

On the convergence of the model of migration of competing species in multizone environment

Doanh Nguyen-Ngoc¹, Kévin Perrot², Thi Ha Duong Phan^{3*}

¹ *JEAI WARM team, UMMISCO, IRD/Sorbonnes University, France and
Thuyloi University, 175 Tay Son, Dong Da, Hanoi, Vietnam.*

² *Aix Marseille Univ, Univ. de Toulon, CNRS, LIS, UMR 7020, Marseille, France and
Univ. Côte d'Azur, CNRS, I3S, UMR 7271, Sophia Antipolis, France.*

³ *Institute of Mathematics, Vietnam Academy of Science and Technology,
18 Hoang Quoc Viet Road, Hanoi, Vietnam.*

* *Corresponding author: phanhaduong@math.ac.vn*

November 10, 2019

Abstract. Many factors govern the relationship between migration of species individuals among multizone environment, and their distribution over time. In order to understand the effect of species densities on the migration, we propose a general discrete model for migration of many competing species over multizones. We prove that under some assumptions the migration tactics of species lead to an exponentially quick convergence of the system to a stable distribution. Because the number of stable distribution forms is very large, we provide a characterization of the initial conditions under which the system converges to each distribution. At the end, we give a method to calculate the probability of reaching each stable distribution form, applied to the case of three species on three zones.

Keywords: competition, migration, distribution, multizone environment, configuration.

1 Introduction

An important issue in ecology is to understand the effects of the tactics that individuals may adopt at the population and community levels. Individuals migrate because the food is limited, they compete with others, environmental conditions are not good for them (weather, natural calamity, etc) and so on. This leads to various portraits of distribution of species over the environment.

There was also a lot of interest in the relationship between migration of species individuals among multizone environment and distribution species. One of the most common and simple theoretical explanation for effects of individuals' migration on the species distribution is ideal free distribution (IFD) theory. The theory states that the number of individuals that will aggregate (or else clump) in various zones is proportional to the amount of resources available in each. In IFD theory, we accept the prediction that the distribution of individuals on multizone environment will minimize resource competition and maximize fitness ([7, 6, 24, 9, 10]).

Some recent investigations studied another factor leading to individuals' migration and also showed the link between the migration and the species distribution over multizone environment. These investigations showed that when migrations of individuals do not depend on species densities, i.e. migration of species over zones is characterized by numbers which are called constant migration rates, then species distribution over a given zone is proportional to migration rate of the species to that zone ([2, 13, 14, 15]). When migration of individuals depend on species densities then species distribution over a given zone can be represented as a function of species densities. Dependence of individuals' migration on species densities were obviously explained and well presented in the previous studies ([2, 17, 18, 1, 5, 12, 11]). In these studies, the authors assumed that individuals of one species (a prey or an inferior competitor) are likely to leave the patch if there are too many individuals of the others (a predator or a superior competitor, respectively). In this case, one species tried to avoid the other species and this led to the fact that the species (prey or a inferior competitor) distribution over a zone depends on the density of the other species (a predator or a superior competitor, respectively). However, in these previous models, only two species and two zones were considered. There were very few studies involving three species and three zones (see [3]). This study only concerned to independence of migration of individuals on species densities. The aim of this work is to take into account dependence of migration of individuals on species densities in a multi competing species system.

We propose a discrete model approach: time is discrete, and density is continuous. We consider the general case: there are p competing species for territory among m zones. At each time step, we examine the picture in each zone, and then aggregate the zones to get the whole picture. We assume that over each zone individual of a species try to avoid the others species in the sense that if the density of the species is smaller than those of the others then their individuals are likely to leave that zone. The raised question is "*what is the stable distribution of the p species among m zones?*" There is no simple answer to this question. We show in this paper that it depends on migration tactics of individuals as well as the initial distribution of species.

As a first attempt, we are interested in the effects of individuals' migration on the distribution of species over zones, we therefore do not take into account the demographic process. We try to keep important assumptions and predictions of ideal free distribution system, such as: all individuals are competitively equal, so they are all equally able to forage and choose the destination zones to move; individuals are free to move to the chosen zones. However, this can be violated by dominant individuals within a species who may keep a weaker individual from reaching the destination zone. A population of individuals will distribute themselves equally among zones with the same probability. Moreover, in order to carry out the analysis for this general case, we have other following assumption: species have constant amount of individuals. Although this assumption is quite strong, we think it is reasonable because in many models [3, 19] when considering multi time scales, authors usually distinguish two time scales: the fast time scale for the migration dynamics and the slow time scale for the demographic dynamics. So in the fast time scale, we may assume that species have constant amount of individuals.

With these assumptions, individuals can move continuously, but it is surprising that the system will always converge to a stable distribution. The stable distribution here is understood as the set of densities of species in each zone. It is worth noting that in the density stability, there may still be migration of some individuals, although they do not change the picture of density because for the same species on the same zone, there will be individuals coming and individuals going. Our analyzes are based on detailed combinatorial calculations. It also helps us to determine the number of all possible stable distributions, and most importantly, to prove that the system will converge in

a very short time, in an exponential way.

One of the next big issues is how to compute the probability for the system reaching a given stable distribution. This is a complex problem, requiring a lot of calculations. At the end of this paper, we will show this problem for the model of three species and three zones, the calculations are based on calculation of simplexes.

The paper is organized as follows. Section 2 is dedicated to a presentation of the model. Section 3 analyzes all possible stable configuration forms of the model and Section 4 proves that the model will converge exponentially to one of these stable configurations. In Section 5, we will accurately calculate the probability for the system reaching a given stable configuration. Finally, Section 6 is about a short discussion and conclusion. Further, we give two appendices on the probability computation and simulation.

2 Presentation of the model

The system evolves in discrete time and continuous space. We consider the case of p species $\mathcal{S} = \{1, \dots, p\}$ and m zones $\mathcal{Z} = \{1, \dots, m\}$. We call *configuration* at time t a distribution of the individuals of each species into the m zones, composed of a density $n_{qi}(t)$ of individuals of species $q \in \mathcal{S}$ in zone $i \in \mathcal{Z}$ at time t , such that for every species $q \in \mathcal{S}$ we have $\sum_{i \in \mathcal{Z}} n_{qi}(t) = 1$. Formally, a configuration is determined by its *density matrix*:

$$\mathbf{n}(t) = \begin{pmatrix} n_{11} & \dots & n_{1m} \\ \vdots & & \vdots \\ n_{p1} & \dots & n_{pm} \end{pmatrix}.$$

If there is no ambiguity, we will usually omit the dependency on the time t and simply denote \mathbf{n} instead of $\mathbf{n}(t)$. The set of configurations is denoted by \mathcal{C} . To describe the dynamics of the system, we introduce some definitions as follows.

Definition 1. *In a configuration, a species dominates a zone when its density in this zone is strictly greater than the densities of all other species in this zone. Formally, let us denote $q \text{DOM}(t)$ if when for all $q' \in \mathcal{S} \setminus \{q\}$ we have $n_{qi}(t) > n_{q'i}(t)$, and $q \overline{\text{DOM}}(t)$ otherwise.*

Definition 2. *The evolution rule is that if a species dominates a zone at time t then those individuals stay into this zone in the next time step ($t + \Delta t$), and if a species does not dominate a zone then in the next time step they split evenly into the $m - 1$ other zones.*

For a configuration $c(t)$ at time t , we denote by $c(t + k\Delta t)$ the configuration obtained from $c(t)$ after k time steps.

Definition 3. *A stable configuration is a configuration such that its density matrix does not change over time (though individuals may move).*

Definitions are illustrated on Figure 1.

3 Stable configurations

In this section, we study the form of stable configurations. We will describe them explicitly (Theorem 1), then calculate the number of possible stable configuration forms (Corollary 1).

$$\begin{array}{ccc}
\begin{pmatrix} \underline{0.9} & 0.1 & 0 & 0 & 0 \\ 0.08 & \underline{0.35} & \underline{0.21} & \underline{0.35} & 0 \\ 0.8 & 0.16 & 0 & 0 & \underline{0.04} \\ 0.3 & 0.2 & 0.2 & 0.3 & 0 \end{pmatrix} & & \begin{pmatrix} \underline{0.925} & 0 & 0.025 & 0.025 & 0.025 \\ 0 & \underline{0.37} & 0.23 & \underline{0.38} & 0.02 \\ 0.04 & 0.2 & \underline{0.24} & 0.24 & \underline{0.28} \\ 0.175 & 0.2 & 0.2 & 0.175 & 0.25 \end{pmatrix} \\
\mathbf{n}(t) & & \mathbf{n}(t + \Delta t) \\
\\
\begin{pmatrix} \frac{1}{5} & 0 & 0 & 0 & 0 \\ 0 & \underline{0.495} & 0 & \underline{0.505} & 0 \\ 0 & 0 & \underline{0.48} & 0 & \underline{0.52} \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \end{pmatrix} & & \\
\lim_{k \rightarrow \infty} \mathbf{n}(t + k\Delta t) & &
\end{array}$$

Figure 1: Example of evolution step with $p = 4$ species and $m = 5$ zones, from $\mathbf{n}(t)$ to $\mathbf{n}(t + \Delta t)$ (top). Dominating species are underlined, we have 1 $\text{DOM}(t)$ 1, 2 $\text{DOM}(t)$ 2, 2 $\text{DOM}(t)$ 3, 2 $\text{DOM}(t)$ 4 and 3 $\text{DOM}(t)$ 5. Asymptotically, $\mathbf{n}(t)$ converges to a stable configuration (bottom).

Lemma 1. *In a stable configuration, if a species appears but does not dominate a zone, then its density in each zone is equal to $\frac{1}{m}$.*

Proof. Let s be a stable configuration and let $\mathbf{n} = (n_{qi})$ be the density matrix of s . Let s' be the configuration obtain from s after a time step, and let $\mathbf{n}' = (n'_{qi})$ be the density matrix of s' . Because s is a stable configuration then $s' = s$ and $\mathbf{n}' = \mathbf{n}$.

Without loss of generality, we can suppose that species 1 appears but does not dominate zone 1. Then the individuals of species 1 in zone 1 will move to all other $m - 1$ zones in the next time step, therefore $n'_{1i} > 0$ for all $i \neq 1$. This implies that $n_{1i} \neq 0$ for all $i \in \mathcal{Z}$. Let $T \subset \mathcal{Z}$ be the set of zones that species 1 does not dominate. Now let n_{1j} be the maximal of all n_{1i} for $i \in T$. After one time step, we have $n'_{1j} = \sum_{i \in T \setminus \{1\}} \frac{1}{m-1} n_{1i} \leq \sum_{i \in T \setminus \{1\}} \frac{1}{m-1} n_{1j} \leq (m-1) \frac{1}{m-1} n_{1j} = n_{1j}$.

The equality holds if and only if $T = \mathcal{Z}$ and $n_{1i} = n_{1j}$, for all $i \in \mathcal{Z}$. This implies that $n_{1i} = \frac{1}{m}$ for all $i \in \mathcal{Z}$. \square

Theorem 1. *A stable configuration is of the following form: there are $p-k$, $k \in \{0\} \cup \{2, 3, 4, \dots, p\}$, species that appear in all m zones with the density of each species in each zone equal to $\frac{1}{m}$. And the other k species dominate m zones (one species can dominate many zones).*

Proof. Let s be a stable configuration and let $\mathbf{n} = (n_{qi})$ be the density matrix of s . Using Lemma 1, we know that if a species appears but does not dominate a zone then its density in every zone is $\frac{1}{m}$. Let $p - k$ be the number of such species. Without loss of generality, we suppose that these species are $\{k + 1, \dots, p\}$. The k other species do not belong to this set, which means that species i with $i \leq k$ dominates in all zones where it appears.

If k is equal to 0 or 1, every species appear in every zone with density $\frac{1}{m}$. This is a very special stable configuration, where there is no domination, therefore the case $k = 1$ is impossible

If $k = p$, each zone contains only one species and the density of species in their zone is ≤ 1 .

Now, let us consider the case when $2 \leq k \leq p - 1$. Then in each zone, species $k + 1, \dots, p$ appear with density $\frac{1}{m}$, beside there is one dominant species with density greater than $\frac{1}{m}$. Note that one species q can dominate different zones, let say ℓ_q zones, where $1 \leq \ell_q \leq p - 1$. And we have $\ell_1 + \ell_2 + \dots + \ell_k = m$. \square

Corollary 1. *The number of stable configuration forms is equal to the number of partitions of m in which the number of parts is from 2 to p , plus the configuration with every densities equal to $\frac{1}{m}$.*

Proof. According to the previous theorem, apart from the very special case $k = 0$, a stable configuration form is determined by $p - k$, the number of dominated species, and by ℓ_i the number of zones the species q dominates, with $1 \leq q \leq k$. Because we are interested in the form of stable configurations, then the order of species does not matter, and we can suppose that $\ell_1 \geq \ell_2 \geq \dots \geq \ell_k$. Then the form of a stable configuration is determined by a non inscreasing sequence (ℓ_1, \dots, ℓ_k) , such that $2 \leq k \leq p - 1$ and that $\ell_1 + \ell_2 + \dots + \ell_k = m$. This sequence is nothing but a partition of m in k parts.

Note that in the case $k = p$, a stable configuration form corresponds to a partition of m in p parts. Then the number of stable configuration forms is equal to the number of partitions of m with the number of parts less than or equal to p . \square

4 Convergence time

In Subsection 4.2 we prove that the repartition of dominances remains fixed after one iteration (Lemma 4), and as a consequence the evolution converges exponentially quickly to a stable configuration of the form described in Theorem 1 (Theorem 3). However, when cases of equality among densities appear the behavior may never converge, but this bears upon an infinitesimal subset of configurations that we will therefore ignore, as developed in Subsection 4.1.

4.1 Ignoring cases of equalities

In the sequel we will ignore cases of equality, because they appear with probability zero if we take a uniform random initial configuration with densities in \mathbb{R} . Recall that \mathcal{C} denotes the set of all configurations with densities in \mathbb{R} , and let us denote \mathcal{C}^* the set of configurations such that the density of two species in a zone are equal at some time during the evolution. Formally,

$$\mathcal{C}^* = \{c \in \mathcal{C} \mid \exists k \in \mathbb{N}, p \in \mathcal{S}, q \in \mathcal{S}, i \in \mathcal{Z} : n_{pi}(t + k\Delta t) = n_{qi}(t + k\Delta t)\}$$

with \mathbf{n} the density matrix of c .

Intuitively, if we consider the set \mathcal{C} which is uncountable (continuous space of densities in $[0; 1] \subset \mathbb{R}$) then a case of equality in \mathcal{C}^* corresponds to the restriction of an uncountably large degree of liberty to a countable one, hence the following result.

Theorem 2. $\frac{|\mathcal{C}^*|}{|\mathcal{C}|} = 0$.

Proof. A uniform random initial configuration is given by a density matrix, *i.e.* for each species $q \in \mathcal{S}$ one has to choose non-negative n_{qi} for all $i \in \mathcal{Z}$, such that $\sum_{i \in \mathcal{Z}} n_{qi}(t) = 1$. For each species this corresponds to choosing $x_{q1}, x_{q2}, \dots, x_{q(m-1)} \in [0; 1]$ uniformly, with $x_{q1} < x_{q2} < \dots < x_{q(m-1)}$, and setting $n_{q1} = x_{q1}$, $n_{q2} = x_{q2} - x_{q1}$, $n_{q3} = x_{q3} - x_{q2}$, \dots , $n_{q(m-1)} = x_{q(m-1)} - x_{q(m-2)}$ and $n_{qm} = 1 - x_{q(m-1)}$. Now if a case of equality appears during some evolution step, it corresponds to having a linear combination of densities, with rational coefficients (given by the evolution rule), such that two densities are equal. This corresponds to restricting an uncountable degree of liberty within $[0; 1]$ to a countable subset of $[0; 1]$, which has null measure. \square

Ignoring cases of equality is important regarding convergence, because there are configurations in \mathcal{C} which do not converge. For example, with $p = 3$ and $m = 2$, we have the following ultimately periodic evolution (with period 2):

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \rightarrow \dots$$

However, this cannot happen for configurations of $\mathcal{C} \setminus \mathcal{C}^*$, they all converge to a stable configuration, as we will see in the next subsection.

4.2 Exponential convergence

Definition 4. Let c and c' be two configurations with density matrices \mathbf{n} and \mathbf{n}' , respectively. The distance between the two configurations is defined by (uniform norm)

$$d(c, c') = \max_{\substack{i \in \mathcal{Z} \\ q \in \mathcal{S}}} \{|n_{qi} - n'_{qi}|\}.$$

Definition 5. Starting from a configuration $c(t)$, we say that the system converges to a stable configuration s if

$$\forall \epsilon > 0, \exists k(\epsilon) \in \mathbb{N}, \forall k > k(\epsilon) : d(c(t + k\Delta t), s) < \epsilon.$$

Moreover, if $k(\epsilon)$ in $\mathcal{O}(\log_2(\frac{1}{\epsilon}))$ we say that the systems exponentially converges to s .

We consider only configurations of $\mathcal{C} \setminus \mathcal{C}^*$, therefore each zone is dominated by exactly one species.

Definition 6. A repartition of dominances is the association of a dominant species to each zone. Let us denote $\mathbb{D}_q(t)$ (resp. $\overline{\mathbb{D}}_q(t)$) the set of zones where species q dominates (resp. does not dominate) at time t .

Recall that according to the evolution rule we have

$$\begin{aligned} q \text{ DOM}(t) \ i &\implies n_{qi}(t + \Delta t) = n_{qi}(t) + \frac{1}{m-1} \sum_{j \in \overline{\mathbb{D}}_q(t) \setminus \{i\}} n_{qj}(t) \\ \text{and } q \overline{\text{DOM}}(t) \ i &\implies n_{qi}(t + \Delta t) = \frac{1}{m-1} \sum_{j \in \overline{\mathbb{D}}_q(t) \setminus \{i\}} n_{qj}(t) \end{aligned}$$

and when $i \in \overline{\mathbb{D}}_q(t + \Delta t)$, by considering separately $\overline{\mathbb{D}}_q(t + \Delta t) \cap \mathbb{D}_q(t)$ and $\overline{\mathbb{D}}_q(t + \Delta t) \cap \overline{\mathbb{D}}_q(t)$, we get that after two time steps $n_{qi}(t + 2\Delta t)$ equals

$$\frac{1}{m-1} \left[\sum_{j \in \overline{\mathbb{D}}_q(t + \Delta t) \cap \mathbb{D}_q(t) \setminus \{i\}} n_{qj}(t) + \frac{1}{m-1} \sum_{j \in \overline{\mathbb{D}}_q(t + \Delta t) \setminus \{i\}} \sum_{k \in \overline{\mathbb{D}}_q(t) \setminus \{j\}} n_{qk}(t) \right]. \quad (1)$$

We begin with three straightforward facts.

Fact 1. One species $q \in \mathcal{S}$ cannot dominate all zones.

Proof. Otherwise $\sum_{i \in \mathcal{Z}} n_{qi} > \sum_{i \in \mathcal{Z}} n_{ri} = 1$ for any other species $r \in \mathcal{S}$, but the densities of one species in all zones sum to one, a contradiction. \square

Fact 2. If $q \overline{\text{DOM}}(t) i$ then $n_{qi}(t + \Delta t) \leq \frac{1}{m-1}$.

Proof. In this case $n_{qi}(t + \Delta t) = \frac{1}{m-1} \sum_{j \in \overline{\text{D}}_q(t) \setminus \{i\}} n_{qj}(t) \leq \frac{1}{m-1}$ since the densities of one species in all zones sum to one. \square

Fact 3. If $q \text{DOM}(t) i$ for some zone i , then $n_{qh}(t + \Delta t) \geq \frac{1}{m-1}$ and $q \text{DOM}(t + \Delta t) h$ with $h = \arg \max_{k \in \text{D}_q(t)} n_{qk}(t)$.

Proof. If $q \text{DOM}(t) i$ then h is well defined, $q \text{DOM}(t) h$, and

$$\begin{aligned} n_{qh}(t + \Delta t) &= n_{qh}(t) + \frac{1}{m-1} \sum_{j \in \overline{\text{D}}_q(t)} n_{qj}(t) \\ &= \frac{1}{m-1} + n_{qh}(t) - \frac{1}{m-1} \sum_{j \in \text{D}_q(t)} n_{qj}(t) \end{aligned}$$

but thanks to the maximality of h and Fact 1 we have $\sum_{j \in \text{D}_q(t)} n_{qj}(t) \leq (m-1)n_{qh}(t)$. With the above development this gives the first part of the result, and the second part comes from Fact 2. \square

The following lemma is an important observation in order to prove Lemma 4.

Lemma 2. If a species does not dominate a zone for two consecutive time steps, then it will never dominate it: $q \overline{\text{DOM}}(t) i$ and $q \overline{\text{DOM}}(t + \Delta t) i$ implies $q \overline{\text{DOM}}(t + 2 \Delta t) i$.

Proof. We split the proof into two cases, depending on whether $\text{D}_q(t) = \emptyset$ or not.

Case $\text{D}_q(t) = \emptyset$. We claim that $n_{qi}(t + 2 \Delta t)$ is either smaller than $n_{qi}(t)$ or than $n_{qi}(t + \Delta t)$. Then, let r (resp. s) be the species dominating zone i at time t (resp. $t + \Delta t$). When a species dominates a zone its density obviously increases in that zone, hence

$$\begin{aligned} n_{qi}(t + 2 \Delta t) &\leq n_{qi}(t) < n_{ri}(t) \leq n_{ri}(t + \Delta t) \leq n_{si}(t + \Delta t) \leq n_{si}(t + 2 \Delta t) \\ \text{or } n_{qi}(t + 2 \Delta t) &\leq n_{qi}(t + \Delta t) < n_{si}(t + \Delta t) \leq n_{si}(t + 2 \Delta t) \end{aligned}$$

i.e. $q \overline{\text{DOM}}(t + 2 \Delta t) i$. Let us now prove the claim. Given that $n_{qi}(t + 2 \Delta t)$ is upper bounded by the case $\text{D}_q(t + \Delta t) = \emptyset$, from Equation (1) we have

$$\begin{aligned} n_{qi}(t + 2 \Delta t) &\leq \frac{1}{(m-1)^2} \sum_{j \in \mathcal{Z} \setminus \{i\}} \sum_{k \in \mathcal{Z} \setminus \{j\}} n_{qk}(t) \\ &= \frac{1}{(m-1)^2} \left[(m-1) n_{qi}(t) + (m-2) \sum_{j \in \mathcal{Z} \setminus \{i\}} n_{qj}(t) \right] \\ &= \frac{1}{m-1} n_{qi}(t) + \frac{m-2}{m-1} \left(\frac{1}{m-1} \sum_{j \in \mathcal{Z} \setminus \{i\}} n_{qj}(t) \right) \\ &= \frac{1}{m-1} n_{qi}(t) + \frac{m-2}{m-1} n_{qi}(t + \Delta t). \end{aligned}$$

So $n_{qi}(t + 2 \Delta t)$ is either smaller than $n_{qi}(t)$ or than $n_{qi}(t + \Delta t)$.

Case $D_q(t) \neq \emptyset$. If $D_q(t) \neq \emptyset$ then, letting $h = \arg \max_{k \in D_q(t)} n_{qk}(t)$ and applying Fact 3 gives $h \in D_q(t + \Delta t)$ and $n_{qh}(t + \Delta t) \geq \frac{1}{m-1}$. We begin with the same development as in the case $D_q(t) = \emptyset$, and keep track of the additional terms. Starting again from Equation (1) we have

$$\begin{aligned}
n_{qi}(t + 2\Delta t) &= \underbrace{\frac{1}{m-1} \sum_{j \in D_q(t) \cap \bar{D}_q(t+\Delta t)} n_{qj}(t)}_a + \left[\frac{1}{(m-1)^2} \sum_{j \in \mathcal{Z} \setminus \{i\}} \sum_{k \in \mathcal{Z} \setminus \{j\}} n_{qk}(t) \right] \\
&\quad + \frac{1}{(m-1)^2} \left(\underbrace{-|m - \bar{D}_q(t + \Delta t)|}_b + \underbrace{\sum_{j \in D_q(t+\Delta t)} n_{qj}(t)}_c \right. \\
&\quad \left. - \underbrace{|\bar{D}_q(t + \Delta t) - 1| \sum_{j \in D_q(t)} n_{qj}(t)}_d + \underbrace{\sum_{j \in D_q(t) \cap D_q(t+\Delta t)} n_{qj}(t)}_e \right).
\end{aligned}$$

We can merge some terms as follows,

$$\begin{aligned}
a - \frac{b}{(m-1)^2} - \frac{d}{(m-1)^2} &= -\frac{1}{m-1} \sum_{j \in D_q(t) \cap D_q(t+\Delta t)} n_{qj}(t) - \frac{|m - \bar{D}_q(t + \Delta t)|}{(m-1)^2} \sum_{j \in \bar{D}_q(t)} n_{qj}(t) \\
\frac{c}{(m-1)^2} + \frac{e}{(m-1)^2} &= \frac{1}{(m-1)^2} - \frac{1}{(m-1)^2} \sum_{j \in \bar{D}_q(t) \cap \bar{D}_q(t+\Delta t)} n_{qj}(t)
\end{aligned}$$

which leads to

$$\begin{aligned}
n_{qi}(t + 2\Delta t) &= \left[\frac{1}{m-1} n_{qi}(t) + \frac{m-2}{m-1} n_{qi}(t + \Delta t) \right] + \frac{1}{(m-1)^2} - \frac{1}{m-1} \sum_{j \in D_q(t) \cap D_q(t+\Delta t)} n_{qj}(t) \\
&\quad - \frac{|\bar{D}_q(t+\Delta t)|}{(m-1)^2} \sum_{j \in \bar{D}_q(t)} n_{qj}(t) - \frac{1}{(m-1)^2} \sum_{j \in \bar{D}_q(t) \cap \bar{D}_q(t+\Delta t)} n_{qj}(t).
\end{aligned}$$

Now, if the sum of terms appart from the square bracket is smaller than zero then we are done as in the case $D_q(t) = \emptyset$. So we suppose the contrary in order to reach a contradiction, *i.e.*,

$$\begin{aligned}
&\sum_{j \in D_q(t) \cap D_q(t+\Delta t)} n_{qj}(t) + \frac{1}{m-1} \sum_{j \in \bar{D}_q(t) \cap \bar{D}_q(t+\Delta t)} n_{qj}(t) \\
&< \frac{1}{m-1} - \frac{|\bar{D}_q(t+\Delta t)|}{m-1} + \frac{|\bar{D}_q(t+\Delta t)|}{m-1} \sum_{j \in D_q(t)} n_{qj}(t) \\
&\leq \frac{1}{m-1} \sum_{j \in D_q(t)} n_{qj}(t) + \frac{|\bar{D}_q(t+\Delta t)|-1}{m-1} \sum_{j \in D_q(t)} n_{qj}(t) - \frac{|\bar{D}_q(t+\Delta t)|-1}{m-1} \\
&\leq \frac{1}{m-1} \sum_{j \in D_q(t)} n_{qj}(t).
\end{aligned}$$

Forgetting one more term on the left side leads to

$$(m-1) \sum_{j \in D_q(t) \cap D_q(t+\Delta t)} n_{qj}(t) < \sum_{j \in D_q(t)} n_{qj}(t) = \sum_{j \in D_q(t) \cap D_q(t+\Delta t)} n_{qj}(t) + \sum_{j \in D_q(t) \cap \bar{D}_q(t+\Delta t)} n_{qj}(t)$$

and therefore

$$(m-2) \sum_{j \in \mathcal{D}_q(t) \cap \mathcal{D}_q(t+\Delta t)} n_{qj}(t) < \sum_{j \in \mathcal{D}_q(t) \cap \overline{\mathcal{D}}_q(t+\Delta t)} n_{qj}(t).$$

Let us show that this is impossible.

- Since $h \in \mathcal{D}_q(t) \cap \mathcal{D}_q(t+\Delta t)$ we have $(m-2)n_{qh}(t)$ smaller or equal to the left hand side.
- With $g = \arg \max_{j \in \mathcal{D}_q(t) \cap \overline{\mathcal{D}}_q(t+\Delta t)} n_{qj}(t)$ the zone maximizing the density of species q among the zones where $q \text{DOM}(t) j$ and $q \overline{\text{DOM}}(t+\Delta t) j$, we obtain that the right hand side is smaller or equal to $|\mathcal{D}_q(t) \cap \overline{\mathcal{D}}_q(t+\Delta t)| n_{qg}(t)$ which in turn is smaller or equal to $(m-2)n_{qg}(t)$ since at least zones i and j do not belong to $\mathcal{D}_q(t) \cap \overline{\mathcal{D}}_q(t+\Delta t)$.

We conclude that $n_{qh}(t) < n_{qg}(t)$, a contradiction to the definition of h as $\arg \max_{k \in \mathcal{D}_q(t)} n_{qk}(t)$ because $g \in \mathcal{D}_q(t)$. \square

From Lemma 2 above we also deduce the following.

Lemma 3. *If a species dominates a zone for two consecutive time steps, then it will dominate it forever: $q \text{DOM}(t) i$ and $q \text{DOM}(t+\Delta t) i$ implies $q \text{DOM}(t+2\Delta t) i$.*

The next result is stronger than Lemmas 2 and 3. It is the key observation in the proof of Theorem 3.

Lemma 4. *Starting from any initial configuration, the repartition of dominances remains fixed after one iteration: $q \text{DOM}(t+\Delta t) i$ implies $q \text{DOM}(t+k\Delta t) i$ for all $k \geq 1$.*

Proof. Let us first prove the lemma when the number of zones is $m=1$ or $m=2$. When $m=1$ the dynamic is trivial, no species dominate zone 1 ($n_{q1}(t+k\Delta t) = 1$ for all $q \in \mathcal{S}$ and all $k \in \mathbb{N}$), and the configuration does not belong to $\mathcal{C} \setminus \mathcal{C}^*$ (though the lemma holds). When $m=2$, each species dominates at most one zone (for a species cannot dominate all zones by Fact 1), hence if $q \text{DOM}(t) i$ then all the individuals of species q from the other zone are moving to zone i , which implies $n_{qi}(t+\Delta t) = 1$ and q dominates zone i at time $t+\Delta t$ and forever (recall that there is no case of equality among densities, we are in $\mathcal{C} \setminus \mathcal{C}^*$), *i.e.*, the lemma holds.

Now, when $m \geq 3$, from Lemma 2, our only chance to contradict the lemma is for two species to alternate dominance in one zone. For convenience with the notations, let us suppose, without loss of generality, that

$$q \text{DOM}(t) 1 \quad \text{and} \quad r \text{DOM}(t+\Delta t) 1.$$

We will prove that

$$n_{q1}(t+2\Delta t) < n_{r1}(t+2\Delta t) \tag{2}$$

therefore $q \text{DOM}(t+2\Delta t) 1$, and it implies $r \text{DOM}(t+2\Delta t) 1$ by Lemma 2 applied to each species $s \neq r$ (at $t+\Delta t$ and $t+2\Delta t$ for species q , and at t and $t+\Delta t$ for others). The lemma follows by induction. We have

$$\begin{aligned} n_{q1}(t+2\Delta t) &= \frac{1}{m-1} \sum_{i \in \overline{\mathcal{D}}_q(t+\Delta t) \setminus \{1\}} n_{qi}(t+\Delta t). \\ n_{r1}(t+2\Delta t) &= n_{r1}(t+\Delta t) + \frac{1}{m-1} \sum_{j \in \overline{\mathcal{D}}_r(t+\Delta t) \setminus \{1\}} n_{rj}(t+\Delta t). \end{aligned}$$

We establish a case disjunction on zones through three claims (proofs are postponed), and then deduce Inequality (2).

Claim 1. If $q \text{DOM}(t) i$ and $s \text{DOM}(t + \Delta t) i$ for some species $s \notin \{q, r\}$, then $n_{qi}(t + \Delta t) < n_{r1}(t + \Delta t) + n_{ri}(t + \Delta t)$.

Claim 2. If $q \text{DOM}(t) i$ and $r \text{DOM}(t + \Delta t) i$, then $n_{qi}(t + \Delta t) \leq n_{r1}(t + \Delta t) + \frac{1}{m-1} n_{q1}(t)$.

Claim 3. If $q \overline{\text{DOM}}(t) i$ then $n_{qi}(t + \Delta t) < n_{r1}(t + \Delta t) - n_{q1}(t)$.

Starting from $n_{q1}(t + 2\Delta t)$,

- developing the case disjunction,
- relaxing the condition on the third term (which preserves the inequality as densities are non-negative),
- then applying Claims 1, 2 and 3 to their respective case,
- and finally reorganizing the sum:
 - factorize at most $m-1$ terms $n_{r1}(t + \Delta t)$ since the three sums represent a case disjunction in $i \in \mathcal{Z}$,
 - cancel terms $n_{q1}(t)$ with at most $m-1$ positive terms (with factor $\frac{1}{m-1}$) and at least one negative term by Fact 1,

we obtain Inequality (2) as follows.

$$\begin{aligned}
& n_{q1}(t + 2\Delta t) \\
&= \frac{1}{m-1} \sum_{i \in \overline{\text{D}}_q(t+\Delta t) \setminus \{1\}} n_{qi}(t + \Delta t) \\
&= \frac{1}{m-1} \left(\sum_{\substack{i \in \overline{\text{D}}_q(t+\Delta t) \setminus \{1\} \\ q \text{DOM}(t) i \\ s \text{DOM}(t+\Delta t) i \\ s \notin \{q, r\}}} n_{qi}(t + \Delta t) + \sum_{\substack{i \in \overline{\text{D}}_q(t+\Delta t) \setminus \{1\} \\ q \text{DOM}(t) i \\ r \text{DOM}(t+\Delta t) i}} n_{qi}(t + \Delta t) + \sum_{i \in \overline{\text{D}}_q(t+\Delta t) \setminus \{1\}} n_{qi}(t + \Delta t) \right) \\
&\leq \frac{1}{m-1} \left(\sum_{\substack{i \in \overline{\text{D}}_q(t+\Delta t) \setminus \{1\} \\ q \text{DOM}(t) i \\ s \text{DOM}(t+\Delta t) i \\ s \notin \{q, r\}}} n_{qi}(t + \Delta t) + \sum_{\substack{i \in \overline{\text{D}}_q(t+\Delta t) \setminus \{1\} \\ q \text{DOM}(t) i \\ r \text{DOM}(t+\Delta t) i}} n_{qi}(t + \Delta t) + \sum_{\substack{i \in \mathcal{Z} \\ q \overline{\text{DOM}}(t) i}} n_{qi}(t + \Delta t) \right) \\
&< \frac{1}{m-1} \left(\sum_{\substack{i \in \overline{\text{D}}_q(t+\Delta t) \setminus \{1\} \\ q \text{DOM}(t) i \\ s \text{DOM}(t+\Delta t) i \\ s \notin \{q, r\}}} (n_{ri}(t + \Delta t) + n_{r1}(t + \Delta t)) \right. \\
&\quad + \sum_{\substack{i \in \overline{\text{D}}_q(t+\Delta t) \setminus \{1\} \\ q \text{DOM}(t) i \\ r \text{DOM}(t+\Delta t) i}} (n_{r1}(t + \Delta t) + \frac{1}{m-1} n_{q1}(t)) \\
&\quad \left. + \sum_{\substack{i \in \mathcal{Z} \\ q \overline{\text{DOM}}(t) i}} (n_{r1}(t + \Delta t) - n_{q1}(t)) \right)
\end{aligned}$$

$$\begin{aligned} &\leq n_{r1}(t + \Delta t) + \frac{1}{m-1} \sum_{\substack{i \in \bar{D}_q(t + \Delta t) \setminus \{1\} \\ q \text{ DOM}(t) i \\ s \text{ DOM}(t + \Delta t) i \\ s \notin \{q, r\}}} n_{ri}(t + \Delta t) \\ &\leq n_{r1}(t + 2\Delta t) \end{aligned}$$

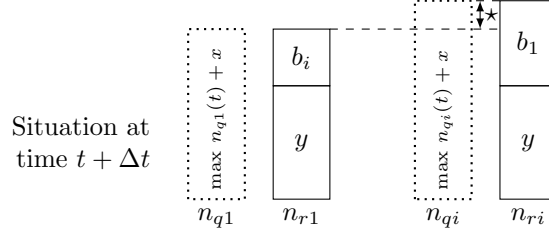
□

Proof of Claim 1. Our hypothesis are

$$\begin{aligned} &q \text{ DOM}(t) 1 \quad q \text{ DOM}(t) i \\ &r \text{ DOM}(t + \Delta t) 1 \quad s \text{ DOM}(t + \Delta t) i \quad r \neq s. \end{aligned}$$

In this case, $n_{qi}(t + \Delta t) < n_{si}(t + \Delta t)$, so let us concentrate on upper bounding $n_{si}(t + \Delta t)$ by $n_{r1}(t + \Delta t) + n_{ri}(t + \Delta t)$, which proves the claim. In order to ease the developments below, let us present and illustrate some notations, regarding what happens from time t to $t + \Delta t$.

$$\begin{aligned} x &= \frac{1}{m-1} \sum_{j \in \bar{D}_q(t)} n_{qj}(t) && : \text{What } q \text{ receives in zones 1 and } i. \\ y &= \frac{1}{m-1} \sum_{j \in \bar{D}_r(t) \setminus \{1, i\}} n_{rj}(t) && : \text{What } r \text{ receives in common into zones 1 and } i. \\ b_1 &= \frac{1}{m-1} n_{r1}(t) && : \text{What } r \text{ in zone 1 gives to zone } i. \\ b_i &= \frac{1}{m-1} n_{ri}(t) && : \text{What } r \text{ in zone } i \text{ gives to zone 1.} \\ z &= \frac{1}{m-1} \sum_{j \in \bar{D}_s(t) \setminus \{1, i\}} n_{sj}(t) && : \text{What } s \text{ receives in common into zones 1 and } i. \\ c_1 &= \frac{1}{m-1} n_{s1}(t) && : \text{What } s \text{ in zone 1 gives to zone } i. \\ c_i &= \frac{1}{m-1} n_{si}(t) && : \text{What } s \text{ in zone } i \text{ gives to zone 1.} \end{aligned}$$



On the last graphic, the highlighted \star is the quantity we are interested in, $n_{si}(t + \Delta t) - n_{r1}(t + \Delta t)$, and we will demonstrate that it is smaller than $n_{ri}(t + \Delta t)$, thus proving the Claim. Our goal is therefore to show that

$$(y + b_1) - (z + c_1) + (y + b_i) > 0.$$

Since $r \text{ DOM}(t + \Delta t) 1$ and $q \text{ DOM}(t) 1$, we can upper bound c_1 with

$$\begin{aligned} n_{r1}(t + \Delta t) &> n_{q1}(t + \Delta t) = n_{q1}(t) + x > n_{s1}(t) + x = (m-1)c_1 + x \\ &\iff \\ \frac{1}{m-1}(y + b_i - x) &> c_1. \end{aligned} \tag{3}$$

Similarly, we can upper bound b_i with

$$\frac{1}{m-1} (z + c_1 - x) > b_i \quad (4)$$

which is useful to lower bound y , as $r \text{DOM}(t + \Delta t) 1$ and from Inequality (4) we have

$$\begin{aligned} y + \frac{1}{m-1} (z + c_1 - x) > y + b_i = n_{r1}(t + \Delta t) > n_{s1}(t + \Delta t) = z + c_i \\ \iff \\ y > \frac{m-2}{m-1} z + c_i - \frac{1}{m-1} c_1 + \frac{1}{m-1} x. \end{aligned} \quad (5)$$

We are now ready to develop the quantity of interest, from Inequalities (5) and then (3).

$$\begin{aligned} y + b_1 + y + b_i - z - c_1 &> b_1 + b_i + y + c_i + \frac{1}{m-1} x - \frac{1}{m-1} (z + c_1) - c_1 \\ &> \frac{m-2}{m-1} (y + b_i) - \frac{1}{m-1} (z + c_1) + \left[b_1 + c_i + \frac{2}{m-1} x \right] \end{aligned} \quad (6)$$

In order to prove that the formula of Inequality (6) is positive, we perform a case disjunction.

- If $m \geq 4$, then the bracketed part is positive, and the other part is negative only if $(y + b_i) = n_{r1}(t + \Delta t)$ is at least twice smaller than $(z + c_1) = n_{s1}(t + \Delta t)$. Let us demonstrate that this is impossible.

$$r \text{DOM}(t + \Delta t) 1 \implies y + b_i > z + c_i > z \quad (7)$$

$$r \text{DOM}(t + \Delta t) 1 \implies y + b_i > n_{q1}(t + \Delta t) \geq n_{q1}(t) \quad (8)$$

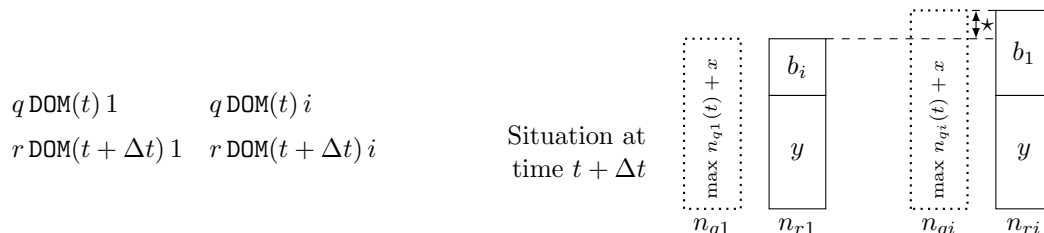
$$q \text{DOM}(t) 1 \implies n_{q1}(t) > (m-1) c_1 \geq c_1 \quad (9)$$

Inequalities (7),(8) and then (9) imply that $2(y + b_i) > z + n_{q1}(t) > z + c_1$, therefore the formula of Inequality (6) is positive.

- If $m = 3$, then the hypothesis are impossible to fulfill, which does not alter the conclusions of Lemma 4. Indeed, all individuals of species q in zone $j \notin \{1, i\}$ (*i.e.* $\mathcal{Z} \setminus \{1, i\} = \{j\}$) are on the go (because q cannot dominate all zones, Fact 1), and no individual of species q is going to zone j . Consequently $n_{qj}(t + \Delta) = 0$ and it follows that in one of the two zones 1 or i , the density of species q is greater than $\frac{1}{2}$. However, it is not possible for another species that does not dominate at time t (neither r nor s) to have a density above $\frac{1}{2}$ at time $t + \Delta t$, therefore q cannot lose dominance in both zones 1 and i .

□

Proof of Claim 2. We use the same notations for y, b_1 and b_i as in the proof of Claim 1.



It follows that

$$n_{qi}(t + \Delta t) - n_{r1}(t + \Delta t) < n_{ri}(t + \Delta t) - n_{r1}(t + \Delta t) \leq b_1 = \frac{1}{m-1} n_{r1}(t) < \frac{1}{m-1} n_{q1}(t).$$

□

Proof of Claim 3. Recall that $q \text{DOM}(t) \perp$ and $r \text{DOM}(t + \Delta t) \perp$. If $q \overline{\text{DOM}}(t) i$ then

$$\begin{aligned} n_{qi}(t + \Delta t) &= \frac{1}{m-1} \sum_{j \in \overline{\text{D}}_q(t) \setminus \{i\}} n_{qj}(t) \\ &\leq \frac{1}{m-1} \sum_{j \in \overline{\text{D}}_q(t)} n_{qj}(t) \\ &= n_{q1}(t + \Delta t) - n_{q1}(t) \\ &< n_{r1}(t + \Delta t) - n_{q1}(t). \end{aligned}$$

□

Thanks to Lemma 4 we can straightforwardly deduce the exponential convergence to a stable configuration.

Theorem 3. *Starting from any initial configuration in $\mathcal{C} \setminus \mathcal{C}^*$, the system converges exponentially to a stable configuration of the form described in Theorem 1.*

Proof. From Lemma 4, after one iteration the repartition of dominances remains fixed. Let us denote by \mathbf{n} the density matrix of the configuration obtained after one time step, and let us denote t that time. Let $q_i \in \mathcal{S}$ be the species dominating zone $i \in \mathcal{Z}$ at time t (and forever), i.e. $\text{D}_q(t) = \{i \mid q = q_i\}$. Then the system converges exponentially to the configuration of density matrix \mathbf{m} such that

$$\mathbf{m}_{qi} = \begin{cases} \frac{1}{m} & \text{if } \{i \mid q = q_i\} = \emptyset \\ 0 & \text{if } q \neq q_i \text{ and } \{i \mid q = q_i\} \neq \emptyset \\ n_{qi} + \frac{1}{u} \sum_{j \notin \{i \mid q = q_i\}} n_{qj} & \text{if } q = q_i \end{cases}$$

with $u = |\{i \mid q = q_i\}|$ the number of zones dominated by q , by Fact 1 we have $1 \leq u \leq m-1$. Indeed, for each time step $k \in \mathbb{N}$ and each species $q \in \mathcal{S}$, we have the following case disjunction.

- If species q dominates no zone (such that $\{i \mid q = q_i\} = \emptyset$), then for every zone $i \in \mathcal{Z}$ we have

$$\begin{aligned} \frac{1}{m} - n_{qi}(t + (k+1)\Delta t) &= \frac{1}{m} - \frac{1}{m-1} \sum_{j \in \mathcal{Z} \setminus \{i\}} n_{qj}(t + k\Delta t) \\ &= \frac{1}{m} - \frac{1}{m-1} (1 - n_{qi}(t + k\Delta t)) \\ &= \frac{1}{m-1} \left(-\frac{1}{m} + n_{qi}(t + k\Delta t) \right) \end{aligned}$$

that is, the density of q in every zone convergence exponentially to $\frac{1}{m}$.

- If species q dominates some zone (such that $\{i \mid q = q_i\} \neq \emptyset$), then let us consider the total density outside the dominated zones:

$$\begin{aligned} \sum_{j \notin \{i \mid q = q_i\}} n_{qj}(t + (k+1)\Delta t) &= \sum_{j \notin \{i \mid q = q_i\}} \frac{1}{m-1} \sum_{\ell \notin \{i \mid q = q_i\} \setminus \{j\}} n_{q\ell}(t + k\Delta t) \\ &= \frac{p-u-1}{m-1} \sum_{j \notin \{i \mid q = q_i\}} n_{qj}(t + \Delta t). \end{aligned}$$

As a consequence, for all $i \notin \{i \mid q = q_i\}$ we have an exponential convergence to density zero, i.e. to m_{qi} . Regarding $i \in \{i \mid q = q_i\}$, we have

$$n_{qi}(t + (k+1)\Delta t) = n_{qi}(t + \Delta t) + \frac{1}{m-1} \sum_{j \notin \{i \mid q = q_i\}} n_{qj}(t + \Delta t)$$

that is, the density gained from time $t + k\Delta t$ to time $t + (k+1)\Delta t$ does not depend on i . In other terms, all such i receive the same amount of density from $\mathcal{Z} \setminus \{i \mid q = q_i\}$. Since we have just seen that the density in zones $\mathcal{Z} \setminus \{i \mid q = q_i\}$ tend exponentially to zero, it implies that the quantity

$$\sum_{j \notin \{i \mid q = q_i\}} n_{qj}(t + (k+1)\Delta t)$$

is shared uniformly among $\{i \mid q = q_i\}$, at an exponential rate. The number of such zones i is precisely u , therefore we deduce that n_{qi} converges exponentially to m_{qi} .

Remark that the configuration of density matrix \mathbf{m} is in accordance with Theorem 1. □

5 Probabilities for the stable configurations in the case of 3 species and 3 zones

In this section we push the analysis of the dynamics a step forward in the case $p = m = 3$, by computing analytically and checking numerically the probabilities to converge to each form of stable configurations. In this case there are only two forms of stable configurations, as was already proved in [21]. We reproduce the result, with $\mathcal{S} = \{A, B, C\}$ and $\mathcal{Z} = \{1, 2, 3\}$.

Definition 7. We call a one-each configuration, denoted by c_{OE} , a configuration such that each zone contains only one species.

Definition 8. We call a one-two configuration, denoted by c_{OT} , a configuration such that one species is of density 1 in one zone, another species dominates and is split in the two other zones, and the last species is evenly split into the 3 zones with a density $1/3$ in each.

Theorem 4 ([21]). Let c be a configuration. Without loss of generality, one can suppose that

$$n_{A1} = \max_{\substack{i \in \mathcal{Z} \\ q \in \mathcal{S}}} \{n_{qi}\} \text{ and } n_{B3} = \max_{q \in \mathcal{S}} \{n_{q3}\}.$$

Then the system exponentially converges to a c_{OT} if c satisfies one of the following conditions, otherwise the system exponentially converges to a c_{OE} .

1. $n_{B2} = \max_{q \in \mathcal{S}} \{n_{q2}\}$ and $n_{B2} + \frac{n_{B1}}{2} > \frac{n_{C1} + n_{C3}}{2}$ and $n_{B3} + \frac{n_{B1}}{2} > \frac{n_{C1} + n_{C2}}{2}$
2. $n_{A2} = \max_{q \in \mathcal{S}} \{n_{q2}\}$ and $n_{A2} + \frac{n_{A3}}{2} > \frac{n_{C1} + n_{C3}}{2}$

We are going to determine the probabilities, given a uniformly chosen configuration, to converge to the stable configurations c_{OE} and c_{OT} according to Theorem 4, and validate the result via numerical simulations. Let us first introduce some definitions.

Definition 9. We denote by BC the set of possible configurations at the beginning.

Definition 10. We denote by BC_{OE} the set of possible configurations at the beginning such that the system converges to c_{OE} , and by p_{OE} the probability that a uniformly chosen configuration among BC belongs to BC_{OE} .

Definition 11. We denote by BC_{OT} the set of possible configurations at the beginning such that the system converges to c_{OT} , and by p_{OT} the probability that a uniformly chosen configuration among BC belongs to BC_{OT} .

According to Theorem 4 we have

$$p_{OE} + p_{OT} = 1. \quad (10)$$

Theorem 5. $p_{OT} = \frac{1}{16200} \sqrt{3} \left(18\sqrt{3} + 2491\sqrt{\frac{1}{3}} \right) \approx 0.1571$ and $p_{OE} = 1 - p_{OT} \approx 0.8429$.

Proof. We will express the sets BC , BC_{OE} and BC_{OT} within the 9-dimensional coordinate space $\mathcal{O}_{n_{A1}, n_{A2}, n_{A3}, n_{B1}, n_{B2}, n_{B3}, n_{C1}, n_{C2}, n_{C3}}$. Since $n_{q1} + n_{q2} + n_{q3} = 1$ for each of the three species $q \in \mathcal{S}$, the set of initial configurations, BC , is a 6-dimensional object in this coordinate space. The sets BC_{OE} and BC_{OT} , which are subsets of BC , are also 6-dimensional. In order to compute the probability for stable configurations, we will compare the 6-dimensional volumes of BC_{OE} and BC_{OT} to that of BC , in this 9-dimensional space.

Let us first argue that it corresponds to the desired uniform probability distribution among BC . Let $\text{Vol}_6(\cdot)$ denote the 6-volume of an object, that is,

$$p_{OE} = \frac{\text{Vol}_6(BC_{OE})}{\text{Vol}_6(BC)} \quad \text{and} \quad p_{OT} = \frac{\text{Vol}_6(BC_{OT})}{\text{Vol}_6(BC)}. \quad (11)$$

For each species $q \in \mathcal{S}$, we have $n_{q1} + n_{q2} + n_{q3} = 1$. A uniform initial configuration corresponds to choosing independently for each of the three species a uniform initial repartition of its individuals into the three zones. This uniform repartition in its turn corresponds to choosing uniformly a point within the equilateral triangle whose vertices are expressed in the coordinate system $\mathcal{O}_{n_{q1}, n_{q2}, n_{q3}}$ as $(1, 0, 0)$ $(0, 1, 0)$ $(0, 0, 1)$. Indeed, it satisfies that the probability distributions for n_{q1} , n_{q2} and n_{q3} are all equal, with an expectation of $\frac{1}{3}$. In order to uniformly choose such a point, one can for example pick two independent random numbers r_1 and r_2 uniformly in $[0, 1]$ and then set $n_{q1} = 1 - \sqrt{r_1}$, $n_{q2} = \sqrt{r_1}(1 - r_2)$ and $n_{q3} = \sqrt{r_1}r_2$.

The 6-volume of BC within the 9-dimensional space is easy to compute: for each species we consider the area (2-dimensional volume) of the equilateral triangle of size $\sqrt{2}$, which is $\frac{\sqrt{3}}{2}$, and BC is the 6-dimensional object formed by the Cartesian product of these three 2-dimensional objects, and its 6-volume is therefore

$$\text{Vol}_6(BC) = \left(\frac{\sqrt{3}}{2} \right)^3 = \frac{3\sqrt{3}}{8}. \quad (12)$$

We now concentrate on calculating $\text{Vol}_6(BC_{OT})$ from Theorem 4. Let us first state clearly the symmetries involved by considering

$$n_{A1} = \max_{\substack{i \in \mathcal{Z} \\ q \in \mathcal{S}}} \{n_{qi}\} \quad \text{and} \quad n_{B3} = \max_{q \in \mathcal{S}} \{n_{q3}\}.$$

We consider that $n_{A1} = \max_{i \in \mathcal{Z} \wedge q \in \mathcal{S}} \{n_{qi}\}$, there are 9 symmetric and independent (not intersecting) cases. Then, either (the two cases are independent):

- Species A dominates another zone. Let say species A dominates zone 2, there are 2 symmetric and independent cases (because species A cannot dominate the three zones by Fact 1). Since species A cannot dominate the three zones, the last zone is dominated by species B or C . Let say species B dominates zone 3, there are 2 symmetric and independent cases. Let us denote by BC_{OT}^1 the set of initial configurations satisfying those conditions.
- Species A does not dominate another zone. Then, either (the two cases are independent):
 - One species dominates the two other zones. Let say it is species B , there are 2 symmetric and independent cases. Let us denote by BC_{OT}^2 the set of initial configurations satisfying those conditions.
 - Each species dominates one zone, in this case the configuration always converges to a COE .

Finally we have $BC_{OT} = BC_{OT}^1 \cup BC_{OT}^2$ and $BC_{OT}^1 \cap BC_{OT}^2 = \emptyset$, but most interestingly

$$\text{Vol}_6(BC_{OT}) = 9 \times (4 \times \text{Vol}_6(BC_{OT}^1) + 2 \times \text{Vol}_6(BC_{OT}^2)). \quad (13)$$

We will now compute the 6-volumes of BC_{OT}^1 and BC_{OT}^2 , which are respectively defined by systems (I) and (II) as follows:

$$(I) \left\{ \begin{array}{l} 0 \leq n_{qi} \leq 1 \text{ for all } q \in \mathcal{S} \text{ and } i \in \mathcal{Z} \\ n_{q1} + n_{q2} + n_{q3} = 1 \text{ for all } q \in \mathcal{S} \\ n_{A1} = \max_{\substack{i \in \mathcal{Z} \\ q \in \mathcal{S}}} \{n_{qi}\} \\ n_{A2} = \max_{q \in \mathcal{S}} \{n_{q2}\} \\ n_{B3} = \max_{q \in \mathcal{S}} \{n_{q3}\} \\ n_{A2} + \frac{n_{A3}}{2} > \frac{n_{C1} + n_{C3}}{2} \end{array} \right. \quad (II) \left\{ \begin{array}{l} 0 \leq n_{qi} \leq 1 \text{ for all } q \in \mathcal{S} \text{ and } i \in \mathcal{Z} \\ n_{q1} + n_{q2} + n_{q3} = 1 \text{ for all } q \in \mathcal{S} \\ n_{A1} = \max_{\substack{i \in \mathcal{Z} \\ q \in \mathcal{S}}} \{n_{qi}\} \\ n_{B2} = \max_{q \in \mathcal{S}} \{n_{q2}\} \\ n_{B3} = \max_{q \in \mathcal{S}} \{n_{q3}\} \\ n_{B2} + \frac{n_{B1}}{2} > \frac{n_{C1} + n_{C3}}{2} \\ n_{B3} + \frac{n_{B1}}{2} > \frac{n_{C1} + n_{C2}}{2} \end{array} \right.$$

where the two first lines of each system corresponds to the conditions (implicit so far) that BC_{OT}^1 and BC_{OT}^2 are subsets of BC .

Seen as 6-dimensional objects embedded in a 9-dimensional space, the two sets BC_{OT}^1 and BC_{OT}^2 are convex polytopes. Indeed, BC is a convex polytope as it is the Cartesian product of three equilateral triangles, and the additional restrictions defining BC_{OT}^1 and BC_{OT}^2 are all made

of inequalities corresponding to the exclusion of half-spaces. Starting from the convex polytope BC and excluding half-spaces preserves the convexity of the polytope, hence BC_{OT}^1 and BC_{OT}^2 are convex polytopes.

We proceed in two steps to compute the volume of a convex polytope (the same technic will be applied to BC_{OT}^1 and BC_{OT}^2): first triangulate the polytope into simplexes, and then sum the volume of simplexes.

The triangulation of convex polytopes is done via the contributive software *sagemath* which makes use of the *TOPCOM* package ([23, 22]). It returns the sets of vertices (the convex polytope is the convex hull of the set of vertices) and simplexes among these vertices, as described in Tables 1 and 2 of Appendix A, in the coordinate system $\mathcal{O}_{n_{A1}, n_{A2}, n_{A3}, n_{B1}, n_{B2}, n_{B3}, n_{C1}, n_{C2}, n_{C3}}$. Note that the simplexes are 6-dimensional, hence defined as the convex hull of 7 vertices.

Now, it remains to compute the 6-volume of each simplex and add them. We follow a method of exact calculation that makes use of the Gramian (see for example [8] Chapter IX, or [4] for a succinct review). Let us explain how to compute the 6-volume of the simplex $s = \langle 0, 1, 2, 3, 4, 7, 11 \rangle$ of BC_{OT}^1 , the same procedure is then used to compute the volume of the other simplexes.

The simplex s has 7 vertices given by the set of 9-dimensional row vectors $\langle v_0, v_1, v_2, v_3, v_4, v_7, v_{11} \rangle$ where $v_0 = (2/3, 1/3, 0, 1/3, 0, 2/3, 2/3, 1/3, 0)$, $v_1 = (1/2, 1/2, 0, 1/2, 0, 1/2, 0, 1/2, 0)$, etc. Let $w_1 = v_1 - v_0, w_2 = v_2 - v_0, \dots, w_5 = v_7 - v_0, w_6 = v_{11} - v_0$ and let W be the 6 by 9 matrix whose rows are the row vectors w_i for $1 \leq j \leq 6$. The *Gram determinant formula* states that

$$\text{Vol}_6(s) = \frac{\sqrt{|\text{Det}(W {}^tW)|}}{6!} = \frac{1}{21600} \sqrt{\frac{1}{3}}$$

where tW is the transpose of W . Table 3 of Appendix A presents the results of those 6-volume computations for all the simplexes of BC_{OT}^1 and BC_{OT}^2 in the same order as they are presented in Table 2. The totals are

$$\text{Vol}_6(BC_{OT}^1) = \frac{1861}{2073600} \sqrt{\frac{1}{3}} \quad \text{and} \quad \text{Vol}_6(BC_{OT}^2) = \frac{1}{14400} \sqrt{3} + \frac{2701}{345600} \sqrt{\frac{1}{3}}.$$

From Equation (13) we therefore deduce that

$$\text{Vol}_6(BC_{OT}) = \frac{1}{800} \sqrt{3} + \frac{2491}{14400} \sqrt{\frac{1}{3}},$$

and the statement follows from Equations (10), (11) and (12):

$$p_{OT} = \frac{1}{16200} \sqrt{3} \left(18 \sqrt{3} + 2491 \sqrt{\frac{1}{3}} \right) \approx 0.157098765432099.$$

□

Let us now proceed to a validation of Theorem 5 via numerical simulations. We ran 1 000 000 simulations from uniform random configurations using a C program. An initial configuration is uniformly chosen by picking 6 random variables $r_{A1}, r_{A2}, r_{B1}, r_{B2}, r_{C1}, r_{C2}$ uniformly in the real interval $[0, 1]$ and setting

$$\begin{aligned} n_{A1} &= 1 - \sqrt{r_{A1}}, & n_{A2} &= \sqrt{r_{A1}} (1 - r_{A2}), & n_{A3} &= \sqrt{r_{A1}} r_{A2}, \\ n_{B1} &= 1 - \sqrt{r_{B1}}, & n_{B2} &= \sqrt{r_{B1}} (1 - r_{B2}), & n_{B3} &= \sqrt{r_{B1}} r_{B2}, \\ n_{C1} &= 1 - \sqrt{r_{C1}}, & n_{C2} &= \sqrt{r_{C1}} (1 - r_{C2}), & n_{C3} &= \sqrt{r_{C1}} r_{C2}. \end{aligned}$$

The program, which performs three evolution steps before classifying the configuration according to the repartition of dominancies (which is enough from Lemma 4), gave the following output:

$$\begin{aligned} p_{OE} &\approx 0.843560 \quad (837\,835 \text{ occurrences}) \\ p_{OT} &\approx 0.156440 \quad (155\,378 \text{ occurrences}) \end{aligned}$$

Over the 1 000 000 runs, 6 787 cases (0.6787%) were discard because a case of equality has been encountered. This is due to computer's finite precision of real numbers, amplified by the fact that there is often two zones where the proportion of two species both tend to 0 exponentially quickly (when the configuration converges to a c_{OE}). A random subset of 1 000 initial configurations that where taken in this experiment is presented on Figure 2 of Appendix B.

The process of choosing independently random uniform configurations is a Bernoulli process, with a probability of success $p = p_{OE} \approx 0.8429$ given by Theorem 5. We can easily perform a Pearson's chi-squared test of the result we encountered ([20]).

$$\chi^2 = \frac{(837\,835 - 993\,213 p)^2}{993\,213 p} + \frac{(155\,378 - 993\,213 (1 - p))^2}{993\,213 (1 - p)} \approx 3.2574$$

This χ^2 value corresponds to the following one-sided p -value (again from [20]):

$$p\text{-value} = \frac{\int_x^\infty e^{-\frac{1}{2} x^2} x dx}{\int_0^\infty e^{-\frac{1}{2} x^2} x dx} \approx 0.1962.$$

It means that the odds are 5.0973 to 1 against the fact that the result of the simulation deviates from the expected theoretical result of Theorem 5 only by chance, which is satisfying.

6 Conclusion

We have demonstrated the convergence of the general model with many species across multiple zones. Moreover, we also analyzed the forms of all possible stable configurations. If the number of species and the number of zones are small, these forms can be described quite explicitly. However, if there numbers slightly increase, the number of stable configurations will exponentially increase because it is based on the number of partitions of integer. Determining the probabilities for the system reaching a given form stable configuration will thus explode combinatorial calculations as well. With the help of calculation on simplexes, we have computed these probabilities in the case of three species and three zones. Moreover, we also performed a simulation to illustrate these computations (see appendices). In the case the number of species or of zones are greater than three, we think that one can also investigate these computation of probabilities, but one need stronger calculation tools.

Acknowledgments This work was partially supported by the Vietnam National Foundation for Science and Technology Development (NAFOSTED) under the grant number 101.99-2016.16, the Vietnam Institute for Advanced Study in Mathematics (VIASM), the *Young Researcher* project ANR-18-CE40-0002-01 "FANs", the project ECOS-CONICYT C16E01 and the project STIC Am-Sud CoDANet 19-STIC-03 (Campus France 43478PD).

References

- [1] E. Abdllaoui, P. Auger, B.W. Kooi, R. Bravo de la Parra and R. Mchich. *Effects of density-dependent migrations on stability of a two-patch predator-prey model*. Mathematical Biosciences, 210(1):335-354, 2007.
- [2] P. Auger, R. Bravo de la Parra, C. Poggiale, E. Sanchez and T. Nguyen-Huu. *Aggregation of variables and applications to population dynamics*. P. Magal, S. Ruan (Eds.), Structured Population Models in Biology and Epidemiology, Lecture Notes in Mathematics, Vol. 1936, Mathematical Biosciences Subseries. Springer, Berlin,, pages 209–263 , 2008.
- [3] P. Auger and E. Benoit. *A prey-predator model in a multi-patch environment with different time scales*. Journal of Biological Systems, pages 187–197 , 1993.
- [4] N. R. Barth. The Gramian and k-volume in n-space: Some classical results in linear algebra. *Journal of Young Investigators*, 1999.
- [5] K. Dao-Duc, P. Auger and T. Nguyen-Huu. *Predator density dependent prey dispersal in a patchy environment with a refuge for the prey*. South African Journal of Science, pages 180–184 , 2008.
- [6] H. Dreisig. *Ideal free distributions of nectar foraging bumblebees*. Oikos, 72(2), 161-172, 1995.
- [7] S. Fretwell. *Populations in a Seasonal Environment*. Princeton, NJ: Princeton University Press, 1972.
- [8] F.R. Gantmacher. *The Theory of Matrices*. Chelsea Publishing Company, 1959.
- [9] R. Graeme, and S. Humphries. *Multiple ideal free distributions of unequal competitors*. Evolutionary Ecology Research. 1(5): 635-640, 1999.
- [10] J.G. Godin and M.H.A. Keenleyside. *Foraging on patchily distributed prey by a chichlid fish (Teleostei Cichlidae): a test of the ideal free distribution theory*. Animal Behaviour 32: 120-131, 1984.
- [11] L. Křivan, R. Cressman and C. Schneider, The ideal free distribution: A review and synthesis of the game-theoretic perspective. *Theoretical Population Biology*, Volume 73, Issue 3, 2008, Pages 403-425, ISSN 0040-5809, <https://doi.org/10.1016/j.tpb.2007.12.009>.
- [12] R. Mchich, P. Auger and J- C. Poggiale. *Effect of predator density dependent dispersal of prey on stability of a predator-prey system*. Mathematical Biosciences, 343-356 , 2007.
- [13] D. Nguyen-Ngoc, A. Drogoul and P. Auger. *Methodological Steps and Issues When Deriving Individual-Based Models from Equation-Based Models: A Case Study in Population Dynamics*. Proceeding in PRIMA2008, LNAI 5357, 2008, 295-306.
- [14] D. Nguyen-Ngoc, R. Bravo de la Parra, M. A. Zavala and P. Auger. *Competition and species coexistence in a metapopulation model: Can fast asymmetric migration reverse the outcome of competition in a homogeneous environment*. Journal of Theoretical Biology, 266, 2010, 256-263.

- [15] D. Nguyen-Ngoc, T.H. D. Phan, A. Nguyen-Thi-Ngoc, A. Drogoul and J-D. Zucker. *Disk graph-based model: a graph theoretical approach for linking agent-based models and dynamical systems*. IEEE RIVF International Conference on Computing and Communication Technologies, Research, Innovation, and Vision for the Future (RIVF). 2010.
- [16] D. Nguyen-Ngoc, P. Taillandier, A. Drogoul and P. Auger. *Inferring equation-based models from agent-based models: a case study in competition dynamics*. Proceeding in PRIMA Conference, India, 183-190, 2010.
- [17] D. Nguyen-Ngoc, T. Nguyen-Huu and Pierre Auger. *Effects of refuges and density dependent dispersal on interspecific competition dynamics*. International Journal of Bifurcation and Chaos, 1250029, 22(2), 2012.
- [18] D. Nguyen-Ngoc, T. Nguyen-Huu and P. Auger. *Effects of fast density dependent dispersal on pre-emptive competition dynamics*. Ecological Complexity, 26-33,10, 2012.
- [19] De La Parra, R. B., Arino, O., Sánchez, E. and Auger. P. A model for an age-structured population with two time scales. *Mathematical and Computer Modelling* 31, pages 17–26, 2000.
- [20] K. Pearson. *On the criterion that a given system of deviations from the probable in the case of a correlated system of variables is such that it can be reasonably supposed to have arisen from random sampling*. Philosophical Magazine Series 5, 50 (302): 157-175, 1900.
- [21] K. Perrot, D. Nguyen-Ngoc, and H. D. Phan. Effects of Migration of Three Competing Species on Their Distributions in Multizone Environment. *Proceedings of RIVF'2013*, IEEE, pages 227–232, 2013.
- [22] J. Rambau. *TOPCOM: Triangulations of Point Configurations and Oriented Matroids*. Mathematical Software—ICMS 2002, World Scientific, 330-340, 2002.
- [23] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 8.7)*, 2019. <https://www.sagemath.org>.
- [24] W.J. Sutherland, C.R. Townsend and J.M. Patmore. *A test of the ideal free distribution with unequal competitors*. Behavioral Ecology and Sociobiology, 23 (1): 51-53, 1988.

A Polytopes and simplexes for probability calculations

Tables 1, 2 and 3.

B Validation of Theorem 5 via numerical simulations

Figure 2.

Polytope	BC_{OT}^1	BC_{OT}^2
Vertices	0:(2/3,1/3,0,1/3,0,2/3,2/3,1/3,0) 1:(1/2,1/2,0,1/2,0,1/2,1/2,0,1/2) 2:(1/2,1/2,0,1/2,0,1/2,1/2,1/2,0) 3:(2/5,1/5,2/5,2/5,1/5,2/5,2/5,1/5,2/5) 4:(1/2,1/2,0,0,1/2,1/2,1/2,0,1/2) 5:(2/3,1/3,0,0,1/3,2/3,2/3,1/3,0) 6:(1/2,1/2,0,0,1/2,1/2,1/2,1/2,0) 7:(2/3,1/3,0,2/3,0,1/3,2/3,1/3,0) 8:(1/3,1/3,1/3,1/3,1/3,1/3,1/3,1/3,1/3) 9:(2/3,1/3,0,2/3,1/3,0,2/3,1/3,0) 10:(1/2,1/2,0,1/2,1/2,0,1/2,1/2,0) 11:(2/3,1/3,0,0,1/3,2/3,0,1/3,2/3) 12:(1/2,1/2,0,1/2,0,1/2,0,1/2,1/2) 13:(1/2,1/2,0,0,1/2,1/2,0,1/2,1/2) 14:(2/3,1/3,0,1/3,0,2/3,0,1/3,2/3) 15:(2/3,1/3,0,2/3,0,1/3,1/3,1/3,1/3) 16:(1/2,1/4,1/4,1/2,0,1/2,1/4,1/4,1/2) 17:(1/2,1/4,1/4,1/2,0,1/2,1/2,1/4,1/4)	0:(1/2,0,1/2,0,1/2,1/2,1/2,0,1/2) 1:(1/2,0,1/2,0,1/2,1/2,1/2,1/2,0) 2:(1/2,1/2,0,0,1/2,1/2,1/2,1/2,0) 3:(1/2,1/2,0,0,1/2,1/2,1/2,0,1/2) 4:(2/5,1/5,2/5,2/5,1/5,2/5,2/5,1/5,2/5) 5:(2/3,0,1/3,0,1/3,2/3,2/3,1/3,0) 6:(2/3,1/3,0,0,1/3,2/3,2/3,1/3,0) 7:(1,0,0,1,0,0,1,0,0) 8:(1,0,0,0,1/2,1/2,1,0,0) 9:(1/3,1/3,1/3,1/3,1/3,1/3,1/3,1/3,1/3) 10:(2/3,1/3,0,0,2/3,1/3,2/3,0,1/3) 11:(2/3,0,1/3,0,2/3,1/3,2/3,0,1/3) 12:(2/5,2/5,1/5,2/5,2/5,1/5,2/5,2/5,1/5) 13:(2/3,1/3,0,0,1/3,2/3,0,1/3,2/3) 14:(1,0,0,0,1/2,1/2,1/2,0,1/2) 15:(1,0,0,0,2/3,1/3,2/3,0,1/3) 16:(1/2,1/2,0,1/4,1/2,1/4,1/4,1/2,1/4) 17:(1,0,0,0,1/3,2/3,2/3,1/3,0) 18:(1,0,0,0,2/3,1/3,0,2/3,1/3) 19:(1,0,0,0,1/3,2/3,0,1/3,2/3) 20:(2/3,1/3,0,0,2/3,1/3,0,2/3,1/3) 21:(1/2,1/2,0,0,1/2,1/2,0,1/2,1/2) 22:(2/3,0,1/3,0,2/3,1/3,0,2/3,1/3) 23:(2/3,0,1/3,0,1/3,2/3,0,1/3,2/3) 24:(1/2,0,1/2,0,1/2,1/2,0,1/2,1/2) 25:(1/2,0,1/2,1/4,1/4,1/2,1/4,1/4,1/2) 26:(1,0,0,0,1/2,1/2,1/2,1/2,0) 27:(1/2,0,1/2,1/4,1/4,1/2,1/2,1/4,1/4) 28:(1/2,1/2,0,1/4,1/2,1/4,1/2,1/4,1/4)

Table 1: Set of vertices of BC_{OT}^1 and BC_{OT}^2 .

Polytope	BC_{OT}^1	BC_{OT}^2		
Simplexes	<0,1,2,3,4,7,11>	<0,1,2,4,5,7,13>	<0,1,9,11,18,20,22>	<1,2,10,12,13,14,15>
	<0,1,2,3,7,11,12>	<0,1,2,4,7,9,13>	<0,1,9,13,16,18,20>	<1,2,10,12,13,15,16>
	<0,1,2,3,7,12,16>	<0,1,2,5,7,8,13>	<0,1,9,13,18,20,22>	<1,2,10,13,15,16,18>
	<0,1,2,3,7,16,17>	<0,1,2,7,8,10,13>	<0,1,9,13,20,21,22>	<1,2,13,15,16,17,18>
	<0,1,3,7,11,12,14>	<0,1,2,7,9,10,13>	<0,1,9,13,21,22,23>	<1,5,7,8,13,14,17>
	<0,1,3,7,12,14,16>	<0,1,2,9,10,13,16>	<0,1,9,21,22,23,24>	<1,5,7,13,14,17,18>
	<0,2,3,4,5,7,11>	<0,1,2,9,13,16,20>	<0,1,10,11,13,16,18>	<1,7,8,10,11,13,14>
	<1,2,3,4,7,8,11>	<0,1,2,9,13,20,21>	<0,1,10,11,16,18,20>	<1,7,8,10,11,14,15>
	<1,2,3,7,8,11,12>	<0,1,2,10,13,16,18>	<0,2,3,4,5,7,13>	<1,7,9,10,11,12,13>
	<1,2,4,7,8,9,11>	<0,1,2,10,16,18,20>	<0,2,3,4,7,9,13>	<1,7,9,11,12,13,18>
	<1,2,4,8,9,10,11>	<0,1,2,13,16,18,20>	<0,2,3,5,7,8,13>	<1,7,9,11,12,18,22>
	<1,2,4,8,10,11,12>	<0,1,4,5,7,13,18>	<0,2,3,7,8,10,13>	<1,7,10,11,12,13,14>
	<1,2,7,8,9,11,12>	<0,1,4,5,7,18,19>	<0,2,3,7,9,10,13>	<1,7,10,11,12,14,15>
	<1,2,8,9,10,11,12>	<0,1,4,5,7,19,23>	<0,2,3,9,10,13,16>	<1,7,11,12,13,14,18>
	<1,3,4,7,8,9,11>	<0,1,4,5,7,23,25>	<0,2,3,9,13,16,20>	<1,7,11,12,14,15,18>
	<1,3,7,8,9,11,12>	<0,1,4,5,7,25,27>	<0,2,3,9,13,20,21>	<1,7,12,13,14,15,18>
	<1,3,7,9,11,12,14>	<0,1,4,5,13,18,19>	<0,2,3,10,13,16,18>	<1,7,12,13,15,16,18>
	<1,3,7,9,12,14,15>	<0,1,4,5,13,19,23>	<0,2,3,10,16,18,20>	<1,7,13,14,15,17,18>
	<1,3,7,12,14,15,16>	<0,1,4,7,9,13,18>	<0,2,3,13,16,18,20>	<1,7,13,15,16,17,18>
	<2,3,4,5,6,7,11>	<0,1,4,7,9,18,22>	<0,3,7,8,10,13,14>	<1,9,10,11,12,13,16>
	<2,3,4,6,7,8,11>	<0,1,4,7,18,19,22>	<0,4,5,7,13,18,19>	<1,9,11,12,13,16,18>
	<2,4,6,7,8,9,11>	<0,1,4,7,19,22,23>	<0,5,7,13,14,18,19>	<1,9,11,12,16,18,20>
	<2,4,6,8,9,10,11>	<0,1,4,7,22,23,24>	<0,7,8,10,11,13,14>	<1,9,11,12,18,20,22>
	<2,4,6,8,10,11,12>	<0,1,4,7,23,24,25>	<1,2,5,7,8,13,17>	<1,10,11,12,13,14,15>
	<3,4,5,6,7,9,11>	<0,1,4,9,13,18,22>	<1,2,7,8,10,13,14>	<1,10,11,12,13,15,16>
	<3,4,6,7,8,9,11>	<0,1,4,9,13,22,23>	<1,2,7,8,10,14,15>	<1,10,11,13,15,16,18>
	<4,6,8,10,11,12,13>	<0,1,4,9,22,23,24>	<1,2,7,8,13,14,17>	<1,11,12,13,14,15,18>
		<0,1,4,13,18,19,22>	<1,2,7,8,14,15,17>	<1,11,12,13,15,16,18>
		<0,1,4,13,19,22,23>	<1,2,7,8,15,17,18>	<2,3,4,5,6,7,13>
		<0,1,5,7,8,13,14>	<1,2,7,8,17,18,26>	<2,3,5,6,7,8,13>
		<0,1,5,7,13,14,18>	<1,2,7,9,10,12,13>	<2,3,7,9,10,12,13>
		<0,1,7,8,10,11,13>	<1,2,7,10,12,13,14>	<2,3,7,10,12,13,16>
		<0,1,7,8,11,13,14>	<1,2,7,10,12,14,15>	<2,3,7,10,12,16,28>
		<0,1,7,9,10,11,13>	<1,2,7,12,13,14,15>	<2,3,9,10,12,13,16>
		<0,1,7,9,11,13,18>	<1,2,7,12,13,15,16>	<2,3,9,13,16,20,21>
		<0,1,7,9,11,18,22>	<1,2,7,13,14,15,17>	<2,5,6,7,8,13,17>
		<0,1,7,11,13,14,18>	<1,2,7,13,15,16,17>	<2,7,10,12,13,14,15>
		<0,1,9,10,11,13,16>	<1,2,7,15,16,17,18>	<2,7,10,12,13,15,16>
		<0,1,9,11,13,16,18>	<1,2,9,10,12,13,16>	<5,7,13,14,17,18,19>
		<0,1,9,11,16,18,20>		

Table 2: Triangulation of BC_{OT}^1 and BC_{OT}^2 .

Polytope	BC_{OT}^1	BC_{OT}^2		
Simplexes 6-volumes	$\frac{1}{21600} \sqrt{1/3}$	$\frac{1}{7200} \sqrt{1/3}$	$\frac{1}{38880} \sqrt{1/3}$	$\frac{1}{43200} \sqrt{1/3}$
	$\frac{1}{43200} \sqrt{1/3}$	$\frac{1}{14400} \sqrt{1/3}$	$\frac{1}{34560} \sqrt{1/3}$	$\frac{1}{28800} \sqrt{1/3}$
	$\frac{1}{115200} \sqrt{1/3}$	$\frac{1}{5760} \sqrt{1/3}$	$\frac{1}{25920} \sqrt{1/3}$	$\frac{1}{17280} \sqrt{1/3}$
	$\frac{1}{230400} \sqrt{1/3}$	$\frac{1}{5760} \sqrt{1/3}$	$\frac{1}{51840} \sqrt{1/3}$	$\frac{1}{51840} \sqrt{1/3}$
	$\frac{1}{32400} \sqrt{1/3}$	$\frac{1}{8640} \sqrt{1/3}$	$\frac{1}{51840} \sqrt{1/3}$	$\frac{1}{17280} \sqrt{1/3}$
	$\frac{1}{86400} \sqrt{1/3}$	$\frac{1}{69120} \sqrt{1/3}$	$\frac{1}{34560} \sqrt{1/3}$	$\frac{1}{25920} \sqrt{1/3}$
	$\frac{1}{32400} \sqrt{1/3}$	$\frac{1}{69120} \sqrt{1/3}$	$\frac{1}{103680} \sqrt{1/3}$	$\frac{1}{17280} \sqrt{1/3}$
	$\frac{1}{43200} \sqrt{1/3}$	$\frac{1}{34560} \sqrt{1/3}$	$\frac{1}{51840} \sqrt{1/3}$	$\frac{1}{17280} \sqrt{1/3}$
	$\frac{1}{43200} \sqrt{1/3}$	$\frac{1}{69120} \sqrt{1/3}$	$\frac{1}{7200} \sqrt{1/3}$	$\frac{1}{16200} \sqrt{1/3}$
	$\frac{1}{25920} \sqrt{1/3}$	$\frac{1}{34560} \sqrt{1/3}$	$\frac{1}{14400} \sqrt{1/3}$	$\frac{1}{8100} \sqrt{1/3}$
	$\frac{1}{17280} \sqrt{1/3}$	$\frac{1}{23040} \sqrt{1/3}$	$\frac{1}{5760} \sqrt{1/3}$	$\frac{1}{16200} \sqrt{1/3}$
	$\frac{1}{23040} \sqrt{1/3}$	$\frac{1}{21600} \sqrt{1/3}$	$\frac{1}{5760} \sqrt{1/3}$	$\frac{1}{21600} \sqrt{1/3}$
	$\frac{1}{25920} \sqrt{1/3}$	$\frac{1}{14400} \sqrt{1/3}$	$\frac{1}{8640} \sqrt{1/3}$	$\frac{1}{21600} \sqrt{1/3}$
	$\frac{1}{17280} \sqrt{1/3}$	$\frac{1}{21600} \sqrt{1/3}$	$\frac{1}{69120} \sqrt{1/3}$	$\frac{1}{10800} \sqrt{1/3}$
	$\frac{1}{32400} \sqrt{1/3}$	$\frac{1}{28800} \sqrt{1/3}$	$\frac{1}{69120} \sqrt{1/3}$	$\frac{1}{10800} \sqrt{1/3}$
	$\frac{1}{32400} \sqrt{1/3}$	$\frac{1}{230400} \sqrt{3}$	$\frac{1}{34560} \sqrt{1/3}$	$\frac{1}{21600} \sqrt{1/3}$
	$\frac{1}{32400} \sqrt{1/3}$	$\frac{1}{21600} \sqrt{1/3}$	$\frac{1}{69120} \sqrt{1/3}$	$\frac{1}{7200} \sqrt{1/3}$
	$\frac{1}{64800} \sqrt{1/3}$	$\frac{1}{32400} \sqrt{1/3}$	$\frac{1}{34560} \sqrt{1/3}$	$\frac{1}{6480} \sqrt{1/3}$
	$\frac{1}{172800} \sqrt{1/3}$	$\frac{1}{10800} \sqrt{1/3}$	$\frac{1}{23040} \sqrt{1/3}$	$\frac{1}{2160} \sqrt{1/3}$
	$\frac{1}{21600} \sqrt{1/3}$	$\frac{1}{21600} \sqrt{1/3}$	$\frac{1}{11520} \sqrt{1/3}$	$\frac{1}{129600} \sqrt{1/3}$
	$\frac{1}{43200} \sqrt{1/3}$	$\frac{1}{14400} \sqrt{1/3}$	$\frac{1}{5400} \sqrt{1/3}$	$\frac{1}{64800} \sqrt{1/3}$
	$\frac{1}{25920} \sqrt{1/3}$	$\frac{1}{14400} \sqrt{1/3}$	$\frac{1}{4320} \sqrt{1/3}$	$\frac{1}{129600} \sqrt{1/3}$
	$\frac{1}{17280} \sqrt{1/3}$	$\frac{1}{28800} \sqrt{1/3}$	$\frac{1}{17280} \sqrt{1/3}$	$\frac{1}{48600} \sqrt{1/3}$
	$\frac{1}{23040} \sqrt{1/3}$	$\frac{1}{115200} \sqrt{3}$	$\frac{1}{8640} \sqrt{1/3}$	$\frac{1}{64800} \sqrt{1/3}$
	$\frac{1}{16200} \sqrt{1/3}$	$\frac{1}{32400} \sqrt{1/3}$	$\frac{1}{11520} \sqrt{1/3}$	$\frac{1}{43200} \sqrt{1/3}$
	$\frac{1}{32400} \sqrt{1/3}$	$\frac{1}{64800} \sqrt{1/3}$	$\frac{1}{11520} \sqrt{1/3}$	$\frac{1}{25920} \sqrt{1/3}$
	$\frac{1}{23040} \sqrt{1/3}$	$\frac{1}{86400} \sqrt{1/3}$	$\frac{1}{11520} \sqrt{1/3}$	$\frac{1}{32400} \sqrt{1/3}$
		$\frac{1}{21600} \sqrt{1/3}$	$\frac{1}{11520} \sqrt{1/3}$	$\frac{1}{21600} \sqrt{1/3}$
		$\frac{1}{21600} \sqrt{1/3}$	$\frac{1}{8640} \sqrt{1/3}$	$\frac{1}{10800} \sqrt{1/3}$
		$\frac{1}{11520} \sqrt{1/3}$	$\frac{1}{11520} \sqrt{1/3}$	$\frac{1}{8640} \sqrt{1/3}$
		$\frac{1}{17280} \sqrt{1/3}$	$\frac{1}{10800} \sqrt{1/3}$	$\frac{1}{21600} \sqrt{1/3}$
		$\frac{1}{8640} \sqrt{1/3}$	$\frac{1}{14400} \sqrt{1/3}$	$\frac{1}{57600} \sqrt{3}$
		$\frac{1}{11520} \sqrt{1/3}$	$\frac{1}{14400} \sqrt{1/3}$	$\frac{1}{230400} \sqrt{3}$
		$\frac{1}{12960} \sqrt{1/3}$	$\frac{1}{28800} \sqrt{1/3}$	$\frac{1}{172800} \sqrt{1/3}$
		$\frac{1}{6480} \sqrt{1/3}$	$\frac{1}{28800} \sqrt{3}$	$\frac{1}{69120} \sqrt{1/3}$
		$\frac{1}{12960} \sqrt{1/3}$	$\frac{1}{17280} \sqrt{1/3}$	$\frac{1}{12960} \sqrt{1/3}$
		$\frac{1}{8640} \sqrt{1/3}$	$\frac{1}{5760} \sqrt{1/3}$	$\frac{1}{43200} \sqrt{1/3}$
		$\frac{1}{103680} \sqrt{1/3}$	$\frac{1}{8640} \sqrt{1/3}$	$\frac{1}{14400} \sqrt{1/3}$
		$\frac{1}{51840} \sqrt{1/3}$	$\frac{1}{86400} \sqrt{1/3}$	$\frac{1}{6480} \sqrt{1/3}$
		$\frac{1}{103680} \sqrt{1/3}$		

Table 3: 6-volumes of the 6-dimensional simplexes described in Table 2.

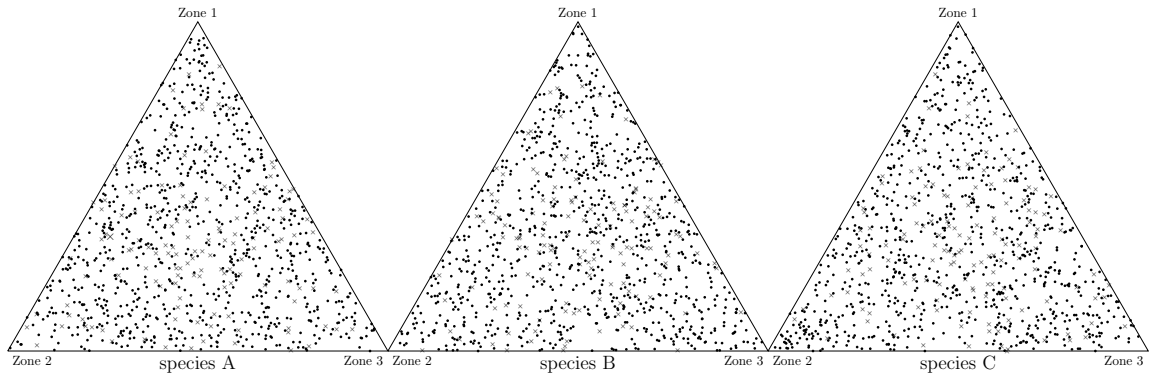


Figure 2: Graphical representation of a random subset of approximately 1 000 initial configurations over the 993 213 that were taken for the simulations. It's aimed at visually confirming (or at least not disapproving) that the initial configurations are randomly and uniformly chosen. Dots (respectively crosses) are configurations that had converged to a c_{OE} (respectively c_{OT}).