VISCOSITY SOLUTIONS TO PARABOLIC COMPLEX MONGE-AMPÈRE EQUATIONS

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Abstract. In this paper, we study the Cauchy-Dirichlet problem for Parabolic complex Monge-Ampère equations on a strongly pseudoconvex domain by the viscosity method. We extend the results in [EGZ15b] on the existence of solution and the convergence at infinity. We also establish the Hölder regularity of the solutions when the Cauchy-Dirichlet data are Hölder continuous.

1. Introduction

In [ST17, ST12], Song and Tian gave a conjectural picture to approach the Minimal Model Program via the Kähler-Ricci flow. In the Song-Tian's program, one need to study the behavior of the Kähler-Ricci flow on mildly singular varieties. This requires a theory of weak solutions for certain degenerate parabolic complex Monge-Ampère equations modelled on

$$(dd^c u)^n = e^{\partial_t u(t,z)+F(t,z,u)}\mu, \tag{1}$$

where μ is a volume form, and u is t-dependent Kähler potential on a compact Kähler manifold.

A viscosity approach for degenerate parabolic Monge-Ampère equations has been developed recently by Eyssidieux-Guedj-Zeriahi [EGZ15b] in domains of $\mathbb{C}^n$ and [EGZ16, EGZ18] on compact Kähler manifolds. The same approach for elliptic Monge-Ampère equations was also established in [EGZ11, EGZ15a, Wan12] (see also [DDT19] for a recent generalization).

In [EGZ15b], Eyssidieux-Guedj-Zeriahi studied a Cauchy-Dirichlet problem for (1) in which the density μ and parabolic boundary condition are independent of time. They proved that in this case the Cauchy-Dirichlet problem for (1) admits a solution if the problem is *admissible* (see below). In this note, we solve a more general Cauchy-Dirichlet problem on a pseudoconvex domain for (1) in which the density μ and the parabolic boundary condition depend on time. In addition, we establish the Hölder regularity of the solutions.

There is a well established pluripotential theory of weak solution to elliptic complex Monge-Ampère equation, following the pionneering work of Bedford and Taylor [BT76, BT82] in local case, but the similar theory for the parabolic side only developed recently [GLZ1, GLZ2]. It is very interesting to compare the viscosity and pluripotential concepts, this requires Theorem 1.2 below. We refer the reader to [GLZ3] for more details.

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We now explain the precise context. Let $\Omega \subset \mathbb{C}^n$ be a smooth bounded strongly pseudoconvex domain and $T \in (0, \infty)$. We consider the Cauchy-Dirichlet problem

$$\begin{cases} e^{\partial_t u + F(t,z,u)}\mu(t,z) = (dd^c u)^n & \text{in} \quad \Omega_T, \\ u = \varphi \quad \text{in} \quad [0,T) \times \partial\Omega, \\ u(0,z) = u_0(z) \quad \text{in} \quad \bar{\Omega}, \end{cases} \tag{2}$$

where

- $\Omega_T = (0,T) \times \Omega$.
- $F(t,z,r)$ is continuous in $[0,T] \times \bar{\Omega} \times \mathbb{R}$ and non-decreasing in r.
- $\mu(t,z) = f(t,z)dV$, where dV is the standard volume form in $\mathbb{C}^n$ and $f \geq 0$ is a bounded continuous function in $[0,T] \times \Omega$.
- $\varphi(t,z)$ is a continuous function in $[0,T] \times \partial\Omega$.
- $u_0(z)$ is continuous in $\bar{\Omega}$ and plurisubharmonic in Ω such that $u_0(z) = \varphi(0,z)$ in $\partial\Omega$.

By [EGZ15b, Definition 5.6], $(u_0, \mu(0,.))$ is called *admissible* if for all $\epsilon > 0$, there exists $u_\epsilon \in PSH(\Omega) \cap C(\bar{\Omega})$ and $C_\epsilon > 0$ such that $u_0 \leq u_\epsilon \leq u_0 + \epsilon$ and $(dd^c u_\epsilon)^n \leq e^{C_\epsilon}\mu(0,z)$ in the viscosity sense. We observe below that the condition $u_\epsilon \in PSH(\Omega)$ is redundant (see Theorem 1.3 (i)). We still use the *admissible* term for the same definition above without this condition.

Definition 1.1. *We say that $(u_0, \mu(0,.))$ is admissible if for all $\epsilon > 0$, there exist $u_\epsilon \in C(\bar{\Omega})$ and $C_\epsilon > 0$ such that $u_0 \leq u_\epsilon \leq u_0 + \epsilon$ and $(dd^c u_\epsilon)^n \leq e^{C_\epsilon}\mu(0,z)$ in the viscosity sense.*

It follows from [EGZ15b] that if φ, μ are independent of t and (u_0, μ) is admissible then (2) admits a solution. In this paper, we extend this result to the case in which φ and μ depend on t as well. Our first main result is the following

Theorem 1.2. *Assume that $\mu = fdV$ where f is independent of t. If $(u_0(z), \mu)$ is admissible then the equation (2) admits a unique solution.*

In fact, we can also obtain the existence result to certain cases where f depends on t as well, we refer to Theorems 4.7, 4.11 and Corollaries 4.8, 4.12.

We are now interested in the notation of admissible data. We obtain the following properties:

Theorem 1.3. *Let $g \geq 0$ be a bounded continuous function in Ω and $\nu = gdV$. Let $\phi \in PSH(\Omega) \cap C(\bar{\Omega})$. Then the following holds:*

(i) *If (ϕ, ν) is admissible then the function u_ϵ in the definition 1.1 can be taken to be psh in Ω.*
(ii) *Admissibility is a local property: if for every $z \in \Omega$, there exists an open neighborhood U of z such that (ϕ, ν) is admissible in U then (ϕ, ν) is admissible in Ω.*
(iii) *If $\int_{\{g=0\}} (dd^c\phi)^n = 0$ then (ϕ, ν) is admissible.*

In particular, when μ is independent of time, we prove that this condition is also necessary (Corollary 3.6 and Remark 4.9). However, we also give a counter-example in which μ depends on t, the Cauchy-Dirichlet problem admits a solution but $(u_0, \mu(0,z))$ is not admissible. In addition, we prove the following local and integral criteria to the admissible condition.

Corollary 1.4. *If $\nu = g(z)dV \geq 0$ with $\{z \in \Omega : g(z) = 0\}$ is a negligible set, then (ϕ, ν) is admissible for every $\phi \in C(\overline{\Omega}) \cap PSH(\Omega)$.*

For degenerate (elliptic) complex Monge-Ampère equations, the Hölder regularity of pluripotential solutions has been studied intensively (we refer to [GKZ08, DDGHKZ] and references therein). Similar results for viscosity solutions can be found in [Lu13, Wan12]. Nevertheless, to the best of our knowledge, no Hölder regularity result to the weak solutions of parabolic complex Monge-Ampère equations has been established in both pluripotential and viscosity senses (in the non-smooth case). In this paper, we make a first step in this direction:

Theorem 1.5. *Assume that $\mu = dV$ and $u(t, z)$ is a viscosity solution to (2). Suppose that there exist $C > 0$, $0 < \alpha < 1$ and $0 < \beta < 1/2$ such that*

$$|\varphi(t, z) - \varphi(s, w)| \leq C(|t - s|^\alpha + |z - w|^{2\beta}),\ \forall z, w \in \partial\Omega, t, s \in [0, T),$$
$$\varphi(t, z) - \varphi(s, z) \leq C(t - s),\ \forall z \in \partial\Omega, 0 < s < t < T,$$

and

$$|u_0(z) - u_0(w)| \leq C|z - w|^\beta,\ \forall z, w \in \bar{\Omega}.$$

Suppose also that, for any $K > 0$, there exists $C_K > 0$ such that,

$$|F(t, z, r) - F(t, w, r)| \leq C_K|z - w|^\beta.$$

for all $z, w \in \Omega$, $t \in [0, T), r \in [-K, K]$. Then, there exists $\tilde{C} > 0$ such that

$$|u(t, z) - u(s, w)| \leq \tilde{C}(|t - s|^\alpha + |z - w|^\beta),$$

for all $z, w \in \Omega, t, s \in [0, T)$.

Moreover, if φ is Lipschitz in t then u is locally Lipschitz in t uniformly in z.

Finally, we prove that the viscosity solution of the Cauchy-Dirichlet problem (2) asymptotically recovers the solution of the corresponding elliptic Dirichlet problem under some assumptions. This also extends the convergence result in [EGZ15b].

Theorem 1.6. *Assume that $T = \infty$, $\varphi(t, z) \rightrightarrows \varphi_\infty(z)$ as $t \to \infty$ and for any $M > 0$, $F(t, z, r) \rightrightarrows F_\infty(z, r)$ in $\bar{\Omega} \times [-M, M]$ as $t \to \infty$, where $\rightrightarrows$ denotes the uniform convergence.*

Suppose that $\sup_{t \geq 0} f(t, z) \in L^p(\Omega)$ for some $p > 1$ and $f(t, z)$ converges almost everywhere to a function $f_\infty(z)$ as $t \to \infty$. Then $u(t, z)$ converges uniformly to $u_\infty(z)$ as $t \to \infty$, where u_∞ is the unique solution of the equation

$$\begin{cases} u_\infty \in PSH(\Omega) \cap C(\overline{\Omega}), \\ (dd^c u_\infty)^n = e^{F_\infty(z, u_\infty)} f_\infty(z) dV(z) \qquad in \qquad \Omega, \\ u_\infty = \varphi_\infty \qquad in \qquad \partial\Omega. \end{cases} \tag{3}$$

The solution u_∞ to the elliptic Dirichlet problem above is well known to exist in the pluripotential sense in [Kol98]. If f_∞ is continuous then the solution in the pluripotential sense is also the solution in the viscosity sense [EGZ11, HL09, Wan12].

In fact, we can also obtain the uniform convergence in capacity when $p = 1$ as well, we refer to Theorem 6.1. In this case, the equation (3) is replaced by the equation (45). The existence of the pluripotential solution to (45) holds due to [Ceg04], [Aha07], [ACCP09] (see also 2.16).

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2. Preliminaries

For the reader's convenience, we recall some basic concepts and well-known results.

2.1. **Viscosity concepts.**

Consider the following parabolic complex Monge-Ampère equations on a bounded domain $\Omega \subset \mathbb{C}^n$

$$e^{\partial_t u + F(t,z,u)}\mu(t,z) = (dd^c u)^n, \tag{4}$$

where

- $\Omega_T = (0,T) \times \Omega$.
- $F(t,z,r)$ is continuous in $[0,T] \times \bar{\Omega} \times \mathbb{R}$ and non-decreasing in r.
- $\mu(t,z) = f(t,z)dV$, where dV is the standard volume form in $\mathbb{C}^n$ and $f \geq 0$ is a bounded continuous function in $[0,T] \times \Omega$.

Definition 2.1. *(Test functions) Let $w : \Omega_T \longrightarrow \mathbb{R}$ be any function defined in Ω_T and $(t_0, z_0) \in \Omega_T$ a given point. An upper test function (resp. a lower test function) for w at the point (t_0, z_0) is a $C^{(1,2)}$-smooth function q in a neighbourhood of the point (t_0, z_0) such that $w(t_0, z_0) = q(t_0, z_0)$ and $w \leq q$ (resp. $w \geq q$) in a neighbourhood of (t_0, z_0). We will write for short $w \leq_{(t_0,z_0)} q$ (resp. $w \geq_{(t_0,z_0)} q$).*

Definition 2.2. *1. A function $u : [0,T) \times \overline{\Omega} \longrightarrow \mathbb{R}$ is said to be a (viscosity) subsolution to the parabolic complex Monge-Ampère equation (4) in $(0,T) \times \Omega$ if u is upper semi-continuous in $[0,T) \times \overline{\Omega}$ and for any point $(t_0, z_0) \in \Omega_T := (0,T) \times \Omega$ and any upper test function q for u at (t_0, z_0), we have*

$$(dd^c q_{t_0}(z_0))^n \geq e^{\partial_t q(t_0,z_0) + F(t_0,z_0,q(t_0,z_0))}\mu(t_0, z_0).$$

In this case, we also say that u satisfies the differential inequality

$$(dd^c u)^n \geq e^{\partial_t u(t,z) + F(t,z,u(t,z))}\mu(t,z),$$

in the viscosity sense in Ω_T.

The function u is called a subsolution to the Cauchy-Dirichlet problem (2) if u is a subsolution to (4) satisfying $u \leq \varphi$ in $[0,T) \times \partial\Omega$ and $u(0,z) \leq u_0(z)$ for all $z \in \Omega$.

2. A function $v : [0,T) \times \overline{\Omega} \longrightarrow \mathbb{R}$ is said to be a (viscosity) supersolution to the parabolic complex Monge-Ampère equation (4) in Ω_T if v is lower semi-continuous in Ω_T and for any point $(t_0, z_0) \in \Omega_T$ and any lower test function q for v at (t_0, z_0) such that $dd^c q_{t_0}(z_0) \geq 0$, we have

$$(dd^c q_{t_0})^n(z_0) \leq e^{\partial_t q(t_0,z_0) + F(t_0,z_0,q(t_0,z_0))}\mu(t_0, z_0).$$

In this case we also say that v satisfies the differential inequality

$$(dd^c v)^n \leq e^{\partial_t v(t,z) + F(t,z,v(t,z))}\mu(t,z),$$

in the viscosity sense in Ω_T.

The function v is called a supersolution to (2) if v is a supersolution to (4) satisfying $v \geq \varphi$ in $[0,T) \times \partial\Omega$ and $v(0,z) \geq u_0(z)$ for all $z \in \Omega$.

3. A function $u : \Omega_T \longrightarrow \mathbb{R}$ is said to be a (viscosity) solution to the parabolic complex Monge-Ampère equation (4) *(respectively,* (2)*) in Ω_T if it is a subsolution and a supersolution to* (4) *(respectively,* (2)*) in Ω_T.*

Definition 2.3. *A discontinuous viscosity solution to the equation* (4) *(resp.* (2)*) is a function $u : \Omega_T \to [+\infty, -\infty]$ such that*

i) the usc envelope u^* of u satisfies $\forall z \in \Omega$, $u^*(t,z) < +\infty$ and is a viscosity subsolution to (4) *(resp.* (2)*),*

ii) the lsc envelope u_* of u satisfies $\forall z \in \Omega$, $u_*(t,z) > -\infty$ and is a viscosity supersolution to the equation (4) *(resp.* (2)*).*

2.2. **Basic properties.** We recall some basic properties of viscosity subsolution and viscosity supersolution.

Lemma 2.4. *Consider the equations*

$$e^{\partial_t u + F_1(t,z,u)}\mu_1(t,z) = (dd^c u)^n \qquad in \qquad (0,T)\times\Omega, \tag{5}$$

and

$$e^{\partial_t u + F_2(t,z,u)}\mu_2(t,z) = (dd^c u)^n \qquad in \qquad (0,T)\times\Omega, \tag{6}$$

where, for $j = 1, 2$,

- *$F_j(t,z,r)$ is continuous in $[0,T)\times\bar{\Omega}\times\mathbb{R}$ and non-decreasing in r.*
- *$\mu_j(t,z) = f_j(t,z)dV$ with f_j is a bounded continuous function in $[0,T)\times\Omega$.*

Assume that $F_1 \geq F_2$ and $\mu_1 \geq \mu_2$. If u_1 is a subsolution to (5) *then u_1 is also a subsolution to* (6)*. Conversely, if u_2 is a supersolution to* (6) *then u_2 is also a supersolution to* (5)*.*

Lemma 2.5. *Let $A > 0$. If $u(t,z)$ is a subsolution (resp. supersolution) to* (4) *in $(0,T)\times\Omega$ then $u_A := \dfrac{1}{A}u(At,z)$ is a subsolution (resp. supersolution) to the equation*

$$\frac{1}{A^n}e^{\partial_t u_A + F(At,z,Au_A)}\mu(At,z) = (dd^c u_A)^n, \tag{7}$$

in $(0,\dfrac{T}{A})\times\Omega$.

Lemma 2.6. [CIL92, IS13, EGZ15b] *Let $\mu^j(t,x) \geq 0$ be a sequence of continuous volume forms converging uniformly to a volume form μ on Ω_T and let F^j be a sequence of continuous functions in $[0,T[\times\Omega\times\mathbb{R}$ converging locally uniformly to a function F. Let (u^j) be a locally uniformly bounded sequence of real valued functions defined in Ω_T.*

1. Assume that for every $j \in \mathbb{N}$, u^j is a viscosity subsolution to the complex Monge-Ampère flow

$$e^{\partial_t u^j + F^j(t,z,u^j)}\mu^j(t,z) - (dd^c u^j_t)^n = 0,$$

associated to (F^j, μ^j) in Ω_T. Then its upper relaxed semi-limit

$$\overline{u} = \limsup{}^*_{j\to+\infty} u^j$$

of the sequence (u^j) is a subsolution to the parabolic Monge-Ampère equation

$$e^{\partial_t u + F(t,z,u)}\mu - (dd^c u)^n = 0,$$

in Ω_T.

2. Assume that for every $j \in \mathbb{N}$, u^j is a viscosity supersolution to the complex Monge-Ampère flow associated to (F^j, μ^j) in Ω_T. Then the lower relaxed semi-limit

$$\underline{u} = \liminf{}_{*j\to+\infty} u^j$$

of the sequence (u^j) is a supersolution to the complex Monge-Ampère flow associated to (F, μ) in Ω_T.

One of applications of Lemma 2.6 is the following

Lemma 2.7. *Let u be a subsolution to the equation*

$$e^{\partial_t w + F_1(t,z,w)} f_1(t,z) dV = (dd^c w)^n, \tag{8}$$

and v be a supersolution to the equation

$$e^{\partial_t w + F_2(t,z,w)} f_2(t,z) dV = (dd^c w)^n, \tag{9}$$

in $(0,T) \times \Omega$. Let p be a negative plurisubharmonic function in Ω and $h : (0,T) \to [0, \infty)$ be a continuous non-decreasing function. Then $\tilde{u}(t,z) = u(t,z) + p(z) - h(t)$ is a subsolution to (8) and $\tilde{v}(t,z) = v(t,z) - p(z) + h(t)$ is a supersolution to (9). Moreover, if $p \in \mathcal{E}(\Omega)$ and there exist $C_1, C_2 > 0$ such that

$$\partial_t u, \partial_t v \leq C_1, \tag{10}$$

in the viscosity sense and

$$\sup F(.,.,\sup v), \sup F(.,.,\sup u) \leq C_2, \tag{11}$$

and

$$(dd^c p)^n \geq e^{C_1+C_2} |f_1(t,z) - f_2(t,z)| dV, \tag{12}$$

in Ω for every $t \in (0,T)$ then $\tilde{u}$ is a subsolution to (9) and $\tilde{v}$ is a supersolution to (8).

Proof. Let $B \Subset \Omega$ be a ball and $0 < a < b < T$. Then, there exist $h_j : (a,b) \to [0, \infty)$ and $p_j \in PSH(B) \cap C^\infty(\overline{B})$ such that

- p_j is smooth and non-decreasing for every $j \in \mathbb{N}$.
- $h_j \searrow h$ in (a,b) and $p_j \searrow p$ in $\overline{B}$ as $j \to \infty$.

By the definition of viscosity subsolution and viscosity supersolution, we get $u(t,z) + p_j(z) - h_j(t)$ is a subsolution to (8) and $v(t,z) - p_j(z) + h_j(t)$ is a supersolution to (9) in $(a,b) \times B$ for every j. Hence, by Lemma 2.6, we have $u(t,z) + p(z) - h(t)$ is a subsolution to (8) and $v(t,z) - p(z) + h(t)$ is a supersolution to (9) in $(a,b) \times B$. Since a, b and B are arbitrary, we obtain the first conclusion.

If (10), (11) and (12) are satisfied then by [EGZ11] (pages 1064-1066) and by using convolution, the functions p_j can be chosen such that

$$(dd^c p_j)^n \geq e^{C_1+C_2} |f_1(t,z) - f_2(t,z)| dV,$$

in B for every j.

Then, by the definition, then $u(t,z) + p_j(z) - h_j(t)$ is a subsolution to (9) and $v(t,z) - p_j(z) + h_j(t)$ is a supersolution to (8) in $(a,b) \times B$ for every j.

Hence, by Lemma 2.6, we obtain the second conclusion. □

2.3. **Comparison principle and Perron envelope.** As is often the case in the viscosity theory and pluripotential theory, one of the main technical tools is the comparison principle:

Theorem 2.8. [EGZ15b] *Let u (resp. v) be a bounded subsolution (resp. supersolution) to the parabolic complex Monge–Ampère equation (4) in Ω_T. Assume that one of the following conditions is satisfied*

a) *$\mu(t,z) > 0$ for every $(t,z) \in (0,T) \times \Omega$.*
b) *μ is independent of t.*
c) *Either u or v is locally Lipschitz in t uniformly in z.*

Then

$$\sup_{\Omega_T}(u-v) \leq \sup_{\partial_P(\Omega_T)}(u-v)_+,$$

where u (resp. v) has been extended as an upper (resp. a lower) semicontinuous function to $\overline{\Omega_T}$.

Lemma 2.9. [EGZ15b] *Given any non empty family S_0 of bounded subsolutions to the parabolic equation (4) which is bounded above by a continuous function, the usc regularization of the upper envelope $\phi_{S_0} = \sup_{\phi \in S_0} \phi$ is a subsolution to (4).*

If S is the family of all subsolutions to the Cauchy-Dirichlet problem (2), its envelope ϕ_S is a discontinuous viscosity solution to (4).

Definition 2.10. *a) A function $u \in USC([0,T) \times \bar{\Omega})$ is called ϵ-subbarrier for (2) if u is subsolution to (4) in the viscosity sense such that $u_0 - \epsilon \leq u_* \leq u \leq u_0$ in $\{0\} \times \bar{\Omega}$ and $\varphi - \epsilon \leq u_* \leq u \leq \varphi$ in $[0,T) \times \partial\Omega$.*
b) A function $u \in LSC([0,T) \times \bar{\Omega})$ is called ϵ-superbarrier for (2) if u is supersolution to (4) in the viscosity sense such that $u_0 + \epsilon \geq u^* \geq u \geq u_0$ in $\{0\} \times \bar{\Omega}$ and $\varphi + \epsilon \geq u^* \geq u \geq \varphi$ in $[0,T) \times \partial\Omega$.

Lemma 2.11. *Assume that for every $\epsilon > 0$, the problem (2) admits a continuous ϵ-superbarrier which is Lipschitz in t and a continuous ϵ-subbarrier. Denote by S the family of all continuous subsolutions to (2). Then $\phi_S = \sup\{v : v \in S\}$ is a discontinuous viscosity solution to (2).*

Lemma 2.12. *Assume that for every $\epsilon > 0$, the problem (2) admits a continuous ϵ-subbarrier which is Lipschitz in t and a continuous ϵ-superbarrier. Denote by S the family of all continuous subsolutions to (2) which is Lipschitz in t. Then $\phi_S = \sup\{v : v \in S\}$ is a discontinuous viscosity solution to (2).*

Using the comparison principle, we have the following L^∞ a priori estimates to the viscosity solution to (2).

Proposition 2.13. *Consider the Cauchy-Dirichlet problem (2) (with Ω is a smooth bounded strongly pseudoconvex domain). If u is a solution to (2) then there exists $C > 0$ depending on Ω, $\sup_{[0,T) \times \partial\Omega} |\varphi|$, $\min_{\bar{\Omega}} u_0$, $\sup_{\Omega} f$, $\sup_{[0,T) \times \bar{\Omega}} F(t,z,\max\varphi)$ such that*

$$|u| \leq C,$$

in $[0,T) \times \bar{\Omega}$.

Proof. Let $\rho \in C^2(\bar{\Omega}) \cap PSH(\Omega)$ such that $\rho|_{\partial\Omega} = 0$ and $(dd^c\rho)^n \geq \mu(t,.)$ for all t. We define

$$\overline{u} = \sup_{[0,T] \times \partial\Omega} \varphi = const,$$

and

$$\underline{u} = m + M\rho,$$

where

$$m = \min\{-\sup_{[0,T)\times\partial\Omega} |\varphi|, \min_{\bar{\Omega}} u_0\}$$

and

$$M = \exp(\sup_{[0,T]\times\bar{\Omega}} \frac{F(t,z,\max\varphi)}{n}).$$

Then $\underline{u}$ is a subsolution and $\overline{u}$ is a supersolution to (2). Moreover, in $\partial_P(\Omega_T)$,

$$\underline{u} \leq u \leq \overline{u}.$$

By the comparison principle, we have

$$\underline{u} \leq u \leq \overline{u},$$

in Ω_T.

Hence, in $[0,T) \times \bar{\Omega}$,

$$|u| \leq C,$$

where $C = \max\{\sup_{[0,T]\times\partial\Omega} \varphi, |m| + M \max_{\bar{\Omega}}(-\rho)\}$. □

2.4. **Regularizing in time.** Given a bounded upper semi-continuous function $u : \Omega_T \longrightarrow \mathbb{R}$, we consider the upper approximating sequence by Lipschitz functions in t,

$$u^k(t,x) := \sup\{u(s,x) - k|s-t|, s \in [0,T)\},\ (t,x) \in \Omega_T.$$

If v is a bounded lower semi-continuous function, we consider the lower approximating sequence of Lipschitz functions in t,

$$v_k(t,x) := \inf\{v(s,x) + k|s-t|, s \in [0,T)\},\ (t,x) \in \Omega_T.$$

Lemma 2.14. [EGZ15b] *For $k \in \mathbb{R}^+$, u^k is an upper semi-continuous function which satisfies the following properties:*

- $u(t,z) \leq u^k(t,z) \leq \sup_{|s-t|\leq A/k} u(s,z)$*, where* $A > 2\, osc_{\Omega_T} u$.
- $|u^k(t,x) - u^k(s,x)| \leq k|s-t|$*, for* $(s,z),(t,z) \in \Omega_T$.
- *For all* $(t_0,z_0) \in [0,T-A/k] \times \Omega$*, there exists* $t_0^* \in [0,T)$ *such that*
$$|t_0^* - t_0| \leq A/k \text{ and } u^k(t_0,z_0) = u(t_0^*,z_0) - k|t_0 - t_0^*|.$$

Moreover, if u satisfies

$$e^{\partial_t u + F(t,z,u)}\mu(t,z) \leq (dd^c u)^n \text{ in } (0,T)\times\Omega, \tag{13}$$

in the viscosity sense then the function u^k is a subsolution of

$$e^{\partial_t w + F_k(t,z,u)}\mu_k(t,z) - (dd^c w)^n = 0 \text{ in } (A/k, T-A/k)\times\Omega,$$

where $F_k(t,z,r) := \inf_{|s-t|\leq A/k}(F(s,z,r)+k|s-t|)$ and $\mu_k(t,z) = \inf_{|s-t|<A/k} \mu(s,z)$. The dual statement is true for a lower semi-continuous function v which is a supersolution.

2.5. **Cegrell's classes.** Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$. The following classes of plurisubharmonic functions were introduced by Cegrell [Ceg98, Ceg04]:

- $\mathcal{E}_0(\Omega)$ is the set of bounded psh function u with $\lim_{z\to\xi} u(z) = 0, \forall \xi \in \partial\Omega$ and $\int_\Omega (dd^c u)^n < +\infty$.
- $\mathcal{E}(\Omega)$ is the set of all $u \in PSH^-(\Omega)$ such that for every $z_0 \in \Omega$, there exist a neighborhood U of z_0 in Ω and a decreasing sequence $h_j \in \mathcal{E}_0(\Omega)$ such that $h_j \searrow u$ on U and $\sup_j \int_\Omega (dd^c h_j)^n < \infty$.
- $\mathcal{F}(\Omega)$ is the set of all $u \in PSH^-(\Omega)$ such that there exists a decreasing sequence $u_j \in \mathcal{E}_0(\Omega)$ such that $u_j \searrow u$ on Ω and $\sup_j \int_\Omega (dd^c u_j)^n < +\infty$.

By [Ceg04, Blo06], $u \in \mathcal{E}(\Omega)$ iff u is a non-positive psh function satisfying the following property: there exists a Borel measure ν such that, if $U \subset \Omega$ and u_j is a sequence of bounded psh functions in U satisfying $u_j \searrow u$ then $(dd^c u_j)^n$ converges weakly to ν in U. In this case, the Monge-Ampère operator of u is defined by $(dd^c u)^n := \nu$.

The class $\mathcal{F}(\Omega)$ satisfies the following property: For every $u \in \mathcal{F}(\Omega)$, for each $z \in \partial\Omega$,

$$\limsup_{\Omega\ni\xi\to z} u(\xi) = 0.$$

Moreover, by [NP09], the comparison principle holds in the class $\mathcal{F}^a(\Omega) = \{u \in \mathcal{F}(\Omega) : (dd^c u)^n$ vanishes on all pluripolar sets $\}$.

The class $\mathcal{F}(\Omega)$ has been generalized as following

Definition 2.15. *Let Ω be a strongly pseudoconvex domain in $\mathbb{C}^n$. Let $\psi \in C(\partial\Omega)$. Then the class $\mathcal{F}(\Omega, \psi)$ is defined by*

$$\mathcal{F}(\Omega,\psi) = \{u \in PSH(\Omega) : \exists v \in \mathcal{F}(\Omega) \text{ such that } U_\psi \geq u \geq v + U_\psi\},$$

where U_ψ is the unique solution to the problem

$$\begin{cases} U_\psi \in C(\overline{\Omega} \cap PSH(\Omega)), \\ (dd^c U_\psi)^n = 0, \\ U_\psi|_{\partial\Omega} = \psi. \end{cases} \tag{14}$$

The class $\mathcal{F}(\Omega,\psi)$ has been used to charaterize the boundary behavior in the Dirichlet problem for Monge-Ampère equation.

Theorem 2.16. [Ceg04, Aha07] *Let Ω be a strongly pseudoconvex domain in $\mathbb{C}^n$. Let ν be a positive Borel measure in Ω and $\psi \in C(\partial\Omega)$. If $\nu(\Omega) < \infty$ and ν vanishes on all pluripolar sets then there exists a unique function $u \in \mathcal{F}(\Omega,\psi)$ such that $(dd^c u)^n = \nu$.*

3. Local regularity in time

In this section, we assume that Ω is a bounded domain in $\mathbb{C}^n$. We will prove some results on the local regularity in time of solution to (2) by using the following comparison principle.

Theorem 3.1. *Let u and v be, respectively, a bounded subsolution and a bounded supersolution to (2).*

a) *Assume that for every $K \Subset \Omega$, for every $0 < R < S < T$ and $\epsilon > 0$, there exists $0 < \delta \ll 1$ such that $(1+\epsilon) f(t+s, z) \geq f(t, z)$ for all $z \in K$, $0 < s < \delta$ and $R < t < S$. Then, for every $0 < R < S < T$, for every $\epsilon > 0$, there exists $0 < \delta \ll 1$ such that*

$$u(t+s,z) < v(t,z) + \epsilon,$$

for every $(t,z) \in (R,S) \times \Omega$ *and* $s \in (0,\delta)$. *In particular, if either* u *or* v *is continuous in* t *then* $u \leq v$.

b) *Assume that for every* $K \Subset \Omega$, *for every* $0 < R < S < T$ *and* $\epsilon > 0$, *there exists* $0 < \delta \ll 1$ *such that* $(1+\epsilon)f(t,z) \geq f(t+s,z)$ *for all* $z \in K$, $0 < s < \delta$ *and* $R < t < S$. *Then, for every* $0 < R < S < T$, *for every* $\epsilon > 0$, *there exists* $0 < \delta \ll 1$ *such that*

$$u(t-s,z) < v(t,z) + \epsilon,$$

for every $(t,z) \in (R,S) \times \Omega$ *and* $s \in (0,\delta)$. *In particular, if either* u *or* v *is continuous in* t *then* $u \leq v$.

Proof. We will prove the part a). The proof of the part b) is similar.

Let $\epsilon > 0$ and $0 < R < S < T$. By the semi-continuity of u, v and by $u \leq v$ in $\partial_P(\Omega_T)$, there exists $\min\{R, T-S\} \gg \delta_1 > 0$ such that

$$u(t,z) \leq v(t,z) + \epsilon, \tag{15}$$

for every $(t,z) \in ([0,2\delta_1] \times \Omega) \cup ([0, S+\delta_1] \times (\overline{\Omega} \setminus \Omega_{\delta_1}))$, where

$$\Omega_{\delta_1} = \{z \in \Omega : dist(z, \partial\Omega) < \delta_1\}.$$

By the assumption, there exists $\delta_2 \in (0,\delta_1)$ such that

$$(1+\epsilon)f(t_1,z) \geq f(t_2,z) \text{ and } |F(t_1,z,r) - F(t_2,z,r)| < \epsilon \tag{16}$$

for every $z \in \Omega_{\delta_1}$, $r \in [-M,M]$, $t_1, t_2 \in (\delta_1, S+\delta_1)$ with $t_2 < t_1 < t_2 + \delta_2$. Here $M = \sup\limits_{[0,T)\times\overline{\Omega}} |u|$.

Denote, for every $(t,z) \in [0,T) \times \overline{\Omega}$,

$$u^k(t,z) = \sup\{u(s,z) - k|t-s| : s \in [0,T])\},$$

then u^k is Lipschitz in t. It follows from Lemma 2.14 and (16) that if $0 < \delta < \delta_2/2$ and $k > \dfrac{(A+1)}{\delta}$, for some $A > 2osc_{\Omega_T} u$ then $u^k(t+\delta,z) - t\log(1+\epsilon) - (1+t)\epsilon$ is a subsolution to

$$e^{\partial_t w + F(t,z,w)}\mu(t,z) = (dd^c w)^n, \tag{17}$$

in $(\delta_1, S) \times \Omega_{\delta_1}$. By using (15) and the comparison principle (Theorem 2.8), we get

$$u^k(t+\delta,z) - t\log(1+\epsilon) - (1+t)\epsilon \leq v(t,z),$$

for every $(t,z) \in (\delta_1, S) \times \Omega$.

Since $u^k \geq u$ and $0 < \log(1+\epsilon) < \epsilon$, we have

$$u(t+\delta,z) - (1+2T)\epsilon \leq v(t,z), \tag{18}$$

for every $(t,z) \in (\delta_1, S) \times \Omega_{\delta_1}$.

Combining (15) and (18), we obtain

$$u(t+s,z) < v(t,z) + (1+2T)\epsilon,$$

for every $(t,z) \in (R,S) \times \Omega$ and $0 < s < \delta$.

The proof is completed. □

In Theorem 3.1, if we assume that u and v are continuous in t then we have $u \leq v$. As a consequence, we have the following results on the Lipschitz regularity in time of viscosity solutions. This kind of regularity is necessary to define parabolic pluripotential solutions (cf. [GLZ1, GLZ2]).

Proposition 3.2. *Assume that μ is non-increasing in t and u is a solution to (2). Suppose that there exists $C_0 > 0$ satisfying*

$$\varphi(t,z) - \varphi(s,z) \geq -C_0(t-s), \quad \forall z \in \partial\Omega, 0 < s < t < T,$$

and for every $m > 0$, there exists $C_m > 0$ satisfying

$$F(t,z,r) - F(s,z,r) \leq C_m(t-s), \quad \forall r \in [-m,m], z \in \partial\Omega, 0 < s < t < T.$$

Denote $M = \sup|u|$, $N = \sup|\varphi|$. Then, for every $0 < B < A < T$,

$$\frac{u(B,z) - u(A,z)}{A-B} \leq \frac{2M}{A} + \max\{C_0, BC_M\} + n + N.B,$$

for all $z \in \bar{\Omega}$.

In particular, $\partial_t u \geq -\dfrac{2M}{t} - \max\{C_0, tC_M\} - n - Nt$ in the viscosity sense.

Proof. The idea of the proof is similar to Theorem 4.2 in [GLZ18].

We consider $u_A = \dfrac{1}{A}u(At,z)$ and $u_B = \dfrac{1}{B}u(Bt,z)$ in $[0,1]\times\bar{\Omega}$.

By Lemma 2.5, in $(0,1)\times\Omega$, we have

$$(dd^c u_A)^n = \frac{1}{A^n} e^{\partial_t u_A + F(At,z,Au_A)} \mu(At,z),$$

and

$$(dd^c u_B)^n = \frac{1}{B^n} e^{\partial_t u_B + F(Bt,z,Bu_B)} \mu(Bt,z) = \frac{1}{A^n} e^{\partial_t u_B + F(Bt,z,Bu_B) + n\log(B/A)} \mu(Bt,z),$$

in the viscosity sense.

By the assumption, we have, for every $(t,z) \in (0,1)\times\Omega$,

$$\mu(Bt,z) \geq \mu(At,z),$$

and

$$\begin{aligned} F(Bt,z,Bu_B) + n\log(B/A) &\geq F(At,z,Bu_B) - C_M(A-B)t - \frac{n(A-B)}{B} \\ &\geq F(At,z,Au_B - N(A-B)) - (C_M + \frac{n}{B})(A-B). \end{aligned}$$

Denote

$$\tilde{u}_B = u_B - (\max\{\frac{C_0}{B}, C_M\} + \frac{n}{B})(A-B)t - (N + \frac{M}{AB})(A-B).$$

We have, by Lemma 2.4,

$$(dd^c \tilde{u}_B)^n \geq \frac{1}{A^n} e^{\partial_t \tilde{u}_B + F(At,z,A\tilde{u}_B)} \mu(At,z),$$

in the viscosity sense in $(0,1)\times\Omega$. Note that $\tilde{u}_B \leq u_A$ in $\partial_P([0,1)\times\Omega)$. Then, by Theorem 3.1, $\tilde{u}_B \leq u_A$ in $[0,1]\times\bar{\Omega}$. In particular, for every $z \in \bar{\Omega}$,

$$\frac{1}{A}u(A,z) \geq \frac{1}{B}u(B,z) - (\max\{\frac{C_0}{B}, C_M\} + \frac{n}{B} + N + \frac{M}{AB})(A-B).$$

Hence,

$$\frac{u(B,z)-u(A,z)}{A-B} \leq \frac{2M}{A} + \max\{C_0, BC_M\} + n + NB,$$

for every $z \in \bar{\Omega}$. □

By the same argument, we have

Proposition 3.3. *Assume that μ is non-decreasing in t and u is a solution to (2). Suppose that there exists $C_0 > 0$ satisfying*

$$\varphi(t,z) - \varphi(s,z) \leq C_0(t-s), \quad \forall z \in \partial\Omega, 0 < s < t < T,$$

and for every $m > 0$, there exists $C_m > 0$ satisfying

$$F(t,z,r) - F(s,z,r) \geq -C_m(t-s), \quad \forall r \in [-m,m], z \in \partial\Omega, 0 < s < t < T.$$

Denote $M = \sup|u|$. Then, for every $0 < B < A < T$,

$$\frac{u(A,z)-u(B,z)}{A-B} \leq \frac{2M}{A} + \max\{C_0, BC_M\} + n + M.B,$$

for all $z \in \bar{\Omega}$.

In particular, $\partial_t u \leq \dfrac{2M}{t} + \max\{C_0, tC_M\} + n + Mt$ in the viscosity sense.

Corollary 3.4. *Assume that μ, F, φ satisfy the conditions in Proposition 3.3. If there exists an open set $U \subset \Omega$ such that u_0 is not a maximal plurisubharmonic function in U and $\lim_{t\to 0^+} t \log \sup_{z\in U} f(t,z) = -\infty$ then (2) does not admit a solution.*

Combining Proposition 3.2 and Proposition 3.3, we have

Corollary 3.5. *Assume that μ is independent of t and u is a solution to (2). Suppose that there exists $C_0 > 0$ satisfying*

$$|\varphi(t,z) - \varphi(s,z)| \leq C_0|t-s|, \quad \forall z \in \partial\Omega, s,t \in [0,T),$$

and for every $m > 0$, there exists $C_m > 0$ satisfying

$$|F(t,z,r) - F(s,z,r)| \leq C_m|t-s|, \quad \forall r \in [-m,m], z \in \partial\Omega, s,t \in [0,T).$$

Denote $M = \sup|u|$. Then, for every $0 < B < A < T$,

$$\frac{|u(A,z)-u(B,z)|}{A-B} \leq \frac{2M}{A} + \max\{C_0, BC_M\} + n + M.B,$$

for all $z \in \bar{\Omega}$.

In particular, $|\partial_t u| \leq \dfrac{2M}{t} + \max\{C_0, tC_M\} + n + Mt$ in the viscosity sense.

We then have the following corollary.

Corollary 3.6. *Assume that μ, F and φ are independent of t. If (2) admits a viscosity solution then for every $0 < t < T$, there exists $C_t > 0$ such that $(dd^c u(t,z))^n \leq C_t \mu(z)$ in the viscosity sense in Ω. In particular, (u_0, μ) is admissible.*

4. The existence of solution

In this section, we prove Theorem 1.2 and Theorem 1.3. Assume that $\Omega \subset \mathbb{C}^n$ is a smooth bounded strongly pseudoconvex domain. Consider the Cauchy-Dirichlet problem spelt out in Introduction

$$\begin{cases} e^{\partial_t u+F(t,z,u)}\mu(t,z)=(dd^c u)^n & \text{in} \quad \Omega_T,\\ u=\varphi \quad \text{in} \quad [0,T)\times\partial\Omega, \\ u(0,z)=u_0(z) \quad \text{in} \quad \bar\Omega. \end{cases} \tag{19}$$

4.1. **The construction of ϵ-subbarrier and ϵ-superbarrier.**

Proposition 4.1. *For all $\epsilon>0$, there exists a continuous ϵ-subbarrier for (2) which is Lipschitz in t.*

Proof. Let $\rho\in C^2(\bar\Omega)\cap PSH(\Omega)$ such that $\rho|_{\partial\Omega}=0$, $\nabla\rho|_{\partial\Omega}\neq 0$ and $dd^c\rho\geq dd^c|z|^2$. Denote $c=\sup_\Omega(-\rho)$. Then, there exists $M_1\gg 1$ such that the function

$$\underline{u}_1=u_0+\frac{\epsilon(\rho-c)}{2c}-M_1 t,$$

is a subsolution to (19) satisfying $\underline{u}_1\leq\varphi$ in $[0,T]\times\partial\Omega$.

Let $\varphi_\epsilon\in C^\infty(\mathbb{R}\times\mathbb{C}^n)$ such that

$$\varphi-\frac{\epsilon}{2}\leq\varphi_\epsilon\leq\varphi,$$

in $[0,T]\times\partial\Omega$. Then, there exists $M_2\gg 1$ such that the function

$$\underline{u}_2=\varphi_\epsilon-\frac{\epsilon}{2}+M_2\rho,$$

is a subsolution to (19) satisfying $\underline{u}_2\leq u_0$ in $\{0\}\times\bar\Omega$.

Now, we define $\underline{u}=\max\{\underline{u}_1,\underline{u}_2\}$. It is clearly that $\underline{u}$ is a continuous ϵ-subbarrier for (19). □

Proposition 4.2. *If $(u_0(z),\mu(0,z))$ is admissible then for all $\epsilon>0$, there exists a continuous ϵ-superbarrier for (19) which is Lipschitz in t.*

Proof. Since $(u_0(z),\mu(0,z)$ is admissible, there exist $u_\epsilon\in C(\bar\Omega)$ and $C_\epsilon>0$ such that $u_0+\frac{\epsilon}{2}\leq u_\epsilon\leq u_0+\epsilon$ and $(dd^c u_\epsilon)^n\leq e^{C_\epsilon}\mu(0,z)$ in the viscosity sense.

Let $\rho\in C^2(\bar\Omega))\cap PSH(\Omega)$ such that $-1\leq\rho\leq 0$ and $dd^c\rho>0$. Then, there exist $0<\delta<T$ and $M_1\gg 1$ such that

$$e^{C_\epsilon}|\mu(t,z)-\mu(0,z)|<\frac{\epsilon^n}{4^n}(dd^c\rho)^n,$$

for every $(t,z)\in(0,\delta]\times\Omega$ and

$$e^{C_\epsilon}\mu(t,z)<(M_1 t)^n(dd^c\rho)^n,$$

for every $(t,z)\in(\delta,T)\times\Omega$.

Let $M_2\gg 1$ such that $M_2 t+\frac{\epsilon}{4}>\varphi(t,z)$ for every $(t,z)\in(0,T)\times\partial\Omega$. We consider the function

$$\overline{u}_1=u_\epsilon-(\frac{\epsilon}{4}+M_1 t)\rho-M_2 t.$$

Then $\overline{u}_1$ is a supersolution to (19) satisfying $\overline{u}_1\geq\varphi$ in $[0,T]\times\partial\Omega$.

Let $\varphi^\epsilon\in C^\infty(\mathbb{R}\times\mathbb{C}^n)$ such that

$$\varphi\leq\varphi^\epsilon\leq\varphi+\epsilon,$$

in $[0,T]\times\partial\Omega$. Let $\overline{u}_2\in C([0,T]\times\bar{\Omega})$ such that $\overline{u}_2=\varphi^\epsilon$ in $[0,T]\times\partial\Omega$ and $\overline{u}_2(t,.)$ is maximal plurisubharmonic in Ω for every $t\in[0,T]$. Then $\overline{u}_2$ is a supersolution to (19) which is Lipschitz in t.

Now, we define $\overline{u}=\min\{\overline{u}_1,\overline{u}_2\}$. It is clearly that $\overline{u}$ is a continuous ϵ-superbarrier for (19). □

Remark 4.3. *The converse statement of Proposition 4.2 is false. For example, if Ω is the unit ball, $u_0=|z|^2-1$, $\varphi=0$, $F=0$, $\mu=tdV$ then for every $T>0$,*

$$\overline{u}_T(t,z)=\min\{0,u_0-t\log t+e^Tt\},$$

is an ϵ-superbarrier for (19) in $[0,T)\times\bar{\Omega}$ for every $\epsilon>0$. But $(u_0,\mu(0,z))=(|z|^2-1,0)$ is not admissible.

Lemma 4.4. 1) *Let $0<\epsilon_0<T$. Let u be a bounded continuous subsolution to (19) in $[0,\epsilon_0)\times\overline{\Omega}$. Then, for every $0<\epsilon<\epsilon_0$, there exists a continuous subsolution $\tilde{u}$ to (19) on $[0,T)\times\Omega$ such that*

$$\begin{cases}\tilde{u}\leq u \text{ in } [0,\epsilon_0)\times\overline{\Omega},\\ \tilde{u}=u \text{ in } [0,\epsilon)\times\overline{\Omega}.\end{cases}$$

Moreover, if u is Lipschitz in t then $\tilde{u}$ is also Lipschitz in t.

2) *Let $0<\epsilon_0<T$. Let v be a bounded continuous supersolution to (19) in $[0,\epsilon_0)\times\overline{\Omega}$. Then, for every $0<\epsilon<\epsilon_0$, there exists a continuous supersolution $\tilde{v}$ to (19) on $[0,T)\times\Omega$ such that*

$$\begin{cases}\tilde{v}\geq v \text{ in } [0,\epsilon_0)\times\overline{\Omega},\\ \tilde{v}=v \text{ in } [0,\epsilon)\times\overline{\Omega}.\end{cases}$$

Moreover, if v is Lipschitz in t then $\tilde{v}$ is also Lipschitz in t.

Proof. 1) Denote

$$M=\sup_{[0,\epsilon_0)\times\overline{\Omega}}u,\ m=\min\{\inf_{[0,\epsilon_0)\times\overline{\Omega}}u(t,z),\min_{[0,T]\times\partial\Omega}\varphi(t,z)\},$$

and

$$M_F=\max\nolimits_{[0,T]\times\overline{\Omega}}F(t,z,m),\ M_f=\sup\nolimits_{[0,T]\times\Omega}f(t,z).$$

Let $\rho\in C^2(\overline{\Omega})\cap PSH^-(\Omega)$ such that $(dd^c\rho)^n\geq e^{M_F}M_fdV$. We define

$$h(t)=\begin{cases}0 & \text{for } t<\epsilon,\\ C(t-\epsilon) & \text{for } t\geq\epsilon,\end{cases}$$

where $C=1+\dfrac{M-m+\max(-\rho)}{\epsilon_0-\epsilon}$.

Then

$$\tilde{u}=\begin{cases}\max\{u(t,z)-h(t),m+\rho\} & \text{in} \quad [0,\epsilon_0)\times\overline{\Omega},\\ m+\rho & \text{in} \quad [\epsilon_0,T)\times\overline{\Omega},\end{cases}$$

is a continuous subsolution to (2) satisfying

$$\begin{cases}\tilde{u}\leq u \text{ in } [0,\epsilon_0)\times\overline{\Omega},\\ \tilde{u}=u \text{ in } [0,\epsilon)\times\overline{\Omega}.\end{cases}$$

2) Denote

$$m = \inf_{[0,\epsilon_0)\times\overline{\Omega}} v \text{ and } M = \max\{\sup_{[0,\epsilon_0)\times\overline{\Omega}} v, \max_{[0,T]\times\partial\Omega} \varphi\}.$$

We define

$$h(t) = \begin{cases} 0 & \text{for } t < \epsilon, \\ C(t-\epsilon) & \text{for } t \geq \epsilon, \end{cases}$$

where $C = 1 + \dfrac{M-m}{\epsilon_0 - \epsilon}$.

Then

$$\tilde{v} = \begin{cases} \min\{v(t,z) + h(t), M\} & \text{in} \quad [0,\epsilon_0) \times \overline{\Omega}, \\ M & \text{in} \quad [\epsilon_0, T) \times \overline{\Omega}, \end{cases}$$

is a continuous supersolution to (2) satisfying

$$\begin{cases} \tilde{v} \geq v \text{ in } [0,\epsilon_0) \times \overline{\Omega}, \\ \tilde{v} = v \text{ in } [0,\epsilon) \times \overline{\Omega}. \end{cases}$$

□

Proposition 4.5. *Let $\epsilon > 0$. If there exists a continuous ϵ-superbarrier u for (2) in $[0,S) \times \overline{\Omega}$ for some $0 < S < T$, then there exists a continuous ϵ-superbarrier $\tilde{u}$ for (2) in $[0,T) \times \overline{\Omega}$. Moreover, if u is Lipschitz in t then $\tilde{u}$ is also Lipschitz in t.*

Proof. By the assumption and by Lemma 4.4, there exists a continuous supersolution u_1 to (19) in $[0,T) \times \overline{\Omega}$ such that $u_0(z) \leq u_1(0,z) \leq u_0(z) + \epsilon$ for all $z \in \Omega$.

Let $\varphi^\epsilon \in C^\infty(\mathbb{R} \times \mathbb{C}^n)$ such that

$$\varphi \leq \varphi^\epsilon \leq \varphi + \epsilon,$$

in $[0,T] \times \partial\Omega$.

Let $u_2 \in C([0,T] \times \bar{\Omega})$ such that $u_2 = \varphi^\epsilon$ in $[0,T] \times \partial\Omega$ and $u_2(t,.)$ is maximal plurisubharmonic in Ω for every $t \in [0,T]$.

Then $\tilde{u} = \min\{u_1, u_2\}$ is a continuous ϵ-superbarrier for (19) in $[0,T) \times \overline{\Omega}$. □

For $j = 1, 2$, we assume that

- $F_j(t,z,r)$ is continuous in $[0,T] \times \bar{\Omega} \times \mathbb{R}$ and non-decreasing in r.
- $\varphi_j(t,z)$ is a continuous function in $[0,T] \times \partial\Omega$ such that $u_0(z) = \varphi_j(0,z)$ in $\partial\Omega$.

We consider the following problems

$$\begin{cases} e^{\partial_t u + F_1(t,z,u)} \mu(t,z) = (dd^c u)^n & \text{in} \quad (0,T) \times \Omega, \\ u = \varphi_1 & \text{in} \quad [0,T) \times \partial\Omega, \\ u(0,z) = u_0(z) & \text{in} \quad \bar{\Omega}, \end{cases} \tag{20}$$

and

$$\begin{cases} e^{\partial_t u + F_2(t,z,u)} \mu(t,z) = (dd^c u)^n & \text{in} \quad (0,T) \times \Omega, \\ u = \varphi_2 & \text{in} \quad [0,T) \times \partial\Omega, \\ u(0,z) = u_0(z) & \text{in} \quad \bar{\Omega}. \end{cases} \tag{21}$$

Proposition 4.6. *Let $\epsilon_0 > 0$. If there exists a continuous ϵ_0-superbarrier $\overline{u}_1$ for the problem (20) then for every $\epsilon > \epsilon_0$, there exists a continuous ϵ-superbarrier $\overline{u}_2$ for the problem (21). Moreover, if $\overline{u}_1$ is Lipschitz in t then $\overline{u}_2$ is Lipschitz in t.*

Proof. Since $u_0(z) = \varphi_1(0, z) = \varphi_2(0, z)$ for every $z \in \partial\Omega$, there exists $\delta > 0$ such that

$$|\varphi_1(t, z) - \varphi_2(t, z)| < \frac{\epsilon - \epsilon_0}{3},$$

for every $(t, z) \in [0, \delta] \times \partial\Omega$. Let $C > 0$ such that

$$e^{\partial_t(\overline{u}_1+Ct)+F_2(t,z,\overline{u}_1+Ct)}\mu(t, z) \geq (dd^c(\overline{u}_1 + Ct)),$$

in the viscosity sense in $[0, \delta) \times \partial\Omega$.

Let $\delta_0 = \min\{\delta, \frac{\epsilon-\epsilon_0}{3C}\}$. Then $\overline{u}_1 + Ct + \frac{\epsilon-\epsilon_0}{3}$ is a continuous ϵ-superbarrier for the problem (21) in $[0, \delta_0) \times \partial\Omega$. Hence, it follows from Proposition 4.5 that there exists a continuous ϵ-superbarrier $\overline{u}_2$ for the problem (21). □

4.2. **The existence of solution.** Now we prove some results about the existence of solution to (2). Theorem 1.2 is an immediate corollary of the following:

Theorem 4.7. *Suppose that for every $\epsilon > 0$, there exists a continuous ϵ-superbarrier for (2). Assume that for every $K \Subset \Omega$, for every $0 < R < S < T$ and $\epsilon > 0$, there exists $0 < \delta \ll 1$ such that*

$$(1 + \epsilon)f(t + s, z) \geq f(t, z), \tag{22}$$

and

$$(1 + \epsilon)f(t, z) \geq f(t + s, z), \tag{23}$$

for all $z \in K$, $0 < s < \delta$ and $R < t < S$.

Then (2) admits a unique solution u. Moreover, $(u(t, z), \mu(t, z))$ is admissible for every $0 < t < T$. If (22) holds in the case $S = T$ then u can be extended continuously to $[0, T] \times \overline{\Omega}$ and $(u(T, z), \mu(T, z))$ is admissible.

Proof. Denote by u the supremum of continuous subsolutions to (2). Then, by Proposition 4.1 and Lemma 2.11, $u = u_*$ is a supersolution to (2) and u^* is a subsolution to (2).

By Theorem 3.1, for every $(t, z) \in (0, T) \times \Omega$, for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$u(t, z) + \epsilon \geq u^*(s, z) \geq u(s, z),$$

for all $0 < |s - t| < \delta$. Hence, by using the fact that u is lower semi-continuous, we get that u is continuous in t.

Then, by Theorem 3.1, for every $\epsilon > 0$ and $(t, z) \in (0, T) \times \Omega$,

$$u^*(t, z) \leq \lim_{s\to t} u(s, z) + \epsilon = u(t, z) + \epsilon.$$

Letting $\epsilon \to 0$, we obtain $u^* = u$.

Thus u is a solution to (2). The uniqueness of solution holds due to Theorem 3.1.

Now, we consider the case where (22) holds in the case $S = T$. For every $t > T$, we define

$$\mu(t, .) = \mu(T, .),\ F(t, ., .) = F(T, ., .) \text{ and } \varphi(t, .) = \varphi(T, .).$$

Denote by $\tilde{u}$ the supremum of continuous subsolutions to the problem

$$\begin{cases} e^{\partial_t u+F(t,z,u)}\mu(t,z)=(dd^c u)^n & \text{in} \quad (0,\infty)\times\Omega, \\ u=\varphi & \text{in} \quad [0,\infty)\times\partial\Omega, \\ u(0,z)=u_0(z) & \text{in} \quad \bar{\Omega}, \end{cases} \tag{24}$$

Then, it follows from Lemma 4.4 that $\tilde{u}=u$ in $[0,T)\times\bar{\Omega}$. By the continuity of u, we also have $\tilde{u}^*=u$ in $[0,T)\times\bar{\Omega}$.

By Lemma 2.11, $\tilde{u}^*$ is a supersolution to (24). Applying the part a) of Theorem 3.1, for every $\epsilon>0$, there exists $\delta_1>0$ such that

$$u(t,z)=\tilde{u}^*(t,z)<\tilde{u}(s,z)+\epsilon=u(s,z)+\epsilon, \tag{25}$$

for every $T-\delta_1<s<t<T$ and $z\in\Omega$. Choose $\rho\in C^2(\overline{\Omega})\cap PSH(\Omega)$ such that $-1\leq\rho\leq 0$ and $dd^c\rho>0$. Choose $C\gg 1$ such that

$$(dd^c\rho)^n\geq e^{-C+\sup F(.,.,\sup\varphi)}\sup f dV.$$

Here, we recall that $\mu(t,z)=f(t,z)dV$.

Then, for every $1>\epsilon>0$, there exists $0<\delta_2<\delta_1$ such that, for every $s\in(T-\delta_2,T)$, the function

$$u(s,z)+\frac{\epsilon}{3}\rho(z)-(C+n\log\frac{2}{\epsilon})(t-s)-\frac{\epsilon}{3},$$

is a subsolution to

$$e^{\partial_t w+F(t,z,w)}\mu(t,z)=(dd^c w)^n, \tag{26}$$

in $(s,T)\times\Omega$.

Choose $\delta_2\ll 1$ such that $|\varphi(t,z)-\varphi(s,z)|<\epsilon/3$ for every $z\in\partial\Omega$ and $t,s\in[T-\delta_2,T]$. Using Theorem 2.8, for every $T-\delta_2<s<t<T$ and $z\in\Omega$,

$$u(s,z)+\frac{\epsilon}{3}\rho(z)-(C+n\log\frac{2}{\epsilon})t-\frac{\epsilon}{3}<u(t,z). \tag{27}$$

Choose $0<\delta_3<\delta_2$ such that $(C+n\log\frac{2}{\epsilon})\delta_3<\frac{\epsilon}{3}$. We have

$$u(s,z)-\epsilon<u(t,z), \tag{28}$$

for every $T-\delta_3<s<t<T$ and $z\in\Omega$.

Combining (25) and (28), we get that $u(t,z)$ converges uniformly to a continuous plurisubharmonic function $u(T,z)$ as $t\nearrow T$. By using the condition (22) and applying Lemma 2.14, we obtain that $(u(T,z),\mu(T,z))$ is admissible. □

Corollary 4.8. *If there exist $0\leq f_0,f_1\in C(\Omega)$ such that (u_0,f_0dV) is admissible and $f(t,z)=tf_1(z)+(T-t)f_0(z)$ then (2) has a unique solution.*

Remark 4.9. *By combining Theorem 4.7 and Lemma 2.14, if μ is independent of t then (2) admits a solution iff (u_0,μ) is admissible. Hence, by Corollary 3.6, Definition 1.1 is equivalent to the definition of the admissible property in* [EGZ15b].

Theorem 4.10. *Let $u\in C^0([0,T]\times\overline{\Omega})$ such that*

$$e^{\partial_t u+F(t,z,u)}\mu(t,z)=(dd^c u)^n, \tag{29}$$

in $((0,S)\cup(S,T))\times\Omega$ in the viscosity sense for some $S\in(0,T)$.

If $(u(S,z),\mu(S,z))$ is admissible then u satisfies (29) in the viscosity sense in $(0,T)\times\Omega$.

Proof. By Lemma 2.6, it remains to show that, for every $\epsilon > 0$, there exist a subsolution $\underline{u}_\epsilon$ and a supersolution $\overline{u}_\epsilon$ to (29) in $(0, T) \times \Omega$ such that

$$|u(t,z) - \underline{u}_\epsilon(t,z)| < \epsilon \text{ and } |u(t,z) - \overline{u}_\epsilon(t,z)| < \epsilon,$$

for every $(t, z) \in (0, T) \times \Omega$.

Let $\rho \in C^2(\overline{\Omega}) \cap PSH(\Omega)$ such that $-1 \leq \rho \leq 0$ and $dd^c\rho > 0$. Let $C_1 \gg 1$ such that

$$(dd^c\rho)^n \geq e^{-C_1 + \sup F(.,.,\sup u)} \sup f dV.$$

Here, we recall that $\mu(t,z) = f(t,z)dV$.

Denote, for every $0 \leq t \leq S$,

$$h(t) = \sup\{|u(S-s,z) - u(S,z)| : z \in \Omega, s \in (0,t)\}.$$

It is obvious that h is a continuous non-decreasing function with $h(0) = 0$.

Let $1 > \epsilon > 0$. Then there exists $\delta_1 \in (0, S)$ such that

$$\max\{h(\delta_1), (C_1 + n\log\frac{3}{\epsilon})\delta_1\} < \frac{\epsilon}{3}.$$

For $(t,z) \in (0,S) \times \overline{\Omega}$, we denote

$$\tilde{h}(t,z) = h(S-t) - (C_1 + n\log\frac{3}{\epsilon})(t - S + \delta_1) + \frac{\epsilon(t - S + \delta_1)}{3\delta_1}\rho(z).$$

For every $(t,z) \in [0,T) \times \overline{\Omega}$, we define

$$h_\epsilon(t,z) = \begin{cases} h(\delta_1) & \text{if } t \in [0, S-\delta_1], \\ \tilde{h}(t,z) & \text{if } t \in [S-\delta_1, S], \\ \dfrac{\epsilon\rho(z)}{3} & \text{if } t \in [S,T), \end{cases}$$

and

$$\underline{u}_\epsilon(t,z) = u(t,z) + h_\epsilon(t,z) - \frac{\epsilon}{3}.$$

Then $\underline{u}_\epsilon$ is a subsolution to (29) in $(0,T) \times \Omega$ satisfying $|u(t,z) - \underline{u}_\epsilon(t,z)| < \epsilon$ for every $(t,z) \in (0,T) \times \Omega$.

Since $(u(S,z), \mu(S,z))$ is admissible, there exist $C_2 > 0$ and $u^\epsilon \in C(\overline{\Omega}) \cap PSH(\Omega)$ such that $u(S,z) + \frac{\epsilon}{3} < u^\epsilon(z) < u(S,z) + \frac{2\epsilon}{3}$ and $(dd^c u^\epsilon)^n \leq e^{C_2 - \inf F(.,.,\inf u)}\mu(S,z)$.

Let $\delta_2 > 0$ such that $C_2\delta_2 < \frac{\epsilon}{3}$ and $\frac{\epsilon}{3} < u(t,z) - u_\epsilon(z) < \frac{2\epsilon}{3}$ for every $(t,z) \in (S-\delta_2, S) \times \Omega$. For every $(t,z) \in [0,T) \times \overline{\Omega}$, we define the function $\overline{u}_\epsilon$ by

$$\begin{cases} u(t,z) & \text{if } t \in [0, S-\delta_2], \\ \min\{u(t,z) + \frac{\epsilon}{\delta_2}(t - S + \delta_2), u_\epsilon(t,z) + C_2(t - S + \delta_2)\} & \text{if } t \in [S-\delta_2, S], \\ \min\{u_\epsilon(t,z) + C_2(t - S + \delta_2), u(t,z) + \epsilon\} & \text{if } t \in [S,T). \end{cases}$$

Then $\overline{u}_\epsilon$ is a supersolution to (29) in $(0,T) \times \Omega$ satisfying $|u(t,z) - \underline{u}_\epsilon(t,z)| < \epsilon$ for every $(t,z) \in (0,T) \times \Omega$. The proof is completed. □

By using Theorem 4.7 and Theorem 4.10, we have the following:

Theorem 4.11. *Assume that there exist* $t_1, ..., t_m$ *with* $0 = t_0 < t_1 < ... < t_m = T$ *satisfying*

i) *For every* $K \Subset \Omega$*, for every* $\epsilon > 0$ *and* $t_{k-1} < R < t_k (1 \leq k \leq m)$*, there exists* $0 < \delta \ll 1$ *such that*

$$(1 + \epsilon) f(t + s, z) \geq f(t, z), \tag{30}$$

for all $z \in K, 0 < s < \delta, R < t < t + s < t_k$.

ii) *For every* $K \Subset \Omega$*, for every* $\epsilon > 0$ *and* $t_{k-1} < R < S < t_k (1 \leq k \leq m)$*, there exists* $0 < \delta \ll 1$ *such that*

$$(1 + \epsilon) f(t, z) \geq f(t + s, z), \tag{31}$$

for all $z \in K, 0 < s < \delta, R < t < t + s < S$.

If for every $\epsilon > 0$*, there exists a continuous* ϵ*-superbarrier for* (19) *then there exists a unique solution to* (19).

In particular, if $(u_0(z), \mu(0, z))$ *is admissible then* (19) *admits a unique solution.*

Proof. By applying Theorem 4.7 and using induction, there exists a unique function $u \in C([0, T) \times \overline{\Omega})$ satisfying

- $e^{\partial_t u + F(t,z,u)} \mu(t, z) = (dd^c u)^n$ in $(t_{k-1}, t_k) \times \Omega$ in the viscosity sense for $k = 1, ..., m$.
- $u = \varphi$ in $(0, T) \times \partial\Omega$.
- $u(0, z) = u_0(z)$ for every $z \in \Omega$.

Moreover, $(u(t_k, z), \mu(t_k, z))$ is admissible for every $k = 1, ..., m - 1$. Then, by using Theorem 4.10, we get that u is the unique solution to (19). ☐

Corollary 4.12. *Assume that there exist* $0 = t_0 < t_1 < ... < t_m = T$ *and* $f_0, ..., f_m \in C(\Omega)$ *such that*

- $0 \leq f_0 \leq f_1 \leq ... \leq f_m$*;*
- $(u_0, f_0 dV)$ *is admissible;*
- $f(t, z) = \dfrac{(t - t_{k-1}) f_k(z) + (t_k - t) f_{k-1}(z)}{t_k - t_{k-1}}$ *for every* $k = 1, ..., m$ *and* $(t, z) \in [t_{k-1}, t_k] \times \Omega$.

Then (2) *has a unique solution.*

4.3. **The proof of Theorem 1.3.** The first conclusion of Theorem 1.3 holds due to Remark 4.9. We now prove that the admisible property is local.

Proposition 4.13. *Let* $g \geq 0$ *be a bounded continuous function in* Ω *and* $\nu = g dV$*. Let* $\phi \in PSH(\Omega) \cap C(\bar{\Omega})$*. If for every* $z \in \Omega$*, there exists an open neighborhood* U *of* z *such that* (ϕ, ν) *is admissible in* U *then* (ϕ, ν) *is admissible in* Ω*.*

Proof. Let $\rho \in C^2(\bar{\Omega}) \cap PSH(\Omega)$ such that $\rho|_{\partial\Omega} = 0$, $\nabla \rho|_{\partial\Omega} \neq 0$ and $dd^c \rho \geq dd^c |z|^2$. For every $r > 0$, we define

$$\Omega_r = \{z \in \Omega | \rho(z) < -r\}.$$

We will show that for all $r > 0$, if $\Omega_{2r} \neq \emptyset$ then (ϕ, ν) is admissible in Ω_{2r}.

By the assumption and by the compactness of $\overline{\Omega_{2r}}$, there exist balls $B(p_1, r_1), ..., B(p_m, r_m)$ such that

- $B(p_k, 2r_k) \subset \Omega_r$ for all $k = 1, ..., m$;
- (ϕ, ν) is admissible in $B(p_k, 2r_k)$ for all $k = 1, ..., m$;

- $\overline{\Omega_{2r}} \subset \bigcup\limits_{k=1}^{m} B(p_k, r_k)$.

Let $\epsilon > 0$. For every $k = 1, ..., m$, there exists $C_{\epsilon,k} > 0$ and $u_{\epsilon,k} \in C(B(p_k, 2r_k))$ such that $\phi \leq \phi_{\epsilon,k} \leq \phi + \epsilon$ and $(dd^c\phi_{\epsilon,k})^n \leq e^{C_{\epsilon,k}}\nu$ in the viscosity sense in $B(p_k, 2r_k)$.

We define

$$\phi_\epsilon(z) = \min\{\phi_{\epsilon,k}(z) + \frac{\epsilon|z-p_k|^2}{r_k^2} : |p_k - z| < 2r_k\} - \frac{\epsilon|z|^2}{\min\limits_{0\leq k\leq m} r_k^2}.$$

Then ϕ_ϵ is a continuous function in Ω_{2r} satisfying $\phi - \dfrac{\epsilon \max_{\overline{\Omega}} |z|^2}{\min_{0\leq k\leq m} r_k} \leq \phi_\epsilon \leq \phi + \epsilon$ and $(dd^c\phi_\epsilon)^n \leq e^{C_\epsilon}\nu$ in the viscosity sense in Ω_{2r}, where $C_\epsilon = \max_{1\leq k\leq m} C_{\epsilon,k}$. Hence, (ϕ, ν) is admissible in Ω_{2r}.

Now, let $\tilde{\phi} \in C(\bar{\Omega}) \cap PSH(\Omega)$ such that $\phi = \tilde{\phi}$ in $\partial\Omega$ and $(dd^c\tilde{\phi})^n = 0$ in Ω. Since $\phi, \tilde{\phi}$ are continuous, for every $\epsilon > 0$, there exists $r_1 > 0$ such that $\phi \leq \tilde{\phi} \leq \phi + \dfrac{\epsilon}{5}$ in $\Omega \setminus \Omega_{r_1}$.

Let $0 < r_2 < \dfrac{r_1}{5}$. Since (ϕ, ν) is admissible in Ω_{r_2}, there exist $\phi_\epsilon \in C(\Omega_{r_2})$ and $C_\epsilon > 0$ such that $\phi + \dfrac{3\epsilon}{5} \leq \phi_\epsilon \leq \phi + \dfrac{4\epsilon}{5}$ and $(dd^c u_\epsilon)^n \leq e^{C_\epsilon}\mu$ in the viscosity sense in Ω_{r_2}. We define

$$\phi_{0,\epsilon} = \begin{cases} \tilde{\phi} - \dfrac{\epsilon\rho}{r_1} & \text{in} \quad \Omega \setminus \Omega_{r_2}, \\ \min\{\tilde{\phi} - \dfrac{\epsilon\rho}{r_1}, \phi_\epsilon\} & \text{in} \quad \Omega_{r_2}. \end{cases}$$

Then $\phi_{0,\epsilon}$ is a continuous function in Ω satisfying $\phi \leq \phi_{0,\epsilon} \leq \phi + \epsilon$ and $(dd^c\phi_{0,\epsilon})^n \leq e^{C_\epsilon}\nu$ in the viscosity sense in Ω. Hence, (ϕ, ν) is admissible in Ω. □

Proposition 4.14. *Let $g \geq 0$ be a bounded continuous function in Ω and $\nu = gdV$. Let $\phi \in PSH(\Omega) \cap C(\bar{\Omega})$. If $\int\limits_{\{g=0\}} (dd^c\phi)^n = 0$ then (ϕ, ν) is admissible.*

Proof. The problem is local by Proposition 4.13.

Let $B \Subset \Omega$ be a ball. Let $\{U_j\}_{j=1}^{\infty}$ be a decreasing sequence of open subsets of B such that

- $\{g = 0\} \cap B \subset U_j$ for all $j \in \mathbb{Z}^+$.
- $\int_{\overline{U_j}} (dd^c\phi)^n < \dfrac{1}{2^j}$ for all $j \in \mathbb{Z}^+$.

Let $\phi_j \in C^\infty(\bar{B}) \cap PSH(B)$ such that $\phi + \dfrac{1}{2^{j+1}} \leq \phi_j \leq \phi + \dfrac{1}{2^j}$ in B and $\int_{U_j} (dd^c\phi_j)^n < \dfrac{1}{2^{j+1}}$. For any j, we define by ψ_j the solution of

$$\begin{cases} \psi_j \in C(\bar{B}) \cap PSH(B), \\ (dd^c\psi_j)^n = \chi_{B\setminus U_j}(dd^c\phi_j)^n & \text{in} \quad B, \\ \psi_j = \phi_j & \text{in} \quad \partial B. \end{cases} \tag{32}$$

Then $\psi_j \geq \phi_j \geq \phi$ and by [Xin96], for any $\epsilon > 0$,

$$\lim_{j\to\infty} Cap(\{\psi_j > \phi_j + \epsilon\}, B) = 0.$$

Hence, for every $\epsilon > 0$ and $k > 0$,

$$\phi \leq (\limsup_{j\to\infty} \psi_j)^* \leq \phi_k + \epsilon \leq \phi + \frac{1}{2^k} + \epsilon.$$

By Hartogs' lemma, ψ_j is uniformly convergent to ϕ in $\bar{B}$. Moreover, (ψ_j, ν) is admissible in B for all j. Hence, (ϕ, ν) is admissible in B. Thus (ϕ, ν) is admissible in Ω. □

Remark 4.15. *The condition "(ϕ, gdV) is admissible" does not imply that*

$$\int\limits_{\{g=0\}} (dd^c\phi)^n = 0.$$

Indeed, if Ω is the unit ball, $g = \max\{|z|^2 - 1/2, 0\}$ and $\phi = \log\max\{|z|^2, 1/2\}$ then (ϕ, gdV) is admissible since $\phi_m = \log\max\{|z|^2, 1/2 + 1/m\}$ is uniformly convergent to ϕ as $m \to \infty$ and (ϕ_m, gdV) is admissible for every $m > 0$. But, it is clearly that $\int\limits_{\{g=0\}} (dd^c\phi)^n > 0$.

5. Hölder continuity of solution

In this section we prove a Hölder regularity for the viscosity solutions to certain degenerate parabolic complex Monge-Ampère equations in smooth bounded strongly pseudoconvex domains.

Proposition 5.1. *Assume that $u(t, z)$ is a viscosity solution to (2). Suppose that there exist $C > 0$ and $0 < \alpha < 1$ such that*

$$|\varphi(t, z) - \varphi(s, z)| \leq C|t - s|^\alpha,$$

for all $z \in \partial\Omega, t, s \in [0, T)$. Then there exists $\tilde{C} > 0$ depending on C, n, Ω, $\sup\limits_{[0,T)\times\Omega} f$, α and $\sup\limits_{[0,T)\times\bar{\Omega}} F(t, z, \sup\varphi)$ such that

$$u(t, z) - u(s, z) \geq -\tilde{C}|t - s|^\alpha, \tag{33}$$

for all $z \in \Omega, 0 \leq s \leq t < T$.

Proof. Since u is bounded, we only need to show (33) in the case $|t - s| < 1$.

Let $0 \leq s_0 < t_0 < T$ such that $t_0 - s_0 = \delta < 1$. Let $\rho \in PSH(\Omega) \cap C^2(\bar{\Omega})$ such that $\rho|_{\partial\Omega} = 0$ and $(dd^c\rho)^n \geq \mu$. Denote

$$C_1 = \max_{\bar{\Omega}}(-\rho), \qquad C_2 = \alpha^{-1}|\sup_{[0,T)\times\bar{\Omega}} F(t, z, \sup\varphi)| + n\sup_{(0,1)}(-r^{1-\alpha}\log r),$$

and

$$u_\delta(t, z) = u(s_0, z) + \delta^\alpha\rho - \max\{C, C_2\}(t - s_0)^\alpha.$$

It is easy to check that

$$(dd^cu_\delta)^n \geq e^{\partial_t u_\delta + F(t,z,u_\delta)}\mu,$$

in viscosity sense in $(s_0, t_0) \times \Omega$. Moreover, $u_\delta \leq u$ in $\partial_P((s_0, t_0) \times \Omega)$. Then, by the comparison principle, $u_\delta \leq u$ in $(s_0, t_0) \times \Omega$. Hence

$$u(t_0, z) \geq u_\delta(t_0, z) \geq u(s_0, z) - (C_1 + \max(C, C_2))\delta^\alpha,$$

for all $z \in \Omega$.

Thus,

$$u(t,z) \geq u(s,z) - (C_1 + \max(C, C_2))|t-s|^\alpha,$$

for every $z \in \Omega$ and $0 \leq s < t < T$ with $t - s < 1$. The proof is completed. □

Proposition 5.2. *Assume that $\mu > 0$ and $u(t,z)$ is a viscosity solution to (2). Suppose that there exist $C > 0$, $0 < \alpha < 1$ and $0 < \beta < 1/2$ such that*

$$|\varphi(t,z) - \varphi(s,z)| \leq C|t-s|^\alpha,$$

for all $z \in \partial\Omega, t, s \in [0,T)$, and

$$|u_0(z) - u_0(w)| \leq C|z-w|^\beta,$$

for all $z, w \in \bar{\Omega}$. Then, there exists $\tilde{C} > 0$ such that

$$u_0(z) - u(t,z) \geq -\tilde{C}t^\alpha,$$

for all $z \in \Omega, 0 \leq t < T$.

Proof. We define on $\mathbb{C}^n$

$$\tilde{u}_0(z) = \max_{\xi \in \bar{\Omega}} (u_0(\xi) - C|z-\xi|^\beta), z \in \mathbb{C}^n.$$

Then $\tilde{u}_0 = u_0$ in $\bar{\Omega}$ and

$$|\tilde{u}_0(z) - \tilde{u}_0(w)| \leq C|z-w|^\beta,$$

for every $z, w \in \mathbb{C}^n$.

Let $\chi \in C^\infty(\mathbb{C}^n, [0,1])$ such that $\chi(z) = 0$ for every $|z| > 2$ and $\int\limits_{\mathbb{C}^n} \chi = 1$. For every $\delta > 0$, we denote

$$u_{\delta,0}(z) = \chi_\delta * \tilde{u}_0(z),$$

where $\chi_\delta(z) = \dfrac{1}{\delta^{2n}}\chi(\dfrac{z}{\delta})$.

Then, there exists $C_1 > 0$ depending only on χ and C such that, for every $\delta > 0$ and $z \in \mathbb{C}^n$,

$$|u_{\delta,0}(z) - \tilde{u}_0| \leq C_1\delta^\beta, |Du_{\delta,0}| \leq C_1\delta^{\beta-1}, |D^2u_{\delta,0}| \leq C_1\delta^{\beta-2}.$$

Since $\mu > 0$, there exists $C_2 > 0$ depending only on C_1 and μ such that

$$(dd^c u_{\delta,0})^n_+ \leq C_2\delta^{-2n+n\beta}\mu.$$

For every $0 < \delta < \min\{1, T\}$, we define

$$u_\delta(t,z) = u_{\delta^{\alpha/\beta},0}(z) + C_1\delta^\alpha + \max\{C, C_3\}t^\alpha,$$

where

$$C_3 = \frac{1}{\alpha}(\log C_2 + |\inf_{[0,T)\times\bar{\Omega}} F(t,z,\inf\varphi)| + \frac{n\alpha(2-\beta)}{\beta}\sup_{(0,1)}(r^{1-\alpha}\log\frac{1}{r})).$$

It is direct to check that

$$(dd^c u_\delta)^n \leq e^{\partial_t u_\delta + F(t,z,u_\delta)}\mu,$$

in viscosity sense in $(0,\delta) \times \Omega$. Moreover, $u_\delta \geq u$ in $\partial_P(0,\delta) \times \Omega)$. Then, by the comparison principle, $u_\delta \geq u$ in $(0,\delta) \times \Omega$. In particular, for every $z \in \Omega$,

$$u(\delta, z) \leq u_\delta(\delta, z) = u_{\delta^{\alpha/\beta},0}(z) + C_1\delta^\alpha + \max\{C, C_3\}\delta^\alpha \leq \\ u_0(z) + (2C_1 + \max\{C, C_3\})\delta^\alpha.$$

Since $0 < \delta < \min\{1, T\}$ is arbitrary, we get

$$u(t, z) \leq u_0(z) + (2C_1 + \max\{C, C_3\})t^\alpha,$$

for every $(t, z) \in (0, \min\{1, T\}) \times \Omega$. Since u is bounded, there exists $C_4 > 0$ depending only on $C_1, C_3, C, \sup|u|$ such that

$$u(t, z) \leq u_0(z) + C_4 t^\alpha,$$

for every $(t, z) \in (0, T) \times \Omega$. The proof is completed. □

Proposition 5.3. *Suppose that there exist $C_1, C_2 > 0$, $0 < \alpha < 1$ and $0 < \beta < 1/2$ such that*

$$|\varphi(t, z) - \varphi(t, w)| \leq C_1|z - w|^\beta, \ \forall z, w \in \partial\Omega, t \in [0, T),$$

and $\forall z \in \partial\Omega, t \mapsto \varphi(t, z) - C_2 t$ is decreasing. Then there exists $\tilde{C} > 0$ such that

$$|u(t, z) - u(t, w)| \leq \tilde{C}|z - w|^\beta.$$

Proof. Let $M = \sup_{[0,T)\times\Omega} F(t, z, u(t, z))$, then u satisfies

$$(dd^c u)^n \leq e^{\partial_t u + M}\mu, \tag{34}$$

in the viscosity sense. Let $v(t, z)$ be the solution of the complex Monge-Ampère equations

$$\begin{cases} (dd^c v)^n = e^{M + C_2}\mu, \\ v(t, z) = \varphi(t, z) \quad \text{on } \partial\Omega, \end{cases} \tag{35}$$

where C_2 satisfies that $\varphi(t, z) - C_2 t$ is decreasing. Then $v(t, z) - C_2 t$ is also the solution of

$$\begin{cases} (dd^c v)^n = e^{M + C_2}\mu, \\ v(t, z) = \varphi(t, z) - C_2 t \quad \text{on } \partial\Omega. \end{cases} \tag{36}$$

Applying the global maximum principle of complex Monge-Ampère operator (see for example [GZ17, Corollary 3.30]) for $v(t, x) - C_2 t$ and the fact that $\partial_t\varphi \leq C$, we have

$$v(t, z) - v(s, z) \leq C_2(t - s), \forall t \geq s. \tag{37}$$

We now have $v(t, s) - C_2 t$ is decreasing in t, so $v(t, z)$ converges, as $t \to 0$, to a psh function v_0 satisfying the equation

$$\begin{cases} (dd^c v_0)^n = e^{M + C_2}\mu, \\ v_0 = \varphi(0, z) \quad \text{on } \partial\Omega. \end{cases} \tag{38}$$

Let $\rho \in C^2(\bar{\Omega}) \cap PSH(\Omega)$ such that $\rho < 0$ on Ω, $\rho|_{\partial\Omega} = 0$ and $(dd^c\rho)^n \geq \mu$. We choose $K > 0$ such that $v - K(-\rho)^\alpha \leq u_0$ on $\bar{\Omega}$. It follows from (37) and (36) that

$$(dd^c(v - K(-\rho)^\alpha))^n \geq (dd^c v)^n = e^{C_2 + M}\mu \geq e^{\partial_t v + M}\mu, \tag{39}$$

in the viscosity sense. Combining with (34) and the parabolic comparison principle yields $u \geq v - K(-\rho)^\alpha$. Moreover, we also have that $v(t, \cdot)$ is uniformly β-Hölder in $\overline{\Omega}$ (cf. [BT76, Cha15])

For the super-barrier, we use the fact that the harmonic extension u_φ of $\varphi(t,z)$ majorizes u from above. Moreover it follows from classical elliptic regularity that

$$|u_\varphi(t,z) - u_\varphi(t,w)| \leq C|z-w|^\beta, \forall t \in [0,T]. \tag{40}$$

Combining both sub/super barriers implies that there exists $B>0$ such that

$$\forall z \in \overline{\Omega}, \forall \xi \in \partial\Omega, \quad |u(t,z) - u(t,\xi)| \leq B|z-\xi|^\beta, \forall t \in [0,T). \tag{41}$$

Consider $\tau \in \mathbb{C}^n$ small with $|\tau|<1$, the function

$$w(t,z) = (1-|\tau|^\beta)u(t,z+\tau) + A_2|\tau|^\beta|z|^2 - A_1|\tau|^\beta$$

is defined on $\Omega_\tau = \{z \in \Omega | z+\tau \in \Omega\}$. Here we choose $A_2 = e^{C_F+M}$, $A_1 = A_2 diam(\Omega) + |u|_{L^\infty} + B$, where C_F is the Hölder constant of F:

$$|F(t,z,r) - F(t,\xi,r)| \leq C_F|z-\xi|^\beta, \forall (t,r) \in [0,T) \times [-\|u\|_{L^\infty}, \|u\|_{L^\infty}].$$

It folllows from (41) that if $z+\tau \in \partial\Omega$ or $z \in \partial\Omega$ then

$$w(t,z) \leq u(t,z) - |\tau|^\beta u(t,z) + B(1-|\tau|^\beta)|\tau|^\beta + A_2 diam(\Omega)|\tau|^\beta - A_1|\tau|^\beta \leq u(t,z).$$

We now prove that $w(t,z) \leq u(t,z)$ on Ω_τ. Assume by contradiction that it is not the case, then consider $U_\tau = \{(t,z) \in [0,T) \times \Omega_\tau | w(t,z) > u(t,z)\}$.

We will show that w is a subsolution for (2) on U_τ. For any (t_0,z_0) and q is an upper test for v at (t_0,z_0), then $\tilde{q} := (1-|\tau|^\beta)^{-1}(q(t,z) - A_2|\tau|^\beta|z|^2 + A_1|\tau|^\beta$ is also a upper test for $u(\cdot, \tau+\cdot)$ at the point (t_0,z_0).

By the definition of viscosity solution $(dd^c\tilde{q})^n \geq e^{\partial_t\tilde{q} + F(t_0,z_0,u(z_0+\tau))}\mu$, so

$$\partial_t q = (1-|\tau|^\beta)\partial_t\tilde{q} \leq (1-|\tau|^\beta)\left(\log\frac{(dd^c\tilde{q})^n}{\mu} - F(t_0, z_0+\tau, u(t_0,z_0+\tau))\right).$$

Combining with the concavity of log det yields

$$\begin{aligned} \log\frac{(dd^c q)^n}{\mu} &= \log\frac{((1-|\tau|^\beta)dd^c\tilde{q} + |\tau|^\beta A_2 dd^c|z|^2)^n}{\mu} \\ &\geq (1-|\tau|^\beta)\log\frac{(dd^c\tilde{q})^n}{\mu} + |\tau|^\beta \log A_2 \\ &= (\partial_t q) + (1-|\tau|^\beta)F(t_0,z_0+\tau,u(t_0,z_0+\tau)) + |\tau|^\beta\log A_2. \end{aligned} \tag{42}$$

This implies that

$$(dd^c q)^n \geq e^{\partial_t q + (1-|\tau|^\beta)F(t_0,z_0+\tau,u(t_0,z_0+\tau)) + |\tau|^\beta \log A_2}\mu. \tag{43}$$

By the monotonicity of F with respect to third variable, on U_τ we have

$$\begin{aligned} F(t_0,z_0+\tau,u(t_0,z_0+\tau)) &\geq F(t_0,z_0+\tau,(1-|\tau|^\beta)u(t_0,z_0+\tau) + A_2|\tau|^\beta|z|^2 - A_1|\tau|^\beta) \\ &= F(t_0,z_0+\tau,v(t_0,z_0)) \\ &\geq F(t_0,z_0+\tau,u(t_0,z_0)). \end{aligned}$$

Combining this with the Hölder continuity in the second variable of F and the choice of A_2, we get

$$(1-|\tau|^\beta)F(t_0,z_0+\tau,u(t_0,z_0+\tau)) + |\tau|^\beta\log A_2 \geq F(t_0,z_0,u(t_0,z_0)).$$

So it follows from (43) that

$$(dd^c q)^n \geq e^{\partial_t q + F(t_0,z_0)}\mu.$$

This implies that v is a viscossity subsolution to (2) on U_τ. Therefore the comparison principle implies that $v \le u$ on U_τ, and we get a contradiction. Hence U_τ is empty. Finally we infer that, for all $z \in \Omega$,

$$u(t, z+\tau) + A_2|\tau|^\beta |z|^2 - A_1|\tau|^\beta \le u(t, z).$$

This implies that u is Hölder in the z variable as required. □

Proof of Theorem 1.5. The Hölder continuity for u on the z-variable is straightforward from Proposition 5.3. In Proposition 5.2, replacing u_0 by u_s and using The Hölder continuity on the z-variable, we infer that, for $0 \le s \le t$,

$$u(t,z) - u(s,z) \le \tilde{C}|t-s|^\alpha. \tag{44}$$

Combining with Proposition 5.1 implies the Hölder continuity of u as required.

In the case where φ is Lipschitz in t, by using Proposition 3.2 and Proposition 3.3, we obtain that u is locally Lipschitz in t uniformly in z. □

6. Convergence

In this section, we prove that the viscosity solution of a parabolic complex Monge-Ampère equation recovers the solution of the corresponding elliptic equation, extending the convergence result in [EGZ15b].

Theorem 6.1. *Consider the problem (2). Assume that $T = \infty$, $\varphi(t,z) \rightrightarrows \varphi_\infty(z)$ as $t \to \infty$ and for any $M > 0$, $F(t,z,r) \rightrightarrows F_\infty(z,r)$ in $\bar{\Omega} \times [-M, M]$ as $t \to \infty$, where $\rightrightarrows$ denotes the uniform convergence.*

Assume that $\sup_{t \ge 0} f(t,z) \in L^1(\Omega)$ and $f(t,z)$ converges almost everywhere to $f_\infty(z) \in L^1(\Omega)$ as $t \to \infty$. If (2) admits a solution u then $u(t,z)$ converges in capacity to $u_\infty(z)$ as $t \to \infty$, where u_∞ is the unique solution of the equation

$$\begin{cases} u_\infty \in \mathcal{F}(\Omega, \varphi_\infty), \\ (dd^c u_\infty)^n = e^{F_\infty(z,u_\infty)} f_\infty(z) dV(z) \qquad in \qquad \Omega, \end{cases} \tag{45}$$

where $\mathcal{F}(\Omega, \varphi_\infty)$ is a Cegrell class (see Definition 2.15).

Moreover, if $\sup_{t \ge 0} f(t,z) \in L^p(\Omega)$ for some $p > 1$ then $u(t,z)$ converges uniformly to $u_\infty(z)$ as $t \to \infty$.

Here *the uniform convergence in capacity* means that, for every $\epsilon > 0$, there exists an open set $U \subset \Omega$ such that $Cap(U, \Omega) < \epsilon$ and $u(t,z)$ converges uniformly to $u_\infty(z)$ in $\Omega \setminus U$ as $t \to \infty$. By the countable subadditivity of capacity, this is equivalent to the following: For every $\epsilon > 0$, there exist an open set $U \subset \Omega$ and $T > 0$ such that $Cap(U, \Omega) < \epsilon$ and $|u(t,z) - u_\infty(z)| < \epsilon$ for every $(t,z) \in (T, \infty) \times (\Omega \setminus U)$.

Proof. Let $1 \gg \epsilon > 0$. For every $T > 0$, we consider the problem

$$\begin{cases} e^{\partial_t w + F_\infty(z,w)}(1+\epsilon^{n+1})\mu_T(z) = (dd^c w)^n \qquad \text{in} \qquad (0,\infty) \times \Omega, \\ w(t,z) = \varphi(T,z) - \epsilon \qquad \text{in} \qquad [0,T) \times \partial\Omega, \\ w(0,z) = u(T,z) - \epsilon \qquad \text{in} \quad \bar{\Omega}, \end{cases} \tag{46}$$

where $\mu_T(z) = \sup_{t \in [T, T+1]} f(t,z) dV$. It follows from Lemma 2.14 that $(u(T,z), \mu_T(z))$ is admissible for every T. Hence, (46) admits a unique solution $u_T(t,z)$.

By Proposition 2.13, we have $\sup |u| = M < \infty$. Let $T_1 > 0$ such that

$$|F(t,z,r) - F_\infty(z,r)| < \log(1+\epsilon^{n+1}), \tag{47}$$

for every $(t, z, r) \in [T_1, \infty) \times \Omega \times [-M, M]$ and

$$|\varphi(t,z) - \varphi_\infty(z)| < \epsilon, \tag{48}$$

for every $(t, z) \in [T_1, \infty) \times \partial\Omega$.

We will find $T_2 > T_1$, $0 < \delta \ll 1$ and $\phi \in \mathcal{F}(\Omega)$ with $Cap(\{\phi < -\epsilon\}, \Omega) = O(\epsilon)$ such that $u_{T_2}(t, z) + \phi$ is a subsolution to the problem

(49)
$$\begin{cases} e^{\partial_t w + F_\infty(z,w)}(1+\epsilon^{n+1}) \sup\limits_{s\in[T_2,T_2+T']} f(s,z)dV = (dd^c w)^n & \text{in} \quad (0,T')\times\Omega, \\ w(t,z) = \varphi(T_2,z) \quad \text{in} \quad [0,T')\times\partial\Omega, \\ w(0,z) = u(T_2,z) \quad \text{in} \quad \bar{\Omega}, \end{cases}$$

and $u_{T_2}(t+\delta, z) - \phi + 2\epsilon$ is a supersolution to the problem

(50)
$$\begin{cases} e^{\partial_t w + F_\infty(z,w)}(1-\epsilon^{n+1}) \inf\limits_{s\in[T_2,T_2+T']} f(s,z)dV = (dd^c w)^n & \text{in} \quad (0,T')\times\Omega, \\ w(t,z) = \varphi(T_2,z) \quad \text{in} \quad [0,T')\times\partial\Omega, \\ w(0,z) = u(T_2,z) \quad \text{in} \quad \bar{\Omega}, \end{cases}$$

for every $T' > \delta$.

By Proposition 5.1, there exists $\delta > 0$ such that

$$u_T(t+s, z) \geq u_T(t,z) - \epsilon, \tag{51}$$

for every $t, T > 0, z \in \Omega$ and $s \in [0, \delta]$. By Corollary 3.5, there exists $C_1 > 0$ such that

$$|\partial_t u_T(t,z)| \leq C_1, \tag{52}$$

for every $T > 0$, $z \in \Omega$ and $t \geq T + \delta$.

By Lebesgue's dominated convergence theorem, $\sup_{s\geq t} f(s,z)$ and $\inf_{s\geq t} f(s,z)$ are convergent to $f_\infty(z)$ in $L^1(\Omega)$ as $t \to \infty$. Hence,

$$\lim_{t\to\infty} \int\limits_\Omega |\sup\nolimits_{s\geq t} f(s,z) - \inf\nolimits_{s\geq t} f(s,z)|dV = 0.$$

Let $T_2 > T_1$ such that

$$\int\limits_\Omega |\sup_{s\geq T_2} f(s,z) - \inf_{s\geq T_2} f(s,z)|dV < \frac{e^{-C_1-C_2}\epsilon^{n+1}}{n!}, \tag{53}$$

where $C_2 = \sup F(.,., \sup \varphi_\infty + \epsilon)$.

Let ϕ be the unique solution to the equation

(54)
$$\begin{cases} \phi \in \mathcal{F}(\Omega), \\ (dd^c\phi)^n = e^{C_1+C_2}|(1+\epsilon^{n+1})\sup_{s\geq T_2} f(s,z) - (1-\epsilon^{n+1})\inf_{s\geq T_2} f(s,z)|dV. \end{cases}$$

Then, by applying Lemma 2.7 for $u_{T_2}(t+\delta, z)$, $\phi(z)$ and the equation

$$e^{\partial_t w + F_\infty(z,w)}(1+\epsilon^{n+1})\mu_{T_2}(z) = (dd^c w)^n,$$

in $(0, T')\times$ for all $T' > \delta$, we get that $u_{T_2}(t+\delta, z) - \phi(z) + 2\epsilon$ is a supersolution to (50) and $u_{T_2}(t, z) + \phi(z)$ is a subsolution to (49).

Note that $u(t+T_2, z)$ is a subsolution to (50) and a supersolution to (49). Since $u_{T_2}(t,z)$ is locally Lipschitz in t uniformly in z, applying Theorem 2.8 and letting $T' \to \infty$, we get

$$u(t+T_2, z) \le u_{T_2}(t+\delta, z) - \phi(z) + 2\epsilon, \tag{55}$$

and

$$u(t+T_2, z) \ge u_{T_2}(t,z) + \phi(z), \tag{56}$$

for every $(t,z) \in (0,\infty)\times\Omega$.

It follows from Theorem 6.2 in [EGZ15b] that $u_{T_2}(t,z)$ converges uniformly to the solution $\tilde{u}$ of the equation

$$\begin{cases} (dd^c w)^n = e^{F_\infty(z,w)}(1+\epsilon^n)\mu_{T_2}(z) & \text{in} \quad \Omega, \\ w = \varphi_\infty - \epsilon \quad \text{in} \quad \partial\Omega. \end{cases} \tag{57}$$

Hence, by (55) and (56), there exists $T_3 > 0$ such that, for every $t > T_3$,

$$\tilde{u}(z) - \epsilon \le u(t,z) \le \tilde{u}(z) - \phi(z) + 3\epsilon. \tag{58}$$

It is easy to check that $\tilde{u}+\phi$ is a subsolution to (45) and $u_\infty + \phi - \epsilon$ is a subsolution to (57). Then

$$\tilde{u} + \phi \le u_\infty \le \tilde{u} - \phi + \epsilon. \tag{59}$$

Combining (58) and (59), we get

$$|u(t,z) - u_\infty(z)| \le -2\phi + 3\epsilon, \tag{60}$$

for every $t > T_3, z \in \Omega$.

Moreover, it follows from Proposition 3.4 in [NP09] that

$$Cap(\{\phi < -\epsilon\}, \Omega) \le (1 + n!2C_3 e^{C_1+C_2})\epsilon,$$

where $C_3 = \int_\Omega \sup_{t>0} f(t,z) fV$.

Hence, $u(t,z)$ converges uniformly in capacity to $u_\infty(z)$ as $t\to\infty$.

If $\sup_{t\ge 0} f(t,z) \in L^p(\Omega)$ for some $p>1$ then we can choose T_2 such that

$$\int\limits_\Omega |\sup_{s\ge T_2} f(s,z) - \inf_{s\ge T_2} f(s,z)|^p dV < \frac{e^{-C_1-C_2}\epsilon^{n+1}}{n!}. \tag{61}$$

Then, by (60) and by using Theorem 1.1 in [GKZ08] for ϕ and 0, we obtain the uniformly convergence of $u(t,z)$ as $t\to\infty$. □

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