Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

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(revised version: December 2019)

by

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Preprint no.: 101 2019

How signatures affect expected return and volatility: a rough model under transaction cost

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Abstract

We develop a general mathematical framework, based on rough path theory, a recent important extension of the classical Itô calculus, that can incorporate the empirically observed nonlinear mean-variance relation of the logarithmic return in a systematic manner. Thus, we propose a stock price model driven by a Hölder continuous noise, understood in the sense of a rough differential equation. This model offers the possibility of additional noises hidden in the signatures of rough paths, hence supporting the idea of mixture of a standard Brownian noise and another source of long memory noise (a fractional Brownian motion for instance), and enabling to account for the multi-scaling phenomenon in financial data. The no-arbitrage principle is then satisfied under the assumption of transaction costs as long as the driving noise is a sticky process. We also discuss the potential risk of model uncertainty where the ambiguity comes from the signatures of rough paths. Our models are supported by empirical evidence from financial data.

Keywords: stock price, expected return, volatility, noise, rough path theory, rough differential equations, no-arbitrage, risk.

1 Introduction

It is well-known that the original Samuelson stock model [35],

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \tag{1.1}$$

for a stock price S_t at time t with growth factor μ, volatility σ and a stochastic integral in the sense of Itô with respect to a standard Brownian motion B_t, does not reproduce certain rather universal features of empirical stock price data (the so-called stylized facts); hence many modifications have been suggested ever since. First, the Hull and White model [24] suggests that the growth factor and the volatility should be time-dependent, leading to models of the form

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t, \tag{1.2}$$

where σ_t satisfies a stochastic differential equation

$$d\log\sigma_t = k(\theta - \log\sigma_t)dt + \gamma dW'_t \tag{1.3}$$

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with parameters k, θ, γ and another Brownian motion W_t', with an instantaneous correlation $dW_t dW_t' = \rho dt$ with a parameter $\rho \in [0,1]$ between the two different Brownian motions. The Heston model [22] proposes that σ_t^2 follows a Cox-Ingersoll-Ross equation

$$d(\sigma_t^2) = k\Big[\theta - (\sigma_t^2)\Big]dt + \lambda \sigma_t dW_t' \tag{1.4}$$

where $k, \theta, \lambda > 0$ are parameters, and the two Brownian motions again possess an instantaneous correlation $dW_t dW_t' = \rho dt$ with $\rho \in [0,1]$.

Still, the stock model (1.2) does not account for certain memory effects in $\log S_t$. This seems to require a more radical solution than simply making the coefficients time dependent, but keeping standard Brownian motion as the underlying stochastic process. This issue was raised already very early in [28], which suggested that the standard Brownian motion in (1.2) should be generalized to self similar processes, including fractional motions B^H, i.e., a family of centered Gaussian processes $B^H = \{B^H(t)\}$, $t \in \mathbb{R}$ or $\mathbb{R}_+$ with continuous sample paths and covariance function

$$R_H(s,t) = \tfrac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}), \quad \forall t, s \in \mathbb{R}.$$

In [11], ordinary Brownian motion is replaced by fractional Brownian motion in the model (1.3) for the variance σ_t, resulting in

$$d\log\sigma_t = k(\theta - \log\sigma_t)dt + \gamma dB_t^H. \tag{1.5}$$

In order to obtain a process with long memory, a Hurst exponent $H > \frac{1}{2}$ is needed. Recent empirical studies [4], [18] however showed that if we assume the model (1.5), the log-volatility behaves essentially as a fractional Brownian motion with Hurst exponent H of order 0.1 at any reasonable time scale, and thus, we do not have a long memory process. This observation motivates a study in [5] on a regularity structure for rough volatility, in which the authors suggest a more general dynamic model of the form

$$\begin{aligned} \frac{dS_t}{S_t} &= f(Z_t)(\rho dW_t + \sqrt{1-\rho^2}dW_t') \\ Z_t &= z + \int_0^t K(s,t)v(Z_s)ds + \int_0^t K(s,t)u(Z_s)dW_s, \end{aligned} \tag{1.6}$$

where K is the kernel and f, u, v are sufficiently smooth functions.

Another important issue is the multi-scaling phenomenon in financial data, which shows that the generalized Hurst exponent varies depending on the time scale (see e.g. [3], [13], [2], [7], [8]). The multi-scaling issue can be explained either by considering a random time change of the Brownian noise B_t through the time change process I_t, i.e. $X_t = B_{I_t}$ for all $t \geq 0$ (see [2]); or by assuming that the noise X_t has the form

$$X_h(t) = \sum_{k=1}^{\lfloor \frac{t}{h} \rfloor} e^{\omega_h(k)}\Big(B_{(k+1)h}^H - B_{kh}^H\Big) \tag{1.7}$$

for some Hurst exponent $H \geq \frac{1}{2}$, $\omega_h(\cdot) \sim \mathcal{N}(0, \lambda^2\log(\frac{L}{h}))$ with the intermittency parameter λ and the autocorrelation length L, and the time scale h such that $\omega_h(k)$ are correlated upto the distance L, i.e.

$$\mathrm{Cov}(\omega_h(k_1), \omega_h(k_2)) = \lambda^2\rho_h(|k_1 - k_2|), \quad \rho_h(|k_1-k_2|) = \begin{cases} \frac{L}{(|k_1-k_2|+1)h} & \text{for} \quad |k_1 - k_2| \leq \frac{L}{h} - 1 \\ 1 & \text{otherwise} \end{cases}$$

(see [3] and [8]).
Studies in [13] and [7] (see also our computations in Subsection 4.1) reveal that the log return of stock indices (for instance Nasdaq, Dow Jones, SP500) or various exchange rates has the Hurst exponents ranging from 0.4 to 0.7 in some period, hence making it unclear whether or not the noise should have long memory, and why the Hurst exponents often look smaller than $\frac{1}{2}$ for data quoted on timescales of minutes to hours but increase to values significantly larger than $\frac{1}{2}$ for daily to monthly quoted data. It is generally agreed, however, that the Hurst exponent for the log return of the stock price is always bigger than $\frac{1}{4}$.

Of course, one can introduce models with many parameters, in order to match the empirical data. Our approach is different. We want to develop a general mathematical framework within which the observed phenomena find a conceptual explanation. In fact, the basic and robust relation that emerges from our numerical investigation of a wide range of stocks and other indices is that when we plot the daily mean-variance relation of the logarithmic return, we find a parabola shaped curve as a lower envelope. In particular, the relation is not linear, as basic models suggest, and the nonlinearity exhibits a clear structure. This asks for a systematic explanation. The mathematical theory we shall draw upon for that purpose is rough path theory [27], [17], [16], a new and very powerful mathematical approach to stochastic processes. The theory can handle rather general driving noises in a systematic manner. The key idea consists in directly incorporating higher order information about the noise to define certain integrals in an algebraic manner.

Thus, we propose a new model using rough path theory, which covers all well-known cases and phenomena. That is, we do not interpret the stochastic integral in (1.2) in the sense of Itô, but in the pathwise sense of a rough integral [21]. In that framework, the information from the driving path $x_t = B_t(\omega)$ is not enough and additional information of a rough path $\mathbf{x} = (x, \mathbb{X}^1, \mathbb{X}^2)$ is required. A rough path approach needs more information than just a driving noise. This is an advantage in our context because it naturally allows for the incorporation of additional sources of noise. It turns out that the theory can be formulated in a unified manner and the data can be matched with a mixture of at most three such noises.

The no-arbitrage principle has been viewed as the fundamental requirement for a model to satisfy the *efficient market hypothesis* (EMH). For models based on standard Brownian motion, this is usually not a problem. When the stochastic noise in the asset price model comes from fractional Brownian motions, which display the long rang dependence observed in empirical data, it has however been shown, e.g. in [34] or in [9] that the model allows for arbitrage. A model in [30] using the Skorohod-Wick-Itô integral shows that the existence of arbitrage can be avoided. Another solution to this arbitrage problem comes from [10], [26], [29, Chapter 5, pp. 305-306] which assumes that the noise is the mixture of a standard Brownian noise B and a fractional Brownian motion B^H for $H \in (\frac{1}{2}, 1)$. Subsequently, the no-arbitrage statement was proved in [6] for the wider class $\exp\{Z_t + \sigma B_t\}$ of geometric mixed noise, where $(B, \mathcal{F})$ is a standard Brownian motion and Z is an $\mathcal{F}$-adapted process independent of B.

In contrast to that approach, in this paper, we follow [19], [20] and assume transaction costs. The reason is that a sufficient condition for no-arbitrage only requires the log-price $\log S_t$ to be a *sticky process*. Also, it turns out that the class of sticky processes is very large, since it contains strong Markov processes or any stochastic process with *conditional full support* (CFS). The stickiness is also studied later in [33] for a larger class of stochastic processes. The CFS criterion was extended in [32] for a class of mixed noise $Z_t + X_t$, where Z is an arbitrary continuous process, and X is a process independent of Z that has CFS. This class includes also mixed forms of mutually independent standard Brownian motions and fractional Brownian motions.

The multi-scaling phenomenon could then be explained by a rough model in Subsection 4.1, where the Hurst exponent is computed from the linear regression method between the logarithms of the variance and the duration. Moreover, our empirical analysis in 4.2 finds that there is a nonlinear mean-variance relation, which could not be explained by using the classical model (1.1) but possibly by using the rough model. It is important to note that the model works under the assumption on the stochasticity of the path $[x,2]$, which is defined from the signatures of the rough path $\mathbf{x} = (x, \mathbb{X}^1, \mathbb{X}^2)$. This assumption has not been observed so far in theoretical studies for the existing types of stochastic integrals (see Example 3.3), thus indicating a possible gap between empirical studies and stochastic calculus applied in finance.

Finally, we show in Section 5 by a simple example that the well-known negative effect of the variance on the expected log-return can indeed come from the path $[x,2]$. Since $[x,2]$ might be stochastic, it leads to the potential appearance of an additional noise which increases the variance of the log-return. We therefore raise the problem of the model risk, in which the volatility increases from stock model ambiguity, and the uncertainty comes from the signatures of the driving rough paths.

2 Rough differential equations

2.1 $\nu \in (\frac{1}{2}, 1)$: Young integrals

We first give a brief introduction to Young integrals. For a compact time interval $I \subset \mathbb{R}$, define $\Delta^2(I) := \{(s,t) : s,t \in I, s \leq t\}$. Let $C(I, \mathbb{R}^d)$ denote the space of all continuous paths $y : I \to \mathbb{R}^d$ equipped with the sup-norm $\| \cdot \|_{\infty,I}$ given by $\|y\|_{\infty,I} = \sup_{t\in I} \|y_t\|$, where $\| \cdot \|$ is the Euclidean norm in $\mathbb{R}^d$. We write $y_{s,t} := y_t - y_s$. For $0 < \alpha < 1$, we denote by $C^\alpha(I, \mathbb{R}^d)$ the space of Hölder continuous functions with exponent α on I equipped with the norm

$$\|y\|_{\alpha,I} := \|y_{\min I}\| + \vert\!\vert\!\vert y \vert\!\vert\!\vert_{\alpha,I} = \|y(a)\| + \sup_{s<t\in I} \frac{\|y_{s,t}\|}{(t-s)^\alpha},$$

For $y \in \mathcal{C}^\alpha(I, R^m \otimes \mathbb{R}^d)$ and $x \in \mathcal{C}^\nu(I, \mathbb{R}^m)$ with $\alpha + \nu > 1$, the Young integral $\int_I y_t dx_t$ can be defined as

$$\int_I y_s dx_s := \lim_{|\Pi|\to 0} \sum_{[u,v]\in\Pi} y_u x_{u,v}, \tag{2.1}$$

where the limit is taken over all finite partitions $\Pi = \{\min I = t_0 < t_1 < \cdots < t_n = \max I\}$ of I with $|\Pi| := \max_{[u,v]\in\Pi} |v-u|$ (see [38, p. 264–265]). This integral satisfies the additivity property by construction, as well as the so-called Young-Loeve estimate [17, Theorem 6.8, p. 116]

$$\left\| \int_s^t y_u dx_u - y_s x_{s,t} \right\| \leq K(\alpha,\nu)|t-s|^{\alpha+\nu} \vert\!\vert\!\vert y \vert\!\vert\!\vert_{\alpha,[s,t]} \vert\!\vert\!\vert x \vert\!\vert\!\vert_{\nu,[s,t]}, \tag{2.2}$$

for all $[s,t] \subset I$, where

$$K(\alpha,\nu) := (1 - 2^{1-\alpha-\nu})^{-1}. \tag{2.3}$$

2.2 $\nu \in (\frac{1}{4}, \frac{1}{3}]$: Rough paths

The basic theory of rough paths covers the case $\nu > \frac{1}{3}$, because in that case, one only needs $\mathbb{X}^1$ to control the path, see [21], [16]. Since we shall also need smaller values of ν, we present here the theory for $\nu \in (\frac{1}{4}, \frac{1}{3}]$, which also requires $\mathbb{X}^2$ for the control. For that purpose, we introduce the construction of the integral using rough paths for the case $y, x \in C^\alpha(I)$ when $\alpha \in (\frac{1}{4}, \nu)$. To do that, we first need to introduce the concept of rough paths. Following [27] and [16], a path $x \in C^\alpha(I, \mathbb{R}^m)$

can be lifted to a *rough path* $\mathbf{x} = (x, \mathbb{X}^1, \mathbb{X}^2)$ with

$$\mathbb{X}^1 \in C_2^{2\alpha}(\Delta^2(I), \mathbb{R}^m \otimes \mathbb{R}^m) := \{\mathbb{X} : \sup_{s<t} \frac{\|\mathbb{X}_{s,t}\|}{|t-s|^{2\alpha}} < \infty\},$$
$$\mathbb{X}^2 \in C_2^{3\alpha}(\Delta^2(I), \mathbb{R}^m \otimes \mathbb{R}^m \otimes \mathbb{R}^m) := \{\mathbb{X} : \sup_{s<t} \frac{\|\mathbb{X}_{s,t}\|}{|t-s|^{3\alpha}} < \infty\},$$

which satisfies Chen's relation, i.e.

$$\mathbb{X}^1_{s,t} = \mathbb{X}^1_{s,u} + \mathbb{X}^1_{u,t} + x_{s,u} \otimes x_{u,t}, \quad \mathbb{X}^2_{s,t} = \mathbb{X}^2_{s,u} + \mathbb{X}^2_{u,t} + x_{s,u} \otimes \mathbb{X}^1_{u,t} + \mathbb{X}^1_{s,u} \otimes x_{u,t}, \tag{2.4}$$

for all $\min I \leq s \leq u \leq t \leq \max I$. As such $\mathbb{X}^1, \mathbb{X}^2$ should be viewed as *postulating* the value of the quantity

$$\int_s^t x_{s,r} \otimes dx_r := \mathbb{X}^1_{s,t}, \quad \int_s^t \int_s^u x_{s,r} \otimes dx_r \otimes dx_u := \mathbb{X}^2_{s,t}, \tag{2.5}$$

where the right hand side is taken as a definition for the left hand side, so that $\int \otimes dx$ is written only symbolically. To illustrate the concepts, let us review below well-known situations for $m = 1$, where the tensor $\otimes$ simply denotes the ordinary multiplication between real numbers.

Example 2.1 1. When x is a realization of a standard Brownian motion B_t, we can choose $\mathbb{X}^1, \mathbb{X}^2$ to be realizations of stochastic processes of the form

$$\mathbb{X}^1_{s,t}(\cdot) := \int_s^t B_{s,u} dB_u, \quad \mathbb{X}^2_{s,t}(\cdot) := \int_s^t \int_s^u B_{s,r} dB_r dB_u, \quad \forall 0 \leq s \leq t \leq T,$$

where the integrals $\int_s^t B_{s,u} dB_u, \int_s^t \mathbb{X}^1_{s,u} dB_u$ are understood in the Itô sense. It is easy to show that

$$\mathbb{X}^1_{s,t}(\cdot) = \frac{1}{2} B^2_{s,t}(\cdot) - \frac{1}{2}(t-s), \quad \mathbb{X}^2_{s,t}(\cdot) = \frac{1}{6} B^3_{s,t}(\cdot) - \frac{1}{2}(t-s) B_{s,t}(\cdot), \quad \forall 0 \leq s \leq t \leq T. \tag{2.6}$$

Hence, in the pathwise sense, it is easy to check from (2.6) that $\mathbf{x} = (x, \mathbb{X}^1, \mathbb{X}^2)$ satisfies Chen's relation (2.4).

2. For a more complex process where X is a local martingale, for instance $X_t = \int_0^t a_s dB_s$, we define the stochastic integrals $\int y dX$ as the integral w.r.t. the local martingale X [15, Section 2.5]. As such, we could apply the Ito formula [15, Section 2.8, p.64]

$$f(X_t) - f(X_s) = \int_s^t f'(X_u) dX_u + \frac{1}{2} \int_s^t f''(X_u) d\langle X \rangle_u$$

for any function $f \in C^2$, where $\langle X \rangle_t$ is the quadratic variance process, to compute explicitly

$$\mathbb{X}^1_{s,t} := \int_s^t X_{s,u} dX_u = \frac{1}{2} X^2_{s,t} - \frac{1}{2}\Big(\langle X \rangle_t - \langle X \rangle_s\Big), \quad \mathbb{X}^2_{s,t} := \int_s^t \mathbb{X}^1_{s,u} dX_u = \frac{1}{6} X^3_{s,t} - \frac{1}{2} X_{s,t}\Big(\langle X \rangle_t - \langle X \rangle_s\Big).$$

In particular if $X_t = \int_0^t a_s dB_s$, then $\langle X \rangle_t = \int_0^t a_u^2 du$.

3. When $X = B^H$ is a fractional Brownian motion which is not a semi-martingale [34], we can not apply the classical Ito calculus, but define the stochastic integral $\int y \delta B^H$ in the sense of Skorohod-Wick-Itô by using the Wick product as in [30, Chapter 5]. Then by using the Wick-Itô formula [30] for the Skorohod-Wick-Itô integral

$$f(B_t^H) - f(B_s^H) = \int_s^t H u^{2H-1} f''(B_u^H) du + \int_s^t f'(B_u^H) \delta B_u^H \tag{2.7}$$

for any function $f \in C^2$, we can compute explicitly

$$\mathbb{X}^1_{s,t} := \int_s^t B^H_{s,u} \delta B^H_u = \frac{1}{2}(B^H_{s,t})^2 - \frac{1}{2}\Big(t^{2H} - s^{2H}\Big),\ \mathbb{X}^2_{s,t} := \int_s^t \mathbb{X}^1_{s,u} \delta B^H_u = \frac{1}{6}(B^H_{s,t})^3 - \frac{1}{2}\Big(t^{2H} - s^{2H}\Big) B^H_{s,t}.$$

In general, the signatures $\mathbb{X}^1, \mathbb{X}^2$ could also be defined for a scalar centered Gaussian process of the form $X_t = \int_0^t K(t,s) dB_s$ where B is a standard Brownian motion, and $K(t,s)$ is a square integrable kernel. In particular, using the Itô-type formula

$$f(X_t) - f(X_s) = \int_s^t f'(X_u) \delta X_u + \frac{1}{2} \int_s^t f''(X_u) dR_u \tag{2.8}$$

for any function $f \in C^2$, where $R_u = E(X_u)^2 = \int_0^s K^2(s,u) du$ and the stochastic integral $\int \delta X$ can be computed as the limit of Riemann sums defined w.r.t. the Wick product [1]. As such $\mathbb{X}^1_{s,t} := \int_s^t X_{s,u} \delta X_u, \mathbb{X}^2_{s,t} := \int_s^t \mathbb{X}^1_{s,u} \delta X_u$ can be computed explicitly. The reader is also referred to [16, Chapter 10] for a detailed construction of $\mathbb{X}^1, \mathbb{X}^2$ of a higher dimensional X with mutually independent components.

4. Next, consider a stochastic process X for which almost surely all realizations belong to a Hölder space C^ν for some $\nu > \frac{1}{2}$. Then $\mathbb{X}^1_{s,t} := \int_s^t X_{s,u} \delta X_u, \mathbb{X}^2_{s,t} := \int_s^t \mathbb{X}^1_{s,u} \delta X_u$ are well defined as Young integrals introduced in Subsection 3.1. Using the chain-rule formula that

$$f(X_t) - f(X_s) = \int_s^t f'(X_u) \delta X_u \tag{2.9}$$

for any function $f \in C^1$, $\mathbb{X}^1, \mathbb{X}^2$ can be computed explicitly as

$$\mathbb{X}^1_{s,t} = \frac{1}{2} x^2_{s,t}, \quad \mathbb{X}^2_{s,t} = \frac{1}{6} x^3_{s,t}.$$

Denote by $\mathcal{C}^\alpha(I) \subset C^\alpha \oplus C_2^{2\alpha} \oplus C_2^{3\alpha}$ the set of all rough paths in I, then $\mathcal{C}^\alpha$ is a closed set but not a linear space, equipped with the rough path semi-norm

$$\|\mathbf{x}\|_{\alpha,I} := \|x\|_{\alpha,I} + \left\|\mathbb{X}^1\right\|_{2\alpha,\Delta^2(I)} + \left\|\mathbb{X}^2\right\|_{3\alpha,\Delta^2(I)} < \infty. \tag{2.10}$$

In general, we will assume that x is a realization of X, where $X(\omega): I \to \mathbb{R}^m, \mathbb{X}^1(\omega): I \times I \to \mathbb{R}^m \otimes \mathbb{R}^m$ and $\mathbb{X}^2(\omega): I \times I \to \mathbb{R}^m \otimes \mathbb{R}^m$ are stochastic processes that satisfy Chen's relation (2.4) and

$$E\|X_{s,t}\| \leq C|t-s|^\nu, \quad E\|\mathbb{X}^1_{s,t}\| \leq C|t-s|^{2\nu}, \quad E\|\mathbb{X}^2_{s,t}\| \leq C|t-s|^{3\nu}, \forall s,t \in I \tag{2.11}$$

for some generic constant C. Then, due to the Kolmogorov criterion for rough paths [17, Appendix A.3] for all $\alpha \in (\frac{1}{3}, \nu)$ there are an ω-wise version of $(x, \mathbb{X}^1, \mathbb{X}^2)$ and random variables $K_\alpha, \mathbb{K}^1_\alpha, \mathbb{K}^2_\alpha$, such that, ω-wise speaking, for all $s,t \in I$,

$$\|x_{s,t}\| \leq K_\alpha |t-s|^\alpha, \quad \|\mathbb{X}^1_{s,t}\| \leq \mathbb{K}^1_\alpha |t-s|^{2\alpha}, \quad \|\mathbb{X}^2_{s,t}\| \leq \mathbb{K}^2_\alpha |t-s|^{3\alpha}.$$

In particular, for every $\alpha \in (\frac{1}{4}, \nu)$ we have $(x, \mathbb{X}^1, \mathbb{X}^2) \in \mathcal{C}^\alpha$. Moreover, we could choose $\frac{1}{4} < \alpha$ smaller such that

$$
\begin{aligned}
& x \in C^{0,\alpha}(I) := \{x \in C^\alpha : \lim_{\delta \to 0} \sup_{0<t-s<\delta} \frac{\|x_{s,t}\|}{|t-s|^\alpha} = 0\}, \\
& \mathbb{X}^1 \in C^{0,2\alpha}(\Delta^2(I)) := \{\mathbb{X} \in C^{2\alpha}(\Delta^2(I)) : \lim_{\delta \to 0} \sup_{0<t-s<\delta} \frac{\|\mathbb{X}_{s,t}\|}{|t-s|^{2\alpha}} = 0\}, \\
& \mathbb{X}^2 \in C^{0,3\alpha}(\Delta^2(I)) := \{\mathbb{X} \in C^{3\alpha}(\Delta^2(I)) : \lim_{\delta \to 0} \sup_{0<t-s<\delta} \frac{\|\mathbb{X}_{s,t}\|}{|t-s|^{3\alpha}} = 0\}.
\end{aligned} \tag{2.12}
$$

Then $\mathcal{C}^\alpha(I) \subset C^{0,\alpha}(I) \oplus C^{0,2\alpha}(\Delta^2(I)) \oplus C^{0,3\alpha}(\Delta^2(I))$ is separable due to the separability of the spaces $C^{0,\alpha}(I), C^{0,2\alpha}(\Delta^2(I))$ and $C^{0,3\alpha}(\Delta^2(I))$ [17, Subsection 5.3.3].

Rough integrals for controlled paths

We are now able to define rough integrals of the form $\int y dx$ for controlling rough paths y, as presented by Gubinelli in [21], but for the case $\frac{1}{3} > \alpha > \frac{1}{4}$. Namely, a path $y \in C^\alpha(I, \mathbb{R}^m \otimes \mathbb{R}^d)$ is *controlled by* $(x, \mathbb{X}^1)$ if there exists a tube $(y', y'', R^y, R^{y'})$ with $y' \in C^\alpha(I, \mathbb{R}^m \otimes \mathbb{R}^m \otimes \mathbb{R}^d), y'' \in C^{2\alpha}(I, \mathbb{R}^m \otimes \mathbb{R}^m \otimes \mathbb{R}^m \otimes \mathbb{R}^d), R^y \in C^{3\alpha}(\Delta^2(I), \mathbb{R}^m \otimes \mathbb{R}^d), R^{y'} \in C^{2\alpha}(\Delta^2(I), \mathbb{R}^m \otimes \mathbb{R}^d)$ such that

$$
y'_{s,t} = y''_s \otimes x_{s,t} + R^{y'}_{s,t}, \quad y_{s,t} = y'_s \otimes x_{s,t} + y''_s \otimes \mathbb{X}^1_{s,t} + R^y_{s,t}, \qquad \forall \min I \leq s \leq t \leq \max I. \tag{2.13}
$$

y', y'' are called respectively the first and the second Gubinelli derivative of y. It is easy to prove that Gubinelli derivatives are uniquely defined as long as x is *truly rough*, i.e. $x \in C^\alpha \setminus C^{2\alpha}$ so that

$$
\limsup_{t \downarrow s} \frac{\|x_{s,t}\|}{|t-s|^{2\alpha}} = \infty,
$$

(see also [16, Proposition 6.4]). According to [16, Theorem 6.6 & Exercise 6.14], almost surely, all realizations of a standard Brownian motion B or a fractional Brownian motion B^H for $H \in (\frac{1}{3}, \frac{1}{2}]$ are truly rough, for any $\alpha \in (\frac{1}{4}, \frac{1}{2})$.
The space $\mathcal{D}^\alpha_{(x,\mathbb{X}^1)}(I)$ of all paths y controlled by $(x, \mathbb{X}^1)$ becomes a Banach space equipped with the norm

$$
\begin{aligned}
\|y\|_{x,2\alpha,I} &:= \|y_{\min I}\| + \|y'_{\min I}\| + \|y''_{\min I}\| + \|\!|y|\!\|_{x,\alpha,I}, \qquad \text{where} \\
\|\!|y|\!\|_{x,\alpha,I} &:= \|\!|y''|\!\|_{\alpha,I} + \left\|\!\left| R^{y'} \right|\!\right\|_{2\alpha,I} + \|\!|R^y|\!\|_{3\alpha,I},
\end{aligned}
$$

where we omit the value space for simplicity of presentation. Now fixing a rough path $\mathbf{x}$ and for any $\mathbf{y} \in \mathcal{D}^\alpha_{(x,\mathbb{X}^1)}(I)$, we define a function $F \in C^\alpha(\Delta^2(I), \mathbb{R}^d)$ by

$$
F_{s,t} := y_s \otimes x_{s,t} + y'_s \otimes \mathbb{X}^1_{s,t} + y''_s \otimes \mathbb{X}^2_{s,t}.
$$

It then follows from (2.13) that

$$
\begin{aligned}
& F_{s,t} - F_{s,u} - F_{u,t} \\
= \;& y_s \otimes x_{s,t} + y'_s \otimes \mathbb{X}^1_{s,t} + y''_s \otimes \mathbb{X}^2_{s,t} - \Big(y_s \otimes x_{s,u} + y'_s \otimes \mathbb{X}^1_{s,u} + y''_s \otimes \mathbb{X}^2_{s,u} \Big) \\
& - \Big(y_u \otimes x_{u,t} + y'_u \otimes \mathbb{X}^1_{u,t} + y''_u \otimes \mathbb{X}^2_{u,t} \Big) \\
= \;& -y_{s,u} \otimes x_{u,t} + y'_s \otimes (\mathbb{X}^1_{s,t} - \mathbb{X}^1_{s,u} - \mathbb{X}^1_{u,t}) - y'_{s,u} \otimes \mathbb{X}^1_{u,t} + y''_s \otimes (\mathbb{X}^2_{s,t} - \mathbb{X}^2_{s,u} - \mathbb{X}^2_{u,t}) - y''_{s,u} \otimes \mathbb{X}^2_{u,t} \\
= \;& -(y'_s \otimes x_{s,u} + y''_s \otimes \mathbb{X}^1_{s,u} + R^y_{s,u}) \otimes x_{u,t} + y'_s \otimes x_{s,u} \otimes x_{u,t} - \Big(y''_s \otimes x_{s,u} + R^{y'}_{s,u} \Big) \otimes \mathbb{X}^1_{u,t} \\
& + y''_s \otimes \Big(x_{s,u} \otimes \mathbb{X}^1_{u,t} + \mathbb{X}^1_{s,u} \otimes x_{u,t} \Big) - y''_{s,u} \otimes \mathbb{X}^2_{u,t} \\
= \;& -R^y_{s,u} \otimes x_{u,t} - R^{y'}_{s,u} \otimes \mathbb{X}^1_{u,t} - y''_{s,u} \otimes \mathbb{X}^2_{u,t}, \quad \forall \min I \leq s \leq u \leq t \leq \max I.
\end{aligned}
$$

As a result,

$$\begin{aligned}\|F_{s,t}-F_{s,u}-F_{u,t}\| &\leq \|R^y_{s,u}\otimes x_{u,t}\|+\|R^{y'}_{s,u}\otimes \mathbb{X}^1_{u,t}\|+\|y''_{s,u}\otimes \mathbb{X}^2_{u,t}\| \\ &\leq |t-s|^{4\alpha}\Big(\left\|\!\left| R^y \right|\!\right\|_{3\alpha} \left\|\!\left| x \right|\!\right\|_{\alpha} + \left\|\!\left| R^{y'} \right|\!\right\|_{2\alpha} \left\|\!\left| \mathbb{X}^1 \right|\!\right\|_{2\alpha} + \left\|\!\left| y'' \right|\!\right\|_{\alpha} \left\|\!\left| \mathbb{X}^2 \right|\!\right\|_{3\alpha}\Big).\end{aligned}$$

Therefore F belongs to the space

$$\begin{aligned}C^{\alpha,4\alpha}_2(I) \quad := \quad \Big\{ & F\in C^\alpha(\Delta^2(I)): F_{t,t}=0 \quad \text{and} \\ & \left\|\!\left| \delta F \right|\!\right\|_{4\alpha,I} := \sup_{\min I\leq s\leq u\leq t\leq \max I} \frac{\|F_{s,t}-F_{s,u}-F_{u,t}\|}{|t-s|^{4\alpha}} <\infty \Big\}.\end{aligned}$$

Thanks to the sewing lemma (see [12], [16, Lemma 4.2]), the integral $\int_s^t y_u dx_u$ can be defined as

$$\int_s^t y_u dx_u := \lim_{|\Pi|\to 0} \sum_{[u,v]\in\Pi} [y_u\otimes x_{u,v}+y'_u\otimes \mathbb{X}^1_{u,v}+y''_u\otimes \mathbb{X}^2_{u,v}] \tag{2.14}$$

where the limit is taken over all finite partitions Π of I with $|\Pi| := \max\limits_{[u,v]\in\Pi}|v-u|$ (see [21]). Moreover, there exists a constant $C_\alpha = C_{\alpha,|I|}>1$ with $|I| := \max I - \min I$, such that

$$\begin{aligned}&\Big\| \int_s^t y_u dx_u - y_s\otimes x_{s,t} + y'_s\otimes \mathbb{X}^1_{s,t}+y''_s\otimes \mathbb{X}^2_{s,t}\Big\| \\ \leq\quad & C_\alpha(|I|)|t-s|^{4\alpha}\Big(\left\|\!\left| R^y \right|\!\right\|_{3\alpha} \left\|\!\left| x \right|\!\right\|_{\alpha} + \left\|\!\left| R^{y'} \right|\!\right\|_{2\alpha} \left\|\!\left| \mathbb{X}^1 \right|\!\right\|_{2\alpha} + \left\|\!\left| y'' \right|\!\right\|_{\alpha} \left\|\!\left| \mathbb{X}^2 \right|\!\right\|_{2\alpha}\Big).\end{aligned} \tag{2.15}$$

where we will simply write $\left\|\!\left| x \right|\!\right\|_{\alpha}, \left\|\!\left| \mathbb{X}^1 \right|\!\right\|_{2\alpha}, \left\|\!\left| \mathbb{X}^2 \right|\!\right\|_{3\alpha}$ regardless of whether the domain is I or $\Delta^2(I)$.

Remark 2.2 In case $\nu\in(\frac{1}{3},\frac{1}{2}]$, the appearance of $\mathbb{X}^2$ can be neglected and we go back to the integral theory for controlling rough paths in [21]. Namely, the $x\in C^\nu$ would then be lifted to a rough path of the form $\mathbf{x}=(x,\mathbb{X}^1)$, where $\mathbb{X}^1\in C^{2\nu}$ satisfies (2.4) and (2.5). For $\alpha\in(\frac{1}{3},\nu)$, a path $y\in C^\alpha(I,\mathbb{R}^m\otimes\mathbb{R}^d)$ is *controlled by* $(x,\mathbb{X}^1)$ if there exists a tube (y',R^y) with $y'\in C^\alpha(I,\mathbb{R}^m\otimes\mathbb{R}^m\otimes\mathbb{R}^d), R^y\in C^{3\alpha}(\Delta^2(I),\mathbb{R}^m\otimes\mathbb{R}^d)$ such that

$$y_{s,t} = y'_s\otimes x_{s,t} + R^y_{s,t}, \qquad \forall \min I\leq s\leq t\leq \max I. \tag{2.16}$$

A similar construction using the sewing lemma leads to the definition of the integral $\int_s^t y_u dx_u$ as

$$\int_s^t y_u dx_u := \lim_{|\Pi|\to 0} \sum_{[u,v]\in\Pi} [y_u\otimes x_{u,v}+y'_u\otimes \mathbb{X}^1_{u,v}] \tag{2.17}$$

where the limit is taken over all finite partitions Π of I with $|\Pi| := \max\limits_{[u,v]\in\Pi}|v-u|$. Moreover, there exists a constant $C_\alpha = C_{\alpha,|I|}>1$ with $|I| := \max I - \min I$, such that

$$\Big\| \int_s^t y_u dx_u - y_s\otimes x_{s,t} + y'_s\otimes \mathbb{X}^1_{s,t}\Big\| \leq C_\alpha(|I|)|t-s|^{3\alpha}\Big(\left\|\!\left| R^y \right|\!\right\|_{3\alpha} \left\|\!\left| x \right|\!\right\|_{\alpha} + \left\|\!\left| y' \right|\!\right\|_{\alpha} \left\|\!\left| \mathbb{X}^1 \right|\!\right\|_{2\alpha}\Big). \tag{2.18}$$

2.3 Rough differential equations

For simplicity of the presentation, from now on, we set $m = 1$ and consider the rough differential equation

$$dy_t = Ay_t dt + Cy_t dx_t, \quad \forall t \in [a,T], y_a \in \mathbb{R}^d, \tag{2.19}$$

on any interval $[a,T]$, where $A \in \mathbb{R}^{d\times d}$ and $C \in \mathbb{R}^{d\times d}$. Equation (2.19) is understood in the integral form

$$y_t = y_a + \int_a^t Ay_u du + \int_a^t Cy_u dx_u, \qquad t \in [a,T], \tag{2.20}$$

where the second integral is understood as the rough integral in the Gubinelli sense, with $\otimes$ being the ordinary multiplication between two matrices (in particular between a matrix and a vector). The following theorem asserts the existence and uniqueness of the solution of a linear rough differential equation in the Gubinelli sense. Note that the same conclusion holds for general dimension m if the solution is understood in the sense of Friz-Victoir, see [17, Section 10.7].

Theorem 2.3 (Existence and uniqueness of the solution) *There exists a unique solution of equation* (2.19) *on any interval I.*

Proof: The proofs for the cases $\nu \in (\frac{1}{2}, 1)$ and $\nu \in (\frac{1}{3}, \frac{1}{2}]$ are provided in [38] and [14] respectively, hence we only need a proof for the case $\nu \in (\frac{1}{4}, \frac{1}{3}]$. Fix an $\alpha \in (\frac{1}{4}, \nu)$, we first prove the existence and uniqueness of the solution for some small interval $[a,T]$ such that $T - a < 1$, which will be clarified later. Define the Itô-Lyons map

$$G(y)_t := y_a + \int_a^t Ay_u du + \int_a^t Cy_u dx_u, \qquad t \in [a,T].$$

Also denote by $\mathcal{D}_x^{2\alpha}(y_a, Cy_a)$ the set of paths y controlled by $(x, \mathbb{X}^1)$ in $[a,T]$ with initial values y_a and $y'_a = Cy_a, y'' = C^2 y_a$ fixed. Observe from (2.15) that if y is controlled by $(x, \mathbb{X}^1)$ then so is $G(y)$ with $G(y)'_s = Cy_s, G(y)''_s = Cy'_s$. For this reason, the following mapping is well defined

$$\mathcal{M} : \mathcal{D}_x^{2\alpha}(y_a, Cy_a, C^2 y_a) \to \mathcal{D}_x^{2\alpha}(y_a, Cy_a, C^2 y_a), \qquad \mathcal{M}(y)_t := (G(y)_t, Cy_t, Cy'_t).$$

Similar to [21] we are going to estimate $\|\mathcal{M}(y)\|_{x,\alpha} = \interleave Cy' \interleave_\alpha + \interleave R^{Cy} \interleave_{2\alpha} + \interleave R^{G(y)} \interleave_{3\alpha}$ using $\|y\|_{x,\alpha} = \interleave y'' \interleave_\alpha + \left\interleave R^{y'} \right\interleave_{2\alpha} + \interleave R^y \interleave_{3\alpha}$. It follows from (2.13) that

$$\begin{aligned}
\interleave Cy' \interleave_\alpha &\leq \|C\| \interleave y' \interleave_\alpha \leq \|C\| \Big(\|y''\|_\infty \interleave x \interleave_\alpha + (T-a)^\alpha \left\interleave R^{y'} \right\interleave_{2\alpha} \Big) \\
&\leq \|C\| \interleave x \interleave_\alpha \|y''_a\| + \|C\|(T-a)^\alpha \interleave x \interleave_\alpha \interleave y'' \interleave_\alpha + \|C\|(T-a)^\alpha \left\interleave R^{y'} \right\interleave_{2\alpha} \\
&\leq \|C\| \Big(\interleave x \interleave_\alpha + (T-a)^{1-3\alpha} \Big) \Big(\|y''_a\| + \|y\|_{x,\alpha} \Big)
\end{aligned}$$

On the other hand, $\|R^{Cy}_{s,t}\| \leq \|Cy''\| \|\mathbb{X}^1_{s,t}\| + \|CR^y_{s,t}\|$, which results in

$$\begin{aligned}
\interleave R^{Cy} \interleave_{2\alpha} &\leq \|C\| \|y''\|_\infty \interleave \mathbb{X}^1 \interleave_{2\alpha} + \|C\|(T-a)^\alpha \interleave R^y \interleave_{3\alpha} \\
&\leq \|C\| \interleave \mathbb{X}^1 \interleave_{2\alpha} \|y''_a\| + \|C\|(T-a)^\alpha \interleave \mathbb{X}^1 \interleave_{2\alpha} \interleave y'' \interleave_\alpha + \|C\|(T-a)^\alpha \interleave R^y \interleave_{3\alpha} \\
&\leq \|C\| \Big(\interleave \mathbb{X}^1 \interleave_{2\alpha} + (T-a)^{1-3\alpha} \Big) \Big(\|y''_a\| + \|y\|_{x,\alpha} \Big)
\end{aligned}$$

Meanwhile

$$\begin{aligned}
\|R^{G(y)}_{s,t}\| &\leq \Big\| \int_s^t A y_u du \Big\| + \Big\| \int_s^t C y_u dx_u - C y_s x_{s,t} - C y'_s \mathbb{X}^1_{s,t} \Big\| \\
&\leq \|A\| |t-s| \|y\|_{\infty,[s,t]} + \|C\| \|y''\|_{\infty,[s,t]} |\mathbb{X}^2_{s,t}| \\
&\quad + C_\alpha (T-a) |t-s|^{4\alpha} \|C\| \Big(\left\vert\!\left\vert\!\left\vert x \right\vert\!\right\vert\!\right\vert_\alpha \left\vert\!\left\vert\!\left\vert R^y \right\vert\!\right\vert\!\right\vert_{3\alpha} + \left\vert\!\left\vert\!\left\vert \mathbb{X}^1 \right\vert\!\right\vert\!\right\vert_{2\alpha} \left\vert\!\left\vert\!\left\vert R^{y'} \right\vert\!\right\vert\!\right\vert_{2\alpha} + \left\vert\!\left\vert\!\left\vert \mathbb{X}^2 \right\vert\!\right\vert\!\right\vert_{3\alpha} \left\vert\!\left\vert\!\left\vert y'' \right\vert\!\right\vert\!\right\vert_\alpha \Big) \\
&\leq \|A\| |t-s| \|y\|_{\infty,[s,t]} + \|C\| \|y''_a\| \|\mathbb{X}^2_{s,t}\| + \|C\| (t-s)^\alpha \left\vert\!\left\vert\!\left\vert y'' \right\vert\!\right\vert\!\right\vert_\alpha \|\mathbb{X}^2_{s,t}\| \\
&\quad + C_\alpha (T-a) |t-s|^{4\alpha} \|C\| \Big(\left\vert\!\left\vert\!\left\vert x \right\vert\!\right\vert\!\right\vert_\alpha + \left\vert\!\left\vert\!\left\vert \mathbb{X}^1 \right\vert\!\right\vert\!\right\vert_{2\alpha} + \left\vert\!\left\vert\!\left\vert \mathbb{X}^2 \right\vert\!\right\vert\!\right\vert_{3\alpha} \Big) \left\vert\!\left\vert\!\left\vert y \right\vert\!\right\vert\!\right\vert_{x,\alpha}
\end{aligned}$$

where we can choose $T-a<1$ so that C_α can be bounded from above by $C_\alpha(1)$. In addition

$$\|y\|_{\infty,[s,t]} \leq \|y_a\| + \|y'_a\| (T-a)^\alpha \left\vert\!\left\vert\!\left\vert x \right\vert\!\right\vert\!\right\vert_\alpha + \|y''_a\| (T-a)^{2\alpha} \left\vert\!\left\vert\!\left\vert \mathbb{X}^1 \right\vert\!\right\vert\!\right\vert_{2\alpha} + (T-a)^{3\alpha} \left\vert\!\left\vert\!\left\vert R^y \right\vert\!\right\vert\!\right\vert_{2\alpha},$$

thus it follows that

$$\begin{aligned}
&\left\vert\!\left\vert\!\left\vert R^{G(y)} \right\vert\!\right\vert\!\right\vert_{3\alpha} \\
&\leq \|A\| \Big[(T-a)^{1-3\alpha} \|y_a\| + (T-a)^{1-2\alpha} \left\vert\!\left\vert\!\left\vert x \right\vert\!\right\vert\!\right\vert_\alpha \|y'_a\| + (T-a)^{1-\alpha} \left\vert\!\left\vert\!\left\vert \mathbb{X}^1 \right\vert\!\right\vert\!\right\vert_{2\alpha} \|y''_a\| + (T-a) \left\vert\!\left\vert\!\left\vert R^y \right\vert\!\right\vert\!\right\vert_{3\alpha} \Big] \\
&\quad + \|C\| \left\vert\!\left\vert\!\left\vert \mathbb{X}^2 \right\vert\!\right\vert\!\right\vert_{3\alpha} \Big(\|y''_a\| + (T-a)^\alpha \left\vert\!\left\vert\!\left\vert y'' \right\vert\!\right\vert\!\right\vert_\alpha \Big) + C_\alpha \|C\| (T-a)^\alpha \Big(\left\vert\!\left\vert\!\left\vert x \right\vert\!\right\vert\!\right\vert_\alpha + \left\vert\!\left\vert\!\left\vert \mathbb{X}^1 \right\vert\!\right\vert\!\right\vert_{2\alpha} + \left\vert\!\left\vert\!\left\vert \mathbb{X}^2 \right\vert\!\right\vert\!\right\vert_{3\alpha} \Big) \left\vert\!\left\vert\!\left\vert y \right\vert\!\right\vert\!\right\vert_{x,\alpha} \\
&\leq (\|A\| + C_\alpha(1)\|C\|) \Big(|T-a|^{1-3\alpha} + \left\vert\!\left\vert\!\left\vert x \right\vert\!\right\vert\!\right\vert_\alpha + \left\vert\!\left\vert\!\left\vert \mathbb{X}^1 \right\vert\!\right\vert\!\right\vert_{2\alpha} + \left\vert\!\left\vert\!\left\vert \mathbb{X}^2 \right\vert\!\right\vert\!\right\vert_{3\alpha} \Big) \Big(\|y_a\| + \|y'_a\| + \|y''_a\| + \left\vert\!\left\vert\!\left\vert y \right\vert\!\right\vert\!\right\vert_{x,\alpha} \Big)
\end{aligned}$$

Altogether, we have just proved that there exists a constant $K = K(\|A\|, \|C\|, C_\alpha(1)) > 1$ such that

$$\left\vert\!\left\vert\!\left\vert \mathcal{M}(y) \right\vert\!\right\vert\!\right\vert_{x,\alpha} \leq K \Big(|T-a|^{1-3\alpha} + \left\vert\!\left\vert\!\left\vert x \right\vert\!\right\vert\!\right\vert_{\alpha,[a,T]} + \left\vert\!\left\vert\!\left\vert \mathbb{X}^1 \right\vert\!\right\vert\!\right\vert_{2\alpha,[a,T]} + \left\vert\!\left\vert\!\left\vert \mathbb{X}^2 \right\vert\!\right\vert\!\right\vert_{3\alpha,[a,T]} \Big) \|y\|_{x,\alpha,[a,T]}, \tag{2.21}$$

where we choose, for a given $\mu < 1$, a time $T = T(a)$ satisfying

$$(T-a)^{1-3\alpha} + \left\vert\!\left\vert\!\left\vert x \right\vert\!\right\vert\!\right\vert_{\alpha,[a,T]} + \left\vert\!\left\vert\!\left\vert \mathbb{X}^1 \right\vert\!\right\vert\!\right\vert_{2\alpha,[a,T]} + \left\vert\!\left\vert\!\left\vert \mathbb{X}^2 \right\vert\!\right\vert\!\right\vert_{3\alpha,[a,T]} = \frac{\mu}{K} < 1.$$

Therefore, if we restrict to the compact set

$$\mathcal{B} := \Big\{ (y,y') \in \mathcal{D}^\alpha_x(y_a, Cy_a, C^2 y_a) |\, \left\vert\!\left\vert\!\left\vert y \right\vert\!\right\vert\!\right\vert_{x,\alpha} \leq \frac{\mu}{1-\mu} (1 + \|C\| + \|C\|^2) \|y_a\| \Big\}$$

then

$$\left\vert\!\left\vert\!\left\vert \mathcal{M}(y) \right\vert\!\right\vert\!\right\vert_{x,\alpha} \leq \mu \|y\|_{x,\alpha} \leq \Big(\frac{\mu^2}{1-\mu} + \mu \Big) (1 + \|C\| + \|C\|^2) \|y_a\| \leq \frac{\mu}{1-\mu} (1 + \|C\| + \|C\|^2) \|y_a\|,$$

which proves that $\mathcal{M} : \mathcal{B} \to \mathcal{B}$. In addition, for any two paths y and $\bar{y} \in \mathcal{B}$, by a computation similar to (2.21), we get

$$\left\vert\!\left\vert\!\left\vert \mathcal{M}(y) - \mathcal{M}(\bar{y}) \right\vert\!\right\vert\!\right\vert_{x,\alpha} \leq \mu \left\vert\!\left\vert\!\left\vert y - \bar{y} \right\vert\!\right\vert\!\right\vert_{x,\alpha},$$

which shows that $\mathcal{M}$ is a contraction on $\mathcal{B}$. This proves the existence of a solution on $[a,T]$. Next, for any two solutions with the same initial conditions $(y_a, Cy_a, C^2 y_a)$, by the same computation as in (2.21), we get

$$\left\vert\!\left\vert\!\left\vert y - \bar{y} \right\vert\!\right\vert\!\right\vert_{x,\alpha} \leq \mu \left\vert\!\left\vert\!\left\vert y - \bar{y} \right\vert\!\right\vert\!\right\vert_{x,\alpha}$$

which, together with $\mu < 1$, proves the uniqueness of the solution of (2.19) on $[a,T]$.

Finally, we construct the greedy time sequence (2.22)

$$\tau_0 = \min I, \quad \tau_{i+1} := \inf\Big\{t > \tau_i : (t-\tau_i)^{1-3\alpha} + \|\mathbf{x}\|_{\alpha,[\tau_i,t]} = \frac{\mu}{K}\Big\} \wedge \max I, \tag{2.22}$$

and putting $N_{I,\alpha}(\mathbf{x}) := \sup\{i \in \mathbb{N} : \tau_i \le \max I\}$, we deduce from the fact

$$\frac{\mu}{K} < |\tau_{i+1} - \tau_i|^{\nu-\alpha}\Big(1 + \|\mathbf{x}\|_{\nu,I}\Big)$$

that

$$N_{I,\alpha}(\mathbf{x}) < \Big[\frac{K}{\mu}(1 + \|\mathbf{x}\|_{\nu,I})\Big]^{\frac{1}{\nu-\alpha}} + 1.$$

Therefore, we can extend and prove the existence of the unique solution on any interval I. It is easy to see that the solution y_t depends linearly on the initial y_a, hence there exists a solution matrix $\Phi(t,a,\mathbf{x})$ of equation (2.20). □

Corollary 2.4 *Assume $A, C : [0,T] \to \mathbb{R}^{d\times d}$, $A.$ is Lebesgue integrable on $[0,T]$ and there exists $\alpha \in (\frac{1}{3}, \nu)$ such that $C. \in C^{\alpha}([0,T], \mathbb{R}^{d\times d})$. Assume further that $C.$ is controlled by x in the sense of (2.16) in Remark 2.2. Then there exists a unique solution of the time dependent rough differential equation*

$$dy_t = A_t y_t dt + C_t y_t dx_t, \quad y_0 \in \mathbb{R}^d. \tag{2.23}$$

Proof: Note that the solution candidate y of (2.23) should be found in the class of continuous paths that are controlled by x, hence $C.y.$ is also controlled by x. We will omit the details of the proof, as it proceeds analogously to that of Theorem 2.3. □

3 Rough model of stock prices

3.1 Rough stock model

We start with a rough differential equation version of the Black-Scholes model (1.1) for the financial asset price $S_t > 0$ at time t,

$$dS_t = \mu S_t dt + \sigma S_t dx_t, \tag{3.1}$$

where $\mu, \sigma \in \mathbb{R}$ are model parameters. Equation (3.1) is understood in the pathwise sense as the integral form

$$S_t = S_{t_0} + \int_{t_0}^t \mu S_u du + \int_{t_0}^t \sigma S_u dx_u, \tag{3.2}$$

where $x \in C^{\nu}([0,T], \mathbb{R})$ for $\nu > \frac{1}{4}$ is the driving path, which is lifted to a rough path $\mathbf{x} = (x, \mathbb{X}^1, \mathbb{X}^2)$, and the second integral is understood as a rough integral defined in Gubinelli's sense for controlled rough paths as presented in Section 2. Fortunately in the scalar case ($m = 1$) we have the following result.

Lemma 3.1 *Assume $\nu \in (\frac{1}{4}, \frac{1}{3}]$ and define*

$$\begin{aligned} [x,2]_{s,t} &:= x_{s,t}^2 - 2\mathbb{X}_{s,t}^1, \\ [x,3]_{s,t} &:= x_{s,t}^3 - 3x_{s,t}\mathbb{X}_{s,t}^1 + 3\mathbb{X}_{s,t}^2. \end{aligned} \tag{3.3}$$

Then $[x,2], [x,3]$ satisfy the additivity condition, i.e.

$$[x,2]_{s,t} = [x,2]_{s,u} + [x,2]_{u,t}; \quad [x,3]_{s,t} = [x,3]_{s,u} + [x,3]_{u,t}, \quad \forall s \le u \le t; \tag{3.4}$$

hence they can be defined by $[x,2]_{s,t} = [x,2]_{0,t} - [x,2]_{0,s}$, $[x,3]_{s,t} = [x,3]_{0,t} - [x,3]_{0,s}$, where $[x,2]_{0,\cdot} \in C^{2\nu}, [x,3]_{0,\cdot} \in C^{3\nu}$ are Hölder continuous paths. In case $\nu \in (\frac{1}{3}, \frac{1}{2}]$, the conclusions still hold for $[x,2]$.

Proof: The first equality in (3.4) is trivial due to the property of $\mathbb{X}^1$. To prove the second one, we use the Chen relation (2.4) for $\mathbb{X}^2$ and the first equality in (3.4) to obtain

$$
\begin{aligned}
& [x,3]_{s,t} - [x,3]_{s,u} - [x,3]_{u,t} \\
= \;& \Big(x_{s,t}^3 - x_{s,u}^3 - x_{u,t}^3\Big) - 3\Big(x_{s,t}\mathbb{X}^1_{s,t} - x_{s,u}\mathbb{X}^1_{s,u} - x_{u,t}\mathbb{X}^1_{u,t}\Big) + 3\Big(\mathbb{X}^2_{s,t} - \mathbb{X}^2_{s,u} - \mathbb{X}^2_{u,t}\Big) \\
= \;& 3x_{s,u}^2 x_{u,t} + 3x_{s,u}x_{u,t}^2 - 3\Big(x_{s,t}\mathbb{X}^1_{s,t} - x_{s,u}\mathbb{X}^1_{s,u} - x_{u,t}\mathbb{X}^1_{u,t}\Big) + 3\Big(x_{s,u}\mathbb{X}^1_{u,t} + x_{u,t}\mathbb{X}^1_{s,u}\Big) \\
= \;& 3\Big\{x_{s,u}x_{u,t}x_{s,t} - x_{s,t}\mathbb{X}^1_{s,t} + (x_{s,u}+x_{u,t})\Big(\mathbb{X}^1_{s,u} + \mathbb{X}^1_{u,t}\Big)\Big\} \\
= \;& 3x_{s,t}\Big(x_{s,u}x_{u,t} - \mathbb{X}^1_{s,t} + \mathbb{X}^1_{s,u} + \mathbb{X}^1_{u,t}\Big) \\
= \;& 0.
\end{aligned}
$$

□

Remark 3.2 1. As a consequence, $\mathbb{X}^1, \mathbb{X}^2$ can be defined formally based on $[x,2],[x,3]$ as follows

$$
\begin{aligned}
\mathbb{X}^1_{s,t} &= \int_s^t x_{s,u}dx_u := \frac{1}{2}x_{s,t}^2 - \frac{1}{2}[x,2]_{s,t}, \\
\mathbb{X}^2_{s,t} &= \int_s^t \mathbb{X}^1_{s,u}dx_u := x_{s,t}\mathbb{X}^1_{s,t} - \frac{1}{3}x_{s,t}^3 + \frac{1}{3}[x,3]_{s,t} = \frac{1}{6}x_{s,t}^3 - \frac{1}{2}x_{s,t}[x,2]_{s,t} + \frac{1}{3}[x,3]_{s,t}
\end{aligned}
\tag{3.5}
$$

for all $0 \le s \le t \le T$, where the integrals in (3.5) are written symbolically.

2. For $\nu \in (\frac{1}{5}, \frac{1}{4}]$, we could use the same procedure to define $\mathbb{X}^3_{s,t}$ which satisfies

$$\mathbb{X}^3_{s,t} - \mathbb{X}^3_{s,u} - \mathbb{X}^3_{u,t} = x_{s,u}\mathbb{X}^2_{u,t} + \mathbb{X}^1_{s,u}\mathbb{X}^1_{u,t} + \mathbb{X}^2_{s,u}x_{u,t}.$$

Based on that, if we define

$$[x,4]_{s,t} := x_{s,t}^4 - 4x_{s,t}^2\mathbb{X}^1_{s,t} + 2(\mathbb{X}^1_{s,t})^2 + 4x_{s,t}\mathbb{X}^2_{s,t} - 4\mathbb{X}^3_{s,t},$$

then it is also easy to prove that $[x,4]$ satisfies the additivity condition and hence is a $C^{4\nu}$-Hölder continuous path. Moreover, $\mathbb{X}^3$ can be defined based on $x,[x,2],[x,3],[x,4]$.

3. The additivity condition can still hold for x of higher dimension, provided that the components of x are mutually independent. For example, for $m=2$ and $x=(x^{(1)},x^{(2)})$ where $x^{(1)},x^{(2)}$ are independent scalar Gaussian noises, we define

$$x_{s,t}\otimes x_{s,t} = \begin{pmatrix} x^{(1)}_{s,t}\otimes x^{(1)}_{s,t} & x^{(1)}_{s,t}\otimes x^{(2)}_{s,t} \\ x^{(2)}_{s,t}\otimes x^{(1)}_{s,t} & x^{(2)}_{s,t}\otimes x^{(2)}_{s,t} \end{pmatrix} \quad \text{and} \quad \mathbb{X}^1_{s,t} = \begin{pmatrix} \int_s^t x^{(1)}_{s,u}\otimes dx^{(1)}_u & \int_s^t x^{(1)}_{s,u}\otimes dx^{(2)}_u \\ \int_s^t x^{(2)}_{s,u}\otimes dx^{(1)}_u & \int_s^t x^{(2)}_{s,u}\otimes dx^{(2)}_u \end{pmatrix}$$

where the integrals $\int_s^t x^{(1)}_{s,u}\otimes dx^{(2)}_u$ and $\int_s^t x^{(2)}_{s,u}\otimes dx^{(1)}_u$ of two independent Gaussian processes can be defined as the limit in $\mathcal{L}^2$ of the Riemann sum (see e.g. [16, Chapter 10]), while $\int_s^t x^{(1)}_{s,u}\otimes dx^{(1)}_u$ and $\int_s^t x^{(2)}_{s,u}\otimes dx^{(2)}_u$ are defined symbolically as in the scalar case. Since the stochastic integral for independent Gaussian processes satisfies the rule of integration by parts, it is easy to check that

$$\int_s^t x^{(1)}_{s,u}\otimes dx^{(2)}_u + \int_s^t x^{(2)}_{s,u}\otimes dx^{(1)}_u = x^{(1)}_{s,t}\otimes x^{(2)}_{s,t}$$

henceforth

$$[x,2]_{s,t} := x_{s,t}\otimes x_{s,t} - 2\mathrm{Sym}\mathbb{X}^1_{s,t} = \begin{pmatrix} [x^{(1)},2]_{s,t} & 0 \\ 0 & [x^{(2)},2]_{s,t} \end{pmatrix}$$

and the additivity condition of $[x,2]$ follows directly. The arguments for the additivity of $[x,3]$ are similar.

Example 3.3 Returning to Example 2.1, we choose $[x,3]_{s,t} \equiv 0$ for all $0 \leq s \leq t \leq T$ and choose $[x,2]$ as follows

$$[x,2]_{s,t} := \begin{cases} t-s & \text{if} \quad X_t = B_t \\ \langle X\rangle_t - \langle X\rangle_s & \text{if} \quad X_t \quad \text{is a local martiangle} \\ \int_0^t K(t,u)^2 du - \int_0^s K(s,u)^2 du & \text{if} \quad X_t = \int_0^t K(t,s) dB_s \\ 0 & \text{if} \quad X. \in C^\nu, \nu > \frac{1}{2} \end{cases}. \tag{3.6}$$

In particular, for a mixture $X_t = \rho B_t + \sqrt{1-\rho^2}\Xi_t$ for a constant $\rho \in [-1,1]$ and a process $\Xi. \in C^\nu$ with $\nu > \frac{1}{2}$, by using formulae (2.6), (2.9) for the Stratonovich stochastic integral $\int dB$ and the Young integral $\int \delta\Xi$ respectively, we can compute explicitly

$$[x,2]_{s,t}, [x,3]_{s,t} \equiv 0.$$

Note that in all considered cases, $[x,2]$ is non-random and $[x,3] \equiv 0$. We therefore raise a question: *Does there exist a stochastic process X and a type of stochastic integral, such that $\mathbb{X}^1, \mathbb{X}^2$ can be defined and either $[x,2]$ is truly stochastic or $[x,3] \neq 0$?*

Theorem 3.4 *The solution of equation* (3.1) *can be written explicitly in the form $S_t = e^{Y_t}$ where*

$$Y_{a,b} = \begin{cases} \mu(b-a) + \sigma x_{a,b} & \textit{if} \quad \nu \in (\frac{1}{2}, 1), \\ \mu(b-a) + \sigma x_{a,b} - \frac{\sigma^2}{2}[x,2]_{a,b} & \textit{if} \quad \nu \in (\frac{1}{3}, \frac{1}{2}], \\ \mu(b-a) + \sigma x_{a,b} - \frac{\sigma^2}{2}[x,2]_{a,b} + \frac{\sigma^3}{3}[x,3]_{a,b} & \textit{if} \quad \nu \in (\frac{1}{4}, \frac{1}{3}] \end{cases} \tag{3.7}$$

for all $0 \leq a \leq b \leq T$.

Proof: The proof follows directly from Theorem 2.3 for $d = 1$. Since the cases are similar, it suffices to prove the result for the most difficult case $\nu \in (\frac{1}{4}, \frac{1}{3}]$. In fact, since zero is the trivial solution of (3.1), it follows from the existence and uniqueness theorem that $S_t \neq 0$ for all $t \geq s$ whenever $S_s \neq 0$. That implies $S_t > 0$ for all $t \geq s$ whenever $S_s > 0$. Now applying the Taylor expansion for the function $Y_t = V(S_t) = \log S_t$, we obtain from the fact $y \in C^\nu$ that

$$\log S_t = \log S_s + \frac{1}{S_s} S_{s,t} - \frac{1}{2S_s^2} S_{s,t}^2 + \frac{1}{3S_s^3} S_{s,t}^3 + O(|t-s|^{4\alpha}), \tag{3.8}$$

for $0 < t-s < h$ small enough on $[0,T]$. On the other hand, the discretized scheme for equation (3.1) using $S_s' = \sigma S_s, S_s'' = \sigma^2 S_s$ and (2.15) yields

$$S_{s,t} = \mu S_s(t-s) + \sigma S_s x_{s,t} + \sigma^2 S_s \mathbb{X}_{s,t}^1 + \sigma^3 S_s \mathbb{X}_{s,t}^2 + O(|t-s|^{4\alpha}), \tag{3.9}$$

for all $|t-s| < h$ on $[0,T]$. Combining (3.8) and (3.9), and using the fact that $x \in C^\alpha, \mathbb{X}^1 \in C^{2\alpha}, \mathbb{X}^2 \in C^{3\alpha}$, we obtain

$$\begin{aligned} \log \frac{S_t}{S_s} &= \Big(\mu(t-s) + \sigma x_{s,t} + \sigma^2 \mathbb{X}_{s,t}^1 + \sigma^3 \mathbb{X}_{s,t}^2 + O(|t-s|^{4\alpha})\Big) \\ &\quad -\frac{1}{2}\Big(\mu(t-s) + \sigma x_{s,t} + \sigma^2 \mathbb{X}_{s,t}^1 + \sigma^3 \mathbb{X}_{s,t}^2 + O(|t-s|^{4\alpha})\Big)^2 \\ &\quad +\frac{1}{3}\Big(\mu(t-s) + \sigma x_{s,t} + \sigma^2 \mathbb{X}_{s,t}^1 + \sigma^3 \mathbb{X}_{s,t}^2 + O(|t-s|^{4\alpha})\Big)^3 \\ &= \mu(t-s) + \sigma x_{s,t} - \sigma^2 \Big[\frac{1}{2} x_{s,t}^2 - \mathbb{X}_{s,t}^1\Big] + \sigma^3 \Big[\frac{1}{3} x_{s,t}^3 - x_{s,t}\mathbb{X}_{s,t}^1 + \mathbb{X}_{s,t}^2\Big] + O(|t-s|^{4\alpha}) \\ &= \mu(t-s) + \sigma x_{s,t} - \frac{\sigma^2}{2}[x,2]_{s,t} + \frac{\sigma^3}{3}[x,3]_{s,t} + O(|t-s|^{4\alpha}) \end{aligned}$$

for all $|t-s| < h$ on $[0,T]$. Now for any $0 \leq a < b \leq T$, by discretizing the interval $[a,b]$ into sub-intervals of length $h = \frac{b-a}{N}$ with end points $a = t_0 < t_1 < \ldots < t_N = b$ and using (3.4), we obtain

$$\begin{aligned}
Y_{a,b} = \log \frac{S_b}{S_a} &= \sum_{i=0}^{N-1} \log \frac{S_{t_{i+1}}}{S_{t_i}} \\
&= \sum_{i=0}^{N-1} \Big(\mu(t_{i+1}-t_i) + \sigma x_{t_i,t_{i+1}} - \frac{\sigma^2}{2}[x,2]_{t_i,t_{i+1}} + \frac{\sigma^3}{3}[x,3]_{t_i,t_{i+1}} + O(h^{4\alpha}) \Big) \\
&= \mu(b-a) + \sigma x_{a,b} - \frac{\sigma^2}{2}[x,2]_{a,b} + \frac{\sigma^3}{3}[x,3]_{a,b} + O(h^{4\alpha})N \\
&= \mu(b-a) + \sigma x_{a,b} - \frac{\sigma^2}{2}[x,2]_{a,b} + \frac{\sigma^3}{3}[x,3]_{a,b} + (b-a)O(h^{4\alpha-1}). \qquad (3.10)
\end{aligned}$$

Letting $h \to 0$ and using the fact that $\alpha > \frac{1}{4}$, the discretized equation (3.10) proves the logarithm form of (3.7). □

Corollary 3.5 *Assume that $\mu : [0,T] \to \mathbb{R}$ is Lebesgue integrable and there exists $\alpha \in (\frac{1}{3}, \nu)$ such that $\sigma_\cdot \in C([0,T],\mathbb{R})$ and σ is controlled by x in the sense of* (2.16) *of Remark 2.2. Then the time dependent rough differential equation*

$$dS_t = \mu_t S_t dt + \sigma_t S_t dx_t \qquad (3.11)$$

could be solved explicitly as $S_t = e^{Y_t}$ where

$$Y_{s,t} = \int_s^t \mu_u du - \frac{1}{2}\int_s^t \sigma_u d[x,2]_u + \int_s^t \sigma_u dx_u, \qquad (3.12)$$

where the second integral in (3.12) *is a Young integral* (2.1) *while the third integral is understood as a rough integral* (2.17).

Proof: The proof uses similar estimates as in the proof of Theorem 3.4, hence will be omitted here. □

Remark 3.6 Theorem 3.4 shows that the stock model (3.1) is the most general model so far, since the driving noise can be constructed as the linear combination of $x, [x,2], [x,3]$ which are not necessarily of Gaussian type, and are not necessarily mutually independent. This matches with the empirical evidence that the log-return does not follow a normal distribution and can have a heavy tail. Moreover, the long memory effect can also be explained given that $[x,2], [x,3]$ are Hölder continuous of order $2\nu, 3\nu > \frac{1}{2}$.
Let us review some special cases.

- If x is a realization of a standard Brownian motion B, we go back to solve the classical model (1.1) using Itô calculus, so that the log-price has the form

  $$Y_{s,t} = \Big(\mu - \frac{\sigma^2}{2}\Big)(t-s) + \sigma B_{s,t}.$$

  That corresponds to $[x,2]_{a,b} = (b-a), [x,3] \equiv 0$.

- Also, if we choose x to be a realization of a fractional Brownian motion B^H for $H \in (\frac{1}{2}, 1)$, we go back to the model $dS_t = \mu S_t dt + \sigma S_t \delta B_t^H$ with the Skorohod-Wick-Itô integral $\int y \delta B^H$ proposed in [30] as discussed in Example 2.1(iii), hence the solution is solved explicitly [31] as

$$Y_{s,t} = \mu(t-s) - \frac{\sigma^2}{2}(t^{2H} - s^{2H}) + \sigma B^H_{s,t}; \tag{3.13}$$

henceforth $[x,2]_{a,b} := (b^{2H} - a^{2H}), [x,3]_{a,b} := 0$.

- Additionally, if $X_t \in C^\nu$ for $\nu > \frac{1}{2}$, then by solving the Young equation $dS_t = \mu S_t dt + \sigma S_t dX_t$, we obtain the explicit solution

$$Y_{s,t} = \mu(t-s) + \sigma x_{s,t}; \tag{3.14}$$

in this case $[x,2] = [x,3] \equiv 0$.

Similarly, by assigning $X_t := \rho B_t + \sqrt{1-\rho^2}\xi_t$ for $\rho \in [-1,1]$, we go back to the mixed noise model

$$S_t = S_s + \int_s^t \mu S_u du + \int_s^t \sigma\rho S_u dB_u + \int_s^t \sigma\sqrt{1-\rho^2} S_u d\xi_u, \tag{3.15}$$

where the first stochastic integral is understood in the Stratonovich sense, and the second stochastic integral is understood in the pathwise sense as a Young integral, due to the fact that $\xi \in C^{2\nu}$ with $2\nu > \frac{1}{2}$, see e.g. [29, Chapter 5]. The explicit solution of equation (3.15) is given by

$$Y_{s,t} = \mu(t-s) + \sigma\rho B_{s,t} + \sigma\sqrt{1-\rho^2}\xi_{s,t}, \quad \forall 0 \le s \le t \le T, \tag{3.16}$$

which corresponds to $[x,2]_{s,t}, [x,3]_{s,t} \equiv 0$. We emphasize here however that, in (3.16) the effect of mixed noises is only linear in σ which comes from the noise $x_{a,b}$, but in general higher order terms of σ could come from $[x,2], [x,3]$ as in (3.7).

3.2 No arbitrage under transaction costs

Motivated by the discussions in Remark 3.6 on different types of stochastic integrals as well as Example 3.3, we propose the following additional hypotheses in this section.

Hypothesis A $[x,2], [x,3]$ are Hölder continuous paths of the form

$$[x,2]_{s,t} = \alpha_{s,t} + \xi_{s,t}, \qquad [x,3]_{s,t} = \beta_{s,t} + \gamma_{s,t}, \tag{3.17}$$

for all $s,t \in [0,T]$, where $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ are deterministic functions of bounded variation; and

Hypothesis B x, ξ, γ are realizations of mutually independent stationary stochastic processes X_t, Ξ_t, Γ_t on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ which are $\mathcal{F}_t$-adapted and satisfy

$$\mathbb{E}X_{s,t} = \mathbb{E}\Xi_{s,t} = \mathbb{E}\Gamma_{s,t} = 0, \qquad \forall 0 \le s \le t, \tag{3.18}$$

and such that almost surely all realizations of X, Ξ, Γ belong to the Hölder space $C^\nu, C^{2\nu}, C^{3\nu}$ respectively for some constant $\nu \in (\frac{1}{4}, \frac{1}{2})$.

It is important to note that from Hypothesis A, the pathwise integral $\int y dx$ for controlling rough paths y w.r.t. $\mathbf{x}$ of the form

$$y_{s,t} = y'_s x_{s,t} + \frac{1}{2} y''_s (x^2_{s,t} - \xi_{s,t}) + R^y_{s,t}, \quad y'_{s,t} = y''_s x_{s,t} + R^{y'}_{s,t}$$

can be computed as

$$\int_s^t y_u dx_u = \lim_{|\Pi|\to 0} \sum_{[u,v]\in\Pi} \Big[y_u x_{u,v} + \frac{1}{2}y'_u(x^2_{u,v} - \xi_{u,v}) + \frac{1}{6}y''_u(x^3_{u,v} - 3x_{u,v}\xi_{u,v} + 2\gamma_{u,v})\Big] + \int_s^t \frac{1}{3}y''_u d\beta_u - \int_s^t \frac{1}{2}y'_u d\alpha_u,$$

where the limit is taken on all the finite partitions Π of I with $|\Pi| := \max_{[u,v]\in\Pi} |v-u|$, and the last two integrals are understood in the Riemann-Stieltjes sense for functions α, β of bounded variation. Then the logarithm price process $Y_t := \log S_t$ can be written explicitly in the pathwise sense as

$$Y_{s,t} = \Big[\mu(t-s) - \frac{\sigma^2}{2}\alpha_{s,t} + \frac{\sigma^3}{3}\beta_{s,t}\Big] + \sigma X_{s,t} - \frac{\sigma^2}{2}\Xi_{s,t} + \frac{\sigma^3}{3}\Gamma_{s,t}, \quad \forall s,t \in [0,T]. \tag{3.19}$$

To avoid the no arbitrage problem we follow [19] to consider a realistic assumption on transaction costs. Namely, consider the model with a riskless asset price process A_t and a risky asset price process $(S_t)_{t\in[0,T]}$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, where the filtration $\mathcal{F}_t$ satisfies the usual assumptions of right continuity and saturatedness and S_t is a càdlàg (right continuous with left limits) path, strictly positive almost surely, adapted and quasi-left continuous w.r.t. $\mathcal{F}_t$. An investor trades in the risky asset according to the strategy $(\theta_t)_{t\in[0,T]}$, which represents the number of shares held at time t, and each unit traded in the risky asset generates a transaction cost of k units, which is charged to the riskless asset account. Consider a simple strategy θ which requires a finite number of transactions at stopping times $(\tau_i)_{i=1}^n$, then $\theta = \sum_{i=1}^n \theta_i 1_{]\tau_{i-1},\tau_i]}$ for some random variables $(\theta_i)_{i=1}^n$, where θ_i is $\mathcal{F}_{\tau_i}-$ measurable, and $\theta_0 = 0$ conventionally. The liquidation value of a portfolio with zero initial capital is

$$V_t(\theta) = \sum_{i=1}^n \theta_i(S_{\tau_i\wedge t} - S_{\tau_{i-1}\wedge t}) - k\sum_{\tau_i \le t} S_{\tau_i}|\theta_i - \theta_{i-1}| - kS_t|\theta_t|. \tag{3.20}$$

This discrete model is then proved to converge to the following continuous model

$$V_t(\theta) = \langle \theta, S\rangle_t - k\int_{[0,t]} S_u d\|\theta\|_u - kS_t|\theta_t|, \tag{3.21}$$

where $\|\theta\|_t$ is the total variation of θ on $[0,t]$ and $\langle\theta, S\rangle_t$ is a certain type of pathwise integral. According to [19], a strategy θ is *admissible* if $V_t(\theta) \ge -M$ a.s. for some $M > 0$ and for all $t > 0$. It is called an *arbitrage opportunity* on $[0,T]$ if it is admissible with $V_T(\theta) \ge 0$ a.s. and $\mathbb{P}(V_T(\theta) > 0) > 0$. A market is *arbitrage free on* $[0,T]$ if, for all admissible strategies θ, $V_T(\theta) \ge 0$ a.s. only if $V_T(\theta) = 0$ a.s. The market is arbitrage free with transaction costs k if S_t satisfies the condition that for all stopping times τ such that $\mathbb{P}(\tau < T) > 0$, we have

$$\mathbb{P}\Big(\sup_{t\in[\tau,T]}\Big|\frac{S_\tau}{S_t} - 1\Big| < k, \tau < T\Big) > 0. \tag{3.22}$$

Condition (3.22) is satisfied when the asset logarithm price process Y_t is *sticky* w.r.t. the filtration $\mathcal{F}_t$, i.e. for all $\epsilon, T > 0$ and all stopping times τ such that $\mathbb{P}(\tau < T) > 0$, one has

$$\mathbb{P}(\sup_{t\in[\tau,T]} |Y_\tau - Y_t| < \epsilon, \tau < T) > 0 \tag{3.23}$$

According to [19], any *strong Markov process*, i.e. for every finite $\mathcal{F}_\tau$-stopping time τ, under the conditional law $\mathbb{P}(\cdot|X_\tau = y)$, the process $(X_{\tau+t})_{t\ge 0}$ is independent of $\mathcal{F}_\tau$ and has the law $\mathbb{P}_y$, is sticky. Another sticky class consists of adapted stochastic processes w.r.t. filtration $(\mathcal{F}_t)_{t\in[0,T]}$ that have *conditional full support* (CFS), i.e.

$$\forall t \in [0,T),\ (\mathbb{P}\text{-a.s.})\ \forall \omega \in \Omega: \quad \operatorname{supp}\Big(\mathrm{law}\big[(X_u)_{u\in[t,T]}|\mathcal{F}_t\big](\omega)\Big) = C_{X_t(\omega)}([t,T],I), \tag{3.24}$$

where $C_\eta([t,T],I)$ is the space of continuous functions $f \in C([t,T],I)$ taking values in an open interval $I \subset \mathbb{R}$ such that $f(t) = \eta$, and we regard $\mathrm{law}\big[(X_u)_{u\in[t,T]}|\mathcal{F}_t\big]$ as a regular conditional law (a random Borel probability measure) on $C([t,T],I)$ [25, pp. 106-107]. Furthermore, any stochastic process with CFS is proved in [20, Theorem 1.2] to admit an ϵ-consistent pricing system for all $\epsilon > 0$, i.e. there exists a pair $(\tilde{S},\tilde{P})$ where $\tilde{P}$ is an equivalent probability w.r.t. $\mathbb{P}$ and $\tilde{S}$ is a $\tilde{P}$-martingale (adapted to $\mathcal{F}_t$) such that

$$\frac{1}{1+\epsilon} \leq \frac{\tilde{S}_t}{S_t} \leq 1+\epsilon, \quad \forall t \in [0,T].$$

We therefore need another assumption.

Hypothesis C Either X_t, Ξ_t, Γ_t are all strong Markov processes, or X has CFS in the sense of (3.24).

Theorem 3.7 *Under Hypotheses A, B, C and the situation of transaction costs k, the logarithm price process Y_t in* (3.19) *is sticky. Hence S_t is arbitrage free under transaction costs k on any interval* $[0,T]$.

Proof: In case X, Ξ, Γ are strong Markov processes, the stickiness follows from [19, Proposition 3.1]. For a more general case, we could apply the method in [20], [32]; in particular the CFS criterion was proved in [32, Theorem 3.3] for an extended class of mixed noises $Z_t + X_t$, where Z is an arbitrary continuous process and X is the adapted process with CFS and independent of Z.

□

Remark 3.8 (Towards a time dependent rough model) Let us consider the case $\nu > \frac{1}{3}$ and the time dependent model (3.12)

$$Y_{s,t} = \int_s^t \mu_u du - \frac{1}{2}\int_s^t \sigma_u d[x,2]_u + \int_s^t \sigma_u dx_u,$$

where $\sigma.$ is a $\mathcal{F}_t$- adapted stochastic process that is independent of X, Ξ and $\sigma_t \neq 0$ for all $t \in [0,T]$ almost surely. We need to impose the condition for X so that Y_t is sticky. In this case the process Γ in Hypotheses A and B could be neglected. The conclusion on stickiness of Y_t should then hold if X, Ξ are strong Markov processes, as seen in Theorem 3.7. However, in order to prove that Y has CFS once X has CFS, we need to modify the CFS condition in Hypothesis C to match with the rough integral $\int_s^t \sigma_u dx_u$.
To do that, observe from Hypothesis B that for a fixed $\alpha \in (\frac{1}{3},\nu)$ almost all trajectories of X belong to the separable space $C^{0,\alpha}([0,T],\mathbb{R})$ as in (2.12). Since

$$\begin{aligned} Y_{s,t} &= \int_s^t \mu_u du - \frac{1}{2}\int_s^t \sigma_u d[x,2]_u + \lim_{|\Pi|\to 0} \sum_{[u,v]\in\Pi} [\sigma_u x_{u,v} + \frac{1}{2}\sigma'_u x^2_{u,v}] - \frac{1}{2}\int_s^t \sigma'_u d[x,2]_u \\ &= \int_s^t \mu_u du - \frac{1}{2}\int_s^t \Big(\sigma_u + \sigma'_u\Big) d[x,2]_u + \lim_{|\Pi|\to 0} \sum_{[u,v]\in\Pi} [\sigma_u x_{u,v} + \frac{1}{2}\sigma'_u x^2_{u,v}], \end{aligned}$$

we should define the CFS condition for X in terms of Hölder spaces as

$$\forall t \in [0,T),\ (\mathbb{P}\text{-a.s.})\ \forall \omega \in \Omega: \quad \operatorname{supp}\Big(\mathrm{law}\big[(X_u)_{u\in[t,T]}|\mathcal{F}_t\big](\omega)\Big) = C^{0,\alpha}_{X_t(\omega)}([t,T],I), \tag{3.25}$$

In this situation, we expect to prove that: *If X has CFS on $[0,T]$ in the sense of* (3.25)*, then so does the process $\kappa_\cdot = \int_0^\cdot \sigma_u dx_u$*. The proof is similar to [20, Lemma 4.5] (see also [32, Theorem 3.3]), except the estimates w.r.t. the supremum norms should be replaced by the estimates with α-Hölder norms. In particular, one important task in the proof is to approximate κ by a function $f \in C^1([0,T],\mathbb{R})$ so that

$$\int_s^t \sigma_u dx_u - f_{s,t} = \int_s^t \sigma_u dx_u - \int_s^t \sigma_u d\Big(\int_s^u \frac{f'_r}{\sigma_r} dr\Big) = \int_s^t \sigma_u d\Big(x_u - \int_s^u \frac{f'_r}{\sigma_r} dr\Big)$$

where all the integrals could be understood in the rough sense. As such the norm $\|\kappa - f\|_{\alpha,[t,T]}$ can be estimated using (2.18).
Note that for $\sigma_\cdot$ to be a piecewise constant function, we go back to model (3.19) without β, Γ, hence there is no need to use the α-Hölder norm but only the supremum norm, and Theorem 3.7 can be applied to obtain the CFS in the sense of (3.24) for Y.

4 Some empirical evidence

4.1 Estimating Hurst exponents

Table 1 shows how the Hurst exponents vary w.r.t. different time scales h, for $h = 1$ corresponding to daily quotes, by using the rescale range test [R/S] with minimal size of 20 to avoid numerical error in estimating linear regression [13]. As seen in Tables 1,2,3, the Hurst exponents for daily, weekly and monthly data are still smaller than $\frac{2}{3}$, implying that there is no effect of the third noise Γ for a Hurst exponent $H_3 > \frac{3}{4}$. On the other hand, the Hurst exponents are bigger but very close to 0.5 for a time scale $h \ll 1$ (minute quotes), implying that the effect of standard Brownian motion dominates $X_t = B_t$ and thus $H_1 = \frac{1}{2}$.
Tables 2 and 3 show the results for the same data but for different methods of computing the fractal dimension, namely using spectral analysis and the Higuchi method [23]. We see that quite often the spectral method gives Hurst exponents smaller than $\frac{1}{2}$ for smaller time-scales. In contrast, the Higuchi method gives quite stable results which are comparable with the rescale range method in Table 1.

In addition, our numerical computations, in Table 5 and Figure 1 with data from stock index SP500 for different timescales, show a non-linear dependence of the variance on the time duration,

$$\text{Var } Y_{t,t+\tau} = \sigma^2 \tau^{2H} \Leftrightarrow \log\Big(\text{Var } Y_{t,t+\tau}\Big) = 2H \log \tau + 2 \log \sigma, \tag{4.1}$$

where $1 > H > 0$ for all data of timescales from minute to monthly quotes. Moreover, H seems to depend increasingly on time scale h. Relation (4.1) is tested by choosing $\tau = 2^k, k = 0, \ldots, m$ where $m = \log_2 \frac{N}{100}$ and N is the length of the data. The variance Var $Y_{t,t+\tau}$ can be computed based on the sequence $\{Y_{i\tau,(i+1)\tau}\}$ with length no less than 100 to neglect potential noises from small data.

It is not clear how this nonlinearity can arise from the time dependent model (1.2) with additional assumptions on the stationarity of the processes μ_t and σ_t. However, we can give a simple explanation for the numerical results in Table 4 and Figure 1 by using model (3.7). Indeed, we can simply assume that $[x,2]_{s,t}$ is a realization of a stochastic process Ξ and let $\beta, \Gamma \equiv 0$. Then it follows from (3.19) that

$$Y_{t,t+h} = \mu h - \frac{\sigma^2}{2}\alpha_{t,t+h} + \sigma X_{t,t+h} - \frac{\sigma^2}{2}\Xi_{t,t+h}, \tag{4.2}$$

	1M	5M	15M	30M	1H	4H	Daily	Weekly	Monthly
Sp500	0.5158	0.5103	0.5234	0.5218	0.5201	0.5278	0.5613	0.5816	0.6105
Dow Jones	0.5199	0.5177	0.5301	0.5410	0.5437	0.5534	0.5764	0.5952	0.5990
Nasdaq100	0.5177	0.5193	0.5245	0.5256	0.5324	0.5490	0.5694	0.6069	0.6305

Table 1: Hurst exponents for different time scales using [R/S] analysis. Data source: Dukascopy Bank SA

	1M	5M	15M	30M	1H	4H	Daily	Weekly	Monthly
Sp500	0.4898	0.4911	0.4969	0.4879	0.4838	0.4818	0.5649	0.6006	0.5284
Dow Jones	0.4937	0.4952	0.5021	0.4909	0.4927	0.4951	0.5049	0.5386	0.5015
Nasdaq100	0.4907	0.4964	0.5048	0.4974	0.5031	0.4755	0.5188	0.5651	0.4649

Table 2: Hurst exponents for different time scales using Spectral analysis. Data source: Dukascopy Bank SA

	1M	5M	15M	30M	1H	4H	Daily	Weekly	Monthly
Sp500	0.5202	0.5202	0.5157	0.5197	0.5181	0.5149	0.5549	0.5410	0.5483
Dow Jones	0.5198	0.5208	0.5174	0.5127	0.5161	0.5081	0.5651	0.5734	0.5655
Nasdaq100	0.5271	0.5294	0.5224	0.5185	0.5307	0.5288	0.6112	0.6490	0.6491

Table 3: Hurst exponents for different time scales using Higuchi method. Data source: Dukascopy Bank SA

where X and Ξ are independent. As a result,

$$\mathbb{E}Y_{t,t+h} = \mu h - \frac{\sigma^2}{2}\alpha_{t,t+h}, \quad \text{Var } Y_{t,t+h} = \sigma^2 \text{Var } X_{t,t+h} + \frac{\sigma^4}{4}\text{Var } \Xi_{t,t+h}. \tag{4.3}$$

Now assume further that

$$\text{Var } X_{t,t+h} = E|X_{t,t+h}|^2 = Ch, \quad \text{Var } \Xi_{t,t+h} = E|\Xi_{t,t+h}|^2 = C_H h^{2H} \tag{4.4}$$

for some $H \in (\frac{1}{2}, 1)$ and constants C, C_H (this assumption (4.4) will be satisfied if $X = B$ and $\Xi = B^H$). Then it follows from (4.3) that

$$\log\Big(\text{Var } Y_{t,t+h}\Big) = \log\Big(C\sigma^2 h + \frac{C_H}{4}\sigma^4 h^{2H}\Big) \approx \begin{cases} \log(h) + \log(C\sigma^2) & \text{if} \quad h \ll 1 \\ 2H\log(h) + \log(\frac{C_H}{4}\sigma^4) & \text{if} \quad h \gg 1 \end{cases}. \tag{4.5}$$

Therefore for different time scales h ranging from $\frac{1}{1440}$ for minute quotes to 30 for monthly quotes, the regression coefficient for the relation between $\log\Big(\text{Var } Y_{t,t+h}\Big)$ and $\log h$ in (4.5) can increase from smaller than (but also close to) $\frac{1}{2}$ to $2H$ for $H > \frac{1}{2}$, as observed in Table 4.

4.2 Upper-parabolic mean-variance relation of the logarithmic return

A drawback of model (1.1) is the fact that

$$\mathbb{E}Y_{t,t+h} = h\Big(\mu - \frac{\sigma^2}{2}\Big), \quad \text{Var } Y_{t,t+h} = \sigma^2 h,$$

	1M	5M	15M	30M	1H	4H	Daily	Weekly	Monthly
σ	0.0003	0.0006	0.0010	0.0014	0.0020	0.0036	0.0121	0.0246	0.0417
H	0.4872	0.4841	0.4810	0.4832	0.4798	0.4673	0.5237	0.5663	0.6191

Table 4: Linear regression coefficients of relation (4.1) for Sp500, from minute to monthly quotes. Data source: Dukascopy Bank SA

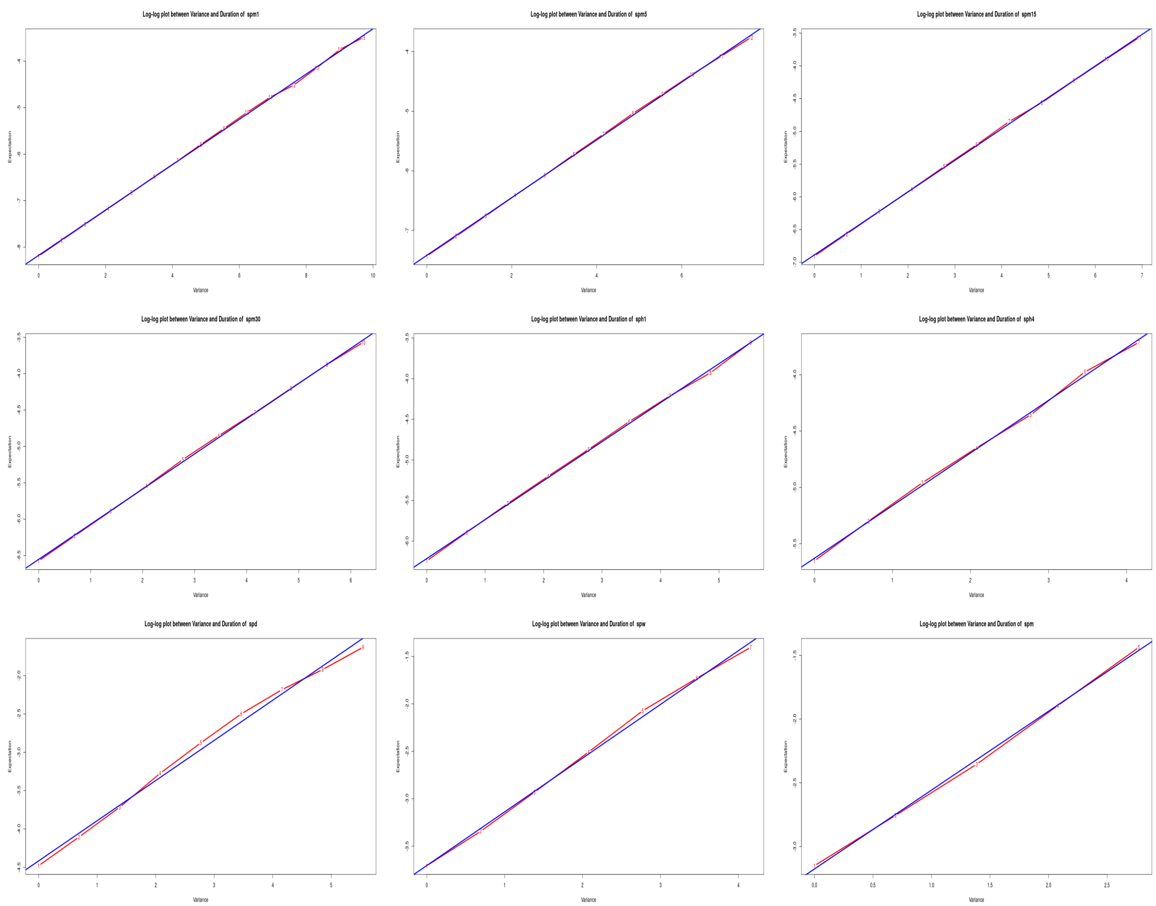

Figure 1: Linear regression of relation (4.1) for Sp500, from minute to monthly quotes. Data source: Dukascopy Bank SA

which implies that the variance depends linearly and negatively on the expected return

$$\text{Var } Y_{t,t+h} = 2\mu h - 2\mathbb{E}Y_{t,t+h}. \tag{4.6}$$

Our numerical computations with empirical data show a different picture. We collect time series $\{Y_j\}_{j=1}^{N}$ of 1 minute logarithmic quotes, so that the time step h is small. Then for any set $Y_k^{(h)} := \{Y_{km+i}\}_{i=1}^{m}$ of daily period where $k = 0 \dots [\frac{N}{m}] - 1$, we calculate the mean $\mathbb{E}Y_k^{(h)} = \frac{1}{m}\sum_{i=1}^{m} Y_{km+i,km+i+1}$ and its variance $\text{Var } Y_k^{(h)} = \frac{1}{m-1}\sum_{i=1}^{m-1}\left(Y_{km+i,km+i+1} - \mathbb{E}Y_k^{(h)}\right)^2$ of the 1-minute logarithmic return during that day. Figures 2 and 3 show that, for all types of financial asset prices from stocks and stock indices to commodities and cryptocurrencies, the set of daily mean-variance $(\mathbb{E}Y_k^{(h)}, \mathrm{Var}\ Y_k^{(h)})$ has a parabola-shaped left envelope, which cannot be explained by model (4.6).

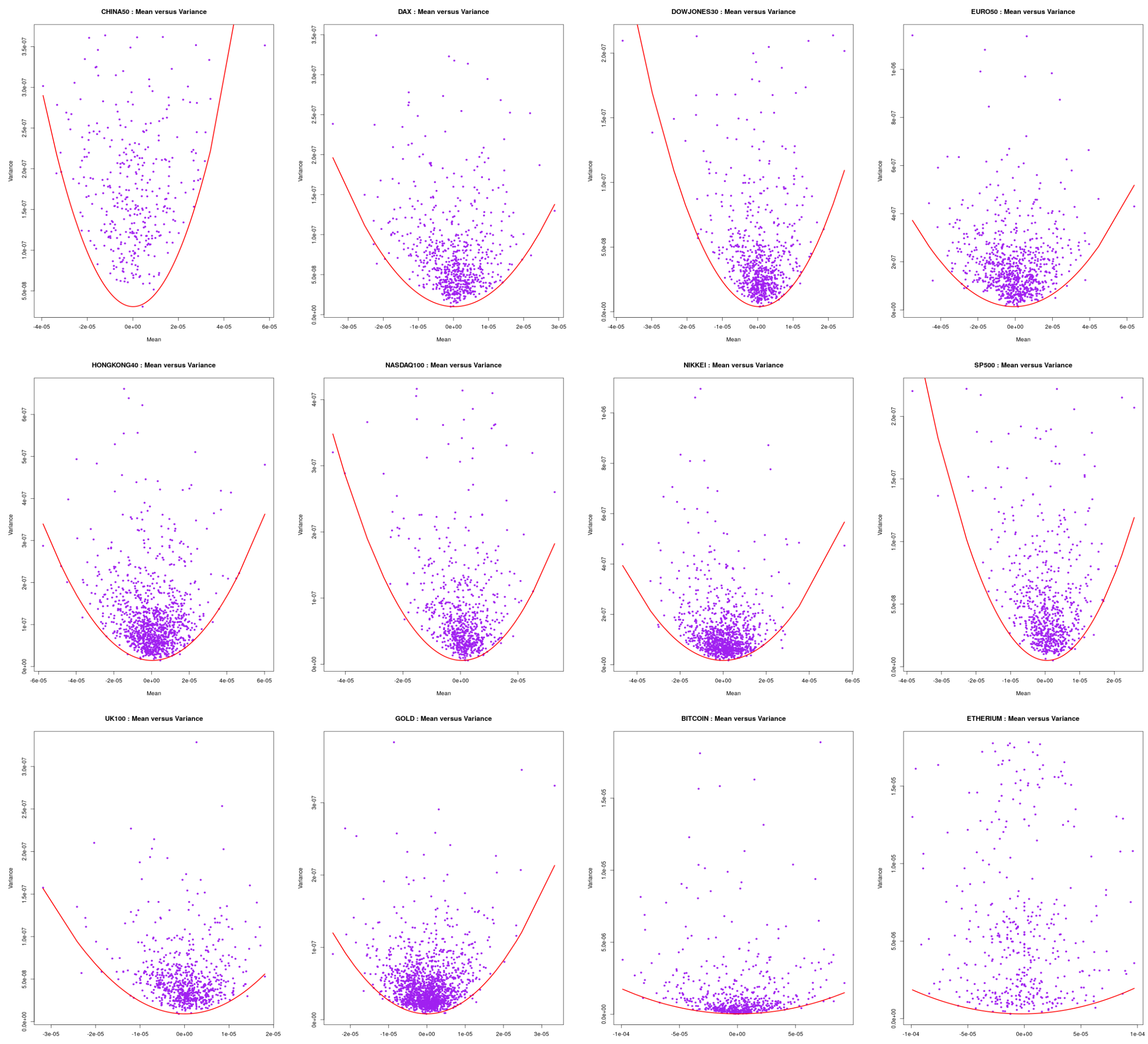

Figure 2: Upper-parabolic mean-variance relation (4.7) for 1-minute quotes of stock indices, commodities and cryptocurrencies. Data: China50, Dax, DowJones30, Euro50, Hongkong40, Nasdaq100, Nikkei, Sp500, Uk100, Gold, Bitcoin, Etherium. Data source: Dukascopy Bank SA

While it seems to be very complicated to theoretically explain this parabolic relation using a time dependent Itô model, our rough model under Hypotheses A-C easily accounts for that. Indeed, consider again model (4.2) with expectation and variance computed in (4.3). By solving for σ^2 in terms of $\mathbb{E}Y_{t,t+h}$ and μ in the first equality and inserting it into the second equality we obtain a parabolic relation

$$\mathrm{Var}\ Y_{t,t+h} = 2\frac{\mathrm{Var}\ X_{t,t+h}}{\alpha_{t,t+h}}\Big(\mu h - \mathbb{E}Y_{t,t+h}\Big) + \frac{\mathrm{Var}\ \Xi_{t,t+h}}{(\alpha_{t,t+h})^2}\Big(\mu h - \mathbb{E}Y_{t,t+h}\Big)^2. \tag{4.7}$$

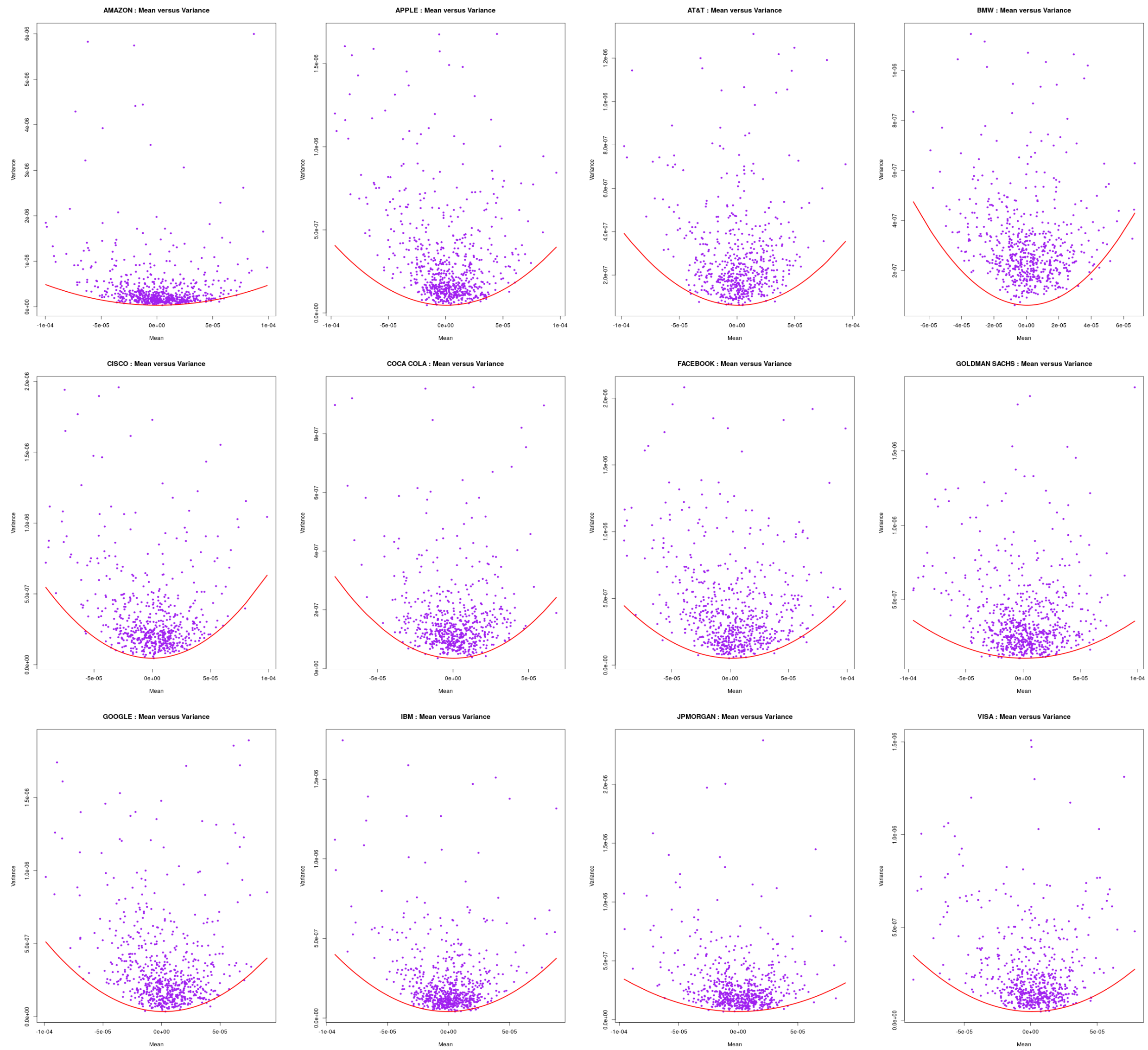

Figure 3: Upper-parabolic mean-variance relation (4.7) for 1-minute quotes of several stocks. Data: Amazon, Apple, AT&T, BMW, Cisco, Coca-cola, Facebook, Goldman Sachs, Google, IBM, JP Morgan, Visa. Data source: Dukascopy Bank SA

In particular, since $\sigma^2 = \frac{2}{\alpha_{t,t+h}}\Big(\mu h - \mathbb{E}Y_{t,t+h}\Big) \geq 0$, it follows that

$$\text{Var } Y_{t,t+h} \geq \frac{\text{Var } \Xi_{t,t+h}}{(\alpha_{t,t+h})^2}\Big(\mu h - \mathbb{E}Y_{t,t+h}\Big)^2 \tag{4.8}$$

$$\Leftrightarrow \quad (\mathbb{E}Y_{t,t+h}, \text{Var } Y_{t,t+h}) \text{ lies inside parabola } \mathcal{P} := \Big\{(x,y) \in \mathbb{R}^2 : y = \frac{\text{Var } \Xi_{t,t+h}}{(\alpha_{t,t+h})^2}(\mu h - x)^2\Big\},$$

which explains Figures 2 and 3, where the symmetry axis of the parabola $\mathcal{P}$ is $x_0 = \mu h \approx 0$ since $\mu h \approx 0$ for small time steps h (due to high frequency data of minute quotes). It is important to note that the parameters of the parabola $\mathcal{P}$ depend only on the noise Ξ and are independent of X.

Moreover, in case Ξ is non-random, Var $\Xi_{t,t+h} \equiv 0$ so the parabola reduces to the flat line

$$\text{Var } Y_{t,t+h} = 2\frac{\text{Var } X_{t,t+h}}{\alpha_{t,t+h}}\Big(\mu h - \mathbb{E}Y_{t,t+h}\Big)$$

which includes the special case (4.6) by assigning $X := B$ and $\alpha_{s,t} := t - s$.

5 Risk under stock model ambiguity: Uncertainty from signatures

The rough model in Section 4 implies that there would be an additional source of noise coming from $[x, 2]$ that affects the asset price. To show how negative the effect of the rough path signatures on the expected log-return could be, let us consider again model (4.2) and assign $h := 1$ for simplicity and denote by $R_t = \log \frac{S_t}{S_{t-h}} = Y_{t-1,t}$ the log-return. It then follows from (4.2) that

$$R_t = Y_{t-1,t} = \mu - \frac{\sigma^2}{2}\alpha_{t-1,t} + \sigma X_{t-1,t} - \frac{\sigma^2}{2}\Xi_{t-1,t}. \tag{5.1}$$

Table 5 shows the expectation and variance of the log-return in the considered models. Observe

	$\mathbb{E}R_t$	Var R_t
Model (1.1)	$\mu - \frac{1}{2}\sigma^2$	σ^2
Model (3.13)	$\mu - \frac{\sigma^2}{2}\Big[t^{2H} - (t-1)^{2H}\Big]$	σ^2
Model (3.1)	$\mu - \frac{\sigma^2}{2}\alpha_{t-1,t}$	$\sigma^2\text{Var } X_{t-1,t} + \frac{1}{4}\sigma^4\text{Var } \Xi_{t-1,t}$

Table 5: Comparison of expected value and variance among models.

that for model (5.1) we have additional nonlinear terms that are quartic in σ and which will be important for our subsequent discussion. Observe from the classical model (1.1) that

$$ER_t = \mu - \frac{1}{2}\text{Var}R_t. \tag{5.2}$$

For the model (3.13) we get

$$\begin{aligned} ER_t &= \mu - \frac{1}{2}\Big[t^{2H} - (t-1)^{2H}\Big]\text{Var}R_t \\ \text{which implies that} \quad & \begin{cases} ER_t > \mu - \frac{1}{2}\text{Var}R_t & \text{if} \quad H < \frac{1}{2} \\ ER_t \leq \mu - \frac{1}{2}\text{Var}R_t & \text{if} \quad H \geq \frac{1}{2} \end{cases}. \end{aligned} \tag{5.3}$$

In comparison, it follows that for the rough model

$$\begin{aligned} ER_t \leq \mu - \frac{1}{2}\text{Var}R_t \quad &\Leftrightarrow \quad \frac{\sigma^2}{2}\text{Var } X_{t-1,t} + \frac{1}{8}\sigma^4\text{Var } \Xi_{t-1,t} \leq \frac{\sigma^2}{2}\alpha_{t-1,t} \\ &\Leftrightarrow \quad \text{Var } X_{t-1,t} + \frac{\sigma^2}{4}\text{Var } \Xi_{t-1,t} \leq \alpha_{t-1,t}. \end{aligned} \tag{5.4}$$

In all three models, the expected return is decreasing when the variance is increasing, which matches with classical results. The negative effect of the variance on the expected return is however very different for all three models.

Indeed, the variance of R_t in the classical model (1.1) or in the mixed model (3.15) is different from that of the rough model (5.1); in fact,

$$\mathrm{Var}\Big[R_t^{\text{Itô}}\Big] = \mathrm{Var}\Big[R_t^{\text{Wick}}\Big] = \sigma^2 \leq \mathrm{Var}\Big[R_t^{\text{Rough}}\Big]. \tag{5.5}$$

Due to that reason, the information is somehow hidden in the signatures, which can increase the uncertainty of the model and result in a bigger risk.

To see how this leads to a model risk, let us now review the strategy for selling an asset in the portfolio, which is discussed in [36] and [37]. Assume that the growth rate μ, which depends mainly on the intrinsic (fundamental) value, is a piecewise constant function, and that the volatility parameter σ is unknown. According to [37], the criterion of the trading strategy is to sell the asset when its expected value is negative. When we follow (5.2), this would mean in practice that we sell when the variance crosses the threshold 2μ

$$\mathbb{E}R_t < 0 \Leftrightarrow 2\mu < \mathrm{Var}R_t. \tag{5.6}$$

When we applied the criterion (5.6) also for the Wick model for $H < \frac{1}{2}$ or for the rough model with $\mathrm{Var}\ X_{t-1,t} + \frac{\sigma^2}{4}\mathrm{Var}\ \Xi_{t-1,t} > \alpha_{t-1,t}$, then because the effect of the variance on the expected return in (5.3) and (5.4) is less negative, we would sell too early. On the other hand, in the case of $H \geq \frac{1}{2}$ in (5.3) for the Wick model or in the case of (5.4) for the rough model, the criterion is not appropriate, because it underestimates the larger effect of the variance on the expected return in (5.3) as well as in (5.4), thus we would sell too late. Failure to use the right model could therefore create a model risk of mis-calculating the expected value, which then affects the trading strategies.

Acknowledgments

The authors would like to thank Dukascopy bank SA for permission to use their data for academic research. This work was carried out at the Max Planck Institute for Mathematics in the Sciences (MPI MIS Leipzig). A part of this work was discussed while L.H.D. visited Vietnam Institute for Advanced Studies in Mathematics (VIASM) for a three month research stay in summer 2019.

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