GLOBAL ŁOJASIEWICZ INEQUALITIES ON COMPARING THE RATE OF GROWTH OF POLYNOMIAL FUNCTIONS

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Abstract. We present a global version of the Łojasiewicz inequality on comparing the rate of growth of two polynomial functions in the case the mapping defined by these functions is (Newton) non-degenerate at infinity. In addition, we show that the condition of non-degeneracy at infinity is generic in the sense that it holds in an open and dense semi-algebraic set of the entire space of input data.

1. Introduction

Let K be a compact semi-algebraic subset of $\mathbb{R}^n$ and let $g, h\colon K \to \mathbb{R}$ be continuous semi-algebraic functions such that the zero set of g is contained in the zero set of h. Then the information concerning the rate of growth of g and h is given by the following *Łojasiewicz inequality*: there exist constants $c > 0$ and $\alpha > 0$ such that for any $x \in K$, we have

$$|g(x)|^{\alpha} \quad \geq \quad c|h(x)|.$$

Note that if K is not compact, the Łojasiewicz inequality does not always hold (see Example 3.1 below). Recently, several versions of the Łojasiewicz inequality have been studied for a special case where h is the distance function to the zero set of g, see [8, 9, 10, 11, 12, 13, 23, 24]. However, the study of the Łojasiewicz inequality on comparing the rate of growth of two *arbitrary* semi-algebraic functions on *non-compact* semi-algebraic sets is barely developed (cf. [30]).

We would like to point out that the Łojasiewicz inequality and its variants play an important role in many branches of mathematics. For example, Łojasiewicz inequalities are very useful in the study of continuous regular functions, a branch of Algebraic Geometry, which has been

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actively developed recently, see [18, 28] for pioneering works and [29] for a survey. Also, Łojasiewicz inequalities, together with Nullstellensätze, are crucial tools for the study of the ring of (bounded) continuous semi-algebraic functions on a semi-algebraic set, see [15, 16, 17].

The purpose of this work is to show that for almost all pairs of polynomial functions, a variant of the Łojasiewicz inequality holds on the entire space. Namely, with the definitions given in Section 2, the following statements hold.

(i) Let $(g, h)\colon \mathbb{R}^n \to \mathbb{R}^2$ be a polynomial map, which is non-degenerate at infinity. Suppose that g is convenient and that the zero set of g is contained in the zero set of h, then there exist some constants $c > 0, \alpha > 0$, and $\beta > 0$ such that

$$|g(x)|^\alpha + |g(x)|^\beta \ \geq \ c|h(x)| \quad \text{for all} \quad x \in \mathbb{R}^n.$$

(ii) The condition of non-degeneracy at infinity is *generic* in the sense that it holds in an open and dense semi-algebraic set of the entire space of input data.

Note that unlike the case where h is the distance function to the zero set of g, estimating the exponents α and β in the first statement is still a delicate problem.

The paper is organized as follows. Section 2 presents some preliminary results from Semi-algebraic Geometry; the condition of non-degeneracy at infinity will be also given there. Section 3 proves the existence of the global Łojasiewicz-type inequality for polynomial maps which are non-degenerate at infinity. Finally, in Section 4, it is shown that the property of being non-degenerate at infinity is generic.

2. Preliminaries

We begin by giving some necessary definitions and notational conventions. Let $\mathbb{R}^n$ denote the Euclidean space of dimension n and $\mathbb{R}^* := \mathbb{R}\backslash\{0\}$. The corresponding inner product (resp., norm) in $\mathbb{R}^n$ is defined by $\langle x, y\rangle$ for any $x, y \in \mathbb{R}^n$ (resp., $\|x\| := \sqrt{\langle x, x\rangle}$ for any $x \in \mathbb{R}^n$). The closure of a set A is denoted by $\overline{A}$. Given a nonempty set $J \subset \{1, \ldots, n\}$, we define

$$\mathbb{R}^J := \{x \in \mathbb{R}^n \ : \ x_j = 0, \text{ for all } j \notin J\}.$$

We denote by $\mathbb{Z}_+$ the set of non-negative integer numbers. If $\kappa = (\kappa_1, \ldots, \kappa_n) \in \mathbb{Z}_+^n$, we denote by x^κ the monomial $x_1^{\kappa_1} \cdots x_n^{\kappa_n}$.

2.1. Semi-algebraic geometry.

In this subsection, we recall some notions and results of semi-algebraic geometry, which can be found in [1, 2, 3, 5, 33].

Definition 2.1. (i) A subset of $\mathbb{R}^n$ is called *semi-algebraic* if it is a finite union of sets of the form

$$\{x \in \mathbb{R}^n \ : \ f_i(x) = 0, i = 1, \ldots, k; f_i(x) > 0, i = k + 1, \ldots, p\}$$

where all f_i are polynomials.

(ii) Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^p$ be semi-algebraic sets. A map $F\colon A \to B$ is said to be *semi-algebraic* if its graph

$$\{(x, y) \in A \times B \ : \ y = F(x)\}$$

is a semi-algebraic subset of $\mathbb{R}^n \times \mathbb{R}^p$.

A major fact concerning the class of semi-algebraic sets is the following Tarski–Seidenberg Theorem.

Theorem 2.1. *The image of a semi-algebraic set by a semi-algebraic map is semi-algebraic.*

Moreover, semi-algebraic sets and functions enjoy a number of remarkable properties:

(i) The class of semi-algebraic sets is closed with respect to Boolean operators; a Cartesian product of semi-algebraic sets is a semi-algebraic set;
(ii) The closure and the interior of a semi-algebraic set is a semi-algebraic set;
(iii) A composition of semi-algebraic maps is a semi-algebraic map;
(iv) The inverse image of a semi-algebraic set under a semi-algebraic map is a semi-algebraic set;
(v) If A is a semi-algebraic set, then the distance function

$$\operatorname{dist}(\cdot, A)\colon \mathbb{R}^n \to \mathbb{R}, \quad x \mapsto \operatorname{dist}(x, A) := \inf\{\|x - a\| \ : \ a \in A\},$$

is also semi-algebraic.

Remark 2.1. As an immediate consequence of Theorem 2.1, we get the semi-algebraicity of any set of the form $\{x \in A \ : \ \exists y \in B, (x, y) \in C\}$, provided that A, B, and C are semi-algebraic sets. It follows also that the set $\{x \in A \ : \ \forall y \in B, (x, y) \in C\}$ is semi-algebraic since its complement is the union of the complement of A and the set $\{x \in A \ : \ \exists y \in B, (x, y) \notin C\}$. Thus, if we have a finite collection of semi-algebraic sets, then any set obtained from them with the help of a finite chain of quantifiers is also semi-algebraic.

Lemma 2.1 (Curve Selection Lemma)**.** *Let $A \subset \mathbb{R}^n$ be a semi-algebraic set, and $x^* \in \overline{A} \setminus A$. Then there exists an analytic semi-algebraic curve*

$$\varphi\colon (-\epsilon, \epsilon) \to \mathbb{R}^n$$

with $\varphi(0) = x^*$ and with $\varphi(t) \in A$ for $t \in (0, \epsilon)$.

Next we give a version of Curve Selection Lemma which will be used in the proof of Theorem 3.1.

Lemma 2.2 (Curve Selection Lemma at infinity)**.** *Let $A \subset \mathbb{R}^n$ be a semi-algebraic set, and let $f := (f_1, \ldots, f_p)\colon \mathbb{R}^n \to \mathbb{R}^p$ be a semi-algebraic map. Assume that there exists a sequence $\{x^\ell\}$ such that $x^\ell \in A$, $\lim_{l\to\infty} \|x^\ell\| = \infty$ and $\lim_{l\to\infty} f(x^\ell) = y \in (\overline{\mathbb{R}})^p$, where $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$. Then there exists an analytic semi-algebraic curve $\varphi\colon (0, \epsilon) \to \mathbb{R}^n$ such that $\varphi(t) \in A$ for all $t \in (0, \epsilon), \lim_{t\to 0} \|\varphi(t)\| = \infty,$ and $\lim_{t\to 0} f(\varphi(t)) = y$.*

Lemma 2.3 (Growth Dichotomy Lemma)**.** *Let $f\colon (0,\epsilon) \to \mathbb{R}$ be a semi-algebraic function with $f(t) \neq 0$ for all $t \in (0,\epsilon)$. Then there exist constants $c \neq 0$ and $q \in \mathbb{Q}$ such that $f(t) = ct^q + o(t^q)$ as $t \to 0^+$.*

2.2. **The (semi-algebraic) transversality theorem with parameters.** Let P, X and Y be some C^∞ manifolds of finite dimension, S be a C^∞ sub-manifold of Y, and $F\colon X \to Y$ be a C^∞ map. Denote by $d_xF : T_xX \to T_{F(x)}Y$, the differential of F at x, where T_xX and $T_{F(x)}Y$ are, respectively, the tangent space of X at x and the tangent space of Y at $F(x)$.

Definition 2.2. The map F is said to be *transverse* to the sub-manifold S, abbreviated by $F \pitchfork S$, if either $F(X) \cap S = \emptyset$ or for each $x \in F^{-1}(S)$, we have

$$d_xF(T_xX) + T_{F(x)}S = T_{F(x)}Y.$$

Remark 2.2. If $\dim X \geq \dim Y$ and $S = \{s\}$, then $F \pitchfork S$ if and only if either $F^{-1}(s) = \emptyset$ or $\text{rank} d_xF = \dim Y$ for all $x \in F^{-1}(s)$. In the case $\dim X < \dim Y$, then $F \pitchfork S$ if and only if $F^{-1}(S) = \emptyset$.

The following result [21, 22] will be useful in the study of the genericity of the condition of non-degeneracy at infinity.

Theorem 2.2 (Transversality Theorem with parameters)**.** *Let $F\colon P \times X \to Y$ be a C^∞ map. For each $p \in P$, consider the map $F_p\colon X \to Y$ defined by $F_p(x) := F(p,x)$. If $F \pitchfork S$, then the set*

$$Q := \{p \in P \;:\; F_p \pitchfork S\}$$

is open and dense in P. Moreover, if $P, X, Y,$ and S are semi-algebraic sets and if F is a semi-algebraic map, then Q is also semi-algebraic.

Proof. The proof of openness and density of Q is done in [21, 22]. The method used also permits to prove that Q is semi-algebraic if P, X, Y, S and F are semi-algebraic. ☐

2.3. **Newton polyhedra and non-degeneracy conditions.**

2.3.1. *Newton polyhedra.* Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a polynomial function. Suppose that f is written as $f = \sum_\kappa c_\kappa x^\kappa$. Then the support of f, denoted by $\text{supp}(f)$, is defined as the set of those $\kappa \in \mathbb{Z}_+^n$ such that $c_\kappa \neq 0$. The *Newton polyhedron (at infinity)* of f, denoted by $\Gamma(f)$, is defined as the convex hull in $\mathbb{R}^n$ of the set $\text{supp}(f)$. The polynomial f is said to be *convenient* if $\Gamma(f)$ intersects each coordinate axis in a point different from the origin 0 in $\mathbb{R}^n$, that is, if for any $j \in \{1, \ldots, n\}$ there exists some $\kappa_j > 0$ such that $\kappa_j e^j \in \Gamma(f)$, where $\{e^1, \ldots, e^n\}$ denotes the canonical basis in $\mathbb{R}^n$. For each (closed) face Δ of $\Gamma(f)$, we will denote by f_Δ the polynomial $\sum_{\kappa \in \Delta} c_\kappa x^\kappa$; if $\Delta \cap \text{supp}(f) = \emptyset$ we let $f_\Delta := 0$.

Given a nonzero vector $q \in \mathbb{R}^n$, we define

$$\begin{aligned} d(q, \Gamma(f)) &:= \min\{\langle q, \kappa\rangle \;:\; \kappa \in \Gamma(f)\}, \\ \Delta(q, \Gamma(f)) &:= \{\kappa \in \Gamma(f) \;:\; \langle q, \kappa\rangle = d(q, \Gamma(f))\}. \end{aligned}$$

By definition, for each nonzero vector $q \in \mathbb{R}^n$, $\Delta(q, \Gamma(f))$ is a closed face of $\Gamma(f)$. Conversely, if Δ is a closed face of $\Gamma(f)$ then there exists a nonzero vector[1] $q \in \mathbb{R}^n$ such that $\Delta = \Delta(q, \Gamma(f))$. The *dimension* of a face Δ is defined to be the minimum of the dimensions of the affine subspaces containing Δ. The faces of Γ of dimension 0 are called the *vertices* of Γ.

Remark 2.3. The following statements follow immediately from definitions:

(i) We have $\Gamma(f) \cap \mathbb{R}^J = \Gamma(f|_{\mathbb{R}^J})$ for all nonempty subset J of $\{1, \ldots, n\}$.

(ii) Let $\Delta := \Delta(q, \Gamma(f))$ for some nonzero vector $q := (q_1, \ldots, q_n) \in \mathbb{R}^n$. By definition, $f_\Delta = \sum_{\kappa \in \Delta} c_\kappa x^\kappa$ is a weighted homogeneous polynomial of type $(q, d := d(q, \Gamma(f)))$, i.e., we have for all $t > 0$ and all $x \in \mathbb{R}^n$,

$$f_\Delta(t^{q_1} x_1, \ldots, t^{q_n} x_n) = t^d f_\Delta(x_1, \ldots, x_n).$$

This implies the Euler relation

$$\sum_{j=1}^{n} q_j x_j \frac{\partial f_\Delta}{\partial x_j}(x) = d \cdot f_\Delta(x).$$

In particular, if $d \neq 0$ and $\nabla f_\Delta(x) = 0$, then $f_\Delta(x) = 0$.

2.3.2. *Non-degeneracy conditions.* In [26] (see also [27]), Khovanskii introduced a condition of non-degeneracy of complex analytic maps $F\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ in terms of the Newton polyhedra of the component functions of F. This notion has been applied extensively to the study of several questions concerning isolated complete intersection singularities (see for instance [4, 7, 20, 32]). We will apply this condition for real polynomial maps. First we need to introduce some notation.

Definition 2.3. Let $F := (f_1, \ldots, f_p)\colon \mathbb{R}^n \to \mathbb{R}^p, 1 \leq p \leq n$, be a polynomial map.

(i) The map F is said to be *Khovanskii non-degenerate at infinity* if for any vector $q \in \mathbb{R}^n$ with $d(q, \Gamma(f_i)) < 0$ for $i = 1, \ldots, p$, the set

$$\{x \in (\mathbb{R}^*)^n \ : \ f_{i,\Delta_i}(x) = 0 \text{ for } i = 1, \ldots, p\}$$

is a reduced smooth complete intersection variety in the torus $(\mathbb{R}^*)^n$, i.e., the system of gradient vectors $\nabla f_{i,\Delta_i}(x)$ for $i = 1, \ldots, p$, is $\mathbb{R}$-linearly independent on this variety, where $\Delta_i := \Delta(q, \Gamma(f_i))$.

(ii) The map F is said to be *non-degenerate at infinity* if for each ascending q-tuple $I := (i_1, \ldots, i_q)$ of integers from the set $\{1, \ldots, p\}$, the polynomial map $\mathbb{R}^n \to \mathbb{R}^q$, $x \mapsto (f_{i_1}(x), \ldots, f_{i_q}(x))$, is Khovanskii non-degenerate at infinity.

Remark 2.4. By definition, the map F is Khovanskii non-degenerate at infinity if and only if for all $q \in \mathbb{R}^n$ with $d(q, \Gamma(f_i)) < 0$ for $i = 1, \ldots, p$ and for all $x \in (\mathbb{R}^*)^n$ with $f_{i,\Delta_i}(x) = 0$ for

[1] Since $\Gamma(f)$ is an integer polyhedron, we can assume that all the coordinates of q are rational numbers.

$i = 1, \dots, p$ we have

$$\operatorname{rank}\begin{pmatrix} x_1 \frac{\partial f_{1,\Delta_1}}{\partial x_1}(x) & \cdots & x_n \frac{\partial f_{1,\Delta_1}}{\partial x_n}(x) \\ \vdots & \cdots & \vdots \\ x_1 \frac{\partial f_{p,\Delta_p}}{\partial x_1}(x) & \cdots & x_n \frac{\partial f_{p,\Delta_p}}{\partial x_n}(x) \end{pmatrix} = p.$$

The map F is non-degenerate at infinity if and only if for all $q \in \mathbb{R}^n$ with $d(q, \Gamma(f_i)) < 0$ for $i = 1, \dots, p$, and for all $x \in (\mathbb{R}^*)^n$, we have

$$\operatorname{rank}\begin{pmatrix} x_1 \frac{\partial f_{1,\Delta_1}}{\partial x_1}(x) & \cdots & x_n \frac{\partial f_{1,\Delta_1}}{\partial x_n}(x) & f_{1,\Delta_1}(x) & & \mathbf{0} \\ \vdots & \cdots & \vdots & & \ddots & \\ x_1 \frac{\partial f_{p,\Delta_p}}{\partial x_1}(x) & \cdots & x_n \frac{\partial f_{p,\Delta_p}}{\partial x_n}(x) & \mathbf{0} & & f_{p,\Delta_p}(x) \end{pmatrix} = p.$$

3. Łojasiewicz inequalities

The main result of this section is the following global Łojasiewicz inequality on comparing the rate of growth of two polynomial functions.

Theorem 3.1. *Let $(g, h)\colon \mathbb{R}^n \to \mathbb{R}^2$ be a polynomial map, which is non-degenerate at infinity. If g is convenient and $g^{-1}(0) \subset h^{-1}(0)$, then there exist some constants $c > 0, \alpha > 0$, and $\beta > 0$ such that*

$$|g(x)|^\alpha + |g(x)|^\beta \ \geq \ c|h(x)| \quad \textit{for all} \quad x \in \mathbb{R}^n.$$

Note that we do not suppose the polynomial h to be convenient. On the other hand, the assumption that the polynomial g is convenient cannot be dropped as shown in the following example.

Example 3.1. Consider the polynomial map

$$(g, h)\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad (x_1, x_2) \mapsto \big((x_1^2 - 1)^2 + (x_1x_2 - 1)^2, (x_1^2 - 1)^2 + (x_2^2 - 1)^2\big).$$

Clearly, (g, h) is non-degenerate at infinity, g is not convenient, and $g^{-1}(0) \subset h^{-1}(0)$. Furthermore, we have

$$\lim_{k\to\infty} g\Big(\frac{1}{k}, k\Big) \ = \ 1 \quad \text{and} \quad \lim_{k\to\infty} h\Big(\frac{1}{k}, k\Big) \ = \ +\infty,$$

and so there are no constants $c > 0, \alpha > 0$, and $\beta > 0$ such that

$$|g(x)|^\alpha + |g(x)|^\beta \ \geq \ c|h(x)| \quad \text{for all} \quad x = (x_1, x_2) \in \mathbb{R}^2.$$

The following simple example shows that the exponents α and β in Theorem 3.1 are different in general.

Example 3.2. Consider the polynomial map

$$(g, h)\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad (x_1, x_2) \mapsto \big(x_1^2 + x_2^4, x_1^2 + x_2^2\big).$$

Clearly, (g,h) is non-degenerate at infinity, g is convenient, and $g^{-1}(0) \subset h^{-1}(0)$. Furthermore, it is not hard to see that there are no constants $c > 0$ and $\alpha > 0$ such that

$$|g(x)|^{\alpha} \ \geq \ c|h(x)| \quad \text{for all} \quad x = (x_1, x_2) \in \mathbb{R}^2.$$

On the other hand, it holds that

$$|g(x)|^{\frac{1}{2}} + |g(x)| \ \geq \ |h(x)| \quad \text{for all} \quad x = (x_1, x_2) \in \mathbb{R}^2.$$

To prove Theorem 3.1, we first need the following definition.

Definition 3.1. Given a polynomial function $f\colon \mathbb{R}^n \to \mathbb{R}$ and a smooth semi-algebraic manifold $X \subset \mathbb{R}^n$ we let

$$\widetilde{K}_{\infty}(f|_X) := \left\{ t \in \mathbb{R} : \begin{cases} \exists \{x^k\} \subset X, \text{ s.t. } \|x^k\| \to +\infty, f(x^k) \to t, \\ \text{and } \|\nabla(f|_X)(x^k)\| \to 0 \end{cases} \right\}.$$

We also set

$$K_0(f|_X) \ := \ \{t \in \mathbb{R} \ : \ \exists x \in X \text{ with } f(x) = t \text{ and } \nabla(f|_X)(x) = 0\}$$

which is the *set of critical values* of the restriction $f|_X$. Note that, by Sard's theorem, $K_0(f|_X)$ is a finite subset of $\mathbb{R}$.

If $X = \mathbb{R}^n$, we write $\widetilde{K}_{\infty}(f)$ and $K_0(f)$ instead of $\widetilde{K}_{\infty}(f|_{\mathbb{R}^n})$ and $K_0(f|_{\mathbb{R}^n})$, respectively.

Lemma 3.1. *Let $(g,h)\colon \mathbb{R}^n \to \mathbb{R}^2$ be a polynomial map, which is non-degenerate at infinity. If g is convenient, then $\widetilde{K}_{\infty}(g) = \emptyset$ and $\widetilde{K}_{\infty}(g|_{\{h=r\}}) = \emptyset$ for $0 < |r| \ll 1$ and for $|r| \gg 1$.*

Proof. For simplicity of notation, we write f_1 and f_2 instead of g and h, respectively. Note that the proof for $\widetilde{K}_{\infty}(f_1) = \emptyset$ can be found in [10, Theorem 1]; since we will use some facts from the proof (and for the sake of completeness), we include the proof of this statement here. By contradiction, suppose that there exists a sequence $\{x^k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$ and a value $y \in \mathbb{R}$ such that

$$\lim_{k\to\infty} \|x^k\| = \infty, \quad \lim_{k\to\infty} f_1(x^k) = y, \quad \text{and} \quad \lim_{k\to\infty} \|\nabla f_1(x^k)\| = 0.$$

By Lemma 2.2, there exists an analytic curve $\varphi\colon (0,\epsilon) \to \mathbb{R}^n, t \mapsto (\varphi_1(t), \ldots, \varphi_n(t))$, such that

(a1) $\lim_{t\to 0} \|\varphi(t)\| = \infty$;
(a2) $\lim_{t\to 0} f_1(\varphi(t)) = y$; and
(a3) $\lim_{t\to 0} \|\nabla f_1(\varphi(t))\| = 0$.

Let $J := \{j \ : \ \varphi_j \not\equiv 0\}$. By Condition (a1), $J \neq \emptyset$. By Lemma 2.3, for each $j \in J$, we can expand the coordinate functions φ_j as follows

$$\varphi_j(t) \ = \ x_j^0 t^{q_j} + \text{ higher order terms in } t,$$

where $x_j^0 \neq 0$ and $q_j \in \mathbb{Q}$. From Condition (a1), we get $\min_{j\in J} q_j < 0$.

Let $q := (q_1, \ldots, q_n) \in \mathbb{R}^n$, where $q_j := M$ for $j \notin J$ with M being sufficiently large and satisfying

$$M \quad > \quad \max\left\{\sum_{j\in J} q_j\kappa_j \ : \ \kappa \in \Gamma(f_1)\right\}.$$

Let d_1 be the minimal value of the linear function $\sum_{j=1}^n q_j\kappa_j$ on $\Gamma(f_1)$ and let Δ_1 be the maximal face of $\Gamma(f_1)$ (maximal with respect to the inclusion of faces) where the linear function takes this value, i.e.,

$$d_1 \ := \ d(q, \Gamma(f_1)) \quad \text{and} \quad \Delta_1 \ := \ \Delta(q, \Gamma(f_1)).$$

Recall that $\mathbb{R}^J := \{\kappa := (\kappa_1, \kappa_2, \ldots, \kappa_n) \in \mathbb{R}^n \ : \ \kappa_j = 0 \text{ for } j \notin J\}$. Since f_1 is convenient, the restriction $f_1|_{\mathbb{R}^J}$ is not constant, and so $\Gamma(f_1) \cap \mathbb{R}^J = \Gamma(f_1|_{\mathbb{R}^J})$ is nonempty and different from $\{0\}$. Furthermore, by definition of the vector q, one has

$$d_1 \ = \ d(q, \Gamma(f_1|_{\mathbb{R}^J})) \quad \text{and} \quad \Delta_1 \ = \ \Delta(q, \Gamma(f_1|_{\mathbb{R}^J})) \ \subset \ \mathbb{R}^J.$$

In particular, for each $j \notin J$, the polynomial f_{1,Δ_1} does not depend on the variable x_j. A direct calculation shows that

$$f_1(\varphi(t)) \ = \ f_{1,\Delta_1}(x^0)t^{d_1} + \text{ higher order terms in } t,$$

where $x^0 := (x_1^0, \ldots, x_n^0)$ with $x_j^0 = 1$ for $j \notin J$. It is easy to see that $d_1 \leq q_{j_*} := \min_{j\in J} q_j$. In fact, since f_1 is convenient, for any $j = 1, \ldots, n$, there exists a natural number $m_j \geq 1$ such that $m_j e^j \in \Gamma(f_1)$. (Recall that $\{e^1, \ldots, e^n\}$ denotes the canonical basic in $\mathbb{R}^n$.) As $q_{j_*} < 0$, it is clear that

$$d_1 \leq q_{j_*} m_{j_*} \leq q_{j_*} < 0.$$

Now, by Condition (a2), we have $f_{1,\Delta_1}(x^0) = 0$.

On the other hand, for $j \in J$, we have

$$\frac{\partial f_1}{\partial x_j}(\varphi(t)) \ = \ \frac{\partial f_{1,\Delta_1}}{\partial x_j}(x^0)t^{d_1 - q_j} + \text{ higher order terms in } t.$$

Since $d_1 \leq \min_{j\in J} q_j$, it follows from (a3) that $\frac{\partial f_{1,\Delta_1}}{\partial x_j}(x^0) = 0$ for all $j \in J$. So this, together with $f_{1,\Delta_1}(x^0) = 0$, implies that f_1 is not Khovanskii non-degenerate at infinity. By definition, the polynomial map (f_1, f_2) is not non-degenerate at infinity, which contradicts our assumption.

Now we will show that $\widetilde{K}_\infty(f_1|_{\{f_2=r\}}) = \emptyset$ for $0 < |r| \ll 1$ and for $|r| \gg 1$.

For $0 < |r| \ll 1$ or $|r| \gg 1$, in view of Sard's theorem, we can make the following assumptions without loss of generality:

(i) $r \notin K_0(f_{2,\Delta_2})$ for any face Δ_2 of $\Gamma(f_2)$;

(ii) If the set $X := \{x \in (\mathbb{R}^*)^n \ : \ f_{1,\Delta_1}(x) = 0, \nabla f_{1,\Delta_1}(x) \neq 0\}$ is not empty[2] for some face Δ_1 of $\Gamma(f_1)$, then $r \notin K_0(f_{2,\Delta_2}|_X)$ for any face Δ_2 of $\Gamma(f_2)$.

[2]Clearly, if the set X is not empty, then it is a semi-algebraic smooth manifold in $\mathbb{R}^n$.

For contradiction, suppose that $\widetilde{K}_\infty(f_1|_{\{f_2=r\}}) \neq \emptyset$ for some $0 < |r| \ll 1$ or $|r| \gg 1$, i.e., there exists a sequence $\{x^k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$ and a value $y \in \mathbb{R}$ such that

$$\lim_{k\to\infty} \|x^k\| = \infty, \quad \lim_{k\to\infty} f_1(x^k) = y, \quad f_2(x^k) = r, \quad \text{and} \quad \lim_{k\to\infty} \|\nabla(f_1|_{\{f_2=r\}})(x^k)\| = 0.$$

By definition, there exists a sequence $\lambda^k \in \mathbb{R}$ such that for all $k \geq 1$, we have

$$\nabla(f_1|_{\{f_2=r\}})(x^k) \ = \ \nabla f_1(x^k) - \lambda^k \nabla f_2(x^k).$$

By Lemma 2.2, there exists an analytic curve $\varphi(t) := (\varphi_1(t), \ldots, \varphi_n(t))$ and an analytic function $\lambda(t)$ with $0 < t \ll 1$ such that

(b1) $\lim_{t\to 0} \|\varphi(t)\| = \infty$;
(b2) $\lim_{t\to 0} f_1(\varphi(t)) = y$;
(b3) $f_2(\varphi(t)) = r$; and
(b4) $\lim_{t\to 0} \|\nabla f_1(\varphi(t)) - \lambda(t)\nabla f_2(\varphi(t))\| = 0$.

Let $J := \{j \ : \ \varphi_j \not\equiv 0\} \neq \emptyset$ and for each $j \in J$, expand φ_j as follows

$$\varphi_j(t) \ = \ x_j^0 t^{q_j} + \text{ higher order terms in } t,$$

where $x_j^0 \neq 0$ and $q_j \in \mathbb{Q}$. Let $q := (q_1, \ldots, q_n) \in \mathbb{R}^n$, where $q_j := M$ for $j \notin J$ with M being sufficiently large and satisfying

$$M \ > \ \max_{i=1,2} \left\{ \sum_{j\in J} q_j \kappa_j \ : \ \kappa \in \Gamma(f_i) \right\}.$$

For each $i = 1, 2$, let d_i be the minimal value of the linear function $\sum_{j=1}^n q_j \kappa_j$ on $\Gamma(f_i)$ and let Δ_i be the maximal face of $\Gamma(f_i)$ (maximal with respect to the inclusion of faces) where the linear function takes this value, i.e.,

$$d_i \ := \ d(q, \Gamma(f_i)) \quad \text{and} \quad \Delta_i \ := \ \Delta(q, \Gamma(f_i)).$$

Since f_1 is convenient, the restriction $f_1|_{\mathbb{R}^J}$ is not constant. Furthermore, we also have the restriction $f_2|_{\mathbb{R}^J}$ is not constant. In fact, if this is not the case, then it follows from (b4) that

$$\lim_{t\to 0} \frac{\partial f_1}{\partial x_j}(\varphi(t)) \ = \ 0 \quad \text{for all} \quad j \in J.$$

Replacing f_1 by the restriction $f_1|_{\mathbb{R}^J}$ and repeating the previous arguments, we can see that f_1 is not Khovanskii non-degenerate at infinity. By definition, then the polynomial map (f_1, f_2) is not non-degenerate at infinity, which contradicts our assumption.

Therefore, the restriction of $f_i, i = 1, 2$, on $\mathbb{R}^J$ is not constant, and so $\Gamma(f_i) \cap \mathbb{R}^J = \Gamma(f_i|_{\mathbb{R}^J})$ is nonempty and different from $\{0\}$. Furthermore, by definition of the vector q, one has

$$d_i \ = \ d(q, \Gamma(f_i|_{\mathbb{R}^J})) \quad \text{and} \quad \Delta_i \ = \ \Delta(q, \Gamma(f_i|_{\mathbb{R}^J})) \ \subset \ \mathbb{R}^J.$$

In particular, for each $j \notin J$, the polynomial f_{i,Δ_i} does not depend on the variable x_j.

By a similar argument as above, we also obtain $d_1 \le q_{j_*} := \min_{j \in J} q_j < 0$ and $f_{1,\Delta_1}(x^0) = 0$, where $x^0 := (x_1^0, \ldots, x_n^0)$ with $x_j^0 = 1$ for $j \notin J$. Observe that for all $j \notin J$, the polynomials f_{1,Δ_1} and f_{2,Δ_2} do not depend on x_j, so

$$\frac{\partial f_{1,\Delta_1}}{\partial x_j}(x^0) \;=\; \frac{\partial f_{2,\Delta_2}}{\partial x_j}(x^0) \;=\; 0.$$

Note that $\lambda(t) \not\equiv 0$, since otherwise $y \in \widetilde{K}_\infty(f_1) = \emptyset$, a contradiction. Hence, we can expand the coordinate $\lambda(t)$ in terms of t as

$$\lambda(t) \;=\; \lambda^0 t^\theta + \text{ higher order terms in } t,$$

where $\lambda^0 \neq 0$ and $\theta \in \mathbb{Q}$. There are three cases to be considered.

Case 1: $d_1 < d_2 + \theta$. For each $j \in J$, we have

$$\frac{\partial f_1}{\partial x_j}(\varphi(t)) - \lambda(t)\frac{\partial f_2}{\partial x_j}(\varphi(t)) \;=\; \frac{\partial f_{1,\Delta_1}}{\partial x_j}(x^0) t^{d_1 - q_j} + \text{ higher order terms in } t.$$

Since $d_1 \le q_{j_*}$, in view of (b4), we have $\frac{\partial f_{1,\Delta_1}}{\partial x_j}(x^0) = 0$ for all $j \in J$. As $f_{1,\Delta_1}(x^0) = 0$, it implies that f_1 is not Khovanskii non-degenerate at infinity. By definition, then the polynomial map (f_1, f_2) is not non-degenerate at infinity, which contradicts our assumption.

Case 2: $d_1 > d_2 + \theta$. For each $j \in J$, we have

$$\frac{\partial f_1}{\partial x_j}(\varphi(t)) - \lambda(t)\frac{\partial f_2}{\partial x_j}(\varphi(t)) \;=\; -\lambda^0 \frac{\partial f_{2,\Delta_2}}{\partial x_j}(x^0) t^{d_2 + \theta - q_j} + \text{ higher order terms in } t.$$

From $d_2 + \theta < d_1 \le q_{j_*}$ and (b4), we get $\frac{\partial f_{2,\Delta_2}}{\partial x_j}(x^0) = 0$ for all $j \in J$. On the other hand, a simple calculation shows that

$$f_2(\varphi(t)) \;=\; f_{2,\Delta_2}(x^0) t^{d_2} + \text{ higher order terms in } t.$$

If $d_2 < 0$ then it follows from (b3) that $f_{2,\Delta_2}(x^0) = 0$ and so f_2 is not Khovanskii non-degenerate at infinity; hence, by definition, the polynomial map (f_1, f_2) is not non-degenerate at infinity, which contradicts our assumption. If $d_2 = 0$, we have $f_{2,\Delta_2}(x^0) = r$ and so $r \in K_0(f_{2,\Delta_2})$, a contradiction. Finally, if $d_2 > 0$, then $r = 0$, which contradicts the assumption $|r| > 0$.

Case 3: $d_1 = d_2 + \theta$. For each $j \in J$, we have

$$\frac{\partial f_1}{\partial x_j}(\varphi(t)) - \lambda(t)\frac{\partial f_2}{\partial x_j}(\varphi(t)) \;=\; \left(\frac{\partial f_{1,\Delta_1}}{\partial x_j}(x^0) - \lambda^0 \frac{\partial f_{2,\Delta_2}}{\partial x_j}(x^0)\right) t^{d_1 - q_j} + \cdots,$$

where the dots stand for the higher-order terms in t. Since $d_1 \le \min_{j \in J} q_j < 0$, it follows from (b4) that

$$\frac{\partial f_{1,\Delta_1}}{\partial x_j}(x^0) - \lambda^0 \frac{\partial f_{2,\Delta_2}}{\partial x_j}(x^0) \;=\; 0 \quad \text{for all} \quad j \in J.$$

Observe that $\frac{\partial f_{1,\Delta_1}}{\partial x_j}(x^0) \neq 0$ for some $j \in J$ since otherwise, we get a contradiction by repeating the arguments in Case 1. Hence the set

$$X \ := \ \{x \in (\mathbb{R}^*)^n \ : \ f_{1,\Delta_1}(x) = 0, \nabla f_{1,\Delta_1}(x) \neq 0\}$$

is a nonempty semi-algebraic smooth manifold in $\mathbb{R}^n$. Moreover, x^0 is a critical point of $f_{2,\Delta_2}|_X$. Finally, by a similar argument as Case 2, we can see that either $d_2 < 0$ and $f_{2,\Delta_2}(x^0) = 0$ which contradicts the assumption that the polynomial map (f_1, f_2) is Khovanskii non-degenerate at infinity, or $d_2 = 0$ and $f_{2,\Delta_2}(x^0) = r$ which contradicts the assumption $r \notin K_0(f_{2,\Delta_2}|_X)$, or $d_2 > 0$ and $r = 0$, which contradicts the assumption $|r| > 0$. □

The following definition is inspired by that proposed in [9].

Definition 3.2. Let $g, h\colon \mathbb{R}^n \to \mathbb{R}$ be polynomial functions. A sequence $\{x^k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$ with $\|x^k\| \to +\infty$ is said to be

(i) a *sequence of the first type* if $g(x^k) \to 0$ and $|h(x^k)| \geq \delta$ for some $\delta > 0$;
(ii) a *sequence of the second type* if the sequence $\{g(x^k)\}$ is bounded and $|h(x^k)| \to +\infty$.

Lemma 3.2. *Let $g, h\colon \mathbb{R}^n \to \mathbb{R}$ be polynomial functions such that $g^{-1}(0) \subset h^{-1}(0)$ and $\widetilde{K}_\infty(g) = \emptyset$. Then following two statements hold:*

(i) *If $\widetilde{K}_\infty(g|_{\{h=r\}}) = \emptyset$ for all $|r| > 0$ sufficiently small, then there are no sequences of the first type.*
(ii) *If $\widetilde{K}_\infty(g|_{\{h=r\}}) = \emptyset$ for all $|r|$ sufficiently large, then there are no sequences of the second type.*

Proof. (i) By contradiction, assume that there exists a real number $\delta > 0$ and a sequence $x^k \in \mathbb{R}^n$, with $\|x^k\| \to +\infty$, such that

$$g(x^k) \to 0 \quad \text{ and } \quad |h(x^k)| \geq \delta.$$

Then $|g(x^k)| > 0$ since $g^{-1}(0) \subset h^{-1}(0)$. By Sard's theorem, the set of critical values of h is a finite subset of $\mathbb{R}$. So we can choose $\delta > 0$ sufficiently small so that each of the level sets $h^{-1}(\pm\delta)$ is either empty or a smooth manifold. Furthermore, by the assumptions, we can suppose that $\widetilde{K}_\infty(g|_{\{h=\pm\delta\}}) = \emptyset$.

Let $X := \{x \in \mathbb{R}^n \ : \ |h(x)| \geq \delta\}$. We have

$$0 \ = \ \inf_{x\in X} |g(x)| \ < \ |g(x^k)| \quad \text{for all} \quad k.$$

Applying the Ekeland variational principle [14] to the function

$$X \to \mathbb{R}, \quad x \mapsto |g(x)|,$$

with data $\epsilon := |g(x^k)| > 0$ and $\lambda := \frac{\|x^k\|}{2} > 0$, there is a point y^k in X such that the following inequalities hold

$$
\begin{aligned}
&|g(y^k)| \le |g(x^k)|,\\
&\|y^k - x^k\| \le \lambda,\\
&|g(y^k)| \le |g(x)| + \frac{\epsilon}{\lambda}\|x - y^k\| \quad \text{for all} \quad x \in X.
\end{aligned}
$$

We deduce easily that $\lim_{k\to\infty} \|y^k\| = +\infty$ and $\lim_{k\to\infty} g(y^k) = 0$. Furthermore, since $g^{-1}(0) \subset h^{-1}(0)$ and $|h(y^k)| \ge \delta > 0$, we have $g(y^k) \ne 0$ for all k. Passing a subsequence and replacing g (resp., h) by $-g$ (resp., $-h$) if necessary, we may assume that for all k the following conditions hold: $g(y^k) > 0$ and either $h(y^k) > \delta$ or $h(y^k) = \delta$. By continuity, g and h are positive in some open neighborhood of y^k. In particular, we have for all x near y^k,

$$|g(x)| = g(x) \quad \text{and} \quad |h(x)| = h(x).$$

Hence y^k is a local minimizer of the function

$$\{x \in \mathbb{R}^n \ : \ h(x) \ge \delta\} \to \mathbb{R}, \quad x \mapsto g(x) + \frac{\epsilon}{\lambda}\|x - y^k\|.$$

Observe that $h^{-1}(\delta)$ is a smooth manifold. Therefore, by Lagrange's multipliers theorem, there exists $\mu_k \le 0$ with $\mu_k(h(y^k) - \delta) = 0$ such that

$$0 \ \in \ \partial\left(g(\cdot) + \frac{\epsilon}{\lambda}(\|\cdot - y^k\|)\right)(y^k) + \mu_k \nabla h(y^k).$$

This implies that

$$0 \ \in \ \nabla g(y^k) + \mu_k \nabla h(y^k) + \frac{\epsilon}{\lambda}\mathbb{B}^n,$$

where $\mathbb{B}^n$ stands for the unit closed ball in $\mathbb{R}^n$. Consequently, we get

$$\|\nabla g(y^k) + \mu_k \nabla h(y^k)\| \ \le \ \frac{\epsilon}{\lambda} \ = \ \frac{2|g(x^k)|}{\|x^k\|}.$$

By letting k tend to infinity, we obtain

$$\lim_{k\to\infty} \|y^k\| = +\infty, \quad \lim_{k\to\infty} g(y^k) = 0, \quad \text{and} \quad \lim_{k\to\infty} \|\nabla g(y^k) + \mu_k \nabla h(y^k)\| = 0.$$

Note that if $h(y^k) > \delta$ then $\mu_k = 0$. Therefore, either $0 \in \widetilde{K}_\infty(g)$ or $0 \in \widetilde{K}_\infty(g|_{\{h=\delta\}})$, and a contradiction follows.

(ii) Suppose for contradiction that there exists a sequence $x^k \in \mathbb{R}^n$, with $\|x^k\| \to +\infty$ such that the sequence $\{g(x^k)\}$ is bounded and $|h(x^k)| \to +\infty$. Then $|g(x^k)| > 0$ from our assumption $g^{-1}(0) \subset h^{-1}(0)$. By Sard's theorem, the set of critical values of h is a finite subset of $\mathbb{R}$. So we can choose $M > 0$ sufficiently large so that each of the level sets $h^{-1}(\pm M)$ is either empty or a smooth manifold. Furthermore, by the assumptions, we can suppose that $\widetilde{K}_\infty(g|_{\{h=\pm M\}}) = \emptyset$.

Let $X := \{x \in \mathbb{R}^n \ : \ |h(x)| \ge M\}$. We have for all k sufficiently large,

$$\inf_{x\in X} |g(x)| \ \le \ |g(x^k)|.$$

By applying the Ekeland variational principle [14] to the function $X \to \mathbb{R}, x \mapsto |g(x)|$, with data $\epsilon := |g(x^k)| > 0$ and $\lambda := \frac{\|x^k\|}{2} > 0$, we get a point y^k in X satisfying the following inequalities

$$\begin{aligned}
&|g(y^k)| \le |g(x^k)|,\\
&\|y^k - x^k\| \le \lambda,\\
&|g(y^k)| \le |g(x)| + \frac{\epsilon}{\lambda}\|x - y^k\| \quad \text{for all} \quad x \in X.
\end{aligned}$$

We deduce easily that

$$\frac{\|x^k\|}{2} \ \le\ \|y^k\| \ \le\ \frac{3\|x^k\|}{2},$$

which yields $\lim_{k\to\infty} \|y^k\| = +\infty$.

Similarly to (i), for k large enough, we can assume that $h(y^k) \ge M > 0$ and $g(y^k) > 0$ from the assumption that $g^{-1}(0) \subset h^{-1}(0)$. Hence, by repeating arguments similar to (i), we have

$$\|\nabla g(y^k) + \mu_k \nabla h(y^k)\| \ \le\ \frac{\epsilon}{\lambda} \ =\ \frac{2|g(x^k)|}{\|x^k\|}$$

for some $\mu_k \le 0$ with $\mu_k(h(y^k) - M) = 0$. Hence,

$$\lim_{k\to\infty} \|\nabla g(y^k) + \mu_k \nabla h(y^k)\| \ =\ 0.$$

On the other hand, since the sequence $\{g(x^k)\}$ is bounded, so is the sequence $\{g(y^k)\}$. Hence, by passing to a subsequence if necessary, we may assume that there exists the limit $t := \lim_{k\to\infty} g(y^k)$. Note that if $h(y^k) > M$ then $\mu_k = 0$. Therefore, either $t \in \widetilde{K}_\infty(g)$ or $t \in \widetilde{K}_\infty(g|_{\{h=M\}})$, which is a contradiction. □

Lemma 3.3. *Let $g, h\colon \mathbb{R}^n \to \mathbb{R}$ be polynomial functions such that $g^{-1}(0) \subset h^{-1}(0)$. The following two conditions are equivalent:*

(i) *there are no sequences of the first and second types.*

(ii) *there exist some constants $c > 0, \alpha > 0$, and $\beta > 0$ such that*

$$|g(x)|^\alpha + |g(x)|^\beta \ \ge\ c|h(x)| \quad \textit{for all} \quad x \in \mathbb{R}^n.$$

Proof. (Cf. [25, Theorem 3.4]).

(ii) $\Rightarrow$ (i): The implication is straightforward.

(i) $\Rightarrow$ (ii): We assume that $h \not\equiv 0$, otherwise the implication is trivial. We only consider the case where $g^{-1}(0) \ne \emptyset$; the case $g^{-1}(0) = \emptyset$ follows similarly. Then for each $t \ge 0$, the set $\{x \in \mathbb{R}^n \ : \ |g(x)| = t\}$ is non-empty. This, together with condition (i), implies that the (semi-algebraic) function $\mu\colon [0, +\infty) \to \mathbb{R}$ given by

$$\mu(t) \ :=\ \sup_{|g(x)|=t} |h(x)|$$

is well-defined. Furthermore, it is not hard to see that $\mu(0) = 0$, $\mu(t) > 0$ for all $t > 0$ small enough and $\mu(t) \to +\infty$ as $t \to +\infty$. By Lemma 2.3, we can write

$$\begin{aligned} \mu(t) &= at^{\alpha} + o(t^{\alpha}) \quad \text{as} \quad t \to 0^+, \\ \mu(t) &= bt^{\beta} + o(t^{\beta}) \quad \text{as} \quad t \to +\infty \end{aligned}$$

for some constants $a \neq 0, b \neq 0, \alpha \geq 0$ and $\beta > 0$. Therefore, we can find constants $c_1 > 0, c_2 > 0, \delta > 0$ and $r > 0$ with $\delta \ll 1 \ll r$ such that the following inequalities hold

$$\begin{aligned} |g(x)|^{\alpha} &\geq c_1|h(x)| \quad \text{for} \quad 0 < |g(x)| \leq \delta, \\ |g(x)|^{\beta} &\geq c_2|h(x)| \quad \text{for} \quad |g(x)| \geq R. \end{aligned}$$

By assumption, we may assume that $\alpha > 0$ so that the first inequality holds for $|g(x)| \leq \delta$. Furthermore, we also may assume $\alpha \leq 1 \leq \beta$ because δ is sufficiently small and r is sufficiently large.

On the other hand, it follows easily from condition (i) that there exists a constant $M > 0$ such that for all $x \in \mathbb{R}^n$ with $\delta \leq |g(x)| \leq R$ we have $|h(x)| \leq M$ and hence

$$|g(x)|^{\alpha} + |g(x)|^{\beta} \geq \delta^{\alpha} + \delta^{\beta} = \frac{\delta^{\alpha} + \delta^{\beta}}{M} M \geq \frac{\delta^{\alpha} + \delta^{\beta}}{M} |h(x)|.$$

Letting $c := \min\{c_1, c_2, \frac{\delta^{\alpha}+\delta^{\beta}}{M}\}$, we get the desired conclusion. □

We now are in position to finish the proof of Theorem 3.1.

Proof of Theorem 3.1. This is a direct consequence of Lemmas 3.1, 3.2, and 3.3. □

The following corollary is inspired by the results in [16].

Corollary 3.1. *Under the assumptions of Theorem 3.1, there exist a positive integer N and a continuous semi-algebraic function $f\colon \mathbb{R}^n \to \mathbb{R}$ such that $h^N = gf$.*

Proof. Clearly, if $\inf_{x\in\mathbb{R}^n} |g(x)| > 0$ then the integer $N := 1$ and the function $f := \frac{h}{g}$ have the desired property. So assume that $\inf_{x\in\mathbb{R}^n} |g(x)| = 0$. By observing the proof of Lemma 3.3, we can find positive constants α and δ with $\alpha \leq 1$ such that

$$|g(x)|^{\alpha} \geq c|h(x)| \quad \text{for} \quad |g(x)| \leq \delta.$$

Let $\ell := \left[\frac{1}{\alpha}\right] + 1 > \frac{1}{\alpha} \geq 1$. It is easy to see that the (semi-algebraic) function $f_0\colon \mathbb{R}^n \to \mathbb{R}$ defined by

$$f_0(x) := \begin{cases} \frac{h^{2\ell}(x)}{g^2(x)} & \text{if } g(x) \neq 0, \\ 0 & \text{otherwise} \end{cases}$$

is continuous. Since $f_0 g^2 = h^{2\ell}$, we deduce that the integer $N := 2\ell$ and the function $f := f_0 g$ have the desired property. □

4. Genericity of non-degenerate at infinity polynomial maps

In this section we show the genericity of the condition of non-degeneracy at infinity for real polynomial maps; actually, we will prove a strong result of this (see Theorem 4.1 below). Note that the genericity of the condition of non-degeneracy for complex polynomial maps has been given in [27] (the case $p = 1$) and in [26, Theorem (Resolution of Singularities)] and [32, Corollary 3.2.1] (the case $p \geq 1$).

For simplicity, we introduce some notation here for this section. Let $\Gamma := (\Gamma_1, \ldots, \Gamma_p)$ with each Γ_i being a Newton polyhedron in $\mathbb{R}^n_+$ and $1 \leq p \leq n$. Let

$$\mathcal{F} \ := \ \{\Delta := (\Delta_1, \ldots, \Delta_p) \ : \ \exists q \in \mathbb{R}^n \text{ s.t. } \Delta_i = \Delta(q, \Gamma_i) \text{ for all } i\}.$$

(Recall that $\Delta(q, \Gamma_i) := \operatorname{argmin}_{\kappa \in \Gamma_i} \langle q, \kappa \rangle$.) Clearly, $\mathcal{F}$ is a finite set as the number of faces of a polyhedron is finite.

For each $i = 1, \ldots, p$, denote by $m_i := \#(\Gamma_i \cap \mathbb{Z}^n)$-the cardinal number of the set $\Gamma_i \cap \mathbb{Z}^n$. For any $c := (c_1, \ldots, c_p) \in \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_p}$ where $c_i = (c_{i,\kappa})_{\kappa \in \Gamma_i \cap \mathbb{Z}^n}$, denote

$$f_i(x, c_i) \ := \ \sum_{\kappa \in \Gamma_i \cap \mathbb{Z}^n} c_{i,\kappa} x^\kappa \in \mathbb{R}[x].$$

For each nonempty set $I := \{i_1, \ldots, i_q\} \subset \{1, \ldots, p\}$ and $\Delta := (\Delta_1, \ldots, \Delta_p) \in \mathcal{F}$, we let

$$\begin{aligned} F_{I,\Delta}(x, c) \ &:= \ (f_{i_1,\Delta_{i_1}}(x, c_{i_1}), \ldots, f_{i_q,\Delta_{i_q}}(x, c_{i_q})), \\ xDF_{I,\Delta}(x, c) \ &:= \ \left(x_j \frac{\partial f_{i,\Delta_i}}{\partial x_j}(x, c_i) \right)_{i \in I, \ j=1,\ldots,n} \end{aligned}$$

and

$$\mathcal{D}_I(\Delta) := \left\{ (c_1, \ldots, c_p) : \begin{cases} c_i = (c_{i,\kappa})_{\kappa \in \Gamma_i \cap \mathbb{Z}^n} \in \mathbb{R}^{m_i}, \ \Gamma(f_i(x, c_i)) = \Gamma_i \text{ for } i = 1, \ldots, p, \\ \text{if } x \in (\mathbb{R}^*)^n \text{ and if } F_{I,\Delta}(x, c) = 0, \text{ then} \\ \operatorname{rank}(xDF_{I,\Delta}(x, c)) = \#I \end{cases} \right\}.$$

The main result of this section is as follows.

Theorem 4.1. *The set $\cap_{I,\Delta} \mathcal{D}_I(\Delta)$ is an open and dense semi-algebraic set in $\mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_p}$, where the intersection is taken over all nonempty sets $I \subset \{1, \ldots, p\}$ and all $\Delta \in \mathcal{F}$.*

Proof. Observe that the number of subsets of $\{1, \ldots, p\}$ is finite, $\mathcal{F}$ is a finite set, and a finite intersection of open dense semi-algebraic sets is open dense semi-algebraic. Now the desired conclusion follows immediately from Propositions 4.1 and 4.2 below. □

Proposition 4.1. *For each nonempty set $I \subset \{1, \ldots, p\}$, the set $\cap_{\Delta \in \mathcal{F}} \mathcal{D}_I(\Delta)$ is open and semi-algebraic.*

Proof. (Cf. [31, Appendix]; see also [6, Proposition 3.1]). Let I be a nonempty subset of $\{1, \ldots, p\}$. By renumbering, we may assume that $I = \{1, \ldots, q\}$ for some $q \leq p$. By definition, for any $(\Delta_1, \ldots, \Delta_p) \in \mathcal{F}$ we have

$$\mathcal{D}_I(\Delta_1, \ldots, \Delta_p) \;=\; \mathcal{D}_I(\Delta_1, \ldots, \Delta_q) \times X$$

where $X := \{(c_{q+1}, \ldots, c_p) \ : \ \Gamma(f_i(x, c_i)) = \Gamma_i \text{ for } i = q+1, \ldots, p\}$. Observe that X is an open (and dense) semi-algebraic subset of $\mathbb{R}^{m_{q+1}} \times \cdots \times \mathbb{R}^{m_p}$ and that X does not depend on the polyhedra $\Gamma_i, i = 1, \ldots, q$. Hence, it suffices to show that $\cap_{(\Delta_1,\ldots,\Delta_q)}\mathcal{D}_I(\Delta_1, \ldots, \Delta_q)$ is an open semi-algebraic subset of $\mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_q}$. In other words, we can assume that $q = p$, i.e., $I = \{1, \ldots, p\}$.

Consider the projection

$$\pi \colon \mathbb{R}^n \times \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_p} \to \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_p}, \quad (x, c_1, \ldots, c_p) \mapsto (c_1, \ldots, c_p),$$

and the union $V^* := \cup_{\Delta \in \mathcal{F}} V(\Delta)$ where

$$V(\Delta) := \left\{ (x, c_1, \ldots, c_p) \ : \ \begin{cases} x \in (\mathbb{R}^*)^n, \ c_i = (c_{i,\kappa})_{\kappa \in \Gamma_i \cap \mathbb{Z}^n} \in \mathbb{R}^{m_i}, \\ \Gamma(f_i(x, c_i)) = \Gamma_i, \ f_{i,\Delta_i}(x, c_i) = 0 \text{ for } i = 1, \ldots, p, \\ \operatorname{rank}(xDF_{I,\Delta}(x, c)) < p \end{cases} \right\}.$$

By definition, $W := \pi(V^*)$ is the complement of $\cap_{\Delta \in \mathcal{F}}\mathcal{D}_I(\Delta)$ in the set

$$\{(c_1, \ldots, c_p) \ : \ c_i = (c_{i,\kappa})_{\kappa \in \Gamma_i \cap \mathbb{Z}^n} \in \mathbb{R}^{m_i}, \ \Gamma(f_i(x, c_i)) = \Gamma_i \text{ for } i = 1, \ldots, p\}.$$

Observe that the latter set is an open dense semi-algebraic subset of $\mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_p}$. In light of Theorem 2.1, W is a semi-algebraic set, and so is $\cap_{\Delta \in \mathcal{F}}\mathcal{D}_I(\Delta)$. Furthermore, showing that $\cap_{\Delta \in \mathcal{F}}\mathcal{D}_I(\Delta)$ is an open set means to prove that W is a closed set.

Assume by contradiction that $\overline{W}$ properly contains W, i.e., we can take a point $(c_1^0, \ldots, c_p^0) \in \overline{W} \setminus W$. By definition, then $(c_1^0, \ldots, c_p^0) \in \overline{\pi(V(\Delta))}$ for some $\Delta := (\Delta_1, \ldots, \Delta_p) \in \mathcal{F}$. In view of Lemma 2.1, there exists a real analytic curve $(\varphi(t), c_1(t), \ldots, c_p(t)) \in V(\Delta)$ defined on a small enough interval $(0, \epsilon)$ such that $\lim_{t \to 0} c_i(t) = c_i^0, i = 1, \ldots, p$. Let us expand $\varphi_j(t), j = 1, \ldots, n$, and $c_i(t)$, $i = 1, \ldots, p$, in terms of the parameter, say

$$\begin{aligned} \varphi_j(t) &= x_j^0 t^{q_j} + \text{ higher order terms in } t, \\ c_i(t) &= c_i^0 + \text{ higher order terms in } t, \end{aligned}$$

where $x_j^0 \neq 0$ and $q_j \in \mathbb{Q}$. Let $q := (q_1, \ldots, q_n)$ and

$$\widetilde{\Delta}_i \;:=\; \Delta(q, \Delta_i) \quad \text{ for all } \quad i = 1, \ldots, p.$$

We have $\widetilde{\Delta} := (\widetilde{\Delta}_1, \ldots, \widetilde{\Delta}_p) \in \mathcal{F}$. In fact, if $q = 0$, then $\widetilde{\Delta} = \Delta$ and there is nothing to prove. So, assume that $q \neq 0$. By definition, we can find a vector $q^0 \in \mathbb{R}^n$ such that $\Delta_i = \Delta(q^0, \Gamma_i)$ for all i. If $\Delta_i = \Gamma_i$ for all i, then it is clear that $\widetilde{\Delta} \in \mathcal{F}$. Otherwise, we have $q^0 \neq 0$ and $I' := \{i \in \{1, \ldots, p\} \ : \ \Delta_i \neq \Gamma_i\} \neq \emptyset$. Then for each $i \in I'$, there exists $\varepsilon_i > 0$ such that for any $\widetilde{q}$ with $\|\widetilde{q} - q^0\| \leq \varepsilon_i$ it holds that

$$\Delta(\widetilde{q}, \Gamma_i) \;\subset\; \Delta(q^0, \Gamma_i) \;=\; \Delta_i.$$

Set $\varepsilon := \min_{i\in I'} \varepsilon_i > 0$ and $\widetilde{q} := q^0 + \varepsilon \frac{q}{\|q\|}$. Clearly $\Delta(\widetilde{q}, \Gamma_i) \subset \Delta_i$. Hence $\Delta(\widetilde{q}, \Gamma_i) = \Delta(\widetilde{q}, \Delta_i)$. Moreover, for any $\kappa \in \Delta_i$, we have

$$\langle \widetilde{q}, \kappa\rangle \ = \ \langle q^0, \kappa\rangle + \frac{\varepsilon}{\|q\|}\langle q, \kappa\rangle \ \geq \ d_i + \frac{\varepsilon}{\|q\|}\widetilde{d}_i,$$

where we put $d_i := \min_{\kappa'\in\Gamma_i}\langle q^0, \kappa'\rangle$ and $\widetilde{d}_i := \min_{\kappa'\in\Delta_i}\langle q, \kappa'\rangle$. Observe that the equality happens if and only if $\kappa \in \widetilde{\Delta}_i$, which yields $\widetilde{\Delta}_i = \Delta(\widetilde{q}, \Delta_i)$. Therefore $\widetilde{\Delta}_i = \Delta(\widetilde{q}, \Gamma_i)$ and so $\widetilde{\Delta} \in \mathcal{F}$.

On the other hand, a simple calculation shows that

$$\begin{aligned} f_{i,\Delta_i}(\varphi(t), c_i(t)) \ &= \ f_{i,\widetilde{\Delta}_i}(x^0, c_i^0)t^{d_i} + \text{ higher order terms in } t, \\ \varphi_j(t)\frac{\partial f_{i,\Delta_i}}{\partial x_j}(\varphi(t), c_i(t)) \ &= \ x_j^0 \frac{\partial f_{i,\widetilde{\Delta}_i}}{\partial x_j}(x^0, c_i^0)t^{d_i} + \text{ higher order terms in } t, \end{aligned}$$

where $x^0 := (x_1^0, \ldots, x_n^0) \in (\mathbb{R}^*)^n$. As $(\varphi(t), c_1(t), \ldots, c_p(t)) \in V(\Delta)$ for all $t \in (0, \epsilon)$, we get easily that $(x^0, c_1^0, \ldots, c_p^0) \in V(\widetilde{\Delta}) \subset V^*$. Thus $(c_1^0, \ldots, c_p^0) \in W$, which contradicts the assumption $(c_1^0, \ldots, c_p^0) \in \overline{W} \setminus W$. □

Proposition 4.2. *For each nonempty set $I \subset \{1, \ldots, p\}$ and each $\Delta \in \mathcal{F}$, $\mathcal{D}_I(\Delta)$ contains an open dense semi-algebraic set in $\mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_p}$.*

Proof. As in the proof of Proposition 4.1, we may assume that $I = \{1, \ldots, p\}$. Furthermore, observe that for any $(c_1, \ldots, c_p) \in \mathcal{D}_I(\Delta)$, all those $c_{i,\kappa}$, with $\kappa \in (\Gamma_i \cap \mathbb{Z}^n) \setminus \Delta_i$ and $i = 1, \ldots, p$, can be replaced by any nonzero real numbers and the resulting $(c_1, \ldots, c_p)$ still belongs to $\mathcal{D}_I(\Delta)$. Consequently, without loss of generality, we may assume that $\Delta_i = \Gamma_i$ for all i. In other words, we need to show that the set

$$\mathcal{D}_I(\Gamma) = \left\{ \ (c_1, \ldots, c_p) \ : \ \begin{cases} c_i = (c_{i,\kappa})_{\kappa\in\Gamma_i\cap\mathbb{Z}^n} \in \mathbb{R}^{m_i}, \ \Gamma(f_i(x, c_i)) = \Gamma_i \text{ for } i = 1, \ldots, p, \\ \text{if } x \in (\mathbb{R}^*)^n \text{ and if } F(x, c) = 0, \text{ then} \\ \operatorname{rank}(xDF(x, c)) = p \end{cases} \right\},$$

contains an open dense semi-algebraic set in $\mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_p}$, where we put

$$\begin{aligned} F(x, c) \ &:= \ (f_1(x, c_1), \ldots, f_p(x, c_p)), \\ xDF(x, c) \ &:= \ \left(x_j \frac{\partial f_i}{\partial x_j}(x, c_i)\right)_{i=1,\ldots,p, \ j=1,\ldots,n}. \end{aligned}$$

It is clear that if there exists an index i_0 such that $m_{i_0} = 1$ (i.e., $f_{i_0}(x, c_{i_0})$ is a monomial), then $\{f_{i_0}(x, c_{i_0}) = 0\} \subset \{x_1 \cdots x_n = 0\}$ for $c_{i_0} \neq 0$; hence $(\mathbb{R}^*)^{m_1} \times \cdots \times (\mathbb{R}^*)^{m_p} \subset \mathcal{D}_I(\Gamma)$ and the problem is trivial. So in what follows we will assume that $m_i > 1$ for every $i = 1, \ldots, p$.

Let $\Gamma_1 + \cdots + \Gamma_p$ be the Minkowski sum and set $d := \dim(\Gamma_1 + \cdots + \Gamma_p)$. By [19, Exercises of page 48], there exist $q^1, \ldots, q^n \in \mathbb{Z}^n$, with $\det(q^1, \ldots, q^n) = 1$, and $d_1, \ldots, d_{n-d} \in \mathbb{R}$ such that the set $\Gamma_1 + \cdots + \Gamma_p$ is contained in the affine space

$$L \ := \ \{\kappa \in \mathbb{R}^n \ : \ \langle q^j, \kappa\rangle = d_j, j = 1, \ldots, n - d\}.$$

We need the following lemma.

Lemma 4.1. *For each $i \in \{1, \ldots, p\}$ there exist $d_{ij}, j = 1, \ldots, n-p$, such that $\Gamma_i \subset L_i$, where*

$$L_i := \{\kappa \in \mathbb{R}^n \ : \ \langle q^j, \kappa \rangle = d_{ij}, j = 1, \ldots, n-d\},$$

i.e., Γ_i is contained in an affine space which is parallel to L.

Proof. For each index $i = 1, \ldots, p$, choose $\kappa^i \in \Gamma_i$. Fix an index i and set $d_{ij} := \langle q^j, \kappa^i \rangle$ for $j = 1, \ldots, n-d$. Take any $\kappa \in \Gamma_i$ with $\kappa \neq \kappa^i$. By definition, $\gamma := \sum_{k=1}^{p} \kappa^k$ and $\gamma' := \gamma + (\kappa - \kappa^i)$ belong to the set $\Gamma_1 + \cdots + \Gamma_p$. We have for all $j = 1, \ldots, n-d$,

$$\langle q^j, \kappa - \kappa^i \rangle \ = \ \langle q^j, \gamma' - \gamma \rangle \ = \ \langle q^j, \gamma' \rangle - \langle q^j, \gamma \rangle \ = \ d_j - d_j \ = \ 0,$$

and so $\langle q^j, \kappa \rangle = \langle q^j, \kappa^i \rangle = d_{ij}$. Consequently, $\kappa \in L_i$. The lemma is proved. □

For each $j = 1, \ldots, n$, let us write $q^j := (q_{j1}, \ldots, q_{jn})$ and set $A := (q_{ij})_{i,j=1,\ldots,n}$. Consider the following change of coordinates

$$\begin{cases} x_1 & = & u_1^{q_{11}} \ldots u_{n-d}^{q_{(n-d)1}} \ldots u_n^{q_{n1}}, \\ \vdots & \vdots & \vdots \\ x_n & = & u_1^{q_{1n}} \ldots u_{n-d}^{q_{(n-d)n}} \ldots u_n^{q_{nn}}. \end{cases} \tag{1}$$

It follows from Lemma 4.1 that for each $\kappa \in \Gamma_i \cap \mathbb{Z}^n$, we have $A\kappa = (d_{i1}, \ldots, d_{i(n-d)}, \gamma_\kappa)$, for some $\gamma_\kappa \in \mathbb{Z}^d$. So in the system of coordinates $u_1, \ldots, u_n$, the polynomial $f_i(x, c_i)$ has the form

$$u_1^{d_{i1}} \ldots u_{n-d}^{d_{i(n-d)}} \sum_{\kappa \in \Gamma_i \cap \mathbb{Z}^n} c_{i,\kappa} u'^{\gamma_\kappa} \in \mathbb{R}[u], \tag{2}$$

where $u' = (u_{n-d+1}, \ldots, u_n)$. Set

$$g_i(u', c_i) = \sum_{\kappa \in \Gamma_i \cap \mathbb{Z}^n} c_{i,\kappa} u'^{\gamma_\kappa} \in \mathbb{R}[u']. \tag{3}$$

Since A is an integer matrix and $\det(A) = 1$, the monomial map (1) admits a unique monomial inverse map, given by A^{-1}. Hence the system $f_1(x, c_1) = \cdots = f_p(x, c_p) = 0$ has solutions in $(\mathbb{R}^*)^n$ if and only if the system $g_1(u', c_1) = \cdots = g_p(u', c_p) = 0$ has solutions in $(\mathbb{R}^*)^d$. There are two cases to consider:

Case 1: $d < p$. Consider the semi-algebraic map

$$G \colon (\mathbb{R}^*)^d \times \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_p} \to \mathbb{R}^p, \quad (u', c_1, \ldots, c_p) \mapsto (g_1(u', c_1), \ldots, g_p(u', c_p)).$$

For $i = 1, \ldots, p$, let $\kappa^i \in \Gamma_i \cap \mathbb{Z}^n$. Then the Jacobian matrix DG of G contains the following diagonal matrix

$$\frac{\partial G}{\partial(c_{1,\kappa^1}, \ldots, c_{p,\kappa^p})} \ = \ \begin{pmatrix} u'^{\gamma_{\kappa^1}} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & u'^{\gamma_{\kappa^p}} \end{pmatrix},$$

which has rank p because of $u' \in (\mathbb{R}^*)^d$. Hence DG is of rank p, which yields $G \pitchfork \{0\}$ (in $\mathbb{R}^p$). By Theorem 2.2, the set

$$P_1 := \{c := (c_1, \ldots, c_p) \in \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_p} \ : \ G(\cdot, c) \pitchfork \{0\}\}$$

is an open dense semi-algebraic set in $\mathbb{R}^{m_1}\times\cdots\times\mathbb{R}^{m_p}$. Since $d<p$, the map $G(\cdot,c)\colon(\mathbb{R}^*)^d\to\mathbb{R}^p$ is transverse to $\{0\}$ if and only if $\mathrm{Im}G(\cdot,c)\cap\{0\}=\emptyset$. We deduce, for each $c\in P_1$, that $\{G(\cdot,c)=0\}\cap(\mathbb{R}^*)^d=\emptyset$, and hence $\{F(\cdot,c)=0\}\cap(\mathbb{R}^*)^n=\emptyset$. This implies that $P_1\subset\mathcal{D}_I(\Gamma)$.

Case 2: $d\geq p$. We first remark that we may assume that $d=n$. In fact, suppose that $d<n$ and fix $c:=(c_1,\ldots,c_p)$. Under the change of coordinates (1), the polynomials $f_i(x,c_i)\in\mathbb{R}[x]$ and $g_i(u',c_i)\in\mathbb{R}[u']$ have the forms (2) and (3), respectively. Recall that $F(x,c)=(f_1(x,c_1),\ldots,f_p(x,c_p))$ and $G(u',c)=(g_1(u',c_1),\ldots,g_p(u',c_p))$. We have seen that $F(x,c)=0$ has solutions in $(\mathbb{R}^*)^n$ if and only if $G(u',c)=0$ has solutions in $(\mathbb{R}^*)^d$.

For any $\kappa\in\mathbb{Z}^n$, let $\kappa=(\kappa_1,\ldots,\kappa_n)$. By a direct calculation, in the system of coordinates $u_1,\ldots,u_n$, the matrix $xDF(x,c)$ has the form

$$\left(u_1^{d_{i1}}\ldots u_{n-d}^{d_{i(n-d)}}\sum_{\kappa\in\Gamma_i\cap\mathbb{Z}^n}\kappa_l c_{i,\kappa}u'^{\gamma_\kappa}\right)_{i=1,\ldots,p,\ l=1,\ldots,n}. \tag{4}$$

Furthermore, let $u'DG(u',c)$ denote the matrix

$$\left(u'_j\frac{\partial g_i}{\partial u'_j}(u',c_i)\right)_{i=1,\ldots,p,\ j=n-d+1,\ldots,n}$$

then we can see that

$$u'DG(u',c)\ =\ \left(\sum_{\kappa\in\Gamma_i\cap\mathbb{Z}^n}\left(\sum_{l=1}^n q_{j,l}\kappa_l\right)c_{i,\kappa}u'^{\gamma_\kappa}\right)_{i=1,\ldots,p,\ j=n-d+1,\ldots,n}.$$

Observe that the columns of $u'DG(u',c)$ are linear combinations of the columns of the matrix

$$\left(\sum_{\kappa\in\Gamma_i\cap\mathbb{Z}^n}\kappa_l c_{i,\kappa}u'^{\gamma_\kappa}\right)_{i=1,\ldots,p,\ l=1,\ldots,n},$$

which has the same rank as the matrix in (4) for any $u\in(\mathbb{R}^*)^n$. As the monomial map (1) admits a unique monomial inverse map, consequently, we have

$$\{x\in(\mathbb{R}^*)^n\ :\ F(x,c)=0\quad\text{and}\quad\mathrm{rank}(xDF(x,c))<p\}\neq\emptyset$$

if and only if

$$\{u'\in(\mathbb{R}^*)^d\ :\ G(u',c)=0\quad\text{and}\quad\mathrm{rank}(u'DG(u',c))<p\}\neq\emptyset.$$

But $G(u',c)$ is a polynomial map in d variables, therefore, the problem is reduced to the case $d=n$.

From now on, we assume that $d=n$. For each $i=1,\ldots,p$, let $v^{i1},\ldots,v^{ir_i}$ be the vertices of Γ_i. Note that $r_i>1$ for every i by the assumption $m_i>1$. Let

$$w^{ij}\ :=\ v^{ij}-v^{ir_i}\quad\text{for}\quad j=1,\ldots,r_i-1.$$

Lemma 4.2. *We have*

$$\mathrm{rank}\{w^{11},\ldots,w^{1(r_1-1)},\ldots,w^{p1},\ldots,w^{p(r_p-1)}\}\ =\ \dim(\Gamma_1+\cdots+\Gamma_p).$$

Proof. Let a, b be two arbitrary points of $\Gamma_1 + \cdots + \Gamma_p$. There exist $a^i, b^i \in \Gamma_i$, $i = 1, \ldots, p$, such that

$$\begin{aligned} a &= a^1 + \cdots + a^p, \\ b &= b^1 + \cdots + b^p. \end{aligned}$$

Observe that

$$\begin{aligned} a^i &= \sum_{j=1}^{r_i} \lambda_{ij} v^{ij}, \quad \text{with} \quad \lambda_{ij} \geq 0 \quad \text{and} \quad \sum_{j=1}^{r_i} \lambda_{ij} = 1, \\ b^i &= \sum_{j=1}^{r_i} \mu_{ij} v^{ij}, \quad \text{with} \quad \mu_{ij} \geq 0 \quad \text{and} \quad \sum_{j=1}^{r_i} \mu_{ij} = 1. \end{aligned}$$

Hence

$$\begin{aligned} b^i - a^i = \sum_{j=1}^{r_i} (\mu_{ij} - \lambda_{ij}) v^{ij} &= \sum_{j=1}^{r_i - 1} (\mu_{ij} - \lambda_{ij}) v^{ij} + (\mu_{ir_i} - \lambda_{ir_i}) v^{ir_i} \\ &= \sum_{j=1}^{r_i - 1} (\mu_{ij} - \lambda_{ij}) v^{ij} - \sum_{j=1}^{r_i - 1} (\mu_{ij} - \lambda_{ij}) v^{ir_i} \\ &= \sum_{j=1}^{r_i - 1} (\mu_{ij} - \lambda_{ij}) w^{ij}. \end{aligned}$$

Consequently, $b - a = \sum_{i=1}^{p} (b^i - a^i)$ is a linear combination of vectors w^{ij}. Since a, b can be chosen arbitrarily, the lemma follows. □

Now we will prove Proposition 4.2 (Case 2 with $d = n$) by induction on p, the number of polynomials. In what follows, $c_{i,j}$ stands for the coefficient of the monomial $x^{v^{ij}}$ in $f_i(x, c_i)$.

Firstly, let $p = 1$ and consider the semi-algebraic map

$$\Phi \colon (\mathbb{R}^*)^n \times \mathbb{R}^{m_1} \to \mathbb{R}^{n+1}, \quad (x, c_1) \mapsto \Big(x_1 \frac{\partial f_1}{\partial x_1}(x, c_1), \ldots, x_n \frac{\partial f_1}{\partial x_n}(x, c_1), f_1(x, c_1) \Big).$$

The Jacobian matrix $D\Phi$ of Φ contains the following matrix

$$\frac{\partial \Phi}{\partial (c_{1,1}, \ldots, c_{1,r_1})} = \begin{pmatrix} x^{v^{11}} v^{11} & \cdots & x^{v^{1(r_1-1)}} v^{1(r_1-1)} & x^{v^{1r_1}} v^{1r_1} \\ x^{v^{11}} & \cdots & x^{v^{1(r_1-1)}} & x^{v^{1r_1}} \end{pmatrix},$$

where v^{1j} $(j = 1, \ldots, r_1)$ are written as column vectors. The rank of $\dfrac{\partial \Phi}{\partial (c_{1,1}, \ldots, c_{1,r_1})}$ is equal to the rank of the following matrix

$$M_1 := \begin{pmatrix} v^{11} & \cdots & v^{1(r_1-1)} & v^{1r_1} \\ 1 & \cdots & 1 & 1 \end{pmatrix}.$$

By some linear operations on the columns of M_1, we obtain the following matrix with the same rank

$$M_2 := \begin{pmatrix} v^{11} - v^{1r_1} & \cdots & v^{1(r_1-1)} - v^{1r_1} & v^{1r_1} \\ 0 & \cdots & 0 & 1 \end{pmatrix} = \begin{pmatrix} w^{11} & \cdots & w^{1(r_1-1)} & v^{1r_1} \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

In light of Lemma 4.2, we know that

$$\operatorname{rank}\{w^{11}, \ldots, w^{1(r_1-1)}\} = \dim \Gamma_1 = d = n.$$

Hence $\operatorname{rank} M_2 = n+1$, and so $\operatorname{rank}(D\Phi) = n+1$. Consequently $\Phi \pitchfork \{0\}$ (in $\mathbb{R}^{n+1}$). By Theorem 2.2, the set

$$P_2 := \{c_1 \in \mathbb{R}^{m_1} \ : \ \Phi(\cdot, c_1) \pitchfork \{0\}\}$$

is an open dense semi-algebraic set in $\mathbb{R}^{m_1}$. Observe that the map $\Phi(\cdot, c_1)\colon (\mathbb{R}^*)^n \to \mathbb{R}^{n+1}$ is transversal to $\{0\}$ if and only if $\operatorname{Im}\Phi(\cdot, c_1) \cap \{0\} = \emptyset$. Hence, for $c_1 \in P_2$, we have $\{\Phi(\cdot, c_1) = 0\} = \emptyset$. Consequently, $P_2 \subset \mathcal{D}_I(\Gamma)$, which completes the proof for the case $p = 1$.

Now assume that $p > 1$. By induction, for each $l = 1, \ldots, p$, the set $\mathcal{D}_{I\setminus\{l\}}(\Gamma)$ contains an open dense semi-algebraic set U_l in $\mathbb{R}^m \times \cdots \times \mathbb{R}^{m_p}$. Consider the semi-algebraic map

$$\begin{aligned} \Psi\colon (\mathbb{R}^*)^n \times (U_1 \cap \cdots \cap U_p) \times (\mathbb{R}^p - \{0\}) &\to \mathbb{R}^n \times \mathbb{R}^p, \\ (x, c_1, \ldots, c_p, \lambda) &\mapsto \Big(\sum_{i=1}^{p} \lambda_i x \nabla f_i(x, c_i), f_1(x, c_1), \ldots, f_p(x, c_p)\Big), \end{aligned}$$

where, for simplicity of notation, we let

$$x\nabla f_i(x, c_i) \ := \ \Big(x_1 \frac{\partial f_i}{\partial x_1}(x, c_i), \ldots, x_n \frac{\partial f_i}{\partial x_n}(x, c_i)\Big).$$

Note that if $(x, c, \lambda) \in \Psi^{-1}(0)$, then $\lambda_1 \cdots \lambda_p \neq 0$. In fact, if $\lambda_l = 0$ for some l, then $\sum_{i \neq l} \lambda_i x \nabla f_i(x, c_i) = 0$, which contradicts the fact that

$$(c_1, \ldots, c_p) \ \in \ U_1 \cap \cdots \cap U_p \ \subset \ U_l \ \subset \ \mathcal{D}_{I\setminus\{l\}}(\Gamma).$$

The Jacobian matrix $D\Psi$ of Ψ contains the matrix

$$M_3 \ := \ \frac{\partial \Psi}{\partial[(c_{1,1}, \ldots, c_{1,r_1}), \ldots, (c_{p,1}, \ldots, c_{p,r_p})]} \ = \ \Big(\ A \ \Big| \ \cdots \ \Big| \ B \ \Big),$$

where

$$A \ := \ \begin{pmatrix} \lambda_1 x^{v^{11}} v^{11} & \cdots & \lambda_1 x^{v^{1(r_1-1)}} v^{1(r_1-1)} & \lambda_1 x^{v^{1r_1}} v^{1r_1} \\ x^{v^{11}} & \cdots & x^{v^{1(r_1-1)}} & x^{v^{1r_1}} \\ 0 & \cdots & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix},$$

and

$$B \;:=\; \begin{pmatrix} \lambda_p x^{v^{p1}} v^{p1} & \cdots & \lambda_p x^{v^{p(r_p-1)}} v^{p(r_p-1)} & \lambda_p x^{v^{pr_p}} v^{pr_p} \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots \\ x^{v^{p1}} & \cdots & x^{v^{p(r_p-1)}} & x^{v^{pr_p}} \end{pmatrix}.$$

Here, v^{ij} are written as column vectors.

If $(x, c, \lambda) \in \Psi^{-1}(0)$, we know that $\lambda_i x^{v^{ij}} \neq 0$ for $i = 1, \ldots, p$, and $j = 1, \ldots, r_i$. Hence, M_3 has the same rank with the matrix

$$M_4 \;:=\; \left(\begin{array}{cccc|c|cccc} v^{11} & \cdots & v^{1(r_1-1)} & v^{1r_1} & \cdots & v^{p1} & \cdots & v^{p(r_p-1)} & v^{pr_p} \\ \hline 1 & \cdots & 1 & 1 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 & \cdots & 1 & 1 \end{array}\right).$$

By some linear operations on the columns of M_4, we obtain

$$\begin{aligned} M_5 \;&:=\; \left(\begin{array}{cccc|c|cccc} v^{11} - v^{1r_1} & \cdots & v^{1(r_1-1)} - v^{1r_1} & v^{1r_1} & \cdots & v^{p1} - v^{pr_p} & \cdots & v^{p(r_p-1)} - v^{pr_p} & v^{pr_p} \\ \hline 0 & \cdots & 0 & 1 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 1 \end{array}\right) \\ &= \left(\begin{array}{cccc|c|cccc} w^{11} & \cdots & w^{1(r_1-1)} & v^{1r_1} & \cdots & w^{p1} & \cdots & w^{p(r_p-1)} & v^{pr_p} \\ \hline 0 & \cdots & 0 & 1 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 1 \end{array}\right). \end{aligned}$$

Rearranging the columns of M_5, we get

$$M_6 \;:=\; \left(\begin{array}{ccccccc|ccc} w^{11} & \cdots & w^{1(r_1-1)} & \cdots & w^{p1} & \cdots & w^{p(r_p-1)} & v^{1r_1} & \cdots & v^{pr_p} \\ \hline & & & & & & & 1 & & \mathbf{0} \\ & & & \mathbf{0} & & & & & \ddots & \\ & & & & & & & \mathbf{0} & & 1 \end{array}\right).$$

In view of Lemma 4.2, we have

$$\mathrm{rank}\{w^{11}, \ldots, w^{1(r_1-1)}, \ldots, w^{p1}, \ldots, w^{p(r_p-1)}\} = d = n.$$

So $\mathrm{rank} M_6 = n + p$ on $\Psi^{-1}(0)$. Thus $D\Psi$ is of maximal rank on $\Psi^{-1}(0)$, namely $\Psi \pitchfork \{0\}$ (in $\mathbb{R}^{n+p}$). Note that $U_1 \cap \cdots \cap U_p$ is an open dense semi-algebraic set in $\mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_p}$. This, together with Theorem 2.2, implies that the set

$$P_3 := \{c \in U_1 \cap \cdots \cap U_p \;:\; \Psi(\cdot, c, \cdot) \pitchfork \{0\}\}$$

is open dense semi-algebraic in $\mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_p}$. Since $\Psi(\cdot, c, \cdot) \colon (\mathbb{R}^*)^n \times (\mathbb{R}^p - \{0\}) \to \mathbb{R}^n \times \mathbb{R}^p$ is a map between two manifolds of same dimension, the transversality condition implies that $\Psi(\cdot, c, \cdot)$ is a local diffeomorphism on $(\Psi(\cdot, c, \cdot))^{-1}(0)$ for each $c \in P_3$.

Let $c \in P_3$. If $(\Psi(\cdot, c, \cdot))^{-1}(0) \neq \emptyset$, there exists $(x, \lambda) \in (\mathbb{R}^*)^n \times (\mathbb{R}^p - \{0\})$ such that $\Psi(x, c, \lambda) = 0$. Note that, for every $t \in \mathbb{R} \setminus \{0\}$, $\Psi(x, c, t\lambda) = 0$. So $\Psi(\cdot, c, \cdot)$ is not a local diffeomorphism at (x, λ), which is a contradiction. Hence $(\Psi(\cdot, c, \cdot))^{-1}(0) = \emptyset$. Consequently, $c \in \mathcal{D}_I(\Gamma)$. Therefore, $P_3 \subset \mathcal{D}_I(\Gamma)$, which ends the proof of the proposition. □

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