On the exponential stability for a class of stochastic differential delay equations with a fractional Brownian noise

Cao Tan Binh [*], Luu Hoang Duc [†], Phan Thanh Hong [‡]

Abstract

We investigate the asymptotic stability of the trivial solution of a stochastic differential delay equation which is driven by a fractional Brownian noise. Under the strong dissipative condition, we prove the exponential stability of the trivial solution under a small intensity of noise.

Keywords: fractional Brownian motion, stochastic differential equations (SDE), Young integral, exponential stability.

1 Introduction

In this paper, we study the asymptotic stability of the following stochastic differential delay equation

$$dy(t) = [Ay(t) + f(y(t-r))]dt + Cy(t)dB^H(t), \tag{1.1}$$

given initial function $\eta \in \mathcal{C}([-r,0],\mathbb{R}^d)$, where $A, C \in \mathbb{R}^{d\times d}$ are matrices, r is a constant delay; $f : \mathbb{R}^d \to \mathbb{R}^d$ is a Lipschitz function, i.e.

$$\|f(y_1) - f(y_2)\| \leq C_f\|y_1 - y_2\|, \tag{1.2}$$

and satisfies $f(0) = 0$. B^H is an one-dimensional fractional Brownian motion [1] on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the Hurst index $H > 1/2$. Equation (1.1) is understood and solved in the pathwise sense as a Young differential equation with delay

$$dy(t) = [Ay(t) + f(y(t-r))]dt + Cy(t)d\omega(t). \tag{1.3}$$

When $r = 0$, system (1.1) has the non-delay form, and the exponential stability of the trivial solution has been studied in recent papers [2–4] for Young equations. The stability problem for the Young differential delay equation (1.1) is however still in a fancy stage and is only studied in a few references, e.g. [5], although solutions of the Young differential equations have been well studied, [4, 6, 7]. Since it is not sure how to apply the semigroup technique developed in [4] for the delay situation, our aim in this paper is to apply the techniques developed in [3] for nondelay equations to prove the exponential stability for the trivial solution of (1.1).

We present here a brief introduction to Young differential equations for Hölder continuous functions to match the delay situation (a detailed explaination can be found in [6, 7]). For each $0 < \alpha < 1$, denote by $C^{\alpha}([a,b],\mathbb{R}^d)$ the space of Hölder continuous functions on $[a,b]$ equipped with the norm

$$\|u\|_{\infty,\alpha,[a,b]} := \|u\|_{\infty,[a,b]} + \left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert u \right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert_{\alpha,[a,b]}$$

[*]Cao T. Binh is with Quy Nhon University, Vietnam & Department of Mathematics, Hanoi National University, Vietnam `caotanbinh@qnu.edu.vn`

[†]Luu H. Duc is with Max Planck Institute for Mathematics in the Sciences, Leipzig, Germany, & Institute of Mathematics, Viet Nam Academy of Science and Technology, Ha Noi, Viet Nam `duc.luu@mis.mpg.de, lhduc@math.ac.vn`

[‡]Phan T. Hong is with Thang Long University, Hanoi, Vietnam `hongpt@thanglong.edu.vn`

where $\|\cdot\|_{\infty,[a,b]}$ is the sup norm of continuous functions on $[a,b]$ and

$$|\!|\!| u |\!|\!|_{\alpha,[a,b]} := \sup_{a\leq s<t\leq b} \frac{\|u(t)-u(s)\|}{(t-s)^\alpha}.$$

We also introduce the space

$$C^{0,\alpha}([a,b],\mathbb{R}^d) := \Big\{ u \in C^\alpha([a,b],\mathbb{R}^d) \Big| \lim_{h\to 0} \sup_{\substack{a\leq s<t\leq b,\\ |t-s|<h}} \frac{\|u(t)-u(s)\|}{(t-s)^\alpha} = 0 \Big\}.$$

Given $y \in C^\beta([a,b],\mathbb{R}^d)$ and $\omega \in C^\nu([a,b],\mathbb{R})$, $\beta+\nu>1$, it is well known that the *Young integral* $\int_a^b y(t)d\omega(t)$ exists (see [8, p. 264–265]). This integral satisfies additive property by the construction, and the so-called Young-Loeve estimate [9, Theorem 6.8, p. 116]

$$\Big\| \int_s^t y(u)d\omega(u) - y(s)[\omega(t)-\omega(s)] \Big\| \leq K|t-s|^{\beta+\nu} |\!|\!| y |\!|\!|_{\beta,[s,t]} |\!|\!| \omega |\!|\!|_{\nu,[s,t]}, \tag{1.4}$$

where

$$K := (1-2^{1-\theta})^{-1}, \qquad \theta := \beta+\nu > 1. \tag{1.5}$$

A result given in [10] shows that, under some regularity conditions of f, i.e. f is Lipchitz continuous, then there exists a unique solution $y(\cdot,\omega,\eta)$ of (1.1) on $\mathcal{C}^\beta([-r,T],\mathbb{R}^d)$ for $H>\beta>1-H$ almost surely. A similar result can be found in [7]. In fact, (1.1) can be solved by induction in each interval $[kr,(k+1)r]$ as a Young differential equation. Namely, given $y_0=\eta \in \mathcal{C}^\beta([-r,0],\mathbb{R}^d)$, then (1.3) derives

$$dy(kr+t) = [Ay(kr+t) + f(y((k-1)r+t))]dt + Cy(kr+t)d\theta_{kr}\omega(t), \quad \forall t \in [0,r].$$

The solution of (1.3) can be written by induction in the explicit form as $y_0(\cdot)=\eta(\cdot)\in\mathcal{C}^\beta$ and

$$y(kr+t) = \Phi(t,\theta_{kr}\omega)y(kr) + \int_0^t \Phi(t-s,\theta_{kr+s}\omega)f(y((k-1)r+s))ds,\ t\in[0,r] \tag{1.6}$$

for all $k\geq 1$, where $\Phi(t,x)$ is the matrix solution of the linear Young differential equation

$$dz(t) = Az(t)dt + Cz(t)d\omega(t) \tag{1.7}$$

and satisfies $\Phi(0,\omega)=Id$.
Due to [11, Remark 2.3] $\Phi(\cdot,\theta_{kr}\omega)y(kr) \in C^\beta([0,r],\mathbb{R}^d)$. Hence (1.6) implies that $y(kr+\cdot) \in C^\beta([0,r],\mathbb{R}^d)$. By induction, one can prove that $y(\cdot)\in C^\beta([-r,T],\mathbb{R}^d)$ for all $T\geq -r$. As a result, $y_t(\cdot)$ belongs to $C^\beta([-r,0],\mathbb{R}^d)$ for all $t\geq 0$.

2 Exponential stability

In this section we would like to study the exponential stability of the zero solution of (1.1). We will work with the canonical space $(\Omega,\mathcal{F},\mathbb{P},\theta)$ of fBm as introduced in [6]. Namely, for fixed $1/2<\nu'<H$, denote by $C_0^{\nu'}(\mathbb{R},\mathbb{R})$ the space of all paths ω which belong to $C^{\nu'}(I,\mathbb{R})$ for all closed interval $I\subset\mathbb{R}$ and receive value 0 at time 0. Then $(\Omega,\mathcal{F},\mathbb{P},\theta)$ is the induced space on $C_0^{\nu'}(\mathbb{R},\mathbb{R})$ of $(C_0(\mathbb{R},\mathbb{R}),\mathcal{B},\mathbb{P},\theta)$ where $(C_0(\mathbb{R},\mathbb{R})$ is the space of all continuous paths ω on $\mathbb{R}$ such that $\omega(0)=0$, $\mathcal{B}$ is the Borel $-\sigma$ algebra generated by the compact open topology, $\mathbb{P}$ is the Wiener measure on $\mathcal{B}$ generated by a fractional Brownian motion and θ is the Wiener shift define by $\theta_t\omega(\cdot)=\omega(t+\cdot)-\omega(\cdot)$. Note that the Wiener shift is ergodic due to [12]. It is proved in [6] that equation (1.1) then generates

a so-called *random dynamical system* $\varphi : \mathbb{R}_+ \times (\Omega, \mathcal{F}, \mathbb{P}, \theta) \times C^{0,\beta}([-r,0], \mathbb{R}^d) \to C^{0,\beta}([-r,0], \mathbb{R}^d)$ given by

$$\varphi(t,\omega,\eta)(s) = y(t+s,\omega,\eta), \quad \forall t \in \mathbb{R}_+, s \in [-r,0], \omega \in \Omega, \eta \in C^{0,\beta}([-r,0], \mathbb{R}^d).$$

From now on, we fix $\nu \in (1/2, \nu')$ and $\beta \in (1-\nu, \nu)$. It is known that $B^H(\omega)|_{[a,b]} \in C^{0,\nu}([a,b], \mathbb{R}),\ \forall \omega \in \Omega$ and $E \left\|\!\left| B^H \right|\!\right\|^m_{\nu,[a,b]}$ is bounded for all $[a,b] \subset \mathbb{R}$ and $m > 0$ (see for instance [5]).
To obtain the main results we need the following lemmas.

Lemma 2.1 *Assume that for fixed T, $y \in C^\beta([0,T], \mathbb{R}^d)$ satisfies*

$$|\!|\!| y |\!|\!|_{\beta,[a,b]} \leq M\Big[(b-a)^{1-\beta} + (b-a)^{\nu-\beta} |\!|\!| \omega |\!|\!|_{\nu,[a,b]}\Big]\Big[L + (b-a)^\beta |\!|\!| y |\!|\!|_{\beta,[a,b]}\Big] \tag{2.1}$$

for all $0 \leq a < b \leq T$, in which M, L are constants. Then

$$|\!|\!| y |\!|\!|_{\beta,[0,T]} \leq L[2 \vee (4M)]^{1+\frac{1}{\nu}} T^{-\beta} \left(1 + T^{1+\frac{1}{\nu}} + T^{1+\nu} |\!|\!| \omega |\!|\!|^{1+\frac{1}{\nu}}_{\nu,[0,T]}\right), \tag{2.2}$$

in which $a \vee b := \max\{a,b\}$.

Proof: Fix $\mu = \frac{1}{2M}$, we construct a sequence $\{t_i\}$ such that $t_0 = 0$ and

$$t_{i+1} = \inf\{t \geq t_i \mid \ (t-t_i) + (t-t_i)^\nu |\!|\!| \omega |\!|\!|_{\nu,[t_i,t]} \geq \mu\}.$$

Since $|\, |\!|\!| \omega |\!|\!|_{\nu,[t_i,t']} - |\!|\!| \omega |\!|\!|_{\nu,[t_i,t]} \,| \leq |\!|\!| \omega |\!|\!|_{\nu,[t,t']}$ for all $t_i \leq t \leq t'$ (see [7, p. 316]) and $\omega \in \mathcal{C}^{0,\nu}(I, \mathbb{R})$ for all closed interval I, $|\, |\!|\!| \omega |\!|\!|_{\nu,[t_i,t']} - |\!|\!| \omega |\!|\!|_{\nu,[t_i,t]} \,| \leq |\!|\!| \omega |\!|\!|_{\nu,[t,t']} \to 0$ as $t' \to t$ (see [9, Corollary 5.31, p. 96]). It follows that for given t_i the function $f(t) = (t-t_i) + (t-t_i)^\nu |\!|\!| \omega |\!|\!|_{\nu,[t_i,t]}$ is continuous on $[t_i, +\infty)$. Moreover, f is strictly increasing, $f(t_i) = 0$ and $\lim_{t \to +\infty} f(t) = +\infty$. Therefore, there exists t_{i+1} such that

$$(t_{i+1} - t_i) + (t_{i+1} - t_i)^\nu |\!|\!| \omega |\!|\!|_{\nu,[t_i,t_{i+1}]} = \mu. \tag{2.3}$$

Assign $N := \sup\{n \mid t_n \leq T\} < \infty$.

For $i = 0, ..., N-1$, (2.3) leads to

$$\mu = (t_{i+1}-t_i)^\nu \Big[(t_{i+1}-t_i)^{1-\nu} + |\!|\!| \omega |\!|\!|_{\nu,[t_i,t_{i+1}]}\Big] \leq (t_{i+1}-t_i)^\nu \left(T^{1-\nu} + |\!|\!| \omega |\!|\!|_{\nu,[0,T]}\right).$$

This implies

$$t_{i+1} - t_i \geq \left(\frac{\mu}{T^{1-\nu} + |\!|\!| \omega |\!|\!|_{\nu,[0,T]}}\right)^{\frac{1}{\nu}},$$

and

$$T \geq \sum_{i=0}^{N-1} (t_{i+1} - t_i) \geq N \left(\frac{\mu}{T^{1-\nu} + |\!|\!| \omega |\!|\!|_{\nu,[0,T]}}\right)^{\frac{1}{\nu}}.$$

It follows that

$$N \leq T \left(\frac{T^{1-\nu} + |\!|\!| \omega |\!|\!|_{\nu,[0,T]}}{\mu}\right)^{\frac{1}{\nu}}. \tag{2.4}$$

By the asumption, for $0 \leq i \leq N-1$ we have

$$|\!|\!| y |\!|\!|_{\beta,[t_i,t_{i+1}]} \leq M\Big[(t_{i+1}-t_i)^{1-\beta} + (t_{i+1}-t_i)^{\nu-\beta} |\!|\!| \omega |\!|\!|_{\nu,[t_i,t_{i+1}]}\Big]\Big[L + (t_{i+1}-t_i)^\beta |\!|\!| y |\!|\!|_{\beta,[t_i,t_{i+1}]}\Big]$$

or

$$|\!|\!| y |\!|\!|_{\beta,[t_i,t_{i+1}]} \leq \frac{ML}{1-M\mu}\Big[T^{1-\beta} + T^{\nu-\beta} |\!|\!| \omega |\!|\!|_{\nu,[0,T]}\Big].$$

We also have

$$\|y\|_{\beta,[t_N,T]} \leq \frac{ML}{1-M\mu}\Big[T^{1-\beta} + T^{\nu-\beta}\,\|\omega\|_{\nu,[0,T]}\Big].$$

Because $[0,T] \subset \cup_{i=0}^{N-1}[t_i,t_{i+1}] \cup [t_N,T]$ and Hölder norm is of the superadditivity, we obtain

$$\begin{aligned}
\|y\|_{\beta,[0,T]} &\leq \sum_{i=0}^{N-1} \|y\|_{\beta,[t_i,t_{i+1}]} + \|y\|_{\beta,[t_N,T]} \\
&\leq (N+1)\frac{ML}{1-M\mu}\Big[T^{1-\beta} + T^{\nu-\beta}\,\|\omega\|_{\nu,[a,b]}\Big]. \qquad (2.5)
\end{aligned}$$

Combining this with (2.4) we get

$$\begin{aligned}
\|y\|_{\beta,[0,T]} &\leq 2ML\Big[1 + T\Big(\frac{T^{1-\nu} + \|\omega\|_{\nu,[0,T]}}{\mu}\Big)^{\frac{1}{\nu}}\Big]\Big[T^{1-\beta} + T^{\nu-\beta}\,\|\omega\|_{\nu,[0,T]}\Big] \\
&\leq L(2M)^{1+\frac{1}{\nu}}T^{-\beta}\Big[\frac{1}{(2M)^{\frac{1}{\nu}}} + \Big(T + T^{\nu}\,\|\omega\|_{\nu,[0,T]}\Big)^{\frac{1}{\nu}}\Big]\Big[T + T^{\nu}\,\|\omega\|_{\nu,[0,T]}\Big] \\
&\leq L[1 \vee (2M)]^{1+\frac{1}{\nu}}T^{-\beta}\Big[1 + \Big(T + T^{\nu}\,\|\omega\|_{\nu,[0,T]}\Big)^{\frac{1}{\nu}}\Big]\Big[T + T^{\nu}\,\|\omega\|_{\nu,[0,T]}\Big] \\
&\leq L[1 \vee (2M)]^{1+\frac{1}{\nu}}T^{-\beta}\Big[T + T^{\nu}\,\|\omega\|_{\nu,[0,T]} + (T + T^{\nu}\,\|\omega\|_{\nu,[0,T]})^{1+\frac{1}{\nu}}\Big]. \qquad (2.6)
\end{aligned}$$

Using the following inequalities

$$x^p + p - 1 \geq px,\ \forall x \geq 0, p \geq 1, \qquad (2.7)$$

and

$$(a+b)^p \leq (a^p + b^p)\max\{1, 2^{p-1}\},\ \forall a, b, p > 0, \qquad (2.8)$$

we have

$$\begin{aligned}
\|y\|_{\beta,[0,T]} &\leq L[1 \vee (2M)]^{1+\frac{1}{\nu}}T^{-\beta}\Big[\frac{(T + T^{\nu}\,\|\omega\|_{\nu,[0,T]})^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}} + \frac{\frac{1}{\nu}}{1+\frac{1}{\nu}} + (T + T^{\nu}\,\|\omega\|_{\nu,[0,T]})^{1+\frac{1}{\nu}}\Big] \\
&\leq L[1 \vee (2M)]^{1+\frac{1}{\nu}}T^{-\beta}\Big[2(T + T^{\nu}\,\|\omega\|_{\nu,[0,T]})^{1+\frac{1}{\nu}} + 1\Big] \\
&\leq L[2 \vee (4M)]^{1+\frac{1}{\nu}}T^{-\beta}\Big(1 + T^{1+\frac{1}{\nu}} + T^{1+\nu}\,\|\omega\|_{\nu,[0,T]}^{1+\frac{1}{\nu}}\Big). \qquad (2.9)
\end{aligned}$$

The proof is completed. □

The following lemma is a Hölder-version of [3, Corrolary 3.5], which facilitates the proof of the stability of the delay system.

Lemma 2.2 *Assuming that in (1.7) A is negative definite, i.e. there exists constant $h_0 > 0$ such that*

$$\langle y, Ay\rangle \leq -h_0\|y\|^2, \quad \forall y \in \mathbb{R}^d. \qquad (2.10)$$

Then for all $0 \leq s < t \leq r$, the following inequalities hold

$$\begin{aligned}
(i) &\quad \|\Phi(t,\omega)\| \leq \exp\{-h_0 t + \|C\|tQ(r,\omega) + 2KG\|C\|\}, \qquad (2.11)\\
(ii) &\quad \|\Phi(t-s,\theta_s\omega)\| \leq \exp\{-h_0(t-s) + \|C\|(t-s)Q(r,\omega) + 2KG\|C\|\}, \qquad (2.12)
\end{aligned}$$

where

$$G := \Big(\max\{2, 8\|A\|, 16K\|C\|\}\Big)^{1+\frac{1}{\nu}} \qquad (2.13)$$

and

$$Q(t,\omega) := 2KG\Big[\frac{\nu(1+2KG)}{2KG}\,\|\omega\|_{\nu,[0,t]}^{\frac{1}{\nu}} + t^{\nu+\frac{1}{\nu}}\,\|\omega\|_{\nu,[0,t]} + t^{2\nu}\,\|\omega\|_{\nu,[0,t]}^{2+\frac{1}{\nu}}\Big]. \qquad (2.14)$$

Proof: (i) In the proof of [3, Theorem 3.4], we obtain

$$\begin{aligned}\log\|\Phi(t,\omega)\| &\leq -h_0t+\Big|\int_0^t\langle y(s),Cy(s)\rangle d\omega(s)\Big|\\ &\leq -h_0t+\|C\|t^\nu \left\|\omega\right\|_{\nu,[0,t]}\Big(1+2Kt^\beta \left\|y\right\|_{\beta,[0,t]}\Big).\end{aligned}$$

in which $y(t)=\frac{\Phi(t,\omega)}{\|\Phi(t,\omega)\|}$ satisfies

$$\left\|y\right\|_{\beta,[s,t]}\leq (2\|A\|\vee 4K\|C\|)\Big[(t-s)^{1-\beta}+(t-s)^{\nu-\beta}\left\|\omega\right\|_{\nu,[s,t]}\Big]\Big[1+(t-s)^\beta\left\|y\right\|_{\beta,[s,t]}\Big],\ \forall s<t.$$

Apply Lemma 2.1 for $M=2\|A\|\vee 4K\|C\|$ and $L=1$, we get

$$\left\|y\right\|_{\beta,[0,t]}\leq Gt^{-\beta}\Big(1+t^{1+\frac{1}{\nu}}+t^{1+\nu}\left\|\omega\right\|_{\nu,[0,t]}^{1+\frac{1}{\nu}}\Big).$$

Therefore, for $0\leq t\leq r$

$$\begin{aligned}\log\|\Phi(t,\omega)\| &\leq -h_0t+\|C\|t^\nu\left\|\omega\right\|_{\nu,[0,t]}\Big[1+2KG\Big(1+t^{1+\frac{1}{\nu}}+t^{1+\nu}\left\|\omega\right\|_{\nu,[0,t]}^{1+\frac{1}{\nu}}\Big)\Big]\\ &\leq -h_0t+\|C\|t^\nu\left\|\omega\right\|_{\nu,[0,t]}\Big[1+2KG+2KGt^{1+\frac{1}{\nu}}+2KGt^{1+\nu}\left\|\omega\right\|_{\nu,[0,t]}^{1+\frac{1}{\nu}}\Big]\\ &\leq -h_0t+(1+2KG)\|C\|t^\nu\left\|\omega\right\|_{\nu,[0,t]}\\ &\quad +2KG\|C\|t^\nu\left\|\omega\right\|_{\nu,[0,t]}\Big(t^{1+\frac{1}{\nu}}+t^{1+\nu}\left\|\omega\right\|_{\nu,[0,t]}^{1+\frac{1}{\nu}}\Big)\\ &\leq -h_0t+(1+2KG)\|C\|\frac{t\left\|\omega\right\|_{\nu,[0,t]}^{\frac{1}{\nu}}+\frac{1}{\nu}-1}{\frac{1}{\nu}}\\ &\quad +2KG\|C\|t^\nu\left\|\omega\right\|_{\nu,[0,t]}\Big(t^{1+\frac{1}{\nu}}+t^{1+\nu}\left\|\omega\right\|_{\nu,[0,t]}^{1+\frac{1}{\nu}}\Big)\\ &\leq -h_0t+2KG\|C\|\\ &\quad +2KG\|C\|t\Big[\frac{\nu(1+2KG)}{2KG}\left\|\omega\right\|_{\nu,[0,t]}^{\frac{1}{\nu}}+t^{\nu+\frac{1}{\nu}}\left\|\omega\right\|_{\nu,[0,t]}+t^{2\nu}\left\|\omega\right\|_{\nu,[0,t]}^{2+\frac{1}{\nu}}\Big].\end{aligned}$$

This implies (2.11).
(ii) For $0\leq s<t\leq r$,

$$\|\Phi(t-s,\theta_s\omega)\| \quad\leq\quad \exp\Big\{-h_0(t-s)+\|C\|(t-s)Q(t-s,\theta_s\omega)+2KG\|C\|\Big\}.$$

On the other hand

$$\begin{aligned}\left\|\theta_s\omega\right\|_{\nu,[0,t-s]} &= \sup_{0\leq v<u\leq t-s}\frac{\|\omega(u+s)-\omega(v+s)\|}{|(u+s)-(v+s)|^\nu}\\ &= \sup_{s\leq v<u\leq t}\frac{\|\omega(u)-\omega(v)\|}{|u-v|^\nu}\leq \left\|\omega\right\|_{\nu,[0,r]}.\end{aligned}$$

As a result $Q(t-s,\theta_s\omega)\leq Q(r,\omega)$, which proves (2.12). □

To state our main result, let us introduce a Gronwall-Bellman type estimation, see the proof in [13].

Proposition 2.3 *Let $t_0\in\mathbb{R}, 0<t_0\leq\infty, c\geq 0$ and $a:[t_0,T]\to\mathbb{R}_+$ be locally integrable. Assume $r\geq 0$, and $\tau:[t_0,T)\to\mathbb{R}_+$ is a measurable function such that $t_0-r\leq t-\tau(t), t_0\leq t<T$. If $x:[t_0-r,T)\to\mathbb{R}_+$ is Borel measurable and locally bounded such that*

$$x(t)\leq c+\int_{t_0}^t a(u)x(u-\tau)du,\ t_0\leq t<T, \tag{2.15}$$

then

$$x(t) \le K \exp\Big\{ \int_{t_0}^{t} \gamma(s)ds \Big\},\ t_0 \le t < T, \tag{2.16}$$

where the function $\gamma : [t_0 - r, T) \to \mathbb{R}_+$ *is locally integrable, and satisfies the characteristic inequality*

$$a(t) \exp\Big\{ - \int_{t-\tau(t)}^{t} \gamma(s)ds \Big\} \le \gamma(t),\ t_0 \le t < T, \tag{2.17}$$

and

$$K := \max\Big\{ c \exp\Big\{ \int_{t_0 - r}^{t_0} \gamma(u)du \Big\}, \sup_{t_0 - r \le s \le t_0} \exp\Big\{ \int_{s}^{t_0} \gamma(u)du \Big\} \Big\}. \tag{2.18}$$

When a and τ are constant functions, we get the following corollary

Corollary 2.4 *Let* $t_0 \in \mathbb{R}, t_0 < T \le \infty$, *and* $c, a, \tau \ge 0$. *If* $y : [t_0 - \tau, T) \to \mathbb{R}_+$ *is Borel measurable and locally bounded such that*

$$y(t) \le c + \int_{t_0}^{t} a y(u - \tau)du,\ t_0 \le t < T, \tag{2.19}$$

then

$$y(t) \le K e^{\gamma(t - t_0)},\ t_0 \le t < T, \tag{2.20}$$

where the nonnegative number γ *satisfies the inequality*

$$a \le \gamma e^{\gamma\tau}, \tag{2.21}$$

and

$$K := \max\Big\{ c e^{\gamma\tau}, \sup_{t_0 - r \le s \le t_0} y(s) e^{\gamma(t_0 - s)} \Big\}. \tag{2.22}$$

Our first main result can be formulated as follows.

Theorem 2.5 *Assume* A *is negative definite, i.e. there exists a* $h_0 > 0$ *such that*

$$\langle y, Ay \rangle \le -h_0 \|y\|^2. \tag{2.23}$$

and $C_f < \frac{h_0}{2} e^{-\frac{h_0 r}{2}}$. *Then there exists* $\varepsilon > 0$ *such that if* $\|C\| < \varepsilon$ *the trivial solution of system* (1.1) *is forward exponentially stable almost surely, i.e. for any solution* y *of* (1.1) *we have*

$$\limsup_{t \to \infty} \frac{1}{t} \log \|y_t(\omega, y_0)\|_{\infty, \beta, [-r, 0]} < 0, \tag{2.24}$$

for almost surely all the realizations ω.

Proof: First, we fix ω, η and denote by $y(t)$, $t \in [-r, \infty)$ the solution of (1.3). Since the assumption of the function f, it follows that

$$\|f(y)\| \le C_f \|y\|. \tag{2.25}$$

Due to (1.6), (2.11), (2.12) and (2.25), for all $t \in [0, r]$

$$\begin{aligned} \|y(kr + t)\| \;\le\;& \exp\{-h_0 t + \|C\| t Q(r, \theta_{kr}\omega) + 2KG\|C\|\} \|y(kr)\| \\ & + C_f \int_0^t \exp\{-h_0(t - s) + \|C\|(t - s) Q(r, \theta_{kr}\omega) + 2KG\|C\|\} \|y(kr + s - r)\| ds. \end{aligned} \tag{2.26}$$

For each $k \in \mathbb{N}$ is fixed. Put

$$z^k(t) = \|y(kr+t)\| \exp\{h_0 t - \|C\| t Q(r, \theta_{kr}\omega)\}, t \in [-r, r]. \tag{2.27}$$

We have

$$\begin{aligned} z^k(t) &\leq e^{2KG\|C\|}\|y(kr)\| + e^{2KG\|C\|} C_f \int_0^t \exp\{h_0 s - \|C\| s Q(r, \theta_{kr}\omega)\} \|y(kr+s-r)\| ds \\ &\leq e^{2KG\|C\|}\|y(kr)\| + C_f \exp\{h_0 r - \|C\| r Q(r, \theta_{kr}\omega) + 2KG\|C\|\} \int_0^t z^k(s-r) ds. \end{aligned} \tag{2.28}$$

By the assumption $C_f < \frac{h_0}{2} e^{-\frac{h_0 r}{2}}$, there exists $\varepsilon_1 > 0$ such that $C_f < \frac{h_0}{2} \exp\{-\frac{h_0 r}{2} - 2KG\varepsilon_1\}$. Then for $\|C\| < \varepsilon_1$

$$C_f \exp\{h_0 r + 2KG\|C\|\} < \frac{h_0}{2} \exp\left\{\frac{h_0 r}{2} + 2KG(\|C\| - \varepsilon_1)\right\} < \frac{h_0}{2} e^{\frac{h_0 r}{2}}.$$

One can choose $\gamma \in (0, \frac{h_0}{2})$ not depending on k and ω such that

$$C_f \exp\{h_0 r - \|C\| r Q(r, \theta_{kr}\omega) + 2KG\|C\|\} \leq C_f \exp\{h_0 r + 2KG\|C\|\} \leq \gamma e^{\gamma r}.$$

Combining this with Corollary 2.4 we have for each $t \in [0, r]$

$$z^k(t) \leq M_k e^{\gamma t}, \ \text{ or } \ \|y(kr+t)\| \leq M_k \exp\{-h_0 t + \gamma t + \|C\| t Q(r, \theta_{kr}\omega)\}, \tag{2.29}$$

where

$$M_k = \max\left\{e^{\gamma r + 2KG\|C\|}\|y(kr)\|, \sup_{s \in [-r,0]} \|y(kr+s)\| \exp\{h_0 s - \gamma s - \|C\| s Q(r, \theta_{kr}\omega)\}\right\}. \tag{2.30}$$

Next, we introduce the function

$$\rho^k(t) = M_k \exp\{-h_0 t + \gamma t + \|C\| t Q(r, \theta_{kr}\omega)\}$$

then $\|y(kr+t)\| \leq \rho^k(t)$ for all $t \in [0, r]$ and

$$\rho^k(r) = M_k \exp\{-h_0 r + \gamma r + \|C\| r Q(r, \theta_{kr}\omega)\}.$$

Next we will prove that

$$M_k \leq \rho^{k-1}(r) \exp\{\gamma r + \|C\| r Q(r, \theta_{kr}\omega) + 2KG\|C\|\}. \tag{2.31}$$

Firstly, by the definition of M_k and (2.29), for $s \in [-r, 0]$,

$$\begin{aligned} &\|y(kr+s)\| \exp\{h_0 s - \gamma s - \|C\| s Q(r, \theta_{kr}\omega)\} \\ \leq\ &\|y((k-1)r + s + r)\| \exp\{h_0(s+r) - \gamma(s+r)\} \exp\{-h_0 r + \gamma r + \|C\| r Q(r, \theta_{kr}\omega)\} \\ \leq\ &\|y((k-1)r + s')\| \exp\{h_0 s' - \gamma s'\} \exp\{-h_0 r + \gamma r + \|C\| r Q(r, \theta_{kr}\omega)\} \\ \leq\ &M_{k-1} \exp\{\|C\| r Q(r, \theta_{(k-1)r}\omega)\} \exp\{-h_0 r + \gamma r + \|C\| r Q(r, \theta_{kr}\omega)\}, \end{aligned} \tag{2.32}$$

in which $0 \leq s + r = s' \leq r$.

On the other hand,

$$M_{k-1} \exp\{-h_0 r + \gamma r + \|C\| r Q(r, \theta_{(k-1)r}\omega)\} = \rho^{k-1}(r)$$

then

$$\sup_{[-r,0]} \|y(ks+r)\| \exp\{h_0 s - \gamma s - \|C\| s Q(r,\theta_{kr}\omega) s\} \le \rho^{k-1}(r) e^{\|C\| r Q(r,\theta_{kr}\omega)}. \tag{2.33}$$

Secondly,

$$e^{\gamma r + 2KG\|C\|}\|y(kr)\| = e^{\gamma r + 2KG\|C\|}\|y((k-1)r + r)\| \le \rho^{k-1}(r) e^{\gamma r + 2KG\|C\|}.$$

Hence

$$M_k \le \rho^{k-1}(r) \exp\{\gamma r + \|C\| r Q(r,\theta_{kr}\omega) + 2KG\|C\|\}. \tag{2.34}$$

This implies

$$\begin{aligned}
\rho^k(r) &= M_k \exp\{-h_0 r + \gamma r + \|C\| r Q(r,\theta_{kr}\omega)\} \\
&\le \rho^{k-1}(r) \exp\{-h_0 r + 2\gamma r + 2\|C\| r Q(r,\theta_{kr}\omega) + 2KG\|C\|\} \\
&\quad \cdots \\
&\le \rho^0(r) \prod_{j=1}^{k} \exp\{-h_0 r + 2\gamma r + 2\|C\| r Q(r,\theta_{jr}\omega) + 2KG\|C\|\} \\
&= \rho^0(r) \exp\Big\{\Big[-(h_0 - 2\gamma) r + 2\|C\| r \frac{1}{k}\sum_{j=1}^{k} Q(r,\theta_{jr}\omega) + 2KG\|C\|\Big] k\Big\}.
\end{aligned} \tag{2.35}$$

Applying Birkhoff ergodic theorem

$$\begin{aligned}
&\lim_{k\to\infty}\Big\{-(h_0 - 2\gamma) r + 2KG\|C\| + \frac{2\|C\| r}{k}\sum_{j=1}^{k} Q(r,\theta_{jr}\omega)\Big\} \\
&= -(h_0 - 2\gamma) r + 2KG\|C\| + 2\|C\| r E[Q(r,\omega)] =: -\lambda
\end{aligned} \tag{2.36}$$

holds for almost all ω. Choose

$$\epsilon = \min\Big\{\epsilon_1, \frac{(h_0 - 2\gamma) r}{2KG + 2rE[Q(r,\omega)]}\Big\}.$$

Then $\lambda > 0$ provided that $\|C\| < \varepsilon$. Moreover, $\rho^0(r) = M_0 \exp\{-h_0 r + \gamma r + \|C\| r Q(r,\omega)\}$, in which

$$\begin{aligned}
M_0 &= \max\left\{e^{\gamma r + 2KG\|C\|}\|\eta(0)\|, \sup_{s\in[-r,0]} \|\eta(s)\| \exp\{h_0 s - \gamma s - \|C\| s Q(r,\omega)\}\right\} \\
&\le \|\eta\|_{\infty,[-r,0]} \exp\Big\{\gamma r + \|C\| Q(r,\omega) + 2KG\|C\|\Big\}.
\end{aligned}$$

Hence

$$\rho^0(r) \le \|\eta\|_{\infty,[-r,0]} \exp\Big\{2\gamma r + 2\|C\| r Q(r,\omega) + 2KG\|C\|\Big\}.$$

Hence when k is large enough, (2.36) deduce

$$\Big[-(h_0 - 2\gamma) r + 2KG\|C\| + 2\|C\| r \frac{1}{k}\sum_{j=1}^{k} Q(r,\theta_{jr}\omega)\Big] k < \frac{-\lambda k}{2}. \tag{2.37}$$

As a result

$$\rho^k(r) \le \rho^0(r) e^{\frac{-\lambda k}{2}}.$$

On the other hand, it is obvious that, for any $0 \le t \le r$

$$\|y(kr + t)\| \le \rho^k(t) \le e^{h_0 r}\rho^k(r) \le e^{-\frac{\lambda k}{2} + h_0 r}\rho^0(r).$$

Therefore

$$\limsup_{t\to\infty} \frac{1}{t} \log \|y_t\|_{\infty,[-r,0]} < -\lambda/2. \tag{2.38}$$

Moreover, for any $0 \leq t_1 < t_2 \leq r$,

$$\begin{aligned}
& \|y(kr+t_2,\omega,\eta) - y(kr+t_1,\omega,\eta)\| \\
\leq\ & \Big\|\Big(\Phi(t_2,\theta_{kr}\omega) - \Phi(t_1,\theta_{kr}\omega)\Big)y(kr,\omega,\eta)\Big\| \\
& + \Big\| \int_{t_1}^{t_2} \Phi(t_2 - s, \theta_{s+kr}\omega) f(y(kr+s-r,\omega,\eta))ds \Big\| \\
& + \Big\| \int_0^{t_1} \Big(\Phi(t_2 - s, \theta_{s+kr}\omega) - \Phi(t_1 - s, \theta_{s+kr}\omega)\Big) f(y(kr+s-r,\omega,\eta))ds \Big\| \\
\leq\ & \|\Phi(t_2,\theta_{kr}\omega) - \Phi(t_1,\theta_{kr}\omega)\| \|y(kr,\omega,\eta)\| \\
& + C_f \int_{t_1}^{t_2} \|\Phi(t_2 - s, \theta_{s+kr}\omega)\| \|y(kr+s-r,\omega,\eta)\| ds \\
& + C_f \int_0^{t_1} \|\Phi(t_2 - s, \theta_{s+kr}\omega) - \Phi(t_1 - s, \theta_{s+kr}\omega)\| \|y(kr+s-r,\omega,\eta)\| ds \\
\leq\ & (t_2 - t_1)^\beta \|y(kr+\cdot,\omega,\eta)\|_{\infty,[-r,0]} \, \vert\!\vert\!\vert \Phi(\cdot,\theta_{kr}\omega) \vert\!\vert\!\vert_{\beta,[0,r]} \\
& + (t_2 - t_1) C_f \|y(kr+\cdot,\omega,\eta)\|_{\infty,[-r,0]} \|\Phi^{-1}(\cdot,\theta_{kr}\omega)\|_{\infty,[0,r]} \|\Phi(\cdot,\theta_{kr}\omega)\|_{\infty,[0,r]} \\
& + C_f \|y(kr+\cdot,\omega,\eta)\|_{\infty,[-r,0]} \int_0^{t_1} \Big\| \Big(\Phi(t_2,\theta_{kr}\omega)) - \Phi(t_1,\theta_{kr}\omega))\Big) \Phi^{-1}(s,\theta_{kr}\omega) \Big\| ds \\
\leq\ & (t_2 - t_1)^\beta \|y(kr+\cdot,\omega,\eta)\|_{\infty,[-r,0]} \, \vert\!\vert\!\vert \Phi(\cdot,\theta_{kr}\omega) \vert\!\vert\!\vert_{\beta,[0,r]} \\
& + (t_2 - t_1) C_f \|y(kr+\cdot,\omega,\eta)\|_{\infty,[-r,0]} \|\Phi^{-1}(\cdot,\theta_{kr}\omega)\|_{\infty,[0,r]} \|\Phi(\cdot,\theta_{kr}\omega)\|_{\infty,[0,r]} \\
& + (t_2 - t_1)^\beta r C_f \|y(kr+\cdot,\omega,\eta)\|_{\infty,[-r,0]} \|\Phi^{-1}(\cdot,\theta_{kr}\omega)\|_{\infty,[0,r]} \, \vert\!\vert\!\vert \Phi(\cdot,\theta_{kr}\omega) \vert\!\vert\!\vert_{\beta,[0,r]} .
\end{aligned} \tag{2.39}$$

We then obtain for k large enough

$$\begin{aligned}
\vert\!\vert\!\vert y_{(k+1)r} \vert\!\vert\!\vert_{\beta,[-r,0]} \leq\ & \|y(kr+\cdot,\omega,\eta)\|_{\infty,[-r,0]} \, \vert\!\vert\!\vert \Phi(\cdot,\theta_{kr}\omega) \vert\!\vert\!\vert_{\beta,[0,r]} \\
& + r^{1-\beta} C_f \|y(kr+\cdot,\omega,\eta)\|_{\infty,[-r,0]} \|\Phi^{-1}(\cdot,\theta_{kr}\omega)\|_{\infty,[0,r]} \|\Phi(\cdot,\theta_{kr}\omega)\|_{\infty,[0,r]} \\
& + r C_f \|y(kr+\cdot,\omega,\eta)\|_{\infty,[-r,0]} \|\Phi^{-1}(\cdot,\theta_{kr}\omega)\|_{\infty,[0,r]} \, \vert\!\vert\!\vert \Phi(\cdot,\theta_{kr}\omega) \vert\!\vert\!\vert_{\beta,[0,r]} \\
\leq\ & e^{-\frac{\lambda(k-1)}{2} + h_0 r} \rho^0(r) \Big[1 + C_f (r^{1-\beta} + r) \|\Phi^{-1}(\cdot,\theta_{kr}\omega)\|_{\infty,[0,r]}\Big] \|\Phi(\cdot,\theta_{kr}\omega)\|_{\infty,\beta,[0,r]}.
\end{aligned} \tag{2.40}$$

On the other hand, due to [11, Proposition 2.4], Φ and $\Psi := \Phi^{-1}$ satisfy

$$\begin{aligned}
d\Phi(t,\omega) &= A\Phi(t,\omega)dt + C\Phi(t,\omega)d\omega(t) \\
d\Psi(t,\omega) &= -A^T\Psi(t,\omega)dt - C^T\Psi(t,\omega)d\omega(t).
\end{aligned}$$

Now, apply [4, Theorem 2.1] we get

$$\begin{aligned}
\|\Phi(\cdot,\omega)\|_{\infty,[0,r]} &\leq e^{2\|A\|r} \left(\frac{K+2}{K+1}\right)^{1 + r[2(K+1)\|C\| \vert\!\vert\!\vert \omega \vert\!\vert\!\vert_{\nu,[0,r]}]^{1/\nu}} \\
\|\Phi^{-1}(\cdot,\omega)\|_{\infty,[0,r]} &\leq e^{2\|A\|r} \left(\frac{K+2}{K+1}\right)^{1 + r[2(K+1)\|C\| \vert\!\vert\!\vert \omega \vert\!\vert\!\vert_{\nu,[0,r]}]^{1/\nu}} .
\end{aligned}$$

Moreover for all $s,t\in[0,r]$

$$\begin{aligned}\left\|\Phi(\cdot,\omega)\right\|_{\beta,[s,t]} &\leq (\|A\|\vee K\|C\|)\Big[(t-s)^{1-\beta}+(t-s)^{\nu-\beta}\left\|\omega\right\|_{\nu,[s,t]}\Big]\times\\ &\quad\times\Big[\|\Phi(\cdot,\omega)\|_{\infty,[0,r]}+(t-s)^{\beta}\left\|\Phi(\cdot,\omega)\right\|_{\beta,[s,t]}\Big]\\ &\leq (\|A\|\vee K\|C\|)\Big[(t-s)^{1-\beta}+(t-s)^{\nu-\beta}\left\|\omega\right\|_{\nu,[s,t]}\Big]\Big[L+(t-s)^{\beta}\left\|y\right\|_{\beta,[s,t]}\Big]\end{aligned}$$

with $L=e^{2\|A\|r}\left(\frac{K+2}{K+1}\right)^{1+r[2(K+1)\|C\|\left\|\omega\right\|_{\nu,[0,r]}]^{1/\nu}}$.
Due to Lemma 2.1

$$\left\|\Phi(\cdot,\omega)\right\|_{\beta,[0,r]}\leq L[2\vee 4\|A\|\vee 4K\|C\|]^{1+\frac{1}{\nu}}r^{-\beta}\left(1+r^{1+\frac{1}{\nu}}+r^{1+\nu}\left\|\omega\right\|_{\nu,[0,r]}^{1+\frac{1}{\nu}}\right).$$

Since $\limsup_{k\to\infty}\frac{\left\|\theta_{kr}\omega\right\|_{\nu,[0,r]}}{k}=0$, it is evident that

$$\limsup_{k\to\infty}\log\left(\Big[1+C_f(r^{1-\beta}+r)\|\Phi^{-1}(\cdot,\theta_{kr}\omega)\|_{\infty,[0,r]}\Big]\|\Phi(\cdot,\theta_{kr}\omega)\|_{\infty,\beta,[0,r]}\right)=0$$

for all ω in a set Ω' with probability 1. Combining this with (2.40), $\left\|y_{(k+1)r}\right\|_{\beta,[-r,0]}\leq e^{-\lambda k/4}$ for k large enough. Now for $t\in[kr,(k+1)r]$

$$\left\|y_t\right\|_{\beta,[-r,0]}\leq\left\|y_{kr}\right\|_{\beta,[-r,0]}+\left\|y_{(k+1)r}\right\|_{\beta,[-r,0]}\leq 2e^{-\lambda(k-1)/4}.$$

Therefore

$$\limsup_{t\to\infty}\frac{\log\left\|y_t\right\|_{\beta,[-r,0]}}{t}<-\lambda/4 \tag{2.41}$$

From (2.38) and (2.41) we obtain (2.24). The proof is completed. □

In the second main result, we prove that the trivial solution of (1.1) is also exponentially stable almost surely but in the pullback sense.

Theorem 2.6 *Under the assumptions of Theorem 2.5, there exists $\varepsilon>0$ such that if $\|C\|<\varepsilon$ the trivial solution of system* (1.1) *is pullback exponentially stable almost surely with respect to the $\|\cdot\|_{\infty,\beta,[-r,0]}$ norm, i.e.*

$$\limsup_{t\to\infty}\frac{1}{t}\log\|y_t(\theta_{-t}\omega,y_0)\|_{\infty,\beta,[-r,0]}<0\,,$$

for almost surely all the realizations ω.

Proof: Fristly, we write $y(t,\omega,\eta)$ and $\rho^k(t,\omega,\eta)$ to stress the dependence of y, ρ^k in Theorem 2.5 on ω and η. The proof of this theorem follows the arguments of Theorem 2.5 with some modifications. Namely, fix $u\in[kr,(k+1)r]$ and replace ω in (2.35) by $\theta_{-u}\omega$, we obtain

$$\begin{aligned}\rho^k(r,\theta_{-u}\omega,\eta) &\leq \rho^0(r,\theta_{-u}\omega,\eta)\exp\Big\{\Big[-(h_0-2\gamma)r+\frac{2\|C\|r}{k}\sum_{j=1}^{k}Q(r,\theta_{jr-u}\omega)+2KG\|C\|\Big]k\Big\}\\ &\leq \rho^0(r,\theta_{-u}\omega,\eta)\exp\Big\{\Big[-(h_0-2\gamma)r+\frac{2\|C\|r}{k}\sum_{j=0}^{k-1}P(r,\theta_{-jr}\omega)+2KG\|C\|\Big]k\Big\},\end{aligned}$$

where

$$P(r,\omega):=2KG\Big[\frac{\nu(1+2KG)}{2KG}\left\|\omega\right\|_{\nu,[-r,r]}^{\frac{1}{\nu}}+r^{\nu+\frac{1}{\nu}}\left\|\omega\right\|_{\nu,[-r,r]}+r^{2\nu}\left\|\omega\right\|_{\nu,[-r,r]}^{2+\frac{1}{\nu}}\Big].$$

Put

$$\epsilon' = \min\Big\{\epsilon_1, \frac{(h_0-2\gamma)r}{2KG+2rE[P(r,\omega)]}\Big\}$$

and consider $\|C\|<\epsilon'$ then for almost all ω

$$\begin{aligned} & \lim_{k\to\infty}\Big\{-(h_0-2\gamma)r+2KG\|C\|+\frac{2\|C\|r}{k}\sum_{j=0}^{k-1}P(r,\theta_{-jr}\omega)\Big\} \\ = & \quad -(h_0-2\gamma)r+2KG\|C\|+2\|C\|rE[P(r,\omega)] =: -\lambda' < 0. \end{aligned} \tag{2.42}$$

Proof similarly to Theorem 2.5, it is easy to see

$$\rho^0(r,\theta_{-u}\omega,\eta) \le \|\eta\|_{\infty,[-r,0]}\exp\Big\{2\gamma r+2\|C\|rP(r,\theta_{-kr}\omega)+2KG\|C\|\Big\}.$$

It follows that when k is large enough,

$$\Big[-(h_0-2\gamma)r+2KG\|C\|+2\|C\|r\frac{1}{k}\sum_{j=0}^{k-1}P(r,\theta_{-jr}\omega)\Big]k < \frac{-\lambda' k}{2}, \tag{2.43}$$

and

$$\rho^0(r,\theta_{-u}\omega,\eta)\le e^{\frac{\lambda' k}{4}}.$$

As a result

$$\rho^k(r,\theta_{-u}\omega,\eta)\le \rho^0(r,\theta_{-u}\omega,\eta)e^{\frac{-\lambda' k}{2}} \le e^{\frac{-\lambda' k}{4}}.$$

On the other hand, it is obvious that, for any $0\le t\le r$

$$\begin{aligned} \|y(kr+t,\theta_{-u}\omega,\eta)\| & \le \rho^k(t,\theta_{-u}\omega,\eta) \\ & \le e^{h_0 r}\rho^k(r,\theta_{-u}\omega,\eta) \\ & \le e^{-\frac{\lambda' k}{2}+h_0 r}. \end{aligned}$$

Also, one can choose k is greater than $n(\omega)$ large enough so that

$$\|y(\cdot,\theta_{-u}\omega,\eta)\|_{\infty,[(k-2)r,(k+1)r]} \le e^{-\frac{\lambda'(k-2)}{2}+h_0 r}. \tag{2.44}$$

Next, in the analog of (1.6), for $u\in[kr,(k+1)r]$ and $u'=u-r$

$$y(u'+t,\omega,\eta) \quad = \quad \Phi(t,\theta_{u'}\omega)y(u',\omega,\eta)+\int_0^t \Phi(t-s,\theta_{s+u'}\omega)f(y(u'+s-r,\omega,\eta))ds.$$

Therefore, for $k\ge n(\omega)$ and $u\in[kr,(k+1)r]$, we have

$$y(u'+t,\theta_{-u}\omega,\eta) \quad = \quad \Phi(t,\theta_{-r}\omega)y(u',\theta_{-u}\omega,\eta)+\int_0^t \Phi(t-s,\theta_{s-r}\omega)f(y(u'+s-r,\theta_{-u}\omega,\eta))ds.$$

For any $0\le t_1<t_2\le r$, by similar computations to (2.39), we obtain

$$\begin{aligned} & \|y(u'+t_2,\theta_{-u}\omega,\eta)-y(u'+t_1,\theta_{-u}\omega,\eta)\| \\ \le & \Big\|\Big(\Phi(t_2,\theta_{-r}\omega)-\Phi(t_1,\theta_{-r}\omega)\Big)y(u',\theta_{-u}\omega,\eta)\Big\| \\ & +\Big\|\int_{t_1}^{t_2}\Phi(t_2-s,\theta_{s-r}\omega)f(y(u'+s-r,\theta_{-u}\omega,\eta))ds\Big\| \end{aligned}$$

$$
\begin{aligned}
& +\Big\| \int_0^{t_1} \Big(\Phi(t_2-s,\theta_{s-r}\omega)-\Phi(t_1-s,\theta_{s-r}\omega)\Big) f(y(u'+s-r,\theta_{-u}\omega,\eta))ds\Big\| \\
\leq\ & \|\Phi(t_2,\theta_{-r}\omega)-\Phi(t_1,\theta_{-r}\omega)\|\|y(u',\theta_{-u}\omega,\eta)\| \\
& +C_f\int_{t_1}^{t_2}\|\Phi(t_2-s,\theta_{s-r}\omega)\|\|y(u'+s-r,\theta_{-u}\omega,\eta)\|ds \\
& +C_f\int_0^{t_1}\|\Phi(t_2-s,\theta_{s-r}\omega)-\Phi(t_1-s,\theta_{s-r}\omega)\|\|y(u'+s-r,\theta_{-u}\omega,\eta)\|ds \\
\leq\ & (t_2-t_1)^\beta \left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert \Phi(\cdot,\theta_{-r}\omega)\right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert_{\beta,[0,r]} \|y_u(\cdot,\theta_{-u}\omega,\eta)\|_{\infty,[-r,0]} \\
& +(t_2-t_1)C_f\|y_{u-r}(\cdot,\theta_{-u}\omega,\eta)\|_{\infty,[-r,0]}\|\Phi^{-1}(\cdot,\theta_{-r}\omega)\|_{\infty,[0,r]}\|\Phi(\cdot,\theta_{-r}\omega)\|_{\infty,[0,r]} \\
& +C_f\|y_{u-r}(\cdot,\theta_{-u}\omega,\eta)\|_{\infty,[-r,0]}\int_0^{t_1}\Big\|\Big(\Phi(t_2,\theta_{-r}\omega))-\Phi(t_1,\theta_{-r}\omega))\Big)\Phi^{-1}(s,\theta_{-r}\omega)\Big\|ds \\
\leq\ & (t_2-t_1)^\beta \left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert \Phi(\cdot,\theta_{-r}\omega)\right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert_{\beta,[0,r]} \|y_u(\cdot,\theta_{-u}\omega,\eta)\|_{\infty,[-r,0]} \\
& +(t_2-t_1)C_f\|y_{u-r}(\cdot,\theta_{-u}\omega,\eta)\|_{\infty,[-r,0]}\|\Phi^{-1}(\cdot,\theta_{-r}\omega)\|_{\infty,[0,r]}\|\Phi(\cdot,\theta_{-r}\omega)\|_{\infty,[0,r]} \\
& +(t_2-t_1)^\beta rC_f\|y_{u-r}(\cdot,\theta_{-u}\omega,\eta)\|_{\infty,[-r,0]}\|\Phi^{-1}(\cdot,\theta_{-r}\omega)\|_{\infty,[0,r]} \left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert \Phi(\cdot,\theta_{-r}\omega)\right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert_{\beta,[0,r]}.
\end{aligned}
$$

Due to (2.44) it follows that

$$
\begin{aligned}
\|y_u(\cdot,\theta_{-u}\omega,\eta)\|_{\infty,\beta,[-r,0]} &= \|y_u(\cdot,\theta_{-u}\omega,\eta)\|_{\infty,[-r,0]} + \left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert y_u(\cdot,\theta_{-u}\omega,\eta)\right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert_{\beta,[-r,0]} \\
&\leq e^{-\frac{\lambda'(k-2)}{2}+h_0 r}\times\Big[1+\left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert \Phi(\cdot,\theta_{-r}\omega)\right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert_{\beta,[0,r]} \\
&\quad +r^{1-\beta}C_f\|\Phi^{-1}(\cdot,\theta_{-r}\omega)\|_{\infty,[0,r]}\|\Phi(\cdot,\theta_{-r}\omega)\|_{\infty,[0,r]} \\
&\quad +rC_f\|\Phi^{-1}(\cdot,\theta_{-r}\omega)\|_{\infty,[0,r]} \left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert \Phi(\cdot,\theta_{-r}\omega)\right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert_{\beta,[0,r]}\Big].
\end{aligned}
$$

Consequently,

$$
\lim_{u\to+\infty}\frac{1}{u}\log\|y_u(\cdot,\theta_{-u}\omega,\eta)\|_{\infty,\beta,[-r,0]} = -\frac{\lambda'}{2}<0.
$$

That means the trivial solution of system (1.1) is pullback exponentially stable almost surely. □

Acknowledgments

L.H.D would like to thank Vietnam Institute for Advanced Studies in Mathematics (VIASM) for finanical support during three month research stay at the institute in 2019. P.T.H would like to thank the IMU Breakout Graduate Fellowship Program for the financial support.

References

[1] Mandelbrot, B., Ness, J. V., (1968), Fractional Brownian motion, fractional noises and applications. *SIAM Review*, **4**, No. 10, pp. 422–437.

[2] Luu, H. D., Garrido-Atienza, M. J., Neuenkirche, A., Schmalfuß, B., (2018), Exponential stability of stochastic evolution equations driven by small fractional Brownian motion with Hurst parameter in (1/2, 1). *Journal of Differential Equations* **264.2**, pp. 1119-1145.

[3] Luu, H. D., Phan, T. H., Nguyen, D. C., (2019), Asymptotic stability for stochastic dissipative systems with a Hölder noise. *SIAM Journal on Control and Optimization*, **57** (4), pp. 3046-3071.

[4] Luu, H. D., Phan, T. H., (2019), Asymptotic dynamics of Young differential equations: a unified approach. *Preprint: arXiv:1905.04945.*

[5] Phan, T. H., Cao, T. B., (2018), A note on exponential stability of non-autonomous linear stochastic differential delay equations driven by a fractional Brownian motion with Hurst index $> 1/2$ *Statistics & Probability Letters* **138**, pp. 127-136.

[6] Luu, H. D., Schmalfuß, B., Siegmund, S., (2015), A note on the generation of random dynamical systems from fractional stochastic delay differential equations. *Stochastics and Dynamics*, Vol. **15**, No. **3**, pp. 1–13. DOI: 10.1142/S0219493715500185.

[7] Luu, H. D., Phan, T. H., (2019), Young differential delay equations driven by Hölder continuous paths. *Modern Mathematics and Mechanics: Fundamentals, Problems and Challenges. Springer, pp. 313–333.*

[8] Young, L. C., (1936), An integration of Hölder type, connected with Stieltjes integration. *Acta Math.* **67**, pp. 251–282.

[9] Friz, P., Victoir, N., (2010), Multidimensional stochastic processes as rough paths: theory and applications. *Cambridge Studies in Advanced Mathematics, 120. Cambridge Unversity Press, Cambridge.*

[10] Boufoussi, B., Hajji, S., (2011), Functional differential equations driven by a fractional Brownian motion. *Computers and Mathematics with Applications* **62**, No. 1, pp. 746–754.

[11] Nguyen, D. C., Luu, H. D., Phan, T. H., (2019), Lyapunov spectrum of nonautonomous linear Young differential equations. *Journal of Dynamics and Differential Equations*, Vol. **25**, No. **3**, DOI: 10.1007/s10884-019-09780-z.

[12] Garrido-Atienza, M. J., Schmalfuß, B., (2011), Ergodicity of the infinite dimensional fractional Brownian motion . *Journal of Dynamics and Differential Equations*, **23**, pp. 671-681. DOI 10.1007/s10884-011-9222-5.

[13] Gyõri, I., Horváth, L., (2016), Sharp Gronwall-Bellman type integral inequalities with delay. *Electronic Journal of Qualitative Theory of Differential Equations* **2016.111**, pp. 1–25.