p_1, p_2, q_1, q_2 \in \mathbb{N}$ *and* $gcd(p_1, q_1) = gcd(p_2, q_2) = 1$. *According to [20, Theorem 1], in order to test the global attractivity of* (7), *we have to solve the following characteristic equation*

$$s^{\alpha_1+\alpha_2} + 3s^{\alpha_1} + 2s^{\alpha_2} + 8 = 0 \tag{8}$$

and check whether all the solutions s of (5) satisfy the condition

$$|\arg(s)| > \frac{\pi}{2}.$$

While, if we apply the condition presented in [21, Theorem 4], we have to compute all the solutions to the equation

$$s^{m(\alpha_1+\alpha_2)} + 3s^{m\alpha_1} + 2s^{m\alpha_2} + 8 = 0 \tag{9}$$

and check the following condition for all the solutions s of (9)

$$|\arg(s)| > \frac{\pi}{2m}$$

where $m = lcm(q_1, q_2)$. *The above tasks are computationally intensive especially for systems with high dimension, i.e. d is a large number. Moreover, for the case where the fractional orders* α_1 *and* α_2 *in the system* (7) *are irrational, the approaches presented in [20, Theorem 1] and [21, Theorem 4] do not work. Also, we cannot apply the condition presented in [23, Theorem 3.1] to check the global attractivity of* (7) *because the matrix A is not in the triangle form. On the other hand, due to the fact that matrix A is not Metzler, following [22, Theorem 1], this system is not positive. Thus we can not use the criterion [22, Theorem 2] to investigate this example. Moreover, according to [25, Theorem 4], to prove the global attractivity of solutions to* (7), *we have to compute explicitly the coefficient*

$$S(2,\alpha_2,\alpha_1) = \sup_{t\geq 0}\int_0^t (t-s)^{\alpha_2-1}E_{\alpha_2,\alpha_2}(-2(t-s)^{\alpha_2})\,(s+1)^{-\alpha_1}(t+1)^{\alpha_1}\,ds$$

and show that

$$\max\{S(2,\alpha_2,\alpha_1), S(2,\alpha_1,\alpha_1)\} < 1,$$

which is not an easy task to do.

The above discussion shows the advantages of our result over existing ones in the literature. Our result as stated in Theorem 1 *is very easy to apply as we can immediately establish the system* (7) *is globally attractive.*

Finally, we consider the nonlinear mixed-order fractional system (2). This system can be rewritten as

$${}^{C}D^{\hat{\alpha}}_{0+}x(t) = Ax(t) + f(x(t)). \tag{10}$$

Suppose that $f = (f_1,\ldots,f_d))$ is continuous on $\mathbb{R}^d$ and Lipschitz continuous in a neighborhood of the origin and

$$f(0) = 0 \quad \text{and} \quad \lim_{r\to 0}\ell_f(r) = 0, \tag{11}$$

where $\ell_f(r)$ is the Lipschitz coefficient defined in (3). We recall the notions of stability of the trivial solution to (10).

Definition 1. *(i) The trivial solution to* (10) *is called* stable *if for any* $\varepsilon > 0$ *there exists* $\delta = \delta(\varepsilon) > 0$ *such that for every* $\|x_0\| < \delta$ *the solution* $\varphi(t, x_0)$ *exists on* $[0,\infty)$ *and satisfies*

$$\|\varphi(t, x_0)\| < \varepsilon, \quad \forall t \geq 0.$$

(ii) The trivial solution is called asymptotically stable *if it is stable and there exists* $\widehat{\delta} > 0$ *such that* $\lim_{t\to\infty}\varphi(t, x_0) = 0$ *whenever* $\|x_0\| < \widehat{\delta}$.

Using the approach as in Theorem 1 and arguments as in the proof of [9, Theorem 3.1], we can obtain the following result.

Theorem 2 (Linearized stability of nonlinear mixed-order fractional systems)**.** *Consider the system* (10). *Assume that the matrix A satisfies the conditions (C1) and (C2), and the function f is Lipschitz continuous in a neighborhood of the origin such that the condition* (11) *holds. Then the trivial solution of this system is asymptotically stable.*

Remark 3. *Consider the mixed order fractional system*

$${}^{C}D^{\hat{\alpha}}_{0+}x(t) = Ax(t) + f(t, x(t)), \quad t \geq 0, \tag{12}$$

where A satisfies the conditions (C1), (C2) and $f : [0,\infty)\times\mathbb{R}^d \to \mathbb{R}^d$ *is continuous,* $f(t, 0) = 0$ *for all* $t \geq 0$ *and*

$$\|f(t,x) - f(t,y)\| \leq K(t)\|x-y\|, \quad \forall x, y \in \mathbb{R}^d, t \geq 0,$$

with $K : [0,\infty) \to \mathbb{R}_{\geq 0}$. *In addition, one of the following assumptions holds:*

(K1) for any $1 \leq i \leq d$,

$$\sup_{t\geq 0}\int_0^t (t-\tau)^{\alpha_i}E_{\alpha_i,\alpha_i}(-a_{ii}(t-\tau)^{\alpha_i})K(\tau)d\tau < \frac{a_{ii} - \sum_{1\leq j\leq d, j\neq i}|a_{ij}|}{a_{ii}};$$

(K2) there exists $\varepsilon > 0$ small enough such that

$$K(t) \leq \varepsilon, \quad \forall t \geq 0;$$

(K3) $\lim_{t\to\infty} K(t) = 0$.

Then basing on the approach as in the proof of Theorem 2 and Theorem 7 in [27], we see that the trivial solution of (12) is also asymptotic stable.

Remark 4. *Because every norm in $\mathbb{R}^d$ is equivalent, the global attractivity of (4) and the asymptotic stability of the trivial solution to (10) do not depend on the norm endowed in this space.*

Example 2. *Consider the mixed fractional order nonlinear system*

$$^{C}D^{\hat{\alpha}}_{0+}x(t) = Ax(t) + f(x(t)), \quad t \geq 0, \tag{13}$$

where $\hat{\alpha} = (0.5, 0.75)$,

$$A = \begin{pmatrix} -3 & 1 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & -5 \end{pmatrix}$$

and

$$f(x) = \begin{pmatrix} 5x_1^2 - 3x_2^2 \\ x_2^3 + x_3^4 \\ 2x_1^2 + x_2^2 + |x_3|^{1.5} \end{pmatrix}$$

for all $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3$. In this case A satisfies (C1), (C2) and the function f satisfies (11). Thus, by Theorem 2, the trivial solution of (13) is asymptotic stable. Let $\varphi(\cdot, 0.05)$ is the solution to (13) which starts from 0.05 at $t = 0$. The trajectory of this solution is described in Figure.

3 Conclusion

In this study, we have studied asymptotic properties of mixed-order fractional systems. We have derived a condition for global attractivity of mixed-order linear fractional systems. The condition is very easy to check, requiring the system's matrix to be strictly diagonally dominant and its diagonal elements to be negative. The proposed condition offers some advantages over exiting results as it can deal with mixed-order fractional systems of high dimension and irrational fractional orders. Finally, we have presented a linearized stability theorem which ensures asymptotic stability of the trivial solution to a nonlinear mixed-order fractional system linearized along its equilibrium point.

Acknowledgment

This work was supported by the Vietnam Academy of Science and Technology under the grant DLT00.01-20/21.

4 References

1 Bandyopadhyay, B., Kamal, S.: '*Stabilization and Control of Fractional Order Systems: A Sliding Mode Approach*' (Lecture Notes in Electrical Engineering 317) (Springer International Publishing, Switzerland, 2015).

2 Diethelm, K.: '*The analysis of fractional differential equations. An application-oriented exposition using differential operators of Caputo type*' (Lecture Notes in Mathematics, 2004) (Springer-Verlag, Berlin, 2010).

3 Lakshmikantham, V., Leela, S., Devi, J.: '*Theory of fractional dynamic systems*' (Cambridge Scientific Publishers Ltd., England, 2009).

4 Podlubny, I.: '*Fractional differential equations. An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*' (Mathematics in Science and Engineering, 198) (Academic Press, Inc., CA, 1999).

5 Gorenflo, R., Kilbas, A., Mainardi, F., Rogosin, S.: '*Mittag-Leffler Functions, Related Topics and Applications*' (Springer Monographs in Mathematics) (Springer-Verlag Berlin Heidelberg, 2014).

6 Aguila-Camacho, N., Duarte-Mermoud, M., Gallegos, J., Castro-Linares, R.: 'Using general quadratic Lyapunov functions to prove Lyapunov uniform stability for fractional order systems', *Communications in Nonlinear Science and Numerical Simulation*, 2015, **22**, (1-3), pp. 650–659.

7 Badri, V., Sojoodi, M.: 'Robust fixed-order dynamic output feedback controller design for fractional-order systems', *IET Control Theory & Applications*, 2018, **12**, (9), pp. 1236–1243.

8 Chen, W., Dai, H., Song, Y., Zhang, Z.: 'Convex Lyapunov functions for stability analysis of fractional order systems', *IET Control Theory Appl.*, 2017, **11**, (5), pp. 1070–1074.

9 Cong, N.D., Son, D.T., Siegmund, S., Tuan, H.T.: 'Linearized asymptotic stability for fractional differential equations', *Electron. J. Qual. Theory Diff. Equ.*, 2016, **39**, pp. 1–13.

10 Cong, N.D., Son, D.T., Siegmund, S., Tuan, H.T.: 'An instability theorem for nonlinear fractional differential systems', *Dis. Continuous Dyn. Syst., B*, 2017, **22**, (8), pp. 3079–3090.

11 He, B.B., Zhou, H.C., Chen, Y., Kou, C.H.: 'Asymptotical stability of fractional order systems with time delay via an integral inequality', *IET Control Theory Appl.*, 2018, **12**, (12), pp. 1748–1754.

12 Li, Y., Zhao, D., Chen, Y., Podlubny, I., Zhang, C.: 'Finite energy Lyapunov function candidate for fractional order general nonlinear systems', *Communications in Nonlinear Science and Numerical Simulation*, 2019, **78**, 104886.

13 Tuan, H.T., Trinh, H.: 'Stability of fractional-order nonlinear systems by Lyapunov direct method', *IET Control Theory Appl.*, 2018, **12**, (17), pp. 2417–2422.

14 Tuan, H.T., Trinh, H.: 'A linearized stability theorem for nonlinear delay fractional differential equations', *IEEE Trans. Autom. Control*, 2018, **63**, (9), pp. 3180–3186.

15 Li, Y., Chen, Y., Podlubny, I.: 'Mittag-Leffler stability of fractional order nonlinear dynamic systems', *Automatica*, 2009, **45**, pp. 1965–1969.

16 Gorenflo, R., Mainardi, F.: 'Fractional calculus: integral and differential equations of fractional order', in: A. Carpinteri, F. Mainardi (Eds.), Fractals and Fractional Calculus in Continuum Mechanics, Springer, Berlin, 1997, pp. 223–276.

17 Torvik, P. J., Bagley, R. L.: 'On the appearance of the fractional derivative in the behavior of real materials', *J. Appl. Mech.*, 1984, **51**, pp. 294–298.

18 Brandibur, O., Kaslik, E.: 'Stability of two-component incommensurate fractional-order systems and applications to the investigation of a FitzHugh-Nagumo neuronal model', *Mathematical Methods in the Applied Sciences*, 2018, **41**, (11), pp. 7182–7194.

19 Škovránek, T., Podlubny, I., Petráš, I.: 'Modelling of the national economies in state-space: A fractional calculus approach', *Econ. Model.*, 2012, **29**, pp. 1322–1327.

20 Deng, W., Li, C., Lu, J.: 'Stability analysis of linear fractional differential system with multiple time delays', *Nonlinear Dyn.*, 2007, **48** (4), pp. 409–416.

21 Petras, I.: 'Stability of fractional-order systems with rational orders: A survey', *Fractional Calculus and Applied Analysis*, 2009, **12** (3), pp. 269–298.

22 Shen, J., Lam, J.: 'Stability and performance analysis for positive fractional-order systems with time-varying delays', *IEEE Trans. Autom. Control*, 2016, **61**, (9), pp. 2676–2681

23 Diethelm, K., Siegmund, S., Tuan, H.T.: 'Asymptotic behavior of solutions of linear multi-order fractional differential systems', *Fractional Calculus and Applied Analysis,* 2017, **20**, (5), pp. 1165–1195.

24 Cong, N.D., Tuan, H.T.: 'Generation of nonlocal fractional dynamical systems by fractional differential equations', *Journal of Integral Equations and Applications*, 2017, **29**, (4), pp. 585–608.

25 Feckan, M., Wang, J.: 'Mixed order fractional differential equations', *Mathematics,* 2017, **5**, (4), 61.

26 Gorenflo, R., Kilbas, A., Mainardy, F., Rogosin, S.: *'Mittag-Leffler functions, related topics and applications*' (Springer Monograph in Mathematics) (Springer-Verlag, Heidelberg, 2014).

27 Cong, N.D., Son, D.T., Tuan, H.T.: 'Asymptotic stability of linear fractional systems with constant coefficients and small time dependent perturbations', *Vietnam Journal of Mathematics*, 2018, **46**, pp. 665–680.