TWISTED ALEXANDER IDEALS AND THE ISOMORPHISM PROBLEM FOR A FAMILY OF PARAFREE GROUPS

DO VIET HUNG AND VU THE KHOI

Abstract. In [3] Baumslag introduced a family of parafree groups $G_{i,j}$ which share many properties with the free group of rank two. The isomorphism problem for the family $G_{i,j}$ is known to be difficult and a few small partial results have been found so far. In this paper, we compute the twisted Alexander ideals of the groups $G_{i,j}$ associated to non-abelian representations into $SL(2, \mathbb{Z}_2)$. Using the twisted Alexander ideals, we can prove that several groups among $G_{i,j}$ are not isomorphic. As a consequences, we are able solve the isomorphism problem for sub-families containing infinitely many groups $G_{i,j}$.

1. Introduction

The isomorphism problem is a fundamental problem in group theory in which one have to decide whether two finitely presented groups are isomorphic. Because the general isomorphism problem is unsolvable, people often restrict the problem to a special class of groups. Recall that a group G is called *parafree* if it is residually nilpotent and has the same nilpotent quotients as a given free group. As parafree groups enjoy many common properties with free groups, the isomorphism problem for parafree groups is known to be difficult.

In [2, 3], Baumslag study the family of parafree groups,

$$G_{i,j} := \left\langle a, b, c \middle| \; a = [c^i, a][c^j, b] \right\rangle,$$

here $[x, y] := x^{-1}y^{-1}xy$. The isomorphism problem for the family $G_{i,j}$ has gained considerable interest. The family $G_{i,j}$ is mentioned in [12] by Magnus and Chandler to demonstrate the difficulty of the isomorphism problem for torsion free one-relator

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groups. They also note that as of 1980 it was unknown if any of the groups $G_{i,j}$ are non-isomorphic. Later on, several approaches to attack the isomorphism problem for the family $G_{i,j}$ were carried out. In 1994, Lewis and Liriano [11] distinguish a number of parafree groups in the family $G_{i,j}$ by counting the homomorphisms between $G_{i,j}$ and the finite groups $SL(2,\mathbb{Z}/4)$ and $SL(2,\mathbb{Z}/5)$. A group-theoretical attack by Fine, Rosenberger, and Stille [6] is able to show that $G_{i,1} \ncong G_{1,1}$ for $i > 1$ and $G_{i,1} \ncong G_{j,1}$ for distinct primes i, j. More recently, by using a computational approach along the line of [11], Baumslag, Cleary and Havas [5] show that all the groups $G_{i,j}, 1 \leq i, j \leq 10$ are distinct.

In our previous work [8], we use the Alexander ideal, an algebraic invariant of groups which was originated from topology, to study the isomorphism problem for families of groups. Our approach can completely solve the isomorphism problem for the Baumslag-Solitar groups and a family of parafree groups $K_{i,j} := \langle a, s, t |\ a^i[s, a] = t^j\rangle$ introduced by Baumslag and Cleary in [4]. However, as noted in [8], the Alexander ideals of all the group $G_{i,j}$ are trivial.

In this paper, we develop our approach in [8] further to attack the isomorphism problem for the family of groups $G_{i,j}$ by using the twisted Alexander ideals. The twisted Alexander ideal is a non-abelian generalization of the classical Alexander polynomial. It turns out that, for certain values of i, j the twisted Alexander ideals of $G_{i,j}$ are non-trivial. By comparing the twisted Alexander ideals, we obtain subfamilies of $G_{i,j}$ which contain infinitely many pairwise non-isomorphic groups.

Theorem 1.1. *(i) Let p, q be two positive odd integers such that $\gcd(p, q) = 1$. For any $d, d' \geq 1$ the following holds*

$$G_{p(2d+1),q(2d+1)} \cong G_{p(2d'+1),q(2d'+1)} \text{ if and only if } d = d'.$$

(ii) Let p, q be two positive integers such that $\gcd(p, q) = 1$ and $3|(p + q)$. Then for any $d, d' \geq 1$ and $3 \nmid d, 3 \nmid d'$ the following holds

$$G_{pd,qd} \cong G_{pd',qd'} \text{ if and only if } d = d'.$$

The rest of this paper consists of three sections. In section 2, we give a sketchy review of the backgrounds on twisted Alexander ideals of a group. Section 3 contains the computation of the twisted Alexander ideals of $G_{i,j}$ associated to non-abelian representations into $SL(2,\mathbb{Z}_2)$. Section 4 is devoted to applications of the twisted Alexander ideals to the isomorphism problem for the family $G_{i,j}$. In particular, we show that several groups among $G_{i,j}$ are non-isomorphic and, as a consequence, we obtain Theorem 1.1 above.

2. Backgrounds on twisted Alexander ideals

The Alexander polynomial (see [1, 7]) is a topological invariant of knots which can be computed from the information on the fundamental group of its complement. The twisted Alexander ideals are non-abelian generalizations of the classical Alexander polynomials. The twisted Alexander ideals for knots were introduced by Lin in [10]. In this paper, we use a version of twisted Alexander ideals by Wada which is defined for a finitely presented group, more details can be found in [13] (see also [9]).

Suppose that $F_k = \langle x_1, \dots, x_k | \rangle$ is the free group on k generators. Let $\epsilon : \mathbb{Z}F_k \to \mathbb{Z}$ be the augmentation homomorphism defined by $\epsilon(\sum n_i g_i) = \sum n_i$. The j^{th} partial Fox derivative is a linear operator $\frac{\partial}{\partial x_j} : \mathbb{Z}F_k \to \mathbb{Z}F_k$ which is uniquely determined by the following rules:

$$\frac{\partial}{\partial x_j}(1) = 0; \ \frac{\partial}{\partial x_j}(x_i) = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j. \end{cases}$$

$$\frac{\partial}{\partial x_j}(uv) = \frac{\partial}{\partial x_j}(u)\epsilon(v) + u\frac{\partial}{\partial x_j}(v).$$

As consequences of the above rules we get:

i) $\dfrac{\partial}{\partial x_i}(x_i^n) = 1 + x_i + x_i^2 + \cdots + x_i^{n-1}$ for all $n \geq 1$.

ii) $\dfrac{\partial}{\partial x_i}(x_i^{-n}) = -x_i^{-1} - x_i^{-2} - \cdots - x_i^{-n}$ for all $n \geq 1$.

Let $G = \langle x_1, \ldots, x_k | r_1, \ldots, r_l\rangle$ be a finitely presented group. We denote by $\mathrm{ab}(G)$ the maximal free abelian quotient of G. From the sequence

$$F_k \xrightarrow{\phi} G \xrightarrow{\alpha} \mathrm{ab}(G)$$

we get the sequence

$$\mathbb{Z}F_k \xrightarrow{\tilde{\phi}} \mathbb{Z}G \xrightarrow{\tilde{\alpha}} \mathbb{Z}[ab(G)].$$

Suppose that we fix an isomorphism $\chi : \mathrm{ab}(G) \to \mathbb{Z}^r$, then the group ring $\mathbb{Z}[\mathrm{ab}(G)]$ can be identified with $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_r^{\pm 1}]$.

Given a homomorphism $\rho : G \to GL_n(R)$, where R is an unique factorization domain. We get the induced homomorphism of group ring $\tilde{\rho} : \mathbb{Z}G \to M_n(R)$.

Denote by $\tilde{\rho}\otimes\tilde{\alpha} : \mathbb{Z}G \to M_n([t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_r^{\pm 1}])$ the tensor product homomorphism and

$$\Phi := (\tilde{\rho} \otimes \tilde{\alpha}) \circ \tilde{\phi} : \mathbb{Z}F_k \to M_n(R[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_r^{\pm 1}]).$$

We regard $\big(\Phi(\frac{\partial}{\partial x_j} r_i)\big)_{i=1,\cdots,\ell,j=1,\cdots,k}$ as an $n\ell \times nk$ matrix whose entries belonging to $R[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_r^{\pm 1}]$ and call it *the twisted Alexander matrix of G associated to the representation ρ*.

The *d^{th} twisted Alexander ideal of G associated to the representation ρ* is defined to be the ideal generated by all the $(k-d)-$minors of the twisted Alexander matrix.

The twisted Alexander ideal does not depend on the choice of the presentation of G. It depends only on the group G, the representation ρ and the choice of the isomorphism χ above. So the twisted Alexander ideal is an invariant of (G, ρ) defined up to a *monomial automorphism* of $R[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_r^{\pm 1}]$. That is an automorphism of the form $\varphi(t_i) = t_1^{a_{i1}} t_2^{a_{i2}} \cdots t_r^{a_{ir}}, i = 1, 2, \cdots r$, where $(a_{ij}) \in GL(n, \mathbb{Z})$.

3. Computations of the twisted Alexander ideals

In this section we will present the computation of the twisted Alexander ideal of the group G_{ij} associated to a representation $\rho : G_{i,j} \to SL(2, \mathbb{Z}_2)$. It is easy to see that, for all i, j, the maximal free abelian quotient of $G_{i,j}$ is generated by the images of b and c. We will fix an identification of $\mathbb{Z}[\mathrm{ab}(G_{i,j})]$ with the ring of Laurent polynomials $\mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$ by mapping b to x and c to y. We denote by L the ring of Laurent polynomials with $\mathbb{Z}_2$ coefficients $\mathbb{Z}_2[x^{\pm 1}, y^{\pm 1}]$.

As each group $G_{i,j}$ is given by three generators and one relation, the twisted Alexander matrix is of size 2×6. We will compute the 4^{th} twisted Alexander ideal of $G_{i,j}$, that is the ideal in L generated by all the $2-$minors of the twisted Alexander matrix.

We only consider the "twisting" given by non-abelian representations since the abelian case reduces to the usual Alexander ideal which is trivial as noted above. Note that by [9] Theorem 2.2, the twisted Alexander ideal only depends on the conjugacy class of ρ. We have the following

Proposition 3.1. *There are exactly 3 conjugacy classes of non-abelian representations $\rho : G_{i,j} \to SL(2, \mathbb{Z}_2)$, for every i, j.*

Proof. It is well-known that the group $SL(2, \mathbb{Z}_2)$ is isomorphic to the symmetric gourp S_3 and its structure is very simple.

If ρ is a respresentation, then $\rho(a) = [\rho(c^i), \rho(a)][\rho(c^j), \rho(b)]$. Because the commutator subgroup of $SL(2, \mathbb{Z}_2)$ is abelian, we get $\rho(a) = [\rho(c^j), \rho(b)][\rho(c^i), \rho(a)]$. From that we easily find that $\rho(a) = \rho(c^i)[\rho(c^j), \rho(b)]\rho(c^{-i})$. Therefore any representation ρ is uniquely specified by the images $\rho(b)$ and $\rho(c)$. Direct calculation gives us that, independent of i, j, there are 3 conjugacy classes each contains 6 representations. The representative of each conjugacy class can be given in the following table

Type of Repr.	$\rho(b)$	$\rho(c)$
1	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
2	$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
3	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

$\square$

Choosing the relation $r = a[c^j, b]^{-1}[c^i, a]^{-1} = ab^{-1}c^{-j}bc^ja^{-1}c^{-i}ac^i$, we compute the Fox derivatives

$$\frac{\partial r}{\partial a} = 1 - a[c^j, b]^{-1}a^{-1} + a[c^j, b]^{-1}a^{-1}c^{-i}.$$

Now, as $r = 1$ in $G_{i,j}$, we can simplify to get

$$\frac{\partial r}{\partial a} = 1 - c^{-i}a^{-1}c^i + c^{-i}a^{-1}.$$

Similarly, we can get the other Fox derivatives,

$$\frac{\partial r}{\partial b} = -ab^{-1} + ab^{-1}c^{-j}.$$

$$\frac{\partial r}{\partial c} = -ab^{-1}(c^{-1} + \cdots + c^{-j}) + ab^{-1}c^{-j}b(1 + c + \cdots + c^{j-1})$$
$$-c^{-i}a^{-1}(1 + c + \cdots + c^{i-1}) + (c^{-1} + \cdots + c^{-i}).$$

In each of the following sub-sections, we will find the twisted Alexander ideal associated to each type of representations for the groups $G_{i,j}$.

3.1. **Representation of type 1.** This sub-section is devoted to the proof of the following.

Proposition 3.2. *Let I be the twisted Alexander ideal of $G_{i,j}$ associated to a representation of type 1 then*

i) $I = L$ in the case j is even or j is odd and i is even;

ii) $I = (f_{2(d-1)})$ in the case both i, j are odd and $4|(i - j)$;

iii) $I = (1 + y^{2d}, f_{2(d-1)} + xyf_{2(d-1)})$ in the case both i, j are odd and $4 \not| (i - j)$.

Here, $d = \gcd(i, j)$ and we define $f_{2n} := 1 + y^2 + \cdots + y^{2n}$ for $n \geq 1$, $f_0 := 1$ and $f_{2n} := 0$ for $n < 0$.

Proof. Case 1: j is even. From the table above, we know $\rho(b)$ and $\rho(c)$. We find that $\rho(a) = \rho(c^i)[\rho(c^j), \rho(b)]\rho(c^{-i}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. We deduce that $\Phi(\frac{\partial r}{\partial a}) = y^{-i}\rho(c)^{-i}$ and therefore $\det(\Phi(\frac{\partial r}{\partial a})) = y^{-i}$.

As y^{-i} is a unit, we see that in this case I is the whole ring L.

Case 2: i is even and j is odd. In this case, we can compute $\rho(a) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Next, we find that

$$\Phi(\tfrac{\partial r}{\partial a}) = \begin{pmatrix} 1 & 1+y^{-i} \\ 1+y^{-i} & y^{-i} \end{pmatrix} \text{ and } \Phi(\tfrac{\partial r}{\partial b}) = x^{-1}\begin{pmatrix} 1 & y^{-j} \\ 1+y^{-j} & 1+y^{-j} \end{pmatrix}$$

Note that multiplying a column by a unit does not affect the twisted Alexander ideal. So for simplicity, we can ignore the factor x^{-1} in $\Phi(\frac{\partial r}{\partial b})$. Consider two 2-minors

$$\det\begin{pmatrix} 1 & y^{-j} \\ 1+y^{-i} & 1+y^{-j} \end{pmatrix} = 1+y^{-i-j},\ \det\begin{pmatrix} 1+y^{-i} & 1 \\ y^{-i} & 1+y^{-j} \end{pmatrix} = 1+y^{-i-j}+y^{-j}.$$

We conclude that the I contains y^{-j}, which is a unit. Therefore part (i) is proved.

Case 3: i and j are odd. We find that $\rho(a) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. From that we get

$$\Phi(\tfrac{\partial r}{\partial a}) = \begin{pmatrix} 1+y^{-i} & 1 \\ 1+y^{-i} & y^{-i} \end{pmatrix} \text{ and } \Phi(\tfrac{\partial r}{\partial b}) = x^{-1}\begin{pmatrix} y^{-j} & 1 \\ 1 & y^{-j} \end{pmatrix}.$$

Now we can write out the matrix of $\Phi(\frac{\partial r}{\partial c})$:

$$\begin{pmatrix} \begin{matrix} x^{-1}y^{-j}f_{j-1}+y^{-j}f_{j-1}+y^{-j+1}f_{j-3}+ \\ y^{-i}f_{i-1}+y^{-i+1}f_{i-3} \end{matrix} & \begin{matrix} x^{-1}y^{-j+1}f_{j-3}+y^{-j+1}f_{j-3}+y^{-j}f_{j-1}+ \\ y^{-i}f_{i-1}+y^{-i+1}f_{i-3} \end{matrix} \\ \\ x^{-1}y^{-j+1}f_{j-3}+y^{-j+1}f_{j-3}+y^{-i+1}f_{i-3} & x^{-1}y^{-j}f_{j-1}+y^{-j}f_{j-1}+y^{-i}f_{i-1} \end{pmatrix}.$$

Now, we perform the elementary operations to simplify the Alexander matrix. Notice that the second column of $\Phi(\frac{\partial r}{\partial a})$ is $\begin{pmatrix} 1 \\ y^{-i} \end{pmatrix}$. We can use this column to make all the entries in the first row of other columns zero. So we can bring the Alexander matrix to the form

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1+y^{-2i} & y^{-i} & 1+y^{-i-j} & y^{-i}+y^{-j} & A & B \end{pmatrix},$$

where

$$A = y^{-i-j}(x^{-1}f_{j-1}+f_{j-1}+yf_{j-3})+y^{-2i}(f_{i-1}+yf_{i-3})+x^{-1}y^{-j+1}f_{j-3}+y^{-j+1}f_{j-3}+y^{-i+1}f_{i-3}.$$

$$B = y^{-i-j}(x^{-1}yf_{j-3}+yf_{j-3}+f_{j-1})+y^{-2i}(f_{i-1}+yf_{i-3})+x^{-1}y^{-j}f_{j-1}+y^{-j}f_{j-1}+y^{-i}f_{i-1}.$$

So $I = (1 + y^{2i}, 1 + y^{i+j}, 1 + y^{i-j}, A, B)$. To reduce the number of generators of I, we first prove a lemma.

Lemma 3.3. *In the ring L, we have*

$$\gcd(1 + y^m, 1 + y^n) = 1 + y^{\gcd(m,n)} \text{ for all positive integer } m, n.$$

Proof. We prove by induction on $m + n$. The case $m + n = 2$ is obviously true. Suppose that the lemma holds for all $m + n < t$, we now show that it also holds for $m + n = t$. If $m = n$ the lemma is also obviously true, so we may assume that $m > n$. We see that

$$\gcd(1 + y^m, 1 + y^n) = \gcd(y^n(1 + y^{m-n}) + (1 + y^n), 1 + y^n) = \gcd(1 + y^{m-n}, 1 + y^n).$$

By induction hypothesis, $gcd(1 + y^{m-n}, 1 + y^n) = 1 + y^{\gcd(m-n,n)} = 1 + y^{\gcd(m,n)}$. So the lemma is proved.

□

Writing $i = kd, j = ld$ where $\gcd(k, l) = 1$, as i, j are odd, we can easily deduce that $\gcd(2i, i + j, i - j) = 2d$. It follows from the Lemma 3.3 that $I = (1 + y^{2d}, A, B)$. We will simplify A and B to write I in a simpler form.

Note that I is unchanged if we replace the generator A by $A' \in L$ which satisfies $A' = A + f$, where f is a multiple of $1 + y^{2d}$. For such an A', we will write $A \equiv A'$.

As $(1 + y^{2i})$ is a multiple of $1 + y^{2d}$, $y^{-2i}g = y^{-2i}(1 + y^{2i})g + g \equiv g$. So we can replace the term y^{-2i} in the generators A or B by 1. Similarly, other terms like $y^{2i}, y^{\pm 2j}, y^{\pm i \pm j}, \ldots$ appearing in A or B can also be replaced by 1.

With this in mind, we find that

$$A \equiv x^{-1}(f_{j-1} + y^{-j+1} f_{j-3}) + (f_{j-1} + y^{-j+1} f_{j-3}) + (f_{i-1} + y^{-i+1} f_{i-3}) + (y f_{i-3} + y f_{j-3}).$$

$$B \equiv x^{-1}(y f_{j-3} + y^{-j} f_{j-1}) + (y f_{j-3} + y^{-j} f_{j-1}) + (y f_{i-3} + y^{-i} f_{i-1}) + (f_{i-1} + f_{j-1}).$$

Multiplying B by a factor y^{-j} and replacing y^{-2j}, y^{-j-i} and y^{i-j} by 1, we get

$$\begin{aligned} B \equiv\ & x^{-1}(y^{-2j} f_{j-1} + y^{-j+1} f_{j-3}) + (y^{-2j} f_{j-1} + y^{-j+1} f_{j-3}) \\ & + (y^{-j-i} f_{i-1} + y^{i-j} y^{-i+1} f_{i-3}) + (y^{i-j} y^{-i} f_{i-1} + y^{-j} f_{j-1}). \end{aligned}$$

$$\equiv x^{-1}(f_{j-1}+y^{-j+1}f_{j-3})+(f_{j-1}+y^{-j+1}f_{j-3})+(f_{i-1}+y^{-i+1}f_{i-3})+(y^{-i}f_{i-1}+y^{-j}f_{j-1}).$$

We can simplify the terms in A and B as follows

$$\begin{aligned} f_{j-1}+y^{-j+1}f_{j-3} &= (1+y^2+\cdots+y^{j-1})+y^{-j+1}(1+y^2+\cdots+y^{j-3}) \\ &\equiv (1+y^2+\cdots+y^{j-1})+y^{j+1}(1+y^2+\cdots+y^{j-3}) \equiv f_{2(j-1)}. \end{aligned}$$

Now since l is odd, we can write

$$f_{2(j-1)} = \frac{1+y^{2j}}{1+y^2} = \frac{(1+y^{2d})(1+y^{2d}+\cdots y^{2(l-1)d})}{1+y^2} = \frac{1+y^{2d}}{1+y^2}+\frac{1+y^{2d}}{1+y^2}(y^{2d}+y^{4d}+\cdots+y^{2(l-1)d})$$

$$= f_{2(d-1)}+(1+y^{2d})\Big[\frac{y^{2d}(1+y^{2d})}{1+y^2}+\frac{y^{6d}(1+y^{2d})}{1+y^2}+\cdots+\frac{y^{2(l-2)d}(1+y^{2d})}{1+y^2}\Big] \equiv f_{2(d-1)}.$$

So we obtain $f_{j-1}+y^{-j+1}f_{j-3} \equiv f_{2(d-1)}$ and similarly, $f_{i-1}+y^{-i+1}f_{i-3} \equiv f_{2(d-1)}$. Moreover, we can write:

$$(yf_{i-3}+yf_{j-3}) = y\frac{(1+y^{i-1})}{1+y^2}+y\frac{(1+y^{j-1})}{1+y^2} = \frac{(y^i+y^j)}{1+y^2}.$$

$$(y^{-i}f_{i-1}+y^{-j}f_{j-1}) = y^{-i}\frac{(1+y^{i+1})}{1+y^2}+y^{-j}\frac{(1+y^{j+1})}{1+y^2} = \frac{(y^{-i}+y^{-j})}{1+y^2} \equiv \frac{(y^i+y^j)}{1+y^2}.$$

Therefore, we obtain

$$A \equiv x^{-1}f_{2(d-1)}+\frac{(y^i+y^j)}{1+y^2} \text{ and } B \equiv x^{-1}f_{2(d-1)}+\frac{(y^i+y^j)}{1+y^2}.$$

To simplify further, we need to divide into two sub-cases.

Sub-case 3a: $4|(i-j)$. It follows that $4|(k-l)$. Without loss of generality, we can assume that $i-j>0$, then

$$\frac{(y^i+y^j)}{1+y^2} = y^j\frac{(1+y^{i-j})}{1+y^2} = y^j\frac{1+y^{2d}}{1+y^2}(1+y^{2d}+y^{4d}\cdots+y^{2(\frac{k-l}{2}-1)d})$$

$$= y^j(1+y^{2d})\Big[\frac{(1+y^{2d})}{1+y^2}+\frac{y^{4d}(1+y^{2d})}{1+y^2}+\cdots+\frac{y^{2(\frac{k-l}{2}-2)d}(1+y^{2d})}{1+y^2}\Big].$$

So in this case $\frac{(y^i+y^j)}{1+y^2}$ is a multiple of $1+y^{2d}$. Therefore $A \equiv B \equiv f_{2(d-1)}$ and we deduce (ii).

Sub-case 3b: $4 \not|(i-j)$. Similar to the previous sub-case,

$$\frac{(y^i+y^j)}{1+y^2} = y^j\frac{(1+y^{i-j})}{1+y^2} = y^j\frac{1+y^{2d}}{1+y^2}(1+y^{2d}+y^{4d}\cdots+y^{2(\frac{k-l}{2}-1)d})$$

$$= y^j\frac{(1+y^{2d})}{1+y^2} + y^j(1+y^{2d})\Big[\frac{y^{2d}(1+y^{2d})}{1+y^2} + \frac{y^{6d}(1+y^{2d})}{1+y^2} + \cdots + \frac{y^{2(\frac{k-l}{2}-2)d}(1+y^{2d})}{1+y^2}\Big].$$

So in this case $A \equiv B \equiv x^{-1}f_{2(d-1)} + y^j f_{2(d-1)}$. Moreover, note that $(1+y^2)f_{2(d-1)} = 1+y^{2d}$, so $y^2 f_{2(d-1)} \equiv f_{2(d-1)}$. As j is odd, we deduce that $y^j f_{2(d-1)} \equiv y f_{2(d-1)}$.

So $I = (1+y^{2d}, f_{2(d-1)} + xyf_{2(d-1)})$ and (iii) follows

□

3.2. **Representation of type 2.** The next proposition allows us to find the twisted Alexander ideal associated to a representation of type 2.

Proposition 3.4. *The twisted Alexander ideal of $G_{i,j}$ associated to a representations of type 2 can be obtained from that of a representation of type 1 by the change of variables $x \mapsto xy^{-1}, y \mapsto y$. .*

Proof. It not hard to check that the map ψ below is a well-defined automorphism of $G_{i,j}$:

$$\psi : G_{i,j} \to G_{i,j} \text{ defined by } \psi(b) = cb,\ \psi(c) = c,\ \psi(a) = a.$$

Notice that if ρ is a representation of type 2 then $\rho \circ \psi^{-1}$ is of type 1. By result in [13], page 246, we deduce the result.

□

3.3. **Representation of type 3.** The case of type 3 representation, we obtain the following result.

Proposition 3.5. *We put $i = kd, j = ld$ where $d = \gcd(i, j)$. Let I be the twisted Alexander ideal of $G_{i,j}$ associated to a representation of type 3 then*

i) $I = L$ in the case $3|j$ or both $3\not| j$ and $3\not| i+j$ hold;

ii) $I = (1+y^d+y^{2d})$ in the case $3\not| j, 3|(i+j)$ and l is even;

iii) $I = (\frac{1+y^d+y^{2d}}{1+y+y^2})$ in the case $3\not| j, 3|(i+j)$ and l is odd.

Proof. Case 1: $3|j$. Since $\rho(c)$ has order 3, we have $\rho(a) = \rho(c^i)[\rho(c^j), \rho(b)]\rho(c^{-i}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Similar to the first case in the proof of Proposition 3.2, we also deduce that $\Phi(\frac{\partial r}{\partial a}) = y^{-i}\rho(c)^{-i}$. So we get $I = L$.

Case 2: $3 \not| j$ and $3 \not| i+j$. We give the detail computations for $j \equiv 1 \mod 3, i \equiv 0 \mod 3$. For other values of i, j, the proof can be carried out in a similar way without any difficulty.

We first find that $\rho(a) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. From that we get

$\Phi(\frac{\partial r}{\partial b}) = x^{-1} \begin{pmatrix} 1+y^{-j} & y^{-j} \\ 1 & 1+y^{-j} \end{pmatrix}$ and $\Phi(\frac{\partial r}{\partial a}) = \begin{pmatrix} y^{-i} & 1+y^{-i} \\ 1+y^{-i} & 1 \end{pmatrix}$.

Ignoring x^{-1} in $\Phi(\frac{\partial r}{\partial b})$, we consider two 2-minors:

$\det \Phi(\frac{\partial r}{\partial b}) = y^{-2j}(1+y^j+y^{2j})$, $\det \begin{pmatrix} 1+y^{-i} & 1+y^{-j} \\ 1 & 1 \end{pmatrix} = y^{-i}(1+y^{i-j})$.

Therefore I contains two polynomials $1+y^j+y^{2j}$ and $1+y^{i-j}$. Next, we will prove a technical lemma

Lemma 3.6. *Suppose that $3 \not| s$, then in L we have*

$$\gcd(1+y^t+y^{2t}, 1+y^s) = 1 \text{ for any } t.$$

Proof. As, $3 \not| s$, using Lemma 3.3 we obtain:

$$\gcd((1+y^t)(1+y^t+y^{2t}), 1+y^s) = \gcd(1+y^{3t}, 1+y^s) = 1+y^{\gcd(3t,s)} = 1+y^{\gcd(t,s)} \quad (1).$$

On the other hand, since $\gcd(1+y^t, 1+y^t+y^{2t}) = 1$, we also have

$$\begin{aligned} \gcd((1+y^t)(1+y^t+y^{2t}), 1+y^s) &= \gcd(1+y^t, 1+y^s)\gcd(1+y^t+y^{2t}, 1+y^s) \\ &= (1+y^{\gcd(t,s)})\gcd(1+y^t+y^{2t}, 1+y^s) \quad (2). \end{aligned}$$

From (1) and (2), it follows that $\gcd(1+y^t+y^{2t}, 1+y^s) = 1$. $\square$

It follows immediately from Lemma 3.6 that $I = L$. So the assertion (i) holds.

Case 3: $j \equiv 1 \mod 3, i \equiv 2 \mod 3$. We compute $\Phi(\frac{\partial r}{\partial b}) = x^{-1} \begin{pmatrix} 1+y^{-j} & y^{-j} \\ 1 & 1+y^{-j} \end{pmatrix}$ and $\Phi(\frac{\partial r}{\partial a}) = \begin{pmatrix} y^{-i} & 1 \\ 1 & 1+y^{-i} \end{pmatrix}$. After some routine computations, we find that the matrix for $\Phi(\frac{\partial r}{\partial c})$ is

$$\begin{pmatrix} \begin{matrix} y^{-i}g_{i-2}+y^{-i+1}g_{i-2}+x^{-1}y^{-j}g_{j-1}+ \\ x^{-1}y^{-j+1}g_{j-4}+y^{-j}g_{j-1}+y^{-j+1}g_{j-4} \end{matrix} & \begin{matrix} y^{-i}g_{i-2}+y^{-i+2}g_{i-5}+x^{-1}y^{-j}g_{j-1}+ \\ x^{-1}y^{-j+2}g_{j-4}+y^{-j}g_{j-1}+y^{-j+2}g_{j-4} \end{matrix} \\ \begin{matrix} y^{-i}g_{i-2}+y^{-i+2}g_{i-5}+x^{-1}y^{-j+1}g_{j-4}+ \\ x^{-1}y^{-j+2}g_{j-4}+y^{-j}g_{j-1}+y^{-j+2}g_{j-4} \end{matrix} & \begin{matrix} y^{-i+1}g_{i-2}+y^{-i+2}g_{i-5}+x^{-1}y^{-j}g_{j-1}+ \\ x^{-1}y^{-j+1}g_{j-4}+y^{-j+1}g_{j-4}+y^{-j+2}g_{j-4} \end{matrix} \end{pmatrix},$$

where $g_{3n} := 1 + y^3 + \cdots + y^{3n}$ for $n \geq 1, g_0 := 1$ and $g_{3n} := 0$ for $n < 0$.

Now, using the elementary operations we can bring the Alexander matrix to the form.

$$\begin{pmatrix} y^{-i} & 1+y^{-i}+y^{-2i} & 1+y^{-i-j} & y^{-i-j}+y^{-i}+y^{-j} & C & D \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Where $C = y^{j-2i}g_{i-2} + y^{j-2i+2}g_{i-5} + y^{j-i}g_{i-2} + y^{j-i+1}g_{i-2} + y^{-i}g_{j-1} + y^{-i+2}g_{j-4} + g_{j-1} + yg_{j-4} + x^{-1}(y^{-i+1}g_{j-4} + y^{-i+2}g_{j-4} + g_{j-1} + yg_{j-4})$ and
$D = y^{j-2i+1}g_{i-2} + y^{j-2i+2}g_{i-5} + y^{j-i}g_{i-2} + y^{j-i+2}g_{i-5} + y^{-i+1}g_{j-4} + y^{-i+2}g_{j-4} + g_{j-1} + yg_{j-4} + x^{-1}(y^{-i+1}g_{j-4} + y^{-i}g_{j-1} + g_{j-1} + y^2g_{j-4})$.

So $I = (1 + y^i + y^{2i}, 1 + y^{i+j}, 1 + y^i + y^j, C, D)$. We need the following lemma.

Lemma 3.7. *Suppose that $m = kd$ and $n = ld$ where $d = \gcd(m, n)$. Assume further that $3\not| k, 3\not| l$, and $3|(k + l)$. Then in the ring L, the following holds*

$$\gcd(1 + y^m + y^{2m}, 1 + y^{m+n}) = 1 + y^d + y^{2d}.$$

Proof. As $3|(k + l)$, By Lemma 3.3

$$\begin{aligned}\gcd((1 + y^m)(1 + y^m + y^{2m}), 1 + y^{m+n}) &= \gcd(1 + y^{3kd}, 1 + y^{(k+l)d}) = 1 + y^{d\gcd(3k,k+l)} \\ &= 1 + y^{3d\gcd(k,k+l)} = 1 + y^{3d}.\end{aligned}$$

On the other hand, as in the proof of Lemma 3.6 above,

$$\gcd((1 + y^m)(1 + y^m + y^{2m}), 1 + y^{m+n}) = (1 + y^d)\gcd(1 + y^m + y^{2m}, 1 + y^{m+n}).$$

It follows that $\gcd(1 + y^m + y^{2m}, 1 + y^{m+n}) = 1 + y^d + y^{2d}$. ☐

Corollary 3.8. *Suppose that $3\not| k$ then $(1+y^d+y^{2d})|(1+y^{kd}+y^{2kd})$ for any positive integer d.*

Proof. We can always reduce to the case k is odd, since if k is even then $1+y^{kd}+y^{2kd}=(1+y^{\frac{kd}{2}}+y^{kd})^2$. The corollary then follows by applying Lemma 3.7 for $m=kd, n=d$ if $k\equiv 2 \mod 3$ and for $m=kd, n=2d$ if $k\equiv 1 \mod 3$.

□

From Lemma 3.7, we know that the greatest common divisor of the first two generators is $1+y^d+y^{2d}$. We now show that the third generator is also a divisible by $1+y^d+y^{2d}$.

In fact, as $3d|(2j-i)$, by Lemma 3.3 we get $(1+y^d)(1+y^d+y^{2d})=(1+y^{3d})|(1+y^{2j-i})$. Moreover Corollary 3.8 implies that $(1+y^d+y^{2d})|(1+y^j+y^{2j})$. Therefore $1+y^i+y^j=(1+y^j+y^{2j})+y^i(1+y^{2j-i})$ is also divisible by $1+y^d+y^{2d}$.

It follows that $I=(1+y^d+y^{2d},C,D)$.

We now proceed by simplifying the generators C and D as we did in Case 3 of Proposition 3.2. Note that adding to C or D a multiple of $1+y^d+y^{2d}$ will not change the twisted Alexander ideal. For the rest of this section, we will use the notation $X\equiv Y$ if $(X-Y)$ is a multiple of $1+y^d+y^{2d}$. By writing $g_{3n}=\frac{1+y^{3n+3}}{1+y^3}$ and putting it into C,D we can simplify them as follows

$$C\equiv \frac{y^{j+1}+y^{j-2i}+y^{j-2i+1}}{1+y+y^2}+\frac{1+y^{2j+1}}{1+y+y^2}+x^{-1}\frac{1+y^j+y^{2j}}{1+y+y^2}.$$

$$D\equiv \frac{y^j+y^{j-2i+1}}{1+y+y^2}+\frac{1+y+y^{2j}}{1+y+y^2}+x^{-1}\frac{(1+y)(1+y^j+y^{2j})}{1+y+y^2}.$$

Note that

$$\begin{aligned}D+y^2C &\equiv \tfrac{(y^j+y^{j+3})+y^{j-2i+1}(1+y+y^2)+(1+y+y^2)+(y^{2j}+y^{2j+3})}{1+y+y^2}+x^{-1}(1+y^j+y^{2j})\\ &= y^j(1+y)+y^{j-2i+1}+1+y^{2j}(1+y)+x^{-1}(1+y^j+y^{2j})\\ &= (1+y)(1+y^j+y^{2j})+y(1+y^{j-2i})+x^{-1}(1+y^j+y^{2j})\equiv 0.\end{aligned}$$

So I is generated by $1+y^d+y^{2d}$ and C only.

We'll need the following technical lemma about polynomials.

Lemma 3.9. *Let m be a positive integer such that $m=ld$ and $3\not| l$. The followings hold in L.*

(i) If l is even then $\frac{1+y^m+y^{2m}}{1+y+y^2}\equiv 0$;

(ii) If $l\equiv 1 \mod 6$ then $\frac{1+y^m+y^{2m}}{1+y+y^2}\equiv\frac{1+y^d+y^{2d}}{1+y+y^2}$;

(iii) If $l \equiv 5 \mod 6$ *then* $\frac{1+y^m+y^{2m}}{1+y+y^2} \equiv y^{-2d}\frac{1+y^d+y^{2d}}{1+y+y^2}$.

Proof. If l is even then $m = 2td$, and

$$\frac{1+y^m+y^{2m}}{1+y+y^2} = \frac{(1+y^{td}+y^{2td})}{1+y+y^2}(1+y^{td}+y^{2td}).$$

It follows from Corollary 3.8 that $1+y^{td}+y^{2td}$ is divisible by both $1+y+y^2$ and $1+y^d+y^{2d}$. So (i) holds. Now if $l = 6t+1$, using Corollary 3.8 repeatedly, we get

$$\frac{1+y^{6td}}{1+y+y^2} = \frac{(1+y^{td})(1+y^{td}+y^{2td})}{1+y+y^2}(1+y^{td})(1+y^{td}+y^{2td}) \equiv 0.$$

So

$$\frac{1+y^m+y^{2m}}{1+y+y^2} = \frac{1+y^d+y^{2d}+y^d(1+y^{6td})+y^{2d}(1+y^{12td})}{1+y+y^2} \equiv \frac{1+y^d+y^{2d}}{1+y+y^2}$$

and (ii) follows. We also get (iii) by similar method. □

We now divide further into 3 cases:

- If l is even, then as $j \equiv 1 \mod 3, i \equiv 2 \mod 3$ we see that $j-2i = (l-2k)d$ is divisible by $6d$. We can write
$$C \equiv \frac{(1+y)(1+y^{j-2i})}{1+y+y^2} + \frac{y(1+y^j+y^{2j})}{1+y+y^2} + x^{-1}\frac{1+y^j+y^{2j}}{1+y+y^2}.$$
Now by Lemma 3.9 and the fact that $\frac{1+y^{6td}}{1+y+y^2} \equiv 0$ we see that $C \equiv 0$. So I is generated by $1+y^d+y^{2d}$.
- If $l = 6t+1$, then $3d \mid j-2i$ but $6d \nmid j-2i$. So $\frac{(1+y^{j-2i})}{1+y+y^2} \equiv \frac{(1+y^{3d})}{1+y+y^2}$. Combining with Lemma 3.9, we deduce that
$$C \equiv \frac{(1+y)(1+y^{3d})}{1+y+y^2} + \frac{y(1+y^d+y^{2d})}{1+y+y^2} + x^{-1}\frac{1+y^d+y^{2d}}{1+y+y^2}$$
$$= \frac{(1+y^d+y^{d+1})}{1+y+y^2}(1+y^d+y^{2d}) + x^{-1}\frac{1+y^d+y^{2d}}{1+y+y^2}.$$
Note that since $l = 6t+1$ and $j \equiv 1 \mod 3$, we have $d \equiv 1 \mod 3$. So $1+y^d+y^{d+1} = 1+y+y^2+y(1+y^{d-1})+y^2(1+y^{d-1})$ is a multiple of $1+y+y^2$.
So in this case I is generated by $\frac{1+y^d+y^{2d}}{1+y+y^2}$

- If $l = 6t+5$, then $3d \mid j-2i$ but $6d \nmid j-2i$. Similar to the previous case, we obtain

$$C \equiv \frac{(1+y)(1+y^{3d})}{1+y+y^2} + \frac{y^{1-2d}(1+y^d+y^{2d})}{1+y+y^2} + x^{-1}y^{-2d}\frac{1+y^d+y^{2d}}{1+y+y^2}$$
$$= \frac{(1+y)(1+y^d)+y^{1-2d}}{1+y+y^2}(1+y^d+y^{2d}) + x^{-1}y^{-d}\frac{1+y^d+y^{2d}}{1+y+y^2}.$$

Note that since $l = 6t+5$ and $j \equiv 1 \mod 3$, we have $d \equiv 2 \mod 3$. So $(1+y)(1+y^d)+y^{1-2d} = 1+y+y^2+y^2(1+y^{d-2})+y^{1-2d}(1+y^{3d})$ is a multiple of $1+y+y^2$.

So in this case I is generated by $\frac{1+y^d+y^{2d}}{1+y+y^2}$

Case 4: $j \equiv 2 \mod 3, i \equiv 1 \mod 3$. In this case we get $\Phi(\frac{\partial r}{\partial b}) = x^{-1}\begin{pmatrix} y^{-j}+1 & 1 \\ y^{-j} & y^{-j}+1 \end{pmatrix}$ and $\Phi(\frac{\partial r}{\partial a}) = \begin{pmatrix} y^{-i}+1 & 1 \\ 1 & y^{-i} \end{pmatrix}$. After some lengthy computation, we get the matrix for $\Phi(\frac{\partial r}{\partial c})$:

$$\begin{pmatrix} \begin{array}{c} y^{-i+2}g_{i-4}+y^{-i+1}g_{i-4}+x^{-1}y^{-j}g_{j-2}+ \\ x^{-1}y^{-j+2}g_{j-5}+y^{-j+1}g_{j-2}+y^{-j+2}g_{j-5} \end{array} & \begin{array}{c} y^{-i}g_{i-1}+y^{-i+1}g_{i-4}+x^{-1}y^{-j+1}g_{j-2}+ \\ x^{-1}y^{-j+2}g_{j-5}+y^{-j}g_{j-2}+y^{-j+1}g_{j-2} \end{array} \\ \\ \begin{array}{c} y^{-i}g_{i-1}+y^{-i+1}g_{i-4}+x^{-1}y^{-j}g_{j-2}+ \\ x^{-1}y^{-j+1}g_{j-2}+y^{-j}g_{j-2}+y^{-j+1}g_{j-2} \end{array} & \begin{array}{c} y^{-i}g_{i-1}+y^{-i+2}g_{i-4}+x^{-1}y^{-j}g_{j-2}+ \\ x^{-1}y^{-j+2}g_{j-5}+y^{-j}g_{j-2}+y^{-j+2}g_{j-5} \end{array} \end{pmatrix}.$$

Now, using the elementary operations we can simplify the Alexander matrix. The Alexander matrix can be brought into the form

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1+y^{-i}+y^{-2i} & y^{-i} & y^{-i-j}+y^{-i}+y^{-j} & 1+y^{-i}+y^{-j} & E & F \end{pmatrix},$$

where

$$E = y^{j-2i+1}g_{i-4}+y^{j-2i+2}g_{i-4}+y^{j-i}g_{i-1}+y^{j-i+1}g_{i-4}+y^{-i+1}g_{j-2}+y^{-i+2}g_{j-5}+$$
$$g_{j-2}+yg_{j-2}+x^{-1}(y^{-i}g_{j-2}+y^{-i+2}g_{j-5}+g_{j-2}+yg_{j-2}),$$
$$F = y^{j-2i}g_{i-1}+y^{j-2i+1}g_{i-4}+y^{j-i}g_{i-1}+y^{j-i+2}g_{i-4}+y^{-i}g_{j-2}+y^{-i+1}g_{j-2}+$$
$$g_{j-2}+y^2g_{j-5}+x^{-1}(y^{-i+1}g_{j-2}+y^{-i+2}g_{j-5}+g_{j-2}+y^2g_{j-5}).$$

By analogous arguments as in the case 3, we deduce that $I = (1+y^d+y^{2d}, E, F)$.

We now proceed by simplifying the generators E and F as we did above.

$$E = \frac{y^j + y^{j+1} + y^{j-2i+1}}{1+y+y^2} + (1+y^{-i-j})(y^{j+1}g_{j-2} + y^{j+2}g_{j-5}) + y^{j+1}g_{j-2} + y^{j+2}g_{j-5} + g_{j-2}$$
$$+yg_{j-2} + x^{-1}((1+y^{-i-j})(y^j g_{j-2} + y^{j+2}g_{j-5}) + y^j g_{j-2} + y^{j+2}g_{j-5} + g_{j-2} + yg_{j-2})$$

$$\equiv \frac{y^j + y^{j+1} + y^{j-2i+1}}{1+y+y^2} + \frac{1+y^{2j}+y^{2j+1}}{1+y+y^2} + x^{-1}\frac{1+y^j+y^{2j}}{1+y+y^2}.$$

$$F = \frac{y^{j+1} + y^{j-2i}}{1+y+y^2} + (1+y^{-i-j})(y^j g_{j-2} + y^{j+1}g_{j-2}) + y^j g_{j-2} + y^{j+1}g_{j-2} + g_{j-2} + y^2 g_{j-5}$$
$$+x^{-1}((1+y^{-i-j})(y^{j+1}g_{j-2} + y^{j+2}g_{j-5}) + y^{j+1}g_{j-2} + y^{j+2}g_{j-5} + g_{j-2} + y^2 g_{j-5})$$

$$\equiv \frac{y^{j+1} + y^{j-2i}}{1+y+y^2} + \frac{1+y+y^{2j+1}}{1+y+y^2} + x^{-1}\frac{(1+y)(1+y^j+y^{2j})}{1+y+y^2}.$$

Note that

$$F + y^2E = \frac{y^{j+1}(1+y+y^2) + y^{j-2i}(1+y^3) + y^{2j+1}(1+y+y^2) + (1+y+y^2)}{1+y+y^2} + x^{-1}(1+y^j+y^{2j})$$
$$= y(1+y^j+y^{2j}) + (1+y)(1+y^{j-2i}) + x^{-1}(1+y^j+y^{2j}) \equiv 0.$$

This follows that I is generated by $1+y^d+y^{2d}$ and F only. We can rewrite F as:

$$F \equiv \frac{y(1+y^j+y^{2j})}{1+y+y^2} + \frac{1+y^{j-2i}}{1+y+y^2} + x^{-1}\frac{(1+y)(1+y^j+y^{2j})}{1+y+y^2}.$$

We can now obtain the twisted Alexander ideal in each of the following cases.

- If l is even, then as $j \equiv 2 \mod 3$, $i \equiv 1 \mod 3$. We see that $j-2i = (l-2k)d$ is divisible by $6d$. By Lemma 3.9, we get $F \equiv 0$ and I is generated by $1+y^d+y^{2d}$.
- If $l = 6t+1$, then $3d \mid j-2i$ but $6d \nmid j-2i$. So $\frac{(1+y^{j-2i})}{1+y+y^2} \equiv \frac{(1+y^{3d})}{1+y+y^2}$. Combining this with Lemma 3.9, we deduce that

$$F \equiv \frac{y(1+y^d+y^{2d})}{1+y+y^2} + \frac{(1+y^{3d})}{1+y+y^2} + x^{-1}(1+y)\frac{1+y^d+y^{2d}}{1+y+y^2}$$
$$= \frac{(1+y+y^d)}{1+y+y^2}(1+y^d+y^{2d}) + x^{-1}(1+y)\frac{1+y^d+y^{2d}}{1+y+y^2}.$$

  Note that since $l = 6t+1$ and $j \equiv 2 \mod 3$, we have $d \equiv 2 \mod 3$. So $1+y+y^d = 1+y+y^2+y^2(1+y^{d-2})$ is a multiple of $1+y+y^2$.

  We deduce that in this case I is generated by $\frac{1+y^d+y^{2d}}{1+y+y^2}$.

- If $l = 6t + 5$, then $3d \mid j - 2i$ but $6d \not| j - 2i$. Also by using Lemma 3.9, we get

$$F \equiv \frac{y^{1-2d}(1 + y^d + y^{2d})}{1 + y + y^2} + \frac{(1 + y^{3d})}{1 + y + y^2} + x^{-1}(1 + y)y^{-2d}\frac{1 + y^d + y^{2d}}{1 + y + y^2}$$

$$= \frac{1 + y^d + y^{1-2d}}{1 + y + y^2}(1 + y^d + y^{2d}) + x^{-1}y^{-2d}\frac{1 + y^d + y^{2d}}{1 + y + y^2}.$$

Note that since $l = 6t + 5$ and $j \equiv 2 \mod 3$, we have $d \equiv 1 \mod 3$. We obtain $1+y^d+y^{1-2d} = y^{-2d}(1+y+y^2+y^2(1+y^{2d-2})+1+y^{3d})$ is a multiple of $1 + y + y^2$.

So in this case I is generated by $\frac{1+y^d+y^{2d}}{1+y+y^2}$

The assertions (ii) and (iii) of the proposition are proved.

□

4. Applications to the isomorphism problem

In this section, we will apply the computation results obtained above to deduce that several groups $G_{i,j}$ are non-isomorphic. For each group $G_{i,j}$, we will denote by $I^1_{i,j}, I^2_{i,j}, I^3_{i,j}$ its twisted Alexander ideals associated to representations of type 1, 2 and 3 respectively. We know that if two groups $G_{i,j}$ and $G_{i',j'}$ are isomorphic then there should be a monomial isomorphism of L under which the set of twisted Alexander ideals of one group is mapped to that of the other. Note that the monomial isomorphism does not need to preserve the representation type, i.e. $I^1_{i,j}$ could be mapped to $I^3_{i',j'}$. From this fact, we obtain the following result.

Proposition 4.1. *Consider the following three disjoint sets*

$$A := \{(i,j)|\ i \textit{ is even or } j \textit{ is even}\} \cup \{(i,j)|\ i,j \textit{ are both odd, } \gcd(i,j) = 1 \textit{ and } 4|(i-j)\}$$

$$B := \{(i,j)|\ i,j \textit{ are both odd, } \gcd(i,j) > 1 \textit{ and } 4|(i-j)\}$$

$$C := \{(i,j)|\ i,j \textit{ are both odd and } 4\not|(i-j)\}.$$

If (i,j) and (i',j') do not belong to the same set among A, B and C then $G_{i,j} \not\cong G_{i',j'}$.

Proof. By Proposition 3.2 and 3.4 if (i,j) belongs to A then the twisted Alexander ideals $I^1_{i,j}$ and $I^2_{i,j}$ both coincides with L. On the other hand if (i',j') belongs to B then the twisted Alexander ideals $I^1_{i',j'}$ and $I^2_{i',j'}$ both are the $(f_{2(d-1)})$. As $d = \gcd(i',j') > 1$, the ideal $(f_{2(d-1)})$ is not the whole ring. It is obvious that there exists no automorphism that maps the set of three ideals, two of which are trivial, to a set of three ideals, two of which are non-trivial. Therefore, we conclude that $G_{i,j} \ncong G_{i',j'}$.

We first need the following claim.

Claim. The ideals $(1+y^2, 1+xy)$ and $(1+y^2, 1+x)$ are neither principal nor the whole ring L.

Proof of the claim. We present the proof for $(1+y^2, 1+xy)$, for $(1+y^2, 1+x)$ the same argument can be applied. Notice that the ideal $(1+y^2, 1+xy)$ does not coincide with L because for any $f(x,y) \in (1+y^2, 1+xy)$ we have $f(1,1) \equiv 0$. Moreover, the ideal $(1+y^2, 1+xy)$ can not be principal because if $(1+y^2, 1+xy) = (g)$ then $g \mid \gcd(1+y^2, 1+xy) = 1$ and this contradicts the fact that the ideal $(1+y^2, 1+xy)$ is not the whole ring. So the claim follows. □

Now suppose that (i',j') belongs to C. By Proposition 3.2 and 3.4, the twisted Alexander ideals $I^1_{i',j'} = (f_{2(d-1)})(1+y^2, 1+xy)$ and $I^2_{i',j'} = (f_{2(d-1)})(1+y^2, 1+x)$. So by the Claim they are neither principal nor the whole ring. On the other hand, if (i,j) belongs to A then two twisted Alexander ideals of $G_{i,j}$ are the whole ring and if (i,j) belongs to B then two twisted Alexander ideals of $G_{i,j}$ are principal. So we deduce that for either (i,j) belongs to A or (i,j) belongs to B then $G_{i,j} \ncong G_{i',j'}$. □

We could not distinguish all the groups $G_{i,j}$ for (i,j) belongs to the same set B or C. However, the following proposition gives a necessary condition for two groups to be isomorphic.

Proposition 4.2. *If i, j, i', j' are positive odd integers then*

$$G_{i,j} \cong G_{i',j'} \text{ implies } \gcd(i,j) = \gcd(i',j').$$

Proof. We put $d := \gcd(i, j)$ and $d' := \gcd(i', j')$. As $G_{i,j} \cong G_{i',j'}$, there must be a monomial automorphism φ of L which maps the set of twisted Alexander ideals of one group to that of the other. By Proposition 4.1, one of the following cases must happen.

Case 1: $4|(i-j)$ *and* $4|(i'-j')$. By Proposition 3.2 and 3.4, the first two twisted Alexander ideals of $G_{i,j}$ and $G_{i',j'}$ are $I^1_{i,j} = I^2_{i,j} = (f_{2(d-1)})$ and $I^1_{i',j'} = I^2_{i',j'} = (f_{2(d'-1)})$ respectively. The following auxiliary result will be used a couple of time below.

Lemma 4.3. *Suppose that $f = 1 + y^{a_1} + \cdots + y^{a_m}$ and $g = 1 + y^{b_1} + \cdots + y^{b_n}$ are Laurent polynomials in L such that they both consist of only non-negative powers of y and have the constant terms equal 1. If there exists a monomial automorphism of L which maps the ideal (f) to the ideal (g) then $a_m = b_n$ and either $f = g$ or $f(y^{-1})y^{a_m} = g$.*

Proof. Suppose that the monomial automorphism is of the form $\varphi(x) = x^a y^b$ and $\varphi(y) = x^u y^v$. From the hypothesis, we get $f(x^a y^b, x^u y^v) = 1 + x^{a_1 u} y^{a_1 v} + \cdots + x^{a_m u} y^{a_m v}$ and g must generate the same ideal. This means that two polynomials only differ by a factor of the form $x^m y^n$. But this can not happen unless $u = 0, v = \pm 1$. So we obtain that either $f = g$ or $f(y^{-1})y^{a_m} = g$. In either case we get $a_m = b_n$. $\square$

From Lemma 4.3 we get $d = d'$.

Case 2: $4 \not| (i-j), 4 \not| (i'-j')$. In this case the first two twisted Alexander ideals of $G_{i,j}$ and $G_{i',j'}$ are $I^1_{i,j} = (1 + y^{2d}, (1 + xy)f_{2(d-1)}), I^2_{i,j} = (1 + y^{2d}, (1 + x)f_{2(d-1)})$ and $I^1_{i',j'} = (1 + y^{2d'}, (1 + xy)f_{2(d'-1)}), I^2_{i',j'} = (1 + y^{2d'}, (1 + x)f_{2(d'-1)})$ respectively. We know that φ maps either $I^1_{i,j}$ or $I^2_{i,j}$ to one of the ideals of the set $\{I^1_{i',j'}, I^2_{i',j'}\}$. So $1 + y^{2d'}$ must belong to the image under φ of either $I^1_{i,j}$ or $I^2_{i,j}$. In any case

$$(1+y^2)f_{2(d'-1)} = f_{2(d-1)}(x^a y^b, x^u y^v)g = (1+x^{2u}y^{2v}+\cdots+x^{2(d-1)u}y^{2(d-1)v})g, \text{ for some } g.$$

Notice that the left-hand side does not contain x and this is impossible unless $u = 0$. Therefore $v = \pm 1$ and we get $f_{2(d-1)}|(1 + y^2)f_{2(d'-1)}$. It is easy to see that

$\gcd(1+y^2, f_{2(d-1)}) = 1$ as d is odd. So $f_{2(d-1)} | f_{2(d'-1)}$ and we deduce that $d \le d'$. The same argument also gives us $d' \le d$. So the proposition is proved.

$\square$

Using the twisted Alexander ideal of a type 3 representation, we can prove the following.

Proposition 4.4. *Let i, j, i', j' be positive integers such that $3|(i+j), 3|(i'+j')$ and $3 \not| j, j'$. The following holds*

$$G_{i,j} \cong G_{i',j'} \text{ implies } \gcd(i,j) = \gcd(i',j').$$

Proof. We put $i = kd, j = ld, i' = k'd', j' = l'd'$ where $\gcd(k,l) = 1, \gcd(k',l') = 1$. We first show that l and l' must have the same parity. Suppose contrary that l is even and l' is odd. Then by Propositions 3.2, 3.4, 3.5 the twisted Alexander ideals of $G_{i,j}$ are $I^1_{i,j} = I^2_{i,j} = L$ and $I^3_{i,j} = (1+y^d+y^{2d})$. Since $G_{i,j} \cong G_{i',j'}$, two of the three twisted Alexander ideals of $G_{i',j'}$ must be trivial. The only way for this to hold is $I^1_{i',j'} = I^2_{i',j'} = L$ and $I^3_{i',j'} = (\frac{1+y^{d'}+y^{2d'}}{1+y+y^2})$. There should be an automorphism which maps $I^3_{i,j}$ to $I^3_{i',j'}$. From the proof of Lemma 4.3, we get $1+y^d+y^{2d} = \frac{1+y^{d'}+y^{2d'}}{1+y+y^2}$ and this is impossible. So we arrive at a contradiction.

So now l and l' have the same parity. Consider the first case where l and l' are both even. By Proposition 3.5 the twisted Alexander ideals of $G_{i,j}$ and $G_{i',j'}$ are $I^1_{i,j} = I^2_{i,j} = L, I^3_{i,j} = (1+y^d+y^{2d})$ and $I^1_{i',j'} = I^2_{i',j'} = L, I^3_{i',j'} = (1+y^{d'}+y^{2d'})$. Using Lemma 4.3 we deduce that $d = d'$.

Now assume that l and l' are both odd. Suppose that $1 \le d < d'$, we will arrive at a contradiction. There exists a twisted Alexander ideal of $G_{i,j}$ which are mapped to $I^3_{i',j'} = (\frac{1+y^{d'}+y^{2d'}}{1+y+y^2})$ by a monomial automorphism. We know that the twisted Alexander ideals of $G_{i,j}$ must be of the following forms: $L, (f_{2(d-1)}), (1+y^{2d}, f_{2(d-1)} + xyf_{2(d-1)}), (1+y^{2d}, f_{2(d-1)} + xf_{2(d-1)})$ and $(\frac{1+y^d+y^{2d}}{1+y+y^2})$. Since $d' > 1$, $I^3_{i',j'}$ is a non-trivial principal ideal, so the twisted Alexander ideal of $G_{i,j}$ which are mapped to $I^3_{i',j'}$ is either $(f_{2(d-1)})$ or $(\frac{1+y^d+y^{2d}}{1+y+y^2})$. In either case, by Lemma 4.3, we arrive at the contradiction that $d = d'$. So the proposition follows. $\square$

Proof of Theorem 1.1. Part (i) and (ii) are immediate corollaries of Proposition 4.2 and Proposition 4.4 respectively. □

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Do Viet Hung, Ha Giang College of Education, Ha Giang, Viet Nam
E-mail address: `viethunghg81@gmail.com`

Vu The Khoi (Corresponding author), Institute of Mathematics, VietNam Academy Of Science And Technology, 18 Hoang Quoc Viet, 10307, Hanoi, Vietnam
E-mail address: `vtkhoi@math.ac.vn`