SOME REMARKS ON THE CEGRELL'S CLASS $\mathcal{F}$

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Abstract. In this paper, we study the near-boundary behavior of functions $u \in \mathcal{F}(\Omega)$ in the case where Ω is strictly pseudoconvex. We also introduce a sufficient condition for belonging to $\mathcal{F}$ in the case where Ω is the unit ball.

Introduction

Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$. By [Ceg04], the class $\mathcal{F}(\Omega)$ is defined as the following: $u \in \mathcal{F}(\Omega)$ iff there exists a sequence of functions $u_j \in \mathcal{E}_0(\Omega)$ such that $u_j \searrow u$ as $j \to \infty$ and $\sup_j \int_\Omega (dd^c u_j)^n < \infty$. Here

$$\mathcal{E}_0(\Omega) = \{u \in PSH(\Omega) \cap L^\infty(\Omega) : \lim_{z\to\partial\Omega} u(z) = 0, \int\limits_\Omega (dd^c u)^n < \infty\}.$$

The class $\mathcal{F}(\Omega)$ has many nice properties. This is a subclass of the domain of definition of Monge-Ampère operator [Ceg04, Blo06]. Moreover, by [Ceg04], for each sequence of functions $u_j \in \mathcal{E}_0(\Omega)$ such that $u_j \searrow u \in \mathcal{F}(\Omega)$ as $j \to \infty$, we have

$$\lim_{j\to\infty} \int\limits_\Omega (dd^c u_j)^n = \int\limits_\Omega (dd^c u)^n.$$

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By [Ceg98, Ceg04], for every pluripolar set $E \subset \Omega$, there exists $u \in \mathcal{F}(\Omega)$ such that $E \subset \{u = -\infty\}$. In [Ceg04], Cegrell also proved some inequalities, a generalized comparison principle and a decomposition of $(dd^c u)^n, u \in \mathcal{F}(\Omega)$. In [NP09], Nguyen and Pham proved a strong version of comparison principle in the class $\mathcal{F}(\Omega)$.

The class $\mathcal{F}(\Omega)$ has been used to characterize the boundary behavior in the Dirichlet problem for Monge-Ampère equation [Ceg04, Aha07]. For every $u \in \mathcal{F}(\Omega)$, for each $z \in \partial\Omega$, we have $\limsup\limits_{\Omega\ni\xi\to z} u(\xi) = 0$ (see [Aha07]). Moreover, if we define by $\mathcal{N}$ the set of functions in the domain of definition of Monge-Ampère operator with smallest maximal plurisubharmonic majorant identically zero then, by the comparison principles in $\mathcal{F}$ and in $\mathcal{N}$ (see [NP09] and [ACCP09]) and by Cegrell's approximation theorem [Ceg04] (see also Lemma 10), we have

$$\mathcal{F}(\Omega) = \{u \in \mathcal{N}(\Omega) : \int\limits_\Omega (dd^c u)^n < \infty\}.$$

In this paper, we study the near-boundary behavior of functions $u \in \mathcal{F}(\Omega)$ in the case where Ω is a bounded strictly pseudoconvex domain, i.e., there exists $\rho \in PSH(\Omega) \cap C(\overline{\Omega})$ such that $\rho|_{\partial\Omega} = 0$, $D\rho|_{\partial\Omega} \neq 0$ and $dd^c\rho \geq c\omega := cdd^c|z|^2$ for some $c > 0$.

Our first main result is the following:

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Theorem 1. *Assume that Ω is a strictly pseudoconvex domain in $\mathbb{C}^n$ and $u \in \mathcal{F}(\Omega)$. Then, there exists $C > 0$ depending only on Ω, n and u such that*

$$Vol_{2n}(\{z \in \Omega | d(z, \partial\Omega) < d, u(z) < -\epsilon\}) \leq \frac{C.d^{n+1-na}}{a^n \epsilon^n}, \tag{1}$$

for any $\epsilon, d > 0$, $a \in (0, 1)$.

For the convenience, we denote $W_d = \{z \in \Omega | d(z, \partial\Omega) < d\}$. By Theorem 1, we have

$$\lim_{d\to 0} \frac{Vol_{2n}(\{z \in W_d | u(z) < -\epsilon\})}{d^t} = 0,$$

for every $0 < t < n + 1$. It helps us to estimate the "density" of the the set $\{u < -\epsilon\}$ near the boundary.

Moreover, by using Theorem 1 for $\epsilon = d^\alpha$ and $0 < a < 1 - \alpha$, we have

Corollary 2. *Assume that Ω is a strictly pseudoconvex domain in $\mathbb{C}^n$ and $u \in \mathcal{F}(\Omega)$. Then, for every $0 < \alpha < 1$,*

$$\lim_{d\to 0} \frac{Vol_{2n}(\{z \in W_d | u(z) < -d^\alpha\})}{d} = 0.$$

When Ω is the unit ball, this result can be improved as following:

Theorem 3. *If $u \in \mathcal{F}(\mathbb{B}^{2n})$ then*

$$\lim_{r\to 1^-} \frac{\int_{\{|z|=r\}} |u(z)| d\sigma(z)}{1 - r} < \infty.$$

In particular, there exists $C > 0$ such that

$$\limsup_{d\to 0^+} \frac{Vol_{2n}(\{z \in \mathbb{B}^{2n} : \|z\| > 1 - d, u(z) < -Ad\})}{d} < \frac{C}{A},$$

for every $A > 0$.

Our second purpose is to find a sharp sufficient condition for u to belong to $\mathcal{F}(\Omega)$ based on the near-boundary behavior of u. We are interested in the following question:

Question 4. *Let Ω be a bounded strictly pseudoconvex domain. Assume that u is a negative plurisubharmonic function in Ω satisfying*

$$\lim_{d\to 0^+} \frac{Vol_{2n}(\{z \in W_d : u(z) < -Ad\})}{d} = 0,$$

for some $A > 0$. Then, do we have $u \in \mathcal{F}(\Omega)$?

In this paper, we answer this question for the case where Ω is the unit ball.

Theorem 5. *Let $u \in PSH^-(\mathbb{B}^{2n})$. Assume that there exists $A > 0$ such that*

$$\lim_{d\to 0^+} \frac{Vol_{2n}(\{z \in \mathbb{B}^{2n} : \|z\| > 1 - d, u(z) < -Ad\})}{d} = 0. \tag{2}$$

Then $u \in \mathcal{F}(\mathbb{B}^{2n})$.

Corollary 6. *Let $u \in \mathcal{N}(\mathbb{B}^{2n})$ such that $\int\limits_{\mathbb{B}^{2n}} (dd^c u)^n = \infty$. Then, for every $A > 0$,*

$$\limsup_{d\to 0^+} \frac{Vol_{2n}(\{z \in \mathbb{B}^{2n} : \|z\| > 1 - d, u(z) < -Ad\})}{d} > 0.$$

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1. Proof of Theorem 1

Since Ω is bounded strictly pseudoconvex, there exists $\rho \in C^2(\bar{\Omega}, [0,1])$ such that $\Omega = \{z : \rho(z) < 0\}$ and

$$|D\rho| > C_1 \text{ in } \bar{\Omega}, \tag{3}$$

and

$$dd^c\rho \geq C_2 dd^c|z|^2 = C_2\omega, \tag{4}$$

where $C_1, C_2 > 0$ are constants.

By (3), there exist $C_3, C_4 > 0$ depending only on Ω and ρ such that

$$C_3 d(z, \partial\Omega) \leq -\rho(z) \leq C_4 d(z, \partial\Omega), \tag{5}$$

for every $z \in \Omega$.

For every $a \in (0,1)$ and $z \in \Omega$, we have

$$dd^c\rho_a(z) := dd^c(-(-\rho(z))^a) = a(1-a)(-\rho)^{a-2}d\rho \wedge d^c\rho + a(-\rho)^{a-1}dd^c\rho.$$

Then

$$(dd^c\rho_a)^n \geq a^n(1-a)(-\rho)^{na-n-1}d\rho \wedge d^c\rho \wedge (dd^c\rho)^{n-1}. \tag{6}$$

Hence, by (3), (4) and (5), there exists $1 \gg d_0 > 0$ depending only on Ω and ρ such that, for every $0 < d < d_0$ and $z \in W_d := \{\xi \in \Omega : d(\xi, \partial\Omega) < d\}$,

$$(dd^c\rho_a)^n \geq C_5 d^{na-n-1}\omega^n. \tag{7}$$

Since $u \in \mathcal{F}(\Omega)$, there exists $\{u_j\}_{j=1}^\infty \subset \mathcal{E}_0(\Omega)$ such that $u_j \searrow u$ and

$$\int\limits_\Omega (dd^c u_j)^n < C_6, \tag{8}$$

for every $j \in \mathbb{Z}^+$, where $C_6 > 0$ depends only on u.

By using (7), (8) and the Bedford-Taylor comparison principle [BT76, BT82] (see also [Kli91]), we have, for every $j \in \mathbb{Z}^+$, $\epsilon, d > 0$ and $a \in (0,1)$,

$$\begin{aligned} C_6 > \int\limits_{\{u_j < \epsilon\rho_a\}} (dd^c u_j)^n &\geq \int\limits_{\{u_j < \epsilon\rho_a\}} (dd^c \epsilon\rho_a)^n \\ &\geq \frac{C_5 a^n \epsilon^n}{d^{n+1-na}} \int\limits_{\{u_j < \epsilon\rho_a\} \cap W_d} \omega^n. \end{aligned}$$

Hence, for every $0 < d < d_0$,

$$Vol_{2n}(\{z \in W_d | u_j(z) < -\epsilon\}) \leq \frac{C_7.d^{n+1-na}}{a^n\epsilon^n},$$

where $C_7 > 0$ depends only on Ω, ρ, n and u.

Letting $j \to \infty$, we get

$$Vol_{2n}(\{z \in W_d | u(z) < -\epsilon\}) \leq \frac{C_7.d^{n+1-na}}{a^n\epsilon^n},$$

for every $0 < d < d_0$.

Denote

$$C = \max\{C_7, \frac{a^n\epsilon^n Vol_{2n}(\Omega)}{d_0^{n+1-na}}\}.$$

We have

$$Vol_{2n}(\{z \in W_d | u(z) < -\epsilon\}) \leq \frac{C.d^{n+1-na}}{a^n\epsilon^n},$$

for every $d > 0$.

This completes the proof of Theorem 1.

2. Proof of Theorem 3

In order to prove Theorem 3, we need the following lemma:

Lemma 7. *Let $\Omega \subset \mathbb{C}^n$ be a bounded hyperconvex domain and (X, d, μ) be a compact metric probability space. Let $u : \Omega \times X \to [-\infty, 0)$ such that*

(i) *For every $a \in X$, $u(., a) \in \mathcal{F}(\Omega)$ and*
$$\int_\Omega (dd^c u(z, a))^n < M,$$
where $M > 0$ is a constant.

(ii) *For every $z \in \Omega$, the function $u(z, .)$ is upper semicontinuous in X.*

Then $\tilde{u}(z) = \int_X u(z, a)d\mu(a) \in \mathcal{F}(\Omega)$.

Proof. It is obvious that $\tilde{u} \in PSH^-(\Omega)$.

Since X is compact, for every $j \in \mathbb{Z}^+$, we can divide X into a finite pairwise disjoint collection of sets of diameter less than $\frac{1}{2^j}$. We denote these sets by $U_{j,1}, ..., U_{j,m_j}$. We can furthermore assume that for every $1 \leq k \leq m_{j+1}$, there exists $1 \leq l \leq m_j$ such that $U_{j+1,k} \subset U_{j,l}$.

For every $j \in Z^+$, we define

$$u_j(z) = \sum_{k=1}^{m_j} \mu(U_{j,k}) \sup_{a \in U_{j,k}} u(z, a) \qquad \text{and} \qquad \tilde{u}_j = (u_j)^*.$$

Then $\tilde{u}_j \in \mathcal{F}(\Omega)$. Moreover, by [Ceg04], we have

$$\int_\Omega (dd^c \tilde{u}_j)^n \leq M,$$

for all $j \in Z^+$.

By the semicontinuity of $u(z, .)$, we get that $\tilde{u}_j$ is decreasing to $\tilde{u}$ as $j \to \infty$. Hence, $\tilde{u} \in \mathcal{F}(\Omega)$ and $\int_\Omega (dd^c \tilde{u})^n \leq M$. □

Recall that if u is a radial plurisubharmonic function then $u(z) = \chi(\log |z|)$ for some convex, increasing function χ. We have the following lemma:

Lemma 8. *Let $u = \chi(\log |z|)$ be a radial plurisubharmonic function in $\mathbb{B}^{2n}$. Then, $u \in \mathcal{F}(\mathbb{B}^{2n})$ iff the following conditions hold*

(i) $\lim_{t \to 0^-} \chi(t) = 0$;

(ii) $\lim_{t \to 0^-} \frac{\chi(t)}{t} < \infty$.

Proof. It is clear that (i) a necessary condition for $u \in \mathcal{F}(\mathbb{B}^{2n})$. We need to show that, when (i) is satisfied, the condition $u \in \mathcal{F}(\mathbb{B}^{2n})$ is equivalent to (ii).

If (ii) is satisfied then there exists $k_0 \gg 1$ such that $k_0 t < \chi(t)$. Hence $u(z) > k_0 \log|z| \in \mathcal{F}(\mathbb{B}^{2n})$. Thus, $u \in \mathcal{F}(\mathbb{B}^{2n})$.

Conversely, if (ii) is not satisfied, we consider the functions $u_k = \max\{u, k\log|z|\}$. Then, for every k, $u_k > u$ near $\partial\mathbb{B}^{2n}$. Hence

$$\int_\Omega (dd^c u)^n \geq \int_\Omega (dd^c u_k)^n = k^n \int_\Omega (dd^c \log|z|)^n \xrightarrow{k\to\infty} \infty.$$

Thus $u \notin \mathcal{F}(\mathbb{B}^{2n})$.

The proof is completed. □

Proof of Theorem 3. Denote by μ the unique invariant probability measure on the unitary group $U(n)$. For every $z \in \mathbb{B}^{2n}$, we define

$$\tilde{u}(z) = \int_{U(n)} u(\phi(z))d\mu(\phi) = \frac{1}{c_{2n-1}|z|^{2n-1}} \int_{\{|w|=|z|\}} u(w)d\sigma(w),$$

where c_{2n-1} is the $(2n-1)$-dimensional volume of $\partial\mathbb{B}^{2n}$.

By Lemma 7, we have $\tilde{u} \in \mathcal{F}(\mathbb{B}^{2n})$. Since $\tilde{u}$ is radial, we have, by Lemma 8,

$$\lim_{|z|\to 1^-} \frac{\tilde{u}(z)}{|z|-1} = \lim_{|z|\to 1^-} \frac{\tilde{u}(z)}{\log|z|} < \infty.$$

Hence

$$\lim_{r\to 1^-} \frac{\int_{\{|z|=r\}} |u(z)|d\sigma(z)}{1-r} = M < \infty.$$

Consequently, we have

$$\limsup_{d\to 0^+} \frac{Vol_{2n}(\{z \in \mathbb{B}^{2n} : \|z\| = 1-d, u(z) < -Ad\})}{d} \leq \frac{M}{A},$$

for all $A > 0$.

By using spherical coordinates to estimate integrals, we get the last assertion of Theorem 3.

The proof is completed. □

3. Proof of Theorem 5

3.1. **An approximation lemma.** In order to prove Theorem 5, we need the following lemma:

Lemma 9. *Let Ω be a hyperconvex domain in $\mathbb{C}^n$ and $u \in PSH^-(\Omega)$. Assume that there are $u_j \in \mathcal{F}(\Omega)$, $j \in \mathbb{N}$, such that u_j converges almost everywhere to u as $j \to \infty$. If $\sup_{j>0} \int_\Omega (dd^c u_j)^n < \infty$ then $u \in \mathcal{F}(\Omega)$.*

This lemma has been proved in [NP09]. For the reader's convenience, we also give the details of the proof. First, we need the following lemmas:

Lemma 10. [Ceg04] *Let $u \in PSH^-(\Omega)$. Then there exists a decreasing sequence of functions $u_j \in \mathcal{E}_0(\Omega) \cap C(\Omega)$ such that $\lim_{j\to\infty} u_j(z) = u(z)$ for every $z \in \Omega$.*

Lemma 11. *Let $u, v \in \mathcal{F}(\Omega)$ be such that $u \leq v$ on Ω. Then*
$$\int_\Omega (dd^c u)^n \geq \int_\Omega (dd^c v)^n.$$

Proof. Let $\{u_j\}_{j\in\mathbb{N}}, \{v_j\}_{j\in\mathbb{N}} \subset \mathcal{E}_0(\Omega)$ be decreasing sequences such that $u_j \searrow u$, $v_j \searrow v$ on Ω and
$$\sup_{j>0} \int_\Omega (dd^c u_j)^n < +\infty, \quad \sup_{j>0} \int_\Omega (dd^c v_j)^n < +\infty.$$
Replacing v_j by $(1 - \dfrac{1}{2^j}) \max\{v_j, u_j\}$, we can assume that $v_j \geq u_j$. By the Bedford-Taylor comparison principle [BT76, BT82] (see also [Kli91]), we obtain, for every j,
$$\int_\Omega (dd^c u_j)^n \geq \int_\Omega (dd^c v_j)^n.$$
Letting $j \to +\infty$, we get
$$\int_\Omega (dd^c u)^n \geq \int_\Omega (dd^c v)^n,$$
as desired. □

Proof of Lemma 9. For every $k \geq 1$, we denote
$$u^k(z) = \sup_{j\geq k} \max\{u, u_j\}.$$
Then, we have

(i) $v_k := (u^k)^* \in PSH^-(\Omega)$ for all $k \geq 1$.
(ii) v_k is a decreasing sequence satisfying $v_k \geq u$ for every $k \geq 1$.
(iii) $v_k = u^k$ almost everywhere and u^k converges to u almost everywhere.

By (iii), we have $\lim_{k\to\infty} v_k = u$ almost everywhere. Since u and $\lim_{k\to\infty} v_k$ are plurisubharmonic, we get $u = \lim_{k\to\infty} v_k$.

Since $0 \geq v_k \geq u_k$, we have $v_k \in \mathcal{F}(\Omega)$. Moreover, by using Lemma 11, we obtain
$$C := \sup_{j>0} \int_\Omega (dd^c u_j)^n \geq \int_\Omega (dd^c v_k)^n,$$
for every $k \geq 1$.

Now, it follows from Lemma 10 that there exists a decreasing sequence $w_k \in \mathcal{E}_0(\Omega) \cap C(\Omega)$ such that $\lim_{j\to\infty} w_j(z) = u(z)$ in Ω. Replacing w_j by $(1-j^{-1})w_j$, we can assume that $w_j(z) > u(z)$ for every $j > 0, z \in \Omega$. Applying Lemma 11, we have
$$\int_{\{v_k < w_j\}} (dd^c w_j)^n \leq \int_{\{v_k < w_j\}} (dd^c v_k)^n \leq C,$$
for every $j, k > 0$.

Letting $k \to \infty$, we get,
$$\int_\Omega (dd^c w_j)^n \leq C,$$
for every $j > 0$.

Thus, $u \in \mathcal{F}(\Omega)$. □

3.2. **Proof of Theorem 5.** For every $0 < a < 1$, we denote $S_a = \{\phi \in U(n) : \|\phi - Id\| < a\}$.

For every $0 < \epsilon, a < 1$ and $z \in \mathbb{B}^{2n}_{1-\epsilon} := \{w \in \mathbb{C}^n : \|w\| < 1 - \epsilon\}$, we define

$$u_{a,\epsilon}(z) = (\sup\{u((1+r)\phi(z)) : \phi \in S_a, 0 \le r \le \epsilon\})^*.$$

Then $u_{a,\epsilon}$ is plurisubharmonic in $\mathbb{B}^{2n}_{1-\epsilon}$ satisfying

$$\lim_{a\to 0^+} \lim_{\epsilon\to 0^+} u_{a,\epsilon}(z) = u(z), \tag{9}$$

for every $z \in \Omega$.

Moreover, for $z \neq 0$,

$$u_{a,\epsilon}(z) = (\sup\{u(\xi) : \xi \in B_{a,\epsilon,z}\})^*, \tag{10}$$

where

$$B_{a,\epsilon,z} = \{\xi \in \mathbb{C}^n : \|\frac{z}{\|z\|} - \frac{\xi}{\|\xi\|}\| < a, \|z\| \le \|\xi\| \le (1+\epsilon)\|z\|\}.$$

It is obvious that there exist $C_1, C_2 > 0$ such that

$$C_1 a^{2n-1}\epsilon < Vol_{2n}(B_{a,\epsilon,z}) < C_2 a^{2n-1}\epsilon, \tag{11}$$

for every $0 < \epsilon, a < 1/2$ and $1/2 < \|z\| < 1 - a$.

By (2), (10) and (11), for every $1/2 > a > 0$, there exists $\epsilon_a > 0$ such that, for every $\epsilon_a > 3\epsilon \ge 1 - \|z\| \ge \epsilon > 0$, we have

$$u_{a,\epsilon}(z) \ge -3A\epsilon. \tag{12}$$

For each $1/2 > a > 0$ and $\epsilon_a > 3\epsilon > 0$, we consider the following function

$$\tilde{u}_{a,\epsilon}(z) = \begin{cases} 3A(-1+|z|^2) & \text{if} \quad 1-\epsilon \le \|z\| \le 1, \\ \max\{3A(-1+|z|^2), u_{a,\epsilon}(z) - 6A\epsilon\} & \text{if} \quad 1-3\epsilon \le \|z\| \le 1-\epsilon, \\ u_{a,\epsilon}(z) - 6A\epsilon & \text{if} \quad \|z\| \le 1-3\epsilon. \end{cases}$$

Then $\tilde{u}_{a,\epsilon} \in \mathcal{F}(\mathbb{B}^{2n})$ and

$$\int_{\mathbb{B}^{2n}} (dd^c \tilde{u}_{a,\epsilon})^n = \int_{\mathbb{B}^{2n}} (dd^c 3A(-1+|z|^2))^n < \infty,$$

for every $1/2 > a > 0$ and $\epsilon_a > 3\epsilon > 0$.

Moreover, $\tilde{u}_{a,\epsilon} \xrightarrow{a.e.} u$ as $a, \epsilon \searrow 0$. Hence, by Lemma 9, we have $u \in \mathcal{F}(\Omega)$.

The proof is completed.

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