TIME DECAY RATES OF THE L^3-NORM FOR STRONG SOLUTIONS TO THE NAVIER-STOKES EQUATIONS IN $\mathbb{R}^3$

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ABSTRACT. Let $u \in C([0, \infty); L^3(\mathbb{R}^3))$ be a strong solution of the Cauchy problem for the 3D Navier-Stokes equations with the initial value u_0. We prove that the time decay rates of u in the L^3-norm coincide with ones of the heat equation with the initial value $|u_0|$. Our proofs use the theory about the exitstence of local strong solutions, time decay rates of strong solutions when the initial value is small enough, and uniqueness arguments.

1. Introduction

We consider the Cauchy problem of the incompressible Navier-Stokes equations in the whole space $\mathbb{R}^3$

$$\begin{cases} u_t - \Delta u + \nabla \cdot \big(u \otimes u\big) + \nabla p = 0, \\ \text{div } u = 0, \\ u(0, x) = u_0. \end{cases} \tag{1.1}$$

The unknown quantities are the velocity $u(t, x) = (u_1(t, x), u_2(t, x), u_3(t, x))$ of the fluid element at time t and position x and the pressure $p(t, x)$.

The global existence of weak solutions goes back to Leray [23] and Hopf [13]. The global well-posedness of strong solutions for small initial data in the critical Sobolev space $\dot{H}^{1/2}$ is due to Fujita and Kato [9], also in [8], Chemin has proved the case of $\dot{H}^s,\ s > 1/2$. In [14], Kato has proved the case of the Lebesgue space L^3 (see also [15, 16, 17, 18, 19]). In [20], Koch and Tataru have proved the case of the space BMO^{-1}.

The L^2 decay for weak solutions is due to Kato [14], he proved that the Leray weak solutions of the Navier-Stokes equations in $\mathbb{R}^n$ exist when $n \leq 4$ and tend to zero in L_2 as $t \to \infty$. The argument of Kato is based on the fact that Leray weak solution becomes a strong solution after a finite time. On the other hand, Wiegner [25] showed that if the solution $e^{t\Delta}u_0$ to the heat equation with the initial data $u_0 \in L^2(\mathbb{R}^n)$ satisfies $\big\|e^{t\Delta}u_0\big\|_2 \leq C(1 + t)^{-\alpha}$ for some $C > 0$ and $\alpha > 0$, then there exists a weak solution u such that $\|u(t)\|_2 \leq C(1 + t)^{-\min\{\alpha,(n+2)/2\}}$.

For the large time behaviour of strong solution, it is well known that if $u \in C([0, \infty), X)$ is a global solution for some divergence-free $u_0 \in X$, where X is either $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ or $L^3(\mathbb{R}^3)$. Then $\lim\limits_{t\to\infty} \big\|u(t)\big\|_X = 0$. These results were proved for $X = \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ in [10] and for $L^3(\mathbb{R}^3)$ in [11]. Benameur [2] proved that if $u \in C([0, \infty), \chi^{-1})$ is a global solution, then $\|u(t)\|_{\chi^{-1}}$ decays to zero as time

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goes to infinity, where

$$\chi^{-1} := \{f \in \mathcal{D}'(\mathbb{R}^3), \int_{\mathbb{R}^3} \frac{|\hat{f}(\xi)|}{|\xi|} d\xi < \infty\}.$$

Our purpose in this paper is to extend the result in [11], we prove that if $u \in C([0,\infty); L^3(\mathbb{R}^3))$ is a strong solution of the Navier-Stokes equations with the initial value u_0 then the time decay rates of u in the L^3-norm coincide with ones of the heat equation with initial value $|u_0|$, see Theorem 2.1, in the particular $\alpha = 0$ then we get back the result in [11]. The content of this paper is as follows: in Section 2, we state our main theorem after introducing some notations. In Section 3, we introduce Lorentz spaces, Besov space, and establish some estimates concerning the heat semigroup with differential. Finally, in Section 4, we will give the proof of the main theorem.

2. Statement of the results

For $T > 0$, we say that u is a mild solution of the Navier-Stokes equations (1.1) on $[0,T]$ corresponding to a divergence-free initial datum u_0 when u solves the integral equation

$$u = e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot\big(u\otimes u\big)ds.$$

Above we have used the following notation: for a tensor $F = (F_{ij})$ we define the vector $\nabla.F$ by $(\nabla.F)_i = \sum_{j=1}^3 \partial_j F_{ij}$ and for two vectors u and v, we define their tensor product $(u\otimes v)_{ij} = u_i v_j$. The operator $\mathbb{P}$ is the Helmholtz-Leray projection onto the divergence-free fields

$$(\mathbb{P}f)_j = f_j + \sum_{1\leq k\leq 3} R_j R_k f_k,$$

where R_j is the Riesz transforms defined as

$$R_j = \frac{\partial_j}{\sqrt{-\Delta}} \quad \text{i.e.} \quad \widehat{R_j g}(\xi) = \frac{i\xi_j}{|\xi|}\hat{g}(\xi)$$

with $\hat{}$ denoting the Fourier transform. The heat kernel $e^{t\Delta}$ is defined as

$$e^{t\Delta}u(x) = ((4\pi t)^{-3/2}e^{-|.|^2/4t} * u)(x).$$

For a space of functions defined on $\mathbb{R}^3$, say $E(\mathbb{R}^3)$, we will abbreviate it as E. We denote by $L^q := L^q(\mathbb{R}^3)$ the usual Lebesgue space for $q \in [1,\infty]$ with the norm $\|.\|_q$, and we do not distinguish between the vector-valued and scalar-valued spaces of functions. Given a Banach space E with norm $\|.\|_E$, we denote by $BC([0,T);E)$, set of bounded continuous functions $f(t)$ defined on $(0,T)$ with values in E such that $\sup_{0<t<T}\|f(t)\|_E < +\infty$. For any collection of Banach spaces $(X_m)_{m=1}^M$ and $X = X_1 \cap ... \cap X_m$, we set $\|g\|_X = \sum_{m=1}^{m=M}\|g\|_{X_m}$. Similarly, for a vector-valued function $f = (f_1, ..., f_M)$, we define $\|f\|_X = \sum_{m=1}^{m=M}\|f_m\|_X$. Throughout the paper, we sometimes use the notation $A \lesssim B$ as an equivalent to $A \leq CB$ with a uniform constant C. The notation $A \simeq B$ means that $A \lesssim B$ and $B \lesssim A$. Now we can state our main results

Theorem 2.1. *Let $u \in C([0,\infty); L^3(\mathbb{R}^3))$ be a mild solution of the Navier-Stokes equations* (1.1) *with the initial value u_0. Then*
(a) *If $\|e^{t\Delta}|u_0|\|_3 = o(t^{-\alpha})$ with $0 \le \alpha < 1$, then $\|u(t)\|_3 = o(t^{-\alpha})$.*
(b) *If $\|e^{t\Delta}|u_0|\|_3 = O(t^{-\alpha})$ with $0 \le \alpha \le 1$, then $\|u(t)\|_3 = O(t^{-\alpha})$.*
(c) *If $u_0 \in L^{p,r}(\mathbb{R}^3)$ with $1 < p < 3,\ 1 \le r < \infty$, then $\|e^{t\Delta}|u_0|\|_3 = o\big(t^{-\frac{1}{2}\left(\frac{3}{p}-1\right)}\big)$.*
(d) *If $|u_0| \in \dot{B}_3^{-2\alpha,\infty}(\mathbb{R}^3)$ with $0 \le \alpha \le 1$, then $\|e^{t\Delta}|u_0|\|_3 = O\big(t^{-\alpha}\big)$.*

3. Tools from harmonic analysis

Definition 3.1. (Lorentz spaces. (See [1].)
Let $\Omega \subseteq \mathbb{R}^3,\ 1 \le p, r \le \infty$. The Lorentz spaces $L^{p,r}(\Omega)$ is defined as follows: A measurable function $f \in L^{p,r}(\Omega)$ if and only if
$\|f\|_{L^{p,r}}(\Omega) := \big(\frac{r}{p}\int_0^\infty (t^{\frac{1}{p}} f^*(t))^r \frac{\mathrm{d}t}{t}\big)^{\frac{1}{r}} < \infty$ when $1 \le r < \infty$,
$\|f\|_{L^{p,\infty}}(\Omega) := \sup_{t>0} t^{\frac{1}{p}} f^*(t) < \infty$ when $r = \infty$,
where $f^*(t) = \inf\{\tau : \mathcal{L}^3(\{x \in \Omega : |f(x)| > \tau\}) \le t\}$, with $\mathcal{L}^3$ being the Lebesgue measure in $\mathbb{R}^3$.

Lemma 3.1. (*Young's inequality for convolution in Lorentz spaces*).
Let $1 < r, p, q < \infty$ and $1 \le \bar{r}, \bar{p}, \bar{q} \le \infty$ satisfy the relations

$$1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \ \text{and}\ \frac{1}{\bar{r}} = \frac{1}{\bar{p}} + \frac{1}{\bar{q}}.$$

Suppose that $f \in L^{p,\bar{p}}(\mathbb{R}^3)$ and $g \in L^{q,\bar{q}}(\mathbb{R}^3)$. Then $f * g \in L^{r,\bar{r}}(\mathbb{R}^3)$ and the following inequality holds

$$\|f * g\|_{L^{r,\bar{r}}} \lesssim \|f\|_{L^{p,\bar{p}}} \|g\|_{L^{q,\bar{q}}}. \tag{3.1}$$

Proof. See Proposition 2.4 in ([21], p. 20). □

In this paper we use the definition of the homogeneous Besov space $\dot{B}_q^{s,p}$ in [3, 4]. The following lemma will provide a different characterization of Besov spaces $\dot{B}_q^{s,p}$ in terms of the heat semigroup.

Lemma 3.2.
Let $1 \le p, q \le \infty$ and $s < 0$. Then the two quantities

$$\Big(\int_0^\infty (t^{-\frac{s}{2}} \|\mathrm{e}^{t\Delta} f\|_{L^q})^p \frac{\mathrm{d}t}{t}\Big)^{\frac{1}{p}} \ \text{and}\ \|f\|_{\dot{B}_q^{s,p}} \ \text{are equivalent.}$$

Proof. See Theorem 5.4 in ([22], p. 45). □

In this section we prepare some auxiliary lemmas, we first establish the $L^p - L^q$ estimate for the heat semigroup with differential.

Lemma 3.3. *Assume that $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3, t > 0$ and $1 \le p \le q \le \infty$. Then for all $f \in L^p$ we have*

$$t^\beta D^\alpha e^{t\Delta} f \in BC([0,\infty); L^q(\mathbb{R}^3)) \text{ and } \|D^\alpha e^{t\Delta} f\|_q \le C_{p,q,\alpha} t^{-\beta} \|f\|_p.$$

where $D^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}, |\alpha| = \alpha_1 + \alpha_2 + \alpha_3, \beta = \frac{3}{2}(\frac{1}{p} - \frac{1}{q}) + \frac{|\alpha|}{2}$, and $C_{p,q,\alpha}$ is a positive constant which depends only on p, q, and α.

Proof. See [24]. □

The main property we use throughout this paper is that the operator $e^{t\Delta}\mathbb{P}$ is a matrix of convolution operators with bounded integrable kernels.

Lemma 3.4. *For $t > 0$, the operator $O_t = e^{t\Delta}\mathbb{P}$ is a convolution operator $O_t f = K_t * f$, where the kernel K_t satisfies $K_t(x) = \frac{1}{t^{\frac{3}{2}}} K\big(\frac{x}{\sqrt{t}}\big)$ for a smooth function K such that*

$$(1+|x|)^{3+|\alpha|} D^\alpha K \ \in L^\infty(\mathbb{R}^3),$$

where $|x| = \big(\sum_{i=1}^3 x_i^2\big)^{1/2}, x = (x_1, x_2, x_3), \alpha = (\alpha_1, \alpha_2, \alpha_3)$.

Proof. See [21]. □

4. Proof of the theorem

In this section we shall give the proofs of Theorem 2.1. We now need some theorems and lemmas

Lemma 4.1. *Assume that $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3, t > 0$ and $1 \le p < q \le \infty$. Then for all $f \in L^p$ we have*

$$t^\beta D^\alpha e^{t\Delta}\mathbb{P}f \in BC([0,\infty); L^q(\mathbb{R}^3)) \text{ and } \big\|D^\alpha e^{t\Delta}\mathbb{P}f\big\|_q \le C_{p,q,\alpha} t^{-\beta}\|f\|_p,$$

where $\beta = \frac{3}{2}(\frac{1}{p} - \frac{1}{q}) + \frac{|\alpha|}{2}$ and $C_{p,q,\alpha}$ is a positive constant which depends only on p, q, and α.

Proof. Applying Lemma 3.4 and Young's inequality in order to obtain

$$\begin{aligned}\big\|D^\alpha e^{t\Delta}\mathbb{P}f\big\|_q &= \frac{1}{t^{\frac{3+|\alpha|}{2}}}\Big\|\partial^\alpha K\Big(\frac{\cdot}{\sqrt{t}}\Big) * f\Big\|_q \le t^{-\frac{3+|\alpha|}{2}}\Big\|\partial^\alpha K\Big(\frac{\cdot}{\sqrt{t}}\Big)\Big\|_{\frac{1}{1+\frac{1}{q}-\frac{1}{p}}}\|f\|_p \\ &= t^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})-\frac{|\alpha|}{2}}\big\|\partial^\alpha K\big\|_{\frac{1}{1+\frac{1}{q}-\frac{1}{p}}}\|f\|_p = Ct^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})-\frac{|\alpha|}{2}}\|f\|_p.\end{aligned}$$

□

Lemma 4.2.
(a) *If $\theta < 1, \gamma < 1$, and $t > 0$ then*

$$\int_0^t (t-s)^{-\gamma}s^{-\theta}\mathrm{d}s = C_1 t^{1-\gamma-\theta}, \text{ where } C_1 = \int_0^1 (1-s)^{-\gamma}s^{-\theta}\mathrm{d}s < \infty.$$

(b) *If $\theta < 1$ then*

$$\int_0^{\frac{t}{2}} (t-s)^{-\gamma}s^{-\theta}\mathrm{d}s = C_2 t^{1-\gamma-\theta}, \text{ where } C_2 = \int_0^{\frac{1}{2}} (1-s)^{-\gamma}s^{-\theta}\mathrm{d}s < \infty.$$

(c) *If $\gamma < 1$ then*

$$\int_{\frac{t}{2}}^t (t-s)^{-\gamma}s^{-\theta}\mathrm{d}s = C_3 t^{1-\gamma-\theta}, \text{ where } C_3 = \int_{\frac{1}{2}}^1 (1-s)^{-\gamma}s^{-\theta}\mathrm{d}s < \infty.$$

The proof of this lemma is elementary and may be omitted. □

Theorem 4.1. *For any $u_0 \in L^3(\mathbb{R}^3)$, $\operatorname{div}(u_0) = 0$, and any $T > 0$, there exists at most one mild solution to the Navier-Stokes equations (1.1) such that $u \in C([0,T); L^3(\mathbb{R}^3))$.*

Proof. See [21, 22]. □

Theorem 4.2. *Let $u \in C\big([0,\infty); L^3(\mathbb{R}^3)\big)$ be a mild solution of the Navier-Stokes equations* (1.1) *then* $\lim_{t\to\infty} \big\|u(t)\big\|_3 = 0$.

Proof. See [11]. $\square$

Theorem 4.3. *Let $u_0 \in L^3$ and* div $(u_0) = 0$. *Then there exists a positive constant δ such that if $\big\|u_0\big\|_3 \le \delta$, then the Navier-Stokes equations has a unique solution u satisfies*

$$t^{\frac{1}{2}\left(1-\frac{3}{q}\right)} u \in BC\big([0,\infty); L^q(\mathbb{R}^3)\ \text{and}\ t^{1-\frac{3}{2q}}\nabla u \in BC\big([0,\infty); L^q(\mathbb{R}^3)\ \text{for all}\ q \ge 3.$$

Proof. See [12]. $\square$

We prove the following result on solutions of a quadratic equation in Banach spaces which is a generalization of Theorem 22.4 in [22], p. 227.

Lemma 4.3. *Let E and F be two normed spaces such that $E \cap F$ is a well-defined Banach space with the norm $\|x\|_{E\cap F} := \|x\|_E + \|x\|_F$. Assume that B is a bounded bilinear operator from $E \times E$ to E, form $E \times F$ to F, and form $F \times E$ to F such that there exists a positive constant $\gamma > 0$ such that*

$$\|B(x,y)\|_E \le \gamma\|x\|_E\|y\|_E,\ \text{for all}\ x \in E\ \text{and}\ y \in E,$$
$$\|B(x,y)\|_F \le \gamma\|x\|_E\|y\|_F,\ \text{for all}\ x \in E\ \text{and}\ y \in F,$$
$$\|B(x,y)\|_F \le \gamma\|x\|_F\|y\|_E,\ \text{for all}\ x \in F\ \text{and}\ y \in E.$$

Then for any fixed $y \in E \cap F$ such that $\|y\|_E < \frac{1}{4\gamma}$, the equation $x = y - B(x,x)$ has a unique solution $\overline{x} \in E \cap F$ satisfying $\|\overline{x}\|_E < \frac{1}{2\gamma}$.

Proof. The uniqueness of $\bar{x}$ in $E \cap F$ is obvious it is even unique in E. Thus, we need to prove the existence of $\bar{x}$ in $E \cap F$. Let x_n be defined by

$$x_0 = y\ \text{and}\ x_{n+1} = y - B(x_n, x_n).$$

By induction, we can easily prove that

$$\|x_n\|_E < 2\|y\|_E$$

for any n. It follows that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in E. We will show that $\{x_n\}_{n=0}^{\infty}$ is also a Cauchy sequence in F. Indeed, we have

$$\|x_1 - x_0\|_F = \|B(y,y)\|_F \le \gamma\|y\|_E\|y\|_F,$$
$$\begin{aligned}\|x_{n+1} - x_n\|_F &= \|B(x_n, x_n - x_{n-1}) + B(x_n - x_{n-1}, x_{n-1})\|_F \\ &\le \gamma\|x_n\|_E\|x_n - x_{n-1}\|_F + \gamma\|x_{n-1}\|_E\|x_n - x_{n-1}\|_F \\ &< 4\gamma\|y\|_E\|x_n - x_{n-1}\|_F,\ \text{with}\ 4\gamma\|y\|_E < 1.\end{aligned}$$

An elementary computation leads to

$$\lim_{m,n\to\infty}\|x_n - x_m\|_F = 0.$$

This proves that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in F. Therefore $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in $E\cap F$, so $\{x_n\}_{n=0}^{\infty}$ converges in $E\cap F$ to an element $\overline{x} \in E\cap F$. We thus obtain, from $\|x_n\|_E < 2\|y\|_E$, that $\|\overline{x}\|_E \le 2\|y\|_E < \frac{1}{2\gamma}$. The proof of Lemma is complete. $\square$

In order to proceed, we use the auxiliary space $\mathcal{K}_T^q$, $3 \le q \le \infty$ which is made up of the functions $u(t,x)$ such that

$$t^{\frac{1}{2}(1-\frac{3}{q})}u(t) \in BC\big([0,T); L^q(\mathbb{R}^3)\big)$$

and

$$\lim_{t\to 0} t^{\frac{1}{2}(1-\frac{3}{q})}\big\|u(t)\big\|_q = 0. \tag{4.1}$$

In the case $q = 3$, it is also convenient to define the space $\mathcal{K}^3$ as the natural subspace of $C([0,T); L^3(\mathbb{R}^3))$.
The space $\mathcal{K}_T^q$ is equipped with the norm

$$\big\|u\big\|_{\mathcal{K}_T^q} := \sup_{0\le t\le T} t^{\frac{1}{2}(1-\frac{3}{q})}\big\|u(t,x)\big\|_q < \infty. \tag{4.2}$$

The space $\mathcal{K}_T^q$ was introduced by Weissler and systematically used by Kato [14] and Cannone [6].

In the following lemmas a particular attention will be devoted to study of the bilinear operator $B(u,v)(t)$ defined by

$$B(u,v)(t) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot\big(u\otimes v\big)\mathrm{d}s.$$

Lemma 4.4. *The bilinear operator B is bicontinuous from $\mathcal{K}_T^q \times \mathcal{K}_T^q$ to $\mathcal{K}_T^p$ for any $3 \le p < \dfrac{3q}{6-q}$ if $3 < q < 6$; any $3 \le p < \infty$ if $q = 6$; and $\dfrac{q}{2} \le p \le \infty$ if $6 < q < \infty$ and the following inequality holds*

$$\|B(u,v)\|_{\mathcal{K}_T^p} \le C\|u\|_{\mathcal{K}_T^q}\|v\|_{\mathcal{K}_T^q} \ \ \textit{for all } u,v\in\mathcal{K}_T^q$$

where C is a positive constant independent of T.

Proof. See [5, 7]. □

Lemma 4.5. *If $u_0 \in L^3(\mathbb{R}^3)$ then $e^{t\Delta}u_0 \in \mathcal{K}_\infty^q$ and $\big\|e^{t\Delta}u_0\big\|_{\mathcal{K}_\infty^q} \le \|u_0\|_3$ for all $q \in (3,\infty]$.*

Proof. See [5, 7]. □

Denote $E_T^q := \mathcal{K}_T^q \cap \mathcal{K}_T^\infty$ with $3 < q < \infty$, we have the following lemma

Lemma 4.6. *Let $6 < q < \infty$, $T > 0$. Then the bilinear operator B is bicontinuous from $E_T^q \times E_T^q$ to E_T^q and the following inequality holds*

$$\|B(u,v)\|_{E_T^q} \le C\ \|u\|_{E_T^q}\|v\|_{E_T^q} \ \ \textit{for all } u,v\in E_T^q, \tag{4.3}$$

where C is a positive constant independent of T.

Proof. Applying the Lemma 4.4, it follows that the bilinear operator B is bicontinuous from $\mathcal{K}_T^q\times\mathcal{K}_T^q$ to $\mathcal{K}_T^q$, from $\mathcal{K}_T^q\times\mathcal{K}_T^q$ to $\mathcal{K}_T^\infty$, and the following inequalities hold

$$\|B(u,v)\|_{K_T^q} \le C_1\|u\|_{K_T^q}\|v\|_{K_T^q} \le C_1\|u\|_{E_T^q}\|v\|_{E_T^q} \ \ \textit{for all } u,v\in E_T^q \tag{4.4}$$

and

$$\|B(u,v)\|_{K_T^\infty} \le C_2\|v\|_{K_T^q}\|u\|_{K_T^q} \le C_2\|v\|_{E_T^q}\|u\|_{E_T^q} \ \ \textit{for all } u,v\in E_T^q, \tag{4.5}$$

where C_1 and C_2 are positive constants independent of T. The estimate (4.3) is deduced from the inequalities (4.4) and (4.5). □

To prove main theorems, we define the auxiliary space G^α, $0 \le \alpha \le 1$ which is made up of the functions $w(x) = (w_1(x), w_2(x), w_3(x)) \in L^3(\mathbb{R}^3)$ such that

$$\sup_{t\ge 0} t^\alpha \big\| e^{t\Delta}|w| \big\|_3 < \infty$$

and

$$\lim_{t\to\infty} t^\alpha \big\| e^{t\Delta}|w| \big\|_3 = 0. \tag{4.6}$$

The norm of the space G^α is defined by

$$\big\|w\big\|_{G^\alpha} := \|w\|_3 + \sup_{t\ge 0} t^\alpha \big\| e^{t\Delta}|w| \big\|_3.$$

where

$$|w(x)| = \big(\sum_{i=1}^{3} w_i^2(x)\big)^{1/2}.$$

Lemma 4.7. *The space G^α is a Banach space which is invariable with translation in the sense that*

$$\big\|w(\cdot - x_0)\big\|_{G^\alpha} = \big\|w\big\|_{G^\alpha} \ \textit{for all} \ \ x_0 \in \mathbb{R}^3.$$

Proof. Firstly, we will show that G^α is a Banach space. Indeed, let $\{w_n\}_{n\ge 1}$ be a Cauchy sequence in G^α, for any $\epsilon > 0$ there exists a large enough N such that

$$\|w_n - w_m\|_3 + \sup_{t\ge 0} t^\alpha \big\| e^{t\Delta}|w_n - w_m| \big\|_3 \le \epsilon \text{ for all } n \ge N, m \ge N.$$

It follows from the above inequality that $\{w_n\}_{n\ge 1}$ is a Cauchy sequence in Banach space L^3, thus there exists $w_0 \in L^3$ such that

$$\lim_{t\to\infty} \big\|w_n - w_0\big\|_3 = 0.$$

We have

$$\Big| t^\alpha \big\| e^{t\Delta}|w_n - w_m| \big\|_3 - t^\alpha \big\| e^{t\Delta}|w_n - w_0| \big\|_3 \Big| \le t^\alpha \big\| e^{t\Delta}|w_n - w_m| - e^{t\Delta}|w_n - w_0| \big\|_3$$
$$\le t^\alpha \big\| e^{t\Delta}|w_m - w_0| \big\|_3 \le t^\alpha \big\|w_m - w_0\big\|_3, \text{ for all } m \ge N, n \ge N, \text{and } t \ge 0.$$

It follows from the above three estimates that

$$\lim_{m\to\infty} t^\alpha \big\| e^{t\Delta}|w_n - w_m| \big\|_3 = t^\alpha \big\| e^{t\Delta}|w_n - w_0| \big\|_3 \le \epsilon \text{ for all } n \ge N, t \ge 0.$$

Thus, we have

$$\sup_{t\ge 0} t^\alpha \big\| e^{t\Delta}|w_n - w_0| \big\|_3 \le \epsilon \text{ for all } n \ge N.$$

From the above inequality we have

$$\lim_{t\to\infty} \big\|w_n - w_0\big\|_{G^\alpha} = 0.$$

Let us now check the validity of condition (4.6) for w_0. For any $\epsilon > 0$ there exists a large enough N such that

$$t^\alpha \big\| e^{t\Delta}|w_n - w_0| \big\|_3 \le \frac{\epsilon}{2} \text{ for all } n \ge N, t \ge 0.$$

On the other hand, there exists a large enough $t_0 = t_0(N)$ such that

$$t^\alpha \big\| e^{t\Delta}|w_N| \big\|_3 \le \frac{\epsilon}{2}.$$

From the above two inequalities we have

$$t^\alpha \big\| e^{t\Delta}|w_0| \big\|_3 \le t^\alpha \big\| e^{t\Delta}|w_N - w_0| \big\|_3 + t^\alpha \big\| e^{t\Delta}|w_N| \big\|_3 \le \epsilon \text{ for all } t \ge t_0.$$

Finally, the property $\big\|w(\cdot - x_0)\big\|_{G^\alpha} = \big\|w\big\|_{G^\alpha}$ is derived from the fact that $e^{t\Delta}|w(\cdot - x_0)|(x) = (4\pi t)^{-\frac{3}{2}} e^{-\frac{|\cdot|^2}{4t}} * |w(\cdot - x_0)|(x) = (4\pi t)^{-\frac{3}{2}} e^{-\frac{|\cdot|^2}{4t}} * |w(\cdot)|(x - x_0)$ and $\big\|u(\cdot - x_0)\big\|_q = \big\|u\big\|_q$ for $u \in L^q, 1 \le q \le \infty$. □

Lemma 4.8. *Let $h \in L^1$ and $w \in G^\alpha$. Then, $h * w \in G^\alpha$ and*

$$\big\|h * w\big\|_{G^\alpha} \le \big\|h\big\|_1 \big\|w\big\|_{G^\alpha}.$$

Proof. Applying the Lemma 4.7, we deduce that

$$\big\|h * w\big\|_{G^\alpha} = \Big\| \int_{\mathbb{R}^3} h(y) w(\cdot - y) dy \Big\|_{G^\alpha} \le \int_{\mathbb{R}^3} |h(y)| \|w(\cdot - y)\|_{G^\alpha} dy$$
$$\le \int_{\mathbb{R}^3} |h(y)| \|w\|_{G^\alpha} dy = \big\|h\big\|_1 \big\|w\big\|_{G^\alpha}.$$

The Lemma is proved. □

Lemma 4.9. *Let $h \in L^\infty$ and $w \in G^\alpha$. Then, $hw \in G^\alpha$ and*

$$\big\|hw\big\|_{G^\alpha} \le \big\|h\big\|_\infty \big\|w\big\|_{G^\alpha}.$$

Proof. The proof is derived directly from the definition of the space G^α. □

We define auxiliary the space F_T^α, $0 \le \alpha < 1$, which is made up of the measured functions $u(t,x)$ such that

$$u(t) \in G^\alpha \text{ for all } t \in [0,T] \text{ and } \sup_{0 \le t \le T} \big\|u(t)\big\|_{G^\alpha} < \infty.$$

We have the following lemma
Lemma 4.10. *Let $p, \alpha, T \in \mathbb{R}$ be such that*

$$0 \le \alpha \le 1, 3 < q < \infty, T > 0.$$

Then the bilinear operator B is bicontinuous from $E_T^q \times F_T^\alpha$ to F_T^α and from $F_T^\alpha \times E_T^q$ to F_T^α and the following inequalities hold

$$\big\|B(u,v)(t)\big\|_{F_T^\alpha} \lesssim \big\|u\big\|_{E_T^q} \big\|v\big\|_{F_T^\alpha} \text{ for all } u \in E_T^q,\ v \in F_T^\alpha \tag{4.7}$$

and

$$\big\|B(u,v)(t)\big\|_{F_T^\alpha} \lesssim \big\|u\big\|_{F_T^\alpha} \big\|v\big\|_{E_T^q} \text{ for all } u \in F_T^\alpha,\ v \in E_T^q. \tag{4.8}$$

Proof. Applying Lemmas 3.4, 4.8, 4.9, and 4.2(a) in order to obtain

$$\begin{aligned}
&\big\|B(u,v)(t)\big\|_{G^\alpha} = \Big\|\int_0^t \frac{1}{(t-s)^2}\nabla K\Big(\frac{\cdot}{\sqrt{t-s}}\Big) * \big(u\otimes v\big)\mathrm{d}s\Big\|_{G^\alpha}\\
&\leq \int_0^t \frac{1}{(t-s)^2}\Big\|\nabla K\Big(\frac{\cdot}{\sqrt{t-s}}\Big) * \big(u\otimes v\big)\Big\|_{G^\alpha}\mathrm{d}s\\
&\leq \int_0^t \frac{1}{(t-s)^2}\Big\|\nabla K\Big(\frac{\cdot}{\sqrt{t-s}}\Big)\Big\|_1\big\|\big(u\otimes v\big)\big\|_{G^\alpha}\mathrm{d}s\\
&\leq \int_0^t (t-s)^{-\frac{1}{2}}\|\nabla K\|_1\big\|\big(u\otimes v\big)\big\|_{G^\alpha}\mathrm{d}s\\
&\leq \int_0^t (t-s)^{-\frac{1}{2}}\|\nabla K\|_1\big\|u(s)\big\|_\infty\big\|v(s)\big\|_{G^\alpha}\mathrm{d}s\\
&\lesssim \int_0^t (t-s)^{-\frac{1}{2}}\big\|u(s)\big\|_\infty\big\|v\big\|_{F_T^\alpha}\mathrm{d}s \quad \leq \ \big\|v\big\|_{F_T^\alpha}\int_0^t (t-s)^{-\frac{1}{2}}s^{-\frac{1}{2}}\big\|u\big\|_{\mathcal{K}_T^\infty}\mathrm{d}s\\
&= \big\|u\big\|_{\mathcal{K}_T^\infty}\big\|v\big\|_{F_T^\alpha}\int_0^t (t-s)^{-\frac{1}{2}}s^{-\frac{1}{2}}\mathrm{d}s = C_1\big\|u\big\|_{\mathcal{K}_T^\infty}\big\|v\big\|_{F_T^\alpha} \leq C_1\big\|u\big\|_{E_T^q}\big\|v\big\|_{F_T^\alpha}.
\end{aligned}$$

This prove the estimate (4.7). By an argument analogous similar to the previous one, we get the estimate (4.8). □

Lemma 4.11. *Let* $3 \leq q < \infty$*,* $T > 0$*, and* $0 \leq \alpha \leq 1$*. Then*
(a) *If* $u_0 \in L^3$ *then* $e^{t\Delta}u_0 \in E_T^q$ *and* $\|e^{t\Delta}u_0\|_{E_T^q} \lesssim \|e^{t\Delta}u_0\|_{K_T^q}$*.*
(b) *If* $u_0 \in G^\alpha$ *then* $e^{t\Delta}u_0 \in F_T^\alpha$ *and* $\big\|e^{t\Delta}u_0\big\|_{F_T^\alpha} \leq \|u_0\|_{G^\alpha}$*.*

Proof.
(a) We have by Lemma 4.5 that $e^{t\Delta}u_0 \in E_\infty^q$. It follows from Lemma 3.3 that

$$\begin{aligned}
\big\|e^{t\Delta}u_0\big\|_{E_T^q} &= \sup_{0\leq t\leq T} t^{\frac{1}{2}}\big\|e^{t\Delta}u_0\big\|_\infty + \sup_{0\leq t\leq T} t^{\frac{1}{2}(1-\frac{3}{q})}\big\|e^{t\Delta}u_0\big\|_q\\
&= \sup_{0\leq t\leq T} t^{\frac{1}{2}}\big\|e^{\frac{t}{2}\Delta}e^{\frac{t}{2}\Delta}u_0\big\|_\infty + \sup_{0\leq t\leq T} t^{\frac{1}{2}(1-\frac{3}{q})}\big\|e^{t\Delta}u_0\big\|_q\\
&\leq C_0\sup_{0\leq t\leq T} t^{\frac{1}{2}(1-\frac{3}{q})}\big\|e^{\frac{t}{2}\Delta}u_0\big\|_q + \sup_{0\leq t\leq T} t^{\frac{1}{2}(1-\frac{3}{q})}\big\|e^{t\Delta}u_0\big\|_q\\
&\leq C_1\sup_{0\leq t\leq T} t^{\frac{1}{2}(1-\frac{3}{q})}\big\|e^{t\Delta}u_0\big\|_q + \sup_{0\leq t\leq T} t^{\frac{1}{2}(1-\frac{3}{q})}\big\|e^{t\Delta}u_0\big\|_q = C_2\|e^{t\Delta}u_0\|_{K_T^q}.
\end{aligned}$$

(b) Using Lemma 4.8, we have

$$\big\|e^{t\Delta}u_0\big\|_{G^\alpha} = \Big\|\frac{1}{(4\pi t)^{\frac{3}{2}}}e^{-\frac{|.|^2}{4t}} * u_0\Big\|_{G^\alpha} \leq \Big\|\frac{1}{(4\pi t)^{\frac{3}{2}}}e^{-\frac{|.|^2}{4t}}\Big\|_1\big\|u_0\big\|_{G^\alpha} = \big\|u_0\big\|_{G^\alpha}.$$

This prove (b). □

Lemma 4.12. *Let* $p, \alpha, T \in \mathbb{R}$ *be such that*

$$6 < q < 12, 0 \leq \alpha \leq 1, \text{ and } T > 0.$$

Then there exists a positive constant $C = C(q, \alpha)$ such that for all $u_0 \in G^\alpha$ with $\operatorname{div}(u_0) = 0$ *satisfying*

$$\sup_{0\le t\le T} t^{\frac{1}{2}(1-\frac{3}{q})} \left\| e^{t\Delta} u_0 \right\|_q < C, \tag{4.9}$$

the Navier-Stokes equations (1.1) *have a solution* $u \in E_T^q \cap F_T^\alpha \cap BC\big([0,T); L^3(\mathbb{R}^3)\big)$. *In particular, for arbitrary $u_0 \in L^3$, there exists $T = T(u_0)$ small enough such that the inequality* (4.9) *holds.*

Proof. Combining Lemma 4.3 with $E = E_T^q$, $F = F_T^\alpha$, Lemma 4.10, Lemma 4.11, it follows that there exists a positive constant C such that if the following inequality

$$\|u_0\|_{K_T^q} = \sup_{0\le t\le T} t^{\frac{1}{2}(1-\frac{3}{q})} \left\| e^{t\Delta} u_0 \right\|_q < C$$

holds then the Navier-Stokes equations (1.1) has a solution $u \in E_T^q \cap F_T^\alpha$. It follows from Lemma 4.4 that $B(u,u) \in \mathcal{K}_T^{\frac{q}{2}}$, we have by Lemma 4.5 that $e^{t\Delta}u_0 \in \mathcal{K}_T^{\frac{q}{2}}$ and therefore $u = e^{\cdot\Delta}u_0 + B(u,u) \in \mathcal{K}_T^{\frac{q}{2}}$. Applying again Lemma 4.4, we obtain $B(u,u) \in \mathcal{K}_T^3 \subseteq BC\big([0,T); L^3(\mathbb{R}^3)\big)$ and therefore $u \in BC\big([0,T); L^3(\mathbb{R}^3)\big)$. The uniqueness of u is deduced from Theorem 4.1. Now we show that the condition (4.9) is valid when T is small enough. Indeed, from the definition of K_T^q, we deduce that the left-hand side of the inequality (4.9) converges to 0 when T tends to 0. Therefore the condition (4.9) holds for arbitrary $u_0 \in L^3$ when $T = T(u_0)$ is small enough.

□

Lemma 4.13. *If $u \in C\big([0,\infty); L^3(\mathbb{R}^3)\big)$ is a mild solution of the Navier-Stokes equations* (1.1) *with the initial value $u_0 \in G^\alpha$, then $u(t) \in G^\alpha$ for all $t > 0$.*

Proof. Applying Lemma 4.12 and Lemma 4.5, there exists a strong solution v of the Navier-Stokes equations (1.1) with the initial value u_0 on some interval $[0, T']$, where $T' > 0$, so that $v(t) \in G^\alpha$ for $t \in [0, T']$. Using Theorem 4.1, we obtain that $u = v$ on $[0, T']$, and so $u(t) \in G^\alpha$ for $t \in [0, T']$. Let

$$T^* = \inf\{t > 0 : u(t) \notin G^\alpha\}.$$

Then $0 < T^* \le \infty$ and $u(t) \in G^\alpha$ for $t \in [0, T^*)$, we only need prove that $T^* = \infty$. Suppose that $T^* < \infty$. Let C be the constant in Lemma 4.12, we have by continuity that there exists $\delta_1 > 0$ such that

$$\left\| u(t) - u(T^*) \right\|_3 \le \frac{C}{2}, \text{ for all } t \in [T^* - \delta_1, T^*].$$

Let $6 < q < 12$, we take a number positive δ_2 enough small such that

$$\sup_{0<t\le \delta_2} t^{\frac{1}{2}(1-\frac{3}{q})} \left\| e^{t\Delta} u(T^*) \right\|_q \le \frac{C}{2}.$$

Let $\delta = \dfrac{1}{2}\min\{\delta_1, \delta_2\}$. It follows from the above two inequalities that

$$\sup_{0<t\le 2\delta} t^{\frac{1}{2}(1-\frac{3}{q})}\big\|e^{t\Delta}u(T^*-\delta)\big\|_q$$
$$\le \sup_{0<t\le 2\delta} t^{\frac{1}{2}(1-\frac{3}{q})}\big\|e^{t\Delta}u(T^*)\big\|_q + \sup_{0<t\le 2\delta} t^{\frac{1}{2}(1-\frac{3}{q})}\big\|e^{t\Delta}\big(u(T^*)-u(T^*-\delta)\big)\big\|_q$$
$$\le \sup_{0<t\le \delta_2} t^{\frac{1}{2}(1-\frac{3}{q})}\big\|e^{t\Delta}u(T^*)\big\|_q + \big\|u(T^*)-u(T^*-\delta)\big\|_3 \le C.$$

From the above inequality, applying Lemma 4.12 and Theorem 4.1, we obtain $u(t) \in G^\alpha$ for all $t \in [T^*-\delta, T^*+\delta]$, which constitutes a contradiction. This completes the proof of Lemma. □

We define the auxiliary space Q^α, $0 \le \alpha \le 1$, which is made up of the measured functions $u(t,x)$ such that

$$\sup_{t\ge 0} t^\alpha \|u(t)\|_3 < \infty$$

and

$$\lim_{t\to\infty} t^\alpha \big\|u(t)\big\|_3 = 0. \tag{4.10}$$

We have the following lemma

Lemma 4.14. *Let $p, \alpha \in \mathbb{R}$ be such that*

$$0 \le \alpha < 1, \max\{0, \frac{2}{3}\big(\alpha - \frac{1}{2}\big)\} < \frac{1}{q} < \frac{1}{3}.$$

Then the bilinear operator B is bicontinuous from $\mathcal{K}^q_\infty \times \mathcal{K}^q_\infty$ to $\mathcal{K}^q_\infty$, from $\mathcal{K}^q_\infty \times Q^\alpha$ to Q^α, from $Q^\alpha \times \mathcal{K}^q_\infty$ to Q^α, and the following estimates hold

$$\big\|B(u,v)(t)\big\|_{\mathcal{K}^q_\infty} \le \gamma\, \big\|u\big\|_{\mathcal{K}^q_\infty}\big\|v\big\|_{\mathcal{K}^q_\infty} \text{ for all } u,\ v \in \mathcal{K}^q_\infty, \tag{4.11}$$

$$\big\|B(u,v)(t)\big\|_{Q^\alpha} \le \gamma\, \big\|u\big\|_{\mathcal{K}^q_\infty}\big\|v\big\|_{Q^\alpha} \text{ for all } u \in \mathcal{K}^q_\infty,\ v \in Q^\alpha, \tag{4.12}$$

$$\big\|B(u,v)(t)\big\|_{Q^\alpha} \le \gamma\, \big\|u\big\|_{Q^\alpha}\big\|v\big\|_{\mathcal{K}^q_\infty} \text{ for all } u \in Q^\alpha,\ v \in \mathcal{K}^q_\infty, \tag{4.13}$$

where γ is a positive constant.

Proof. The estimate (4.11) is deduced from Lemma 4.4. Applying Lemma 4.1, Hölder inequality, and Lemma 4.2(a) in order to obtain

$$\begin{aligned}
\big\|B(u,v)(t)\big\|_3 &\le \int_0^t (t-s)^{-\frac{1}{2}-\frac{3}{2q}}\big\|u\otimes v\big\|_{\frac{1}{\frac{1}{3}+\frac{1}{q}}}\mathrm{d}s \\
&\le \int_0^t (t-s)^{-\frac{1}{2}-\frac{3}{2q}}\|u\|_q\|v\|_3\mathrm{d}s \\
&\le \int_0^t (t-s)^{-\frac{1}{2}-\frac{3}{2q}}s^{-\alpha-\frac{1}{2}\left(1-\frac{3}{q}\right)}s^{\frac{1}{2}(1-\frac{3}{q})}\big\|u(s)\big\|_q s^\alpha\big\|v(s)\big\|_3\mathrm{d}s \\
&\le \big\|u\big\|_{\mathcal{K}^q_\infty}\big\|v\big\|_{Q^\alpha}\int_0^t (t-s)^{-\frac{1}{2}-\frac{3}{2q}}\, s^{-\alpha-\frac{1}{2}\left(1-\frac{3}{q}\right)}\mathrm{d}s \\
&= C_1 t^{-\alpha}\big\|u\big\|_{\mathcal{K}^q_\infty}\big\|v\big\|_{Q^\alpha}.
\end{aligned}$$

The estimate (4.12) is deduced from the above inequality. Let us now check the validity of the condition (4.10) for the bilinear term B. Indeed, we have by the above estimate and the change variable of the variable $s = t\tau$ that

$$t^\alpha \big\|B(u,v)(t)\big\|_3 \le t^\alpha \int_0^t (t-s)^{-\frac{1}{2}-\frac{3}{2q}} \big\|u(s)\big\|_q \big\|v(s)\big\|_3 \mathrm{d}s$$

$$\le t^\alpha \int_0^t (t-s)^{-\frac{1}{2}-\frac{3}{2q}} s^{-\frac{1}{2}(1-\frac{3}{q})} s^{\frac{1}{2}(1-\frac{3}{q})} \big\|u(s)\big\|_q \big\|v(s)\big\|_3 \mathrm{d}s$$

$$\le t^\alpha \big\|u\big\|_{\mathcal{K}^q_\infty} \int_0^t (t-s)^{-\frac{1}{2}-\frac{3}{2q}} s^{-\frac{1}{2}(1-\frac{3}{q})} \big\|v(s)\big\|_3 \mathrm{d}s$$

$$= \big\|u\big\|_{\mathcal{K}^q_\infty} \int_0^1 (1-\tau)^{-\frac{1}{2}-\frac{3}{2q}} \tau^{-\alpha-\frac{1}{2}\left(1-\frac{3}{q}\right)} (t\tau)^\alpha \big\|v(t\tau)\big\|_3 \mathrm{d}\tau.$$

From $\lim\limits_{t\to\infty} (t\tau)^\alpha \big\|v(t\tau)\big\|_3 = 0$ for all $\tau > 0$, applying Lebesgue's convergence theorem, we deduce that

$$\lim_{t\to\infty} t^\alpha \big\|B(u,v)(t)\big\|_3 = 0.$$

By an argument analogous similar to the previous one, we get estimate (4.13). □

Lemma 4.15. *Let $p, \alpha \in \mathbb{R}$ be such that*

$$0 \le \alpha < 1, \max\{0, \frac{2}{3}\big(\alpha - \frac{1}{2}\big)\} < \frac{1}{q} < \frac{1}{3}.$$

Then there exists a positive constant $C = C(q,\alpha)$ such that for all $u_0 \in G^\alpha$ satisfying

$$\sup_{0\le t\le\infty} t^{\frac{1}{2}(1-\frac{3}{q})} \big\|e^{t\Delta}u_0\big\|_q < C, \tag{4.14}$$

the Navier-Stokes equations (1.1) *has a solution $u \in K^q_\infty \cap Q^\alpha \cap BC\big([0,\infty); L^3(\mathbb{R}^3)\big)$.*

Proof. It is easy to see that $e^{t\Delta}u_0 \in Q^\alpha$. Combining Lemma 4.3, Lemma 4.14 with $E = K^q_\infty$ and $F = G^\alpha$, it follows that there exists a positive constant C such that if the following inequality

$$\|u_0\|_{K^q_\infty} = \sup_{0\le t\le\infty} t^{\frac{1}{2}(1-\frac{3}{q})} \big\|e^{t\Delta}u_0\big\|_q < C$$

holds then the Navier-Stokes equations (1.1) has a solution $u \in K^q_\infty \cap Q^\alpha$. By applying Lemma 4.4, Lemma 4.5, and using an argument similar to that of the proof of Lemma 4.12, we obtain $u \in BC\big([0,T); L^3(\mathbb{R}^3)\big)$. The uniqueness of u is deduced from Theorem 4.1. □

Proof of Theorem 2.1

Proof. (a) It is easy to see that $u_0 \in G^\alpha$. Using Lemma 4.13, we get $u(t) \in G^\alpha$ for all $t \ge 0$. Since Theorem 4.2 and Lemma 4.5, it follows that there exists a positive value t_0 large enough such that the condition

$$\sup_{0\le t\le\infty} t^{\frac{1}{2}(1-\frac{3}{q})} \big\|e^{t\Delta}u(t_0)\big\|_q < C$$

holds. Applying Lemma 4.15 we obtain the strong solution v of the Navier-Stokes equations (1.1) with the initial value $u(t_0)$ on the interval $[0,\infty)$ satisfying $v \in K^q_\infty \cap Q^\alpha \cap BC\big([0,\infty); L^3(\mathbb{R}^3)\big)$ and therefore $\big\|v(t)\big\|_3 = o(t^{-\alpha})$. Using Theorem

4.1, we obtain that $u(t)=v(t-t_0)$ for all $t\in[t_0,\infty)$ and so $\big\|u(t)\big\|_3=o(t^{-\alpha})$.
(b) We consider two cases $0\le\alpha<1$ and $\alpha=1$ separately. The proof of the case $0\le\alpha<1$ is deduced from the part (a), we only consider the case $\alpha=1$. Now, we prove that there exists a positive number t_0 large enough such that

$$\nabla u(t)\in L^3 \ for\ t\geqslant t_0 \text{ and } \big\|\nabla u(t)\big\|_3=O(t^{-\frac{5}{4}}). \tag{4.15}$$

Indeed, using Theorems 4.2, 4.3, and 4.1 it follows that there exists $t_1>0$ larger enough such that

$$\|\nabla u(t)\|_6\lesssim t^{-\frac{3}{4}} \quad\text{for } t\ge t_1. \tag{4.16}$$

Applying Theorem 2.1(a) for the case $0\le\alpha<1$ we have

$$\|u(t)\|_3\lesssim(1+t)^{-\frac{3}{4}} \text{ for } t>0. \tag{4.17}$$

For $t\ge 2t_1$ we have

$$\begin{aligned}\big\|\nabla B(u,u)(t)\big\|_3&=\Big\|\nabla\int_0^t e^{(t-s)\Delta}\mathbb{P}(u\cdot\nabla u)\mathrm{d}s\Big\|_3\le\int_0^t\big\|\nabla e^{(t-s)\Delta}\mathbb{P}(u\cdot\nabla u)\big\|_3\mathrm{d}s\\&=\int_{\frac{t}{2}}^t\big\|\nabla e^{(t-s)\Delta}\mathbb{P}(u\cdot\nabla u)\big\|_3\mathrm{d}s+\int_0^{\frac{t}{2}}\big\|\nabla e^{(t-s)\Delta}\mathbb{P}(u\cdot\nabla u)\big\|_3\mathrm{d}s. \end{aligned}\tag{4.18}$$

Firstly, we estimate the first term on the right-hand side of equation (4.18), applying Lemma 4.1, Holder inequality, inequality (4.16), inequality (4.17), and Lemma 4.2(c) in order to obtain

$$\begin{aligned}\int_{\frac{t}{2}}^t\big\|\nabla e^{(t-s)\Delta}\mathbb{P}(u\cdot\nabla u)\big\|_3\mathrm{d}s&\lesssim\int_{\frac{t}{2}}^t(t-s)^{-\frac{3}{4}}\big\|u\cdot\nabla u\big\|_2\mathrm{d}s\\\le\int_{\frac{t}{2}}^t(t-s)^{-\frac{3}{4}}\big\|u\big\|_3\big\|\nabla u\big\|_6\mathrm{d}s&\lesssim\int_{\frac{t}{2}}^t(t-s)^{-\frac{3}{4}}s^{-\frac{3}{2}}\mathrm{d}s=C_3t^{-\frac{5}{4}}. \end{aligned}\tag{4.19}$$

Secondly, we estimate the second term on the right-hand side of the equation (4.18), applying Lemma 4.1, Holder inequality, inequality (4.17) in order to obtain

$$\begin{aligned}\int_0^{\frac{t}{2}}\big\|\nabla e^{(t-s)\Delta}\mathbb{P}(u\cdot\nabla u)\big\|_3\mathrm{d}s&=\int_0^{\frac{t}{2}}\big\|\nabla^2 e^{(t-s)\Delta}\mathbb{P}(u\otimes u)\big\|_3\mathrm{d}s\\&\lesssim\int_0^{\frac{t}{2}}(t-s)^{-\frac{3}{2}}\|u\otimes u\|_{\frac{3}{2}}\mathrm{d}s\le\int_0^{\frac{t}{2}}(t-s)^{-\frac{3}{2}}\big\|u\big\|_3^2\mathrm{d}s\\&\le\Big(\frac{t}{2}\Big)^{-\frac{3}{2}}\int_0^{\frac{t}{2}}(1+s)^{-\frac{3}{2}}\mathrm{d}s\lesssim t^{-\frac{3}{2}}. \end{aligned}\tag{4.20}$$

Combining the inequalities (4.19) and (4.20), we obtain

$$\big\|\nabla B(u,u)(t)\big\|_3=\Big\|\nabla\int_0^t e^{(t-s)\Delta}\mathbb{P}(u\cdot\nabla u)\mathrm{d}s\Big\|_3\lesssim t^{-\frac{5}{4}} \text{ for } t\ge 2t_0. \tag{4.21}$$

On the other hand, we have

$$\big\|\nabla e^{t\Delta}u_0\big\|_3=\big\|\nabla e^{\frac{t}{2}\Delta}e^{\frac{t}{2}\Delta}u_0\big\|_3\lesssim t^{-\frac{1}{2}}\big\|e^{\frac{t}{2}\Delta}u_0\big\|_3\le t^{-\frac{1}{2}}\big\|e^{\frac{t}{2}\Delta}|u_0|\big\|_3\le t^{-\frac{3}{2}}. \tag{4.22}$$

The property (4.15) is deduced from the inequalities (4.21) and (4.22) with $t_0=2t_1$. Now we come back to prove Theorem 2.1(b) for the case $\alpha=1$, we have

$$\big\|B(u,u)\big\|_3 = \Big\| \int_0^t e^{(t-s)\Delta}\mathbb{P}(u\cdot\nabla u)\mathrm{d}s\Big\|_3$$

$$\leq \int_0^{\frac{t}{2}} \big\|\nabla e^{(t-s)\Delta}\mathbb{P}(u\otimes u)\mathrm{d}s\big\|_3 + \int_{\frac{t}{2}}^t \big\|e^{(t-s)\Delta}\mathbb{P}(u\cdot\nabla u)\big\|_3\mathrm{d}s. \tag{4.23}$$

Firstly, we estimate the first term on the right-hand side of equation (4.23). Applying Lemma 4.1, Holder inequality, inequality (4.17), and Lemma 4.2(c) in order to obtain

$$\int_0^{\frac{t}{2}} \big\|\nabla e^{(t-s)\Delta}\mathbb{P}(u\otimes u)\big\|_3\mathrm{d}s \lesssim \int_0^{\frac{t}{2}} (t-s)^{-1}\big\|u\otimes u\big\|_{\frac{3}{2}}\mathrm{d}s \leq \int_0^{\frac{t}{2}}(t-s)^{-1}\big\|u\big\|_3^2\mathrm{d}s$$

$$\leq \Big(\frac{t}{2}\Big)^{-1}\int_0^{\frac{t}{2}}(1+s)^{-\frac{3}{2}}\mathrm{d}s \lesssim t^{-1}. \tag{4.24}$$

Secondly, we estimate the second term on the right-hand side of the equation (4.23). Applying Lemma 4.1, Holder inequality, inequality (4.15), inequality (4.17), Lemma 4.2(c) in order to obtain

$$\int_{\frac{t}{2}}^t \big\|e^{(t-s)\Delta}\mathbb{P}(u\cdot\nabla u)\big\|_3\mathrm{d}s \lesssim \int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2}}\|u\cdot\nabla u\|_{\frac{3}{2}}\mathrm{d}s$$

$$\leq \int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2}}\|u\|_3\|\nabla u\|_3\mathrm{d}s \lesssim \int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2}}s^{-2}\mathrm{d}s \lesssim t^{-\frac{3}{2}} \text{ for } \mathrm{t}\geq 2\mathrm{t}_0. \tag{4.25}$$

Combining the inequalities (4.24) and (4.25), we obtain

$$\big\|B(u,u)(t)\big\|_3 = \Big\|\int_0^t e^{(t-s)\Delta}\mathbb{P}(u\cdot\nabla u)\mathrm{d}s\Big\|_3 \lesssim t^{-1} \text{ for } t\geq 2t_0$$

and therefore $\big\|u(t)\big\|_3 = O(t^{-1})$.
(c) To prove this part, we need this following lemma

Lemma 4.16. *Suppose that $u_0\in L^{p,r}(\mathbb{R}^3)$ with $1\leq p\leq\infty$ and $1\leq r<\infty$. Then $\lim_{n\to\infty}\big\|\mathcal{X}_n u_0\big\|_{L^{p,r}} = 0$, where $\mathcal{X}_n(x)=0$ for $x\in\{x:\ |x|<n\}\cap\{x:\big|u_0(x)\big|<n\}$, and $\mathcal{X}_n(x)=1$ otherwise.*

Proof. See [16]. □

We only need to prove

$$\lim_{t\to\infty} t^{\frac{1}{2}\left(\frac{3}{p}-1\right)}\big\|e^{t\Delta}|u_0|\big\|_3 = 0.$$

We have

$$t^{\frac{1}{2}\left(\frac{3}{p}-1\right)}\|e^{t\Delta}|u_0|\|_3 \leq t^{\frac{1}{2}\left(\frac{3}{p}-1\right)}\big\|e^{t\Delta}(\mathcal{X}_n|u_0|)\big\|_3 + t^{\frac{1}{2}\left(\frac{3}{p}-1\right)}\big\|e^{t\Delta}\big((1-\mathcal{X}_n)|u_0|\big)\big\|_3$$

$$\leq \frac{t^{\frac{3}{2p}-2}}{(4\pi)^{3/2}}\big\|e^{\frac{-|.|^2}{4t}} * (\mathcal{X}_n|u_0|)\big\|_3 + \frac{t^{\frac{3}{2p}-2}}{(4\pi)^{3/2}}\big\|e^{\frac{-|.|^2}{4t}} * \big((1-\mathcal{X}_n)|u_0|\big)\big\|_3. \tag{4.26}$$

For any $\varepsilon > 0$, applying Lemma 3.1 and Lemma 4.16, we have

$$\frac{t^{\frac{3}{2p}-2}}{(4\pi)^{3/2}}\left\|e^{\frac{-|.|^2}{4t}} * (\mathcal{X}_n|u_0|)\right\|_3 \leq \frac{t^{\frac{3}{2p}-2}}{(4\pi)^{3/2}}\left\|e^{\frac{-|.|^2}{4t}} * (\mathcal{X}_n|u_0|)\right\|_{L^{3,1}}$$

$$\leq \frac{t^{\frac{3}{2p}-2}}{(4\pi)^{3/2}}\left\|e^{\frac{-|.|^2}{4t}}\right\|_{L^{\frac{3p}{4p-3},\frac{r}{r-1}}}\left\|\mathcal{X}_n|u_0|\right\|_{L^{p,r}} = \frac{1}{(4\pi)^{3/2}}\left\|e^{\frac{-|.|^2}{4}}\right\|_{L^{\frac{3p}{4p-3},\frac{r}{r-1}}}\left\|\mathcal{X}_n|u_0|\right\|_{L^{p,r}}$$

$$= C\left\|\mathcal{X}_n|u_0|\right\|_{L^{p,r}} < \frac{\varepsilon}{2} \tag{4.27}$$

for large enough n. Fixed one of such n, let p^* be such that $1 < p^* < p$, applying Lemma 3.1 we get

$$\frac{t^{\frac{3}{2p}-2}}{(4\pi)^{3/2}}\left\|e^{\frac{-|.|^2}{4t}} * \left((1-\mathcal{X}_n)|u_0|\right)\right\|_3 \leq \frac{t^{\frac{3}{2p}-2}}{(4\pi)^{3/2}}\left\|e^{\frac{-|.|^2}{4t}}\right\|_{L^{\frac{3p^*}{4p^*-3},\frac{p^*}{p^*-1}}}\left\|(1-\mathcal{X}_n)|u_0|\right\|_{L^{p^*,r}}$$

$$= t^{\frac{3}{2}\left(\frac{1}{p}-\frac{1}{p^*}\right)}\frac{1}{(4\pi)^{3/2}}\left\|e^{\frac{-|.|^2}{4}}\right\|_{L^{\frac{3p^*}{4p^*-3},\frac{p^*}{p^*-1}}}\left\|(1-\mathcal{X}_n)|u_0|\right\|_{L^{p^*,r}}$$

$$\leq C_1 t^{\frac{3}{2}\left(\frac{1}{p}-\frac{1}{p^*}\right)}\left\|n(1-\mathcal{X}_n)\right\|_{L^{p^*,r}} = C_2(n)t^{\frac{3}{2}\left(\frac{1}{p}-\frac{1}{p^*}\right)} < \frac{\varepsilon}{2} \tag{4.28}$$

for all $t > t^*$ with $t^* = \left(\frac{\varepsilon}{2C_2(n)}\right)^{\frac{2pp*}{3(p^*-p)}}$. From the inequalities (4.26), (4.27), and (4.28) we have

$$t^{\frac{1}{2}\left(\frac{3}{p}-1\right)}\|e^{t\Delta}|u_0|\|_3 < \varepsilon \text{ for all } t > t^*.$$

(d) From Lemma 3.2, the two quantities $\left\||u_0|\right\|_{\dot{B}_3^{-2\alpha,\infty}(\mathbb{R}^3)}$ and $\sup_{t\geq 0} t^\alpha\left\|e^{t\Delta}|u_0|\right\|_3$ are equivalent. This prove (d). □

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