

Hopf bifurcation with additive noise

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Abstract

We consider the dynamics of a two-dimensional ordinary differential equation exhibiting a Hopf bifurcation subject to additive white noise and identify three dynamical phases: (I) a random attractor with uniform synchronisation of trajectories, (II) a random attractor with non-uniform synchronisation of trajectories and (III) a random attractor without synchronisation of trajectories. The random attractors in phases (I) and (II) are random equilibrium points with negative Lyapunov exponents while in phase (III) there is a so-called random strange attractor with positive Lyapunov exponent.

We analyse the occurrence of the different dynamical phases as a function of the linear stability of the origin (deterministic Hopf bifurcation parameter) and shear (amplitude-phase coupling parameter). We show that small shear implies synchronisation and obtain that synchronisation cannot be uniform in the absence of linear stability at the origin or in the presence of sufficiently strong shear. We provide numerical results in support of a conjecture that irrespective of the linear stability of the origin, there is a critical strength of the shear at which the system dynamics loses synchronisation and enters phase (III).

Key words. Dichotomy spectrum, Hopf bifurcation, Lyapunov exponent, random attractor, random dynamical system, stochastic bifurcation

Mathematics Subject Classification (2010). 37C75, 37D45, 37G35, 37H10, 37H15.

1 Introduction

The results of this paper are part of an emerging bifurcation theory for random dynamical systems. Earlier attempts to develop such a theory (notably by Ludwig Arnold, Peter Baxendale and coworkers [2, 3, 5, 28] in the 1990s) resulted in notions of so-called *phenomenological* (or "P") bifurcations which consider qualitative changes of stationary distributions and *dynamical* (or "D") bifurcations that concern the bifurcation of an invariant random measure. Our research and that of others suggest that these concepts do not comprehensively capture the intricacies of bifurcation in random dynamical systems. For instance, finite-time behaviour and the *dichotomy spectrum* [8, 13], *minimal invariant sets* [22, 35] and the emergence of *chaotic attractors* [1, 26] are also important. Despite its relevance for many applications of topical interest, a bifurcation theory of random dynamical systems is still in its infancy. Current research in this context mainly concerns case studies of relatively elementary examples. Studies in the context of stochastic Hopf bifurcation have mainly considered the Duffing–Van der Pol oscillator with multiplicative white noise [3, 28, 29]. Most research has produced conjectures based on numerical observations [19], with so far few rigorous results being reported.

The model we consider in this paper is exemplary in the following sense: firstly, it discusses the typical phenomenon of random systems to exhibit a transition between *synchronisation* and *chaos* [34]. Secondly, the normal form is locally equivalent to that of a generic deterministic Hopf

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bifurcation and, hence, at least for small noise, one expects other examples of Hopf bifurcation to feature similar local dynamical behaviour.

In more detail, we consider the two-dimensional stochastic differential equation

$$\begin{aligned} dx &= (\alpha x - \beta y - (ax - by)(x^2 + y^2)) dt + \sigma dW_t^1, \\ dy &= (\alpha y + \beta x - (bx + ay)(x^2 + y^2)) dt + \sigma dW_t^2, \end{aligned} \tag{1.1}$$

where $\sigma \geq 0$ represents the strength of the noise, $\alpha \in \mathbb{R}$ is a parameter equal to the real part of eigenvalues of the linearization of the vector field at $(0, 0)$, $b \in \mathbb{R}$ represents shear strength (amplitude-phase coupling parameter when writing the deterministic part of (1.1) in polar coordinates), $a > 0$, $\beta \in \mathbb{R}$, and W_t^1, W_t^2 denote independent one-dimensional Brownian motions. Throughout the paper we will consider (1.1) for all possible values of the parameters, i.e. all points in the $(a, b, \alpha, \beta, \sigma)$ -parameter space. For illustrations and numerical simulations we will fix the values of a, β and σ and only consider the (b, α) -plane since these are the two crucial bifurcation parameters.

The main aim of this paper is to provide a precise mathematical analysis of the stability regimes of system (1.1) as a function of its parameters and identify associated qualitative dynamical features.

In the limits of small noise limits and small shear, the stability properties of the stochastic system (1.1) was studied before by Deville *et al* [12] as an example of a non-Hamiltonian system perturbed by noise. It was shown that the largest Lyapunov exponent λ_{top} is negative in these limits. We extend these results in the first part of this paper to global parameter space and prove implications for the associated random dynamical system in terms of its random attractor and invariant measure. We establish exponentially fast synchronisation of almost all trajectories when $\lambda_{\text{top}} < 0$. In the second part of this paper we follow ideas from [8, 27] and discuss *finite-time Lyapunov exponents* and *uniform attractivity* and their relation to the dichotomy spectrum. This analysis enables a more refined dynamical characterisation than through Lyapunov exponents only and identifies a route to breaking synchronisation from uniform attractivity, through an intermediate phase with *non-uniform attractivity* and a non-negative dichotomy spectrum where shear-induced instabilities induce arbitrarily large positive finite-time Lyapunov exponents. This route to synchrony-breaking is similar to that obtained before by [27] in the context of a random logistic map.

Numerical investigations by Lin and Young [26], Wieczorek [33] and Deville *et al* [12], including system (1.1), highlighted the observation that shear and noise may cause Lyapunov exponents to turn positive, leading to chaotic random dynamical behaviour without synchronisation. A proof of shear-induced chaos with instantaneous periodic driving (kicks) was obtained by Wang and Young [32]. Only recently, [14] obtained an analytical proof of this phenomenon in an SDE model with continuous-time stochastic forcing. In this paper, we prove the existence of a transition to non-uniform attractivity characterised by a non-negative dichotomy spectrum and positive finite-time Lyapunov exponents which are arbitrarily large for sufficiently strong shear. An analytical result on the breaking of synchronisation for system (1.1) remains out of reach,¹ but our result on positive finite-time Lyapunov exponents is an important step in this direction.

Analysis of system (1.1) from a random dynamical systems perspective

In the absence of noise ($\sigma = 0$), the stochastic differential equation (1.1) is a normal form for the supercritical Hopf bifurcation: when $\alpha \leq 0$ the system has a globally attracting equilibrium at $(x, y) = (0, 0)$ which is exponentially stable until $\alpha = 0$ and, when $\alpha > 0$, the system has a limit cycle at $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = \alpha/a\}$ which is globally attracting on $\mathbb{R}^2 \setminus \{0\}$.

¹The analysis in [14] exploited special properties of the SDE model that do not hold for (1.1).

In the presence of noise ($\sigma \neq 0$), statistical information about the (one point) dynamics of (1.1) can be described by the Fokker–Planck equation and its stationary density. In this case, the stationary density can be calculated analytically, yielding (cf. [12])

$$\frac{2\sqrt{2a}}{\sqrt{\pi}\sigma \operatorname{erfc}(-\alpha/\sqrt{2a\sigma^2})} \exp\left(\frac{2\alpha(x^2 + y^2) - a(x^2 + y^2)^2}{2\sigma^2}\right). \quad (1.2)$$

We note in particular that this density does not depend on the shear parameter b .

We observe a clear relation between the stationary measures in the presence of noise ($\sigma > 0$) and the attractors in the deterministic limit: the stationary density is maximal on attractors of the deterministic limit dynamics and (locally) minimal on its repellers, see Figure 1.

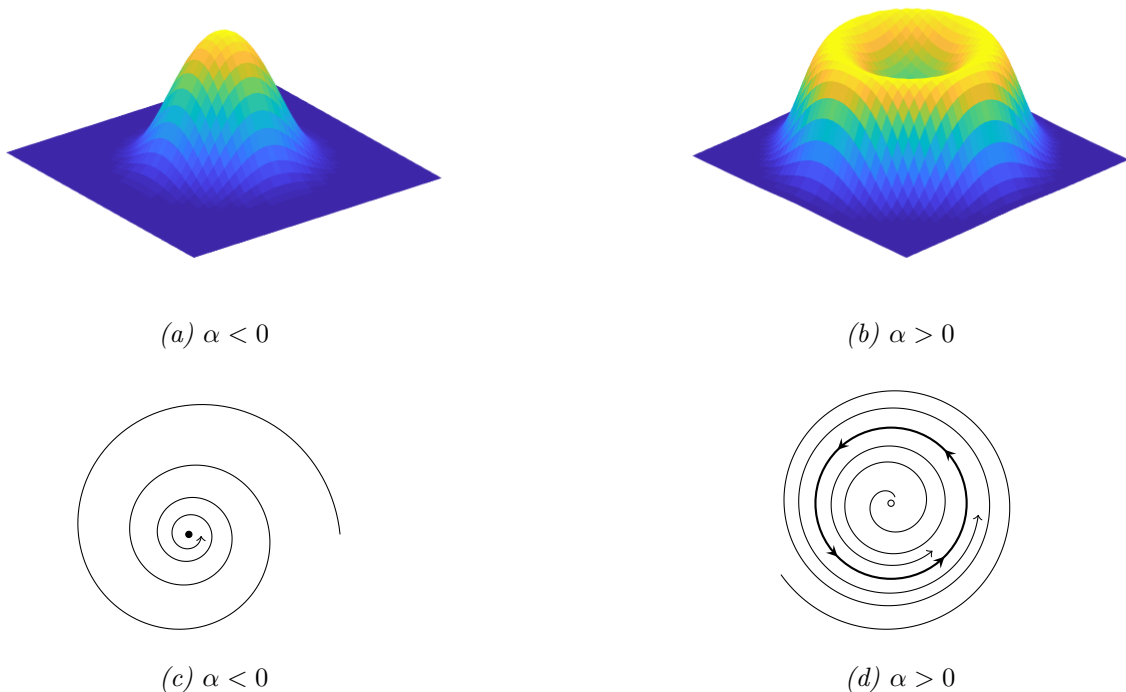


Figure 1: Shape of the stationary density of (1.1) with noise and corresponding phase portraits of the deterministic limit. The qualitative features only depend on the sign of the linear stability parameter α . Figures (a) and (b) present the shapes of the stationary densities in the presence of noise. (a) is characterised by a unique maximum at the origin and (b) by a local minimum at the origin surrounded by a circle of maxima when $\alpha > 0$. Figures (c) and (d) show phase portraits in the deterministic limit $\sigma = 0$ displaying an attracting equilibrium if $\alpha < 0$ and an attracting limit cycle if $\alpha > 0$, precisely where stationary densities have their maxima.

From Figure 1 it is natural to propose that the stochastic differential equation (1.1) has a bifurcation at $\alpha = 0$, represented by the qualitative change of the shape of the stationary density. Such kind of bifurcation is called a phenomenological bifurcation, cf. [2].

In this paper, we consider the system (1.1) with noise from a random dynamical systems point of view: with a canonical model for the noise, (1.1) can be represented as a dynamical system that is driven by a random signal.

While the stationary density (1.2) provides certain statistics about the dynamics of (1.1), by the fact that the underlying Markov process only models probabilistically a single time-series, many relevant dynamical properties cannot be captured, such as a comparison of the trajectories of nearby initial conditions (with the same noise).

As trajectories of random dynamical systems depend on the noise realisation, one does not a priori expect any asymptotic long-term convergent behaviour of individual trajectories to a fixed attractor. An alternative view point that circumvents this problem and often yields convergence, is to consider, for a fixed noise realisation in the past, the flow of a set of initial conditions from time $t = -T$ to a fixed endpoint in time, say $t = 0$, and then take the (pullback) limit $T \rightarrow \infty$. If trajectories of initial conditions converge under this procedure to some set, then this set is called a *pullback attractor* (see e.g. [11, 30]). To illustrate the pullback dynamics of (1.1), in Figure 2 (see [33] for similar pictures), we present some numerical examples². We observe two distinctly different behaviours: either all initial conditions converge to a fixed point, see (a)–(d), or all initial conditions converge to a rather complicated object, see (e)–(h). The former is indicative of the phenomenon of synchronisation (see e.g. [15]), i.e. convergence of all trajectories to a single random equilibrium point, while the latter points to a *random strange attractor* [24].

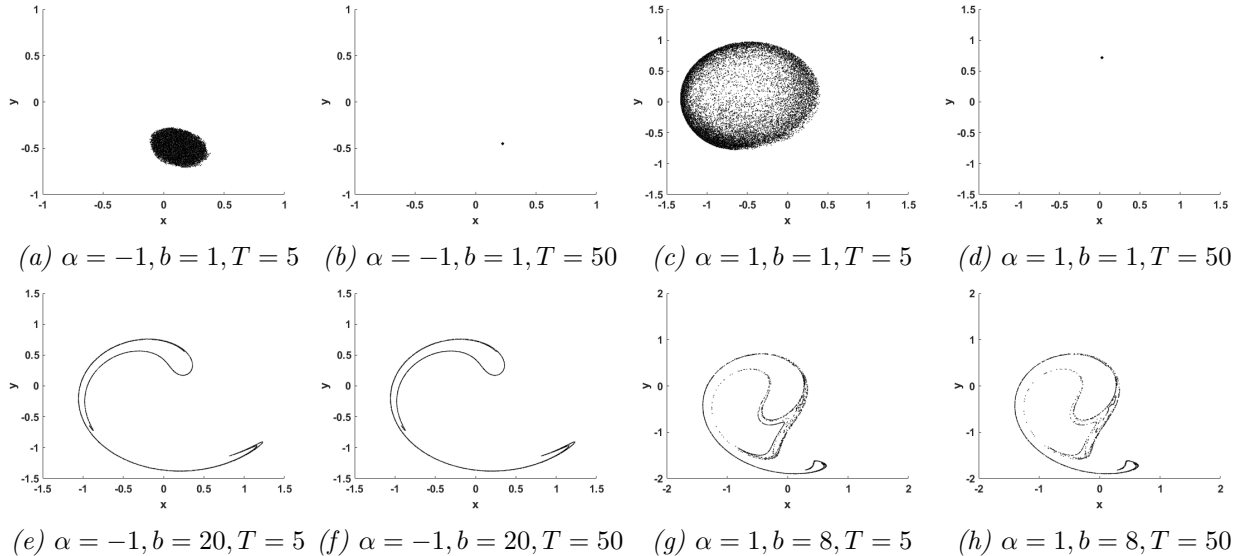


Figure 2: Pullback dynamics of (1.1) with $\sigma = \beta = a = 1$ for initial conditions chosen in approximation of the stationary density. In (a)–(d), in the presence of small shear we observe synchronisation, i.e. pullback convergence of all trajectories to a single point, irrespective of the linear stability at the origin. In (e)–(h), in the presence of sufficiently large shear there is no synchronisation but pullback convergence to a more complicated object (random strange attractor), again irrespective of the linear stability at the origin.

The differences between the types of pullback attractor can also be observed from the Lyapunov exponents, representing the asymptotic long-term average derivative along trajectories. Roughly speaking, random attractors with negative Lyapunov exponents are associated with synchronisation and a positive Lyapunov exponent impedes synchronisation. Accordingly, in Figure 2 (a)–(d) we have negative Lyapunov exponents and in (e)–(h) the largest Lyapunov exponent λ_{top} is positive. In Figure 3 we present a numerical investigation of λ_{top} as a function of the relevant parameters,

²The simulations in this paper are based on an explicit Euler–Maruyama integration of the stochastic differential equation (1.1), usually with time step size 10^{-3} . When we compute Lyapunov exponents, we use an explicit second-order Runge–Kutta method for integrating the variational equation.

similarly to [33] where, however, only positive α are considered. We note that, in contrast with the deterministic and statistical transitions at $\alpha = 0$, the change of sign of the largest Lyapunov exponent is indicative of a dynamical bifurcation, cf. [2], which arises along an altogether different curve in the (b, α) -plane. In particular, we note that as the stationary density is independent of b , different dynamical behaviours underly identical stationary measures, reconfirming our earlier claim that the one-point Markov process and associated stationary measure only provide partial information about the dynamics of a random dynamical system.

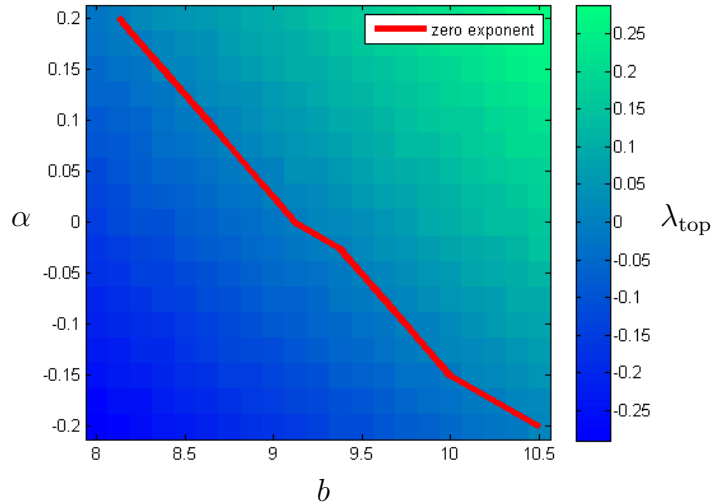


Figure 3: Numerical approximation of the largest Lyapunov exponent λ_{top} for system (1.1) as a function of the linear stability at the origin (α) and strength of shear (b) with $a = \beta = \sigma = 1$. The red curve highlights the border between regions with negative and positive largest Lyapunov exponents, corresponding to synchronisation or random strange attractor, respectively.

Finally, we address a more subtle differentiation between two types of synchronisation that may arise. Synchronisation may be uniform, so that trajectories are guaranteed to approximate each other bounded by upper estimates that are independent of the noise realisation, or non-uniform, when such uniform upper estimates do not exist. In the latter case, the time it takes for two trajectories to converge up to a certain given margin is bounded for any fixed noise realisation, but assessed over all noise realisations these bounds have no maximum. It turns out that the uniformity of the synchronisation is related with the distribution of finite-time Lyapunov exponents, reflecting the average derivatives along trajectories for finite time. The (unique) largest Lyapunov exponent of an attractor is associated with the limit of the distribution of finite-time Lyapunov exponents as the time over which derivatives are averaged goes to infinity. Importantly, while this distribution converges to a Dirac measure concentrated in the largest Lyapunov exponent, the support of this distribution typically converges to a wider range. If this range is contained entirely within the negative real axis, synchronisation is uniform. But it may also happen that the top Lyapunov is negative while the limit of the support of finite-time Lyapunov exponents extends into the positive half line, which results in non-uniform synchronisation. In Figure 5, these scenarios are illustrated with numerical computations. It is natural to find an interface with non-uniform synchronisation in the parameter space between uniform synchronisation regions and regions without synchronisation. For a sketch of the corresponding regions for (1.1) in the (b, α) -plane, see Figure 4.

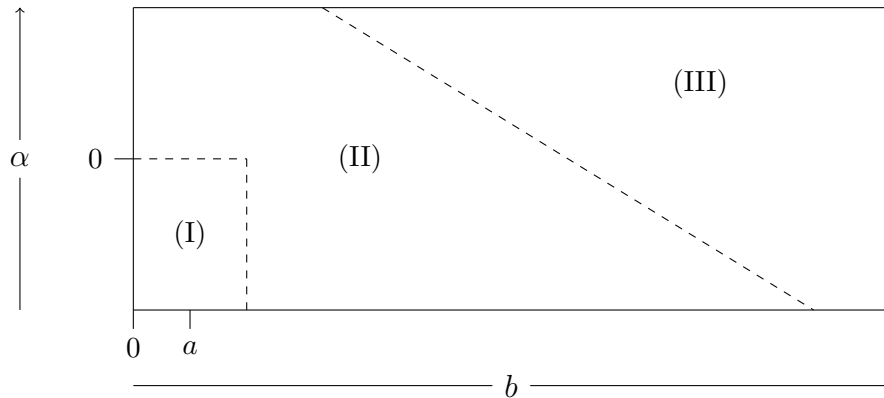


Figure 4: For a, β, σ fixed, we partition the (b, α) -plane associated with (1.1) into three parts with different stability behaviour. Region (I) represents uniform synchronisation, only possible for non-positive α and small b . In region (II), we observe non-uniform synchronisation, i.e. finite-time instabilities occur, but the asymptotic behaviour is exponentially stable for almost all trajectories. (The border between (I) and (II) is described in Theorems E and F.) Region (III) exhibits a positive largest Lyapunov exponent and the absence of synchronisation since the shear is large enough for locally unstable behaviour to prevail (cf. Conjecture D).

The main results

The results of the paper are structured as follows. We first establish (Theorem A) that the stochastic differential equation (1.1) induces a random dynamical system and possesses a random attractor for all choices of parameters. Using results from [15], we show that a negative largest Lyapunov exponent implies the random attractor being a random equilibrium. We then prove (Theorem B) the synchronisation of almost all trajectories from all initial conditions in forward time with exponential speed. We also achieve an explicit upper bound for the shear as a function of other parameters for having a negative largest Lyapunov exponent (Theorem C), extending results in [12] to the full parameter space.

We finally assert (Conjecture D) the appearance of a positive largest Lyapunov exponent beyond a critical shear levels for any given value of α , cf. Figure 3. This would in turn imply the existence of a random strange attractor with positive entropy and SRB sample measures [24]. Based on numerical evidence, we conjecture this scenario also for negative α which is remarkable in view of the fact that in the literature shear-induced chaos is associated with random perturbations of limit cycles and not equilibria.

The second part of this paper focuses on parameter-dependence of finite-time Lyapunov exponents and uniform attractivity and the dichotomy spectrum associated with the linear random dynamical system on the tangent space along trajectories. In the case of small shear, we establish (Theorem E) the existence of a bifurcation at the deterministic Hopf parameter value $\alpha = 0$ from a global uniformly attracting random equilibrium ($\alpha < 0$) to a non-uniformly attracting random equilibrium ($\alpha > 0$). This bifurcation is accompanied by the emergence of positive finite-time Lyapunov exponents and a loss of hyperbolicity of the associated dichotomy spectrum $\Sigma = [-\infty, \alpha]$. This result provides an example of the bifurcation scenario proposed in [8], highlighting the importance of new notions of bifurcation to complement the deterministic ones, by showing that despite the persistence of random equilibria, additive noise does not necessarily "destroy" bifurcations, cf. [10].

Finally, we establish (Theorem F) the relation between shear and positive finite-time Lyapunov

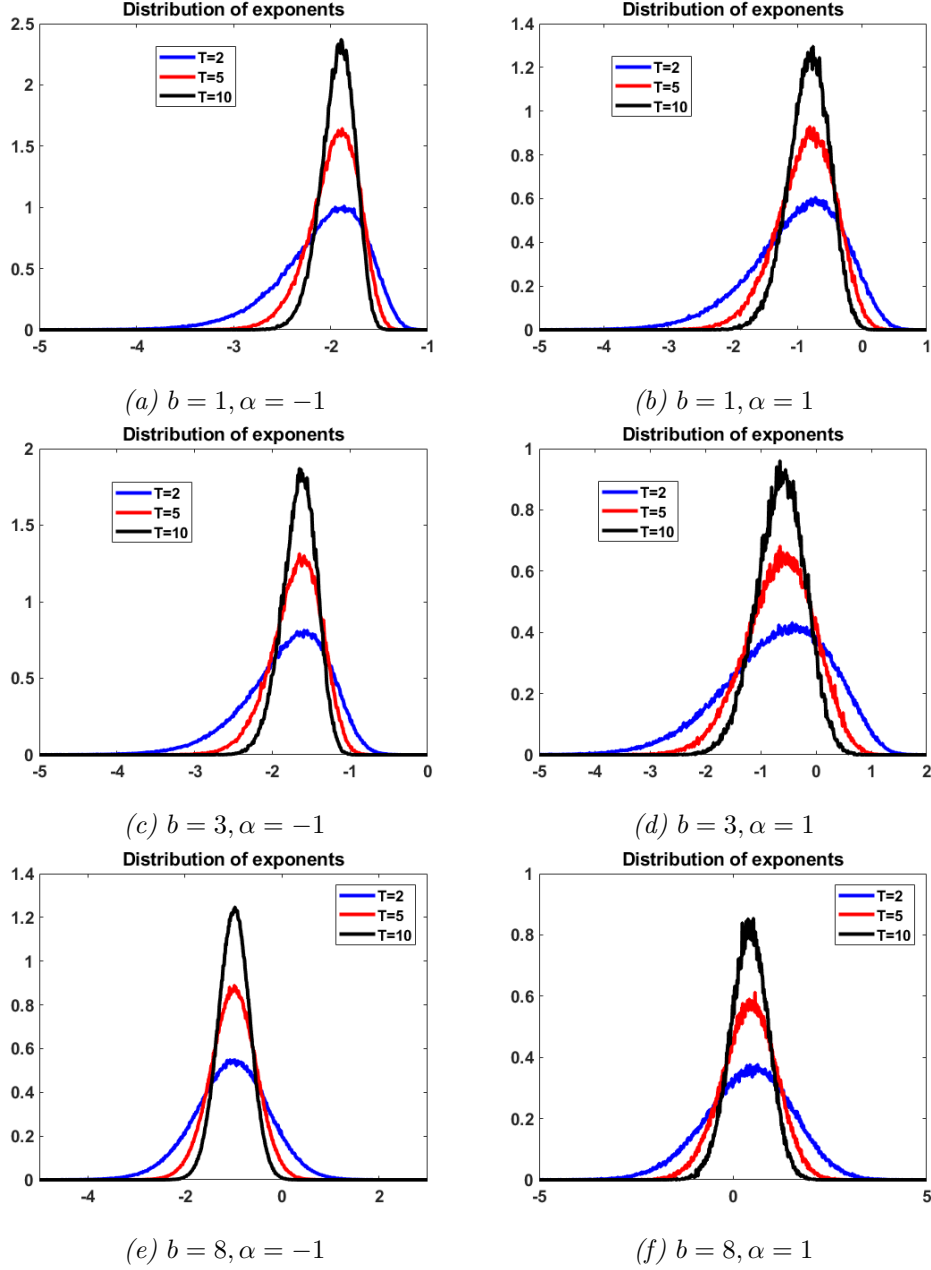


Figure 5: Distribution of finite-time ($T > 0$) Lyapunov exponents of (1.1) with $a = \beta = \sigma = 1$, $\alpha \in \{-1, 1\}$, $T \in \{2, 5, 10\}$ and $b \in \{1, 3, 8\}$, illustrating the type of distributions in phases (I) uniform synchronisation (a), (c); (II) non-uniform synchronisation (b), (d), (e); and (III) absence of synchronisation (f).

exponents. In particular, we show that for any $\alpha \in \mathbb{R}$, there exist arbitrarily large finite-time Lyapunov exponents for sufficiently strong shear intensity b . This is the first analytical result on shear-induced chaos in (1.1). It is in general challenging to obtain lower bounds for the largest Lyapunov exponent in dimension larger than one due to the subadditivity property of matrices, cf. [34]. Therefore analytical results on positive Lyapunov exponents for random dynamical systems have only been achieved in certain special cases, like in simple time-discrete models [25], certain linear models [14] and under special circumstances enabling for stochastic averaging arguments [6, 7]. It remains an open problem to prove Conjecture D.

This paper is organised as follows. Section 2 comprehensively introduces the technical framework and formulates the main results of this paper. Section 3 is dedicated to a detailed proof of Theorem A, establishing the existence of a random attractor for all parameters. In Section 4 we prove Theorems B and C and show some statistical properties of the random equilibrium. In conclusion, Section 5 contains the proofs of Theorems E and F highlighting different aspects of the random bifurcations in α and b . We also provide an Appendix with background material on random dynamical systems comprising the most relevant definitions and results used in this paper.

2 Statement of the main results

The stochastic differential equation (1.1) can be rewritten as

$$dZ_t = f(Z_t)dt + \sigma dW_t, \quad (2.1)$$

where $Z_t = (x_t, y_t)^\top$ and $W_t = (dW_t^1, dW_t^2)^\top$, and the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is defined by

$$f(Z) := \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} Z - (x^2 + y^2) \begin{pmatrix} a & -b \\ b & a \end{pmatrix} Z.$$

To investigate sample path properties of the solutions of (1.1), it is convenient to work with the canonical sample path space of Brownian motions. Let $\Omega = C_0(\mathbb{R}, \mathbb{R}^2)$ be the space of all continuous functions $\omega : \mathbb{R} \rightarrow \mathbb{R}^2$ satisfying that $\omega(0) = 0$. We endow Ω with the compact open topology and denote by $\mathcal{F} = \mathcal{B}(\Omega)$ the Borel σ -algebra on Ω .

It is well known that there exists the so-called *Wiener probability measure* \mathbb{P} on (Ω, \mathcal{F}) which ensures that the two processes $(W_t^1)_{t \in \mathbb{R}}$ and $(W_t^2)_{t \in \mathbb{R}}$, defined by $(W_t^1(\omega), W_t^2(\omega))^\top := \omega(t)$ for $\omega \in \Omega$, are independent one-dimensional Brownian motions. We define the sub σ -algebra $\mathcal{F}_{s,t}$ as the σ -algebra generated by $\omega(u) - \omega(v)$ for $s \leq v \leq u \leq t$. For each $t \in \mathbb{R}$, we define the shift map $\theta_t : \Omega \rightarrow \Omega$ by

$$(\theta_t \omega)(s) = \omega(s+t) - \omega(t) \quad \text{for all } s \in \mathbb{R}.$$

It is well known that $(\theta_t)_{t \in \mathbb{R}}$ is an ergodic flow preserving the probability measure \mathbb{P} , see e.g. [2]. Thus, $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ is an ergodic dynamical system.

2.1 Generation of a random dynamical system with a random attractor

Given $\omega \in \Omega$, an initial value $Z \in \mathbb{R}^2$ and $T > 0$, we say that a continuous function $\varphi(\cdot, \omega, Z) : [0, T] \rightarrow \mathbb{R}^2$ solves the stochastic differential equation (1.1) if it satisfies the integral equation

$$\varphi(t, \omega, Z) = Z + \int_0^t f(\varphi(s, \omega, Z)) ds + \sigma \omega(t) \quad \text{for all } t \in [0, T].$$

The first result in this paper concerns global existence of solutions of (1.1) for almost every sample path, implying that the solutions do not blow up in forward time. We show that the solutions of

(1.1) generate a random dynamical system (θ, φ) (see [2, Definition 1.1.1] for a general definition). This means that the $(\mathcal{B}(\mathbb{R}_0^+) \otimes \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^2), \mathcal{B}(\mathbb{R}^2))$ -measurable mapping $\varphi : \mathbb{R}_0^+ \times \Omega \times \mathbb{R}^2 \rightarrow \mathbb{R}^2, (t, \omega, x) \mapsto \varphi(t, \omega, x)$, is a cocycle over θ , i.e.

$$\varphi(0, \omega, \cdot) \equiv \text{Id} \quad \text{and} \quad \varphi(t+s, \omega, x) = \varphi(t, \theta_s \omega, \varphi(s, \omega, x)) \quad \text{for all } \omega \in \Omega, x \in \mathbb{R}^2 \text{ and } t, s \geq 0.$$

In addition to the generation of a random dynamical system, the following theorem addresses also the existence of a random attractor (see Appendix B for a definition).

Theorem A (Generation of a random dynamical system with a random attractor). *For the stochastic differential equation (1.1), there exists a θ -invariant \mathcal{F} -measurable set $\widehat{\Omega} \subset \Omega$ of full probability such that the following statements hold.*

- (i) *For all $\omega \in \widehat{\Omega}$ and $Z \in \mathbb{R}^2$, the stochastic differential equation (1.1) admits a unique solution $\varphi(\cdot, \omega, Z)$ such that φ forms a cocycle for a random dynamical system on $(\widehat{\Omega}, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$.*
- (ii) *There exists a random attractor $A \in \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^2)$ of the random dynamical system (θ, φ) such that $\omega \mapsto A(\omega)$ is measurable with respect to $\mathcal{F}_{-\infty}^0$, i.e. the past of the system.*

Since the difference of the spaces Ω and $\widehat{\Omega}$ is a set of measure zero, we identify both in the following.

2.2 Negativity of largest Lyapunov exponent and synchronisation

The following results concern the asymptotic behaviour of trajectories, in particular their stability properties. This will give information about the structure of the random attractor A associated with the stochastic differential equation (1.1).

To analyse asymptotic stability, we study the linearisation $\Phi(t, \omega, Z) := D_x \varphi(t, \omega, Z)$. A direct computation yields that $\Phi(0, \omega, Z) = \text{Id}$ and

$$\dot{\Phi}(t, \omega, Z) = Df(\varphi(t, \omega, Z))\Phi(t, \omega, Z). \quad (2.2)$$

It is easy to observe that Φ is a linear cocycle over the skew product flow $(\Theta_t)_{t \in \mathbb{R}_0^+}$ on $\Omega \times \mathbb{R}^2$, defined by

$$\Theta_t(\omega, Z) := (\theta_t \omega, \varphi(t, \omega, Z)).$$

In fact, (Θ, Φ) is a linear random dynamical system, where the ergodic dynamical system $(\theta_t)_{t \in \mathbb{R}}$ is replaced by $(\Theta_t)_{t \in \mathbb{R}_0^+}$. We obtain an ergodic probability measure for the skew product flow $(\Theta_t)_{t \in \mathbb{R}_0^+}$ by using the fact that there exists a one-to-one correspondence between the stationary measure ρ for the Markov semigroup associated to (2.1) and a certain invariant measure of $(\Theta_t)_{t \in \mathbb{R}_0^+}$.

In more detail, recall from (1.2) that the density of the unique stationary distribution ρ reads as

$$p(x, y) = K_{a, \alpha, \sigma} \exp\left(\frac{2\alpha(x^2 + y^2) - a(x^2 + y^2)^2}{2\sigma^2}\right) \quad \text{for all } (x, y) \in \mathbb{R}^2, \quad (2.3)$$

where $K_{a, \alpha, \sigma} > 0$ is the normalisation constant and is given by

$$K_{a, \alpha, \sigma} = \frac{2\sqrt{2a}}{\sqrt{\pi}\sigma \operatorname{erfc}\left(-\frac{\alpha}{\sqrt{2a}\sigma^2}\right)}.$$

The stationary measure ρ gives rise to an invariant measure μ for $(\Theta_t)_{t \in \mathbb{R}_0^+}$ on $\Omega \times \mathbb{R}^2$ in the following sense: the push-forward limit

$$\mu_\omega := \lim_{t \rightarrow \infty} \varphi(t, \theta_{-t} \omega) \rho \quad (2.4)$$

exists for almost all $\omega \in \Omega$ and is an $\mathcal{F}_{-\infty}^0$ -measurable random measure, i.e. $\omega \mapsto \mu_\omega(B)$ is $\mathcal{F}_{-\infty}^0$ -measurable for any $B \in \mathcal{B}(\mathbb{R}^2)$. This defines a *Markov measure* μ on $(\Omega \times \mathbb{R}^2, \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^2))$ via

$$\mu(C) := \int_{\Omega} \mu_\omega(C_\omega) \, d\mathbb{P}(\omega) \quad \text{for all } C \in \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^2),$$

where $C_\omega := \{Z \in \mathbb{R}^2 : (\omega, Z) \in C\}$. μ is invariant under $(\Theta_t)_{t \in \mathbb{R}_0^+}$ (see e.g. [9]). Reversely, the stationary measure ρ is given by

$$\rho(B) = \int_{\Omega} \mu_\omega(B) \, d\mathbb{P}(\omega) \quad \text{for all } B \in \mathcal{B}(\mathbb{R}^2). \quad (2.5)$$

The uniqueness of the stationary measure ρ with density $p(x, y)$ implies that the invariant measure μ is ergodic. We will see in Proposition 4.1 that the linear system Φ defined in (2.2) satisfies the integrability condition

$$\sup_{0 \leq t \leq 1} \ln^+ \|\Phi(t, \omega, Z)\| \in L^1(\mu).$$

Therefore, we can apply Oseledets' Multiplicative Ergodic Theorem [2, Theorem 3.4.1] to obtain the Lyapunov spectrum of the linear random dynamical system (Θ, Φ) (see Appendix A). In particular, the largest Lyapunov exponent λ_{top} is given by

$$\lambda_{\text{top}} = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \|\Phi(t, \omega, Z)\| \quad \text{for } \mu\text{-almost all } (\omega, Z) \in \Omega \times \mathbb{R}^2. \quad (2.6)$$

The largest Lyapunov exponent allows to characterise synchronisation for the random dynamical system generated by (1.1), i.e. if for all $Z_1, Z_2 \in \mathbb{R}^2$, we have

$$\lim_{t \rightarrow \infty} \|\varphi(t, \omega, Z_1) - \varphi(t, \omega, Z_2)\| = 0 \quad \text{for almost all } \omega \in \Omega.$$

Theorem B (Existence of random equilibrium and synchronisation of trajectories). *Suppose that $\lambda_{\text{top}} < 0$. Then the random attractor A for the stochastic differential equation (1.1) is given by a random equilibrium, i.e. $A(\omega)$ is a singleton for almost all $\omega \in \Omega$. In addition, the stochastic differential equation (1.1) admits exponentially fast synchronisation, i.e. for all $Z_1, Z_2 \in \mathbb{R}^2$, we have*

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \ln \|\varphi(t, \omega, Z_1) - \varphi(t, \omega, Z_2)\| < 0 \quad \text{for almost all } \omega \in \Omega.$$

We now aim to determine the region of parameters for which $\lambda_{\text{top}} < 0$. In [12], analytical results are obtained that show that λ_{top} is negative in certain regions of the parameters space, in particular when shear is small. The following theorem extends this result to a larger region in the parameter space.

Theorem C (Small shear implies synchronisation). *For each a, α, β, σ , let*

$$\kappa := a \sqrt{\frac{\pi K_{a,\alpha,\sigma} \sigma^2}{\alpha + \pi K_{a,\alpha,\sigma} \sigma^2} \left(\frac{\pi K_{a,\alpha,\sigma} \sigma^2}{\alpha + \pi K_{a,\alpha,\sigma} \sigma^2} + 2 \right)}.$$

Then the largest Lyapunov exponent λ_{top} is negative if $|b| \leq \kappa$.

Remark 2.1. (i) Note that $K_{a,0,\sigma} = \frac{2\sqrt{2a}}{\sqrt{\pi}\sigma}$, and Theorem C then implies that $\lambda_{\text{top}} < 0$ provided that $|b| < \sqrt{3}a$ and α is sufficiently small. This special case is considered in [12, Proposition 4.1].

(ii) For fixed a and α , we have

$$\lim_{\sigma \rightarrow \infty} \pi K_{a,\alpha,\sigma} \sigma^2 = \lim_{\sigma \rightarrow \infty} \frac{2\sqrt{2\pi a}\sigma}{\operatorname{erfc}\left(-\frac{\alpha}{\sqrt{2a\sigma^2}}\right)} = \infty.$$

Therefore, by Theorem C we have $\lambda_{\text{top}} < 0$ provided that $|b| < \sqrt{3}a$ and the noise intensity σ is sufficiently large.

Numerical evidence from [12] and Figure 3 suggest that large shear leads to positive largest Lyapunov exponent. Unfortunately, we are not able to prove this analytically and formulate this in the following conjecture. Note that in [14], positivity of the largest Lyapunov exponent was analytically established for a two-dimensional system that admits large shear.

Conjecture D (Large shear induces chaos). *Consider the random dynamical system induced by the stochastic differential equation (1.1), and fix $a > 0$ and $\beta \in \mathbb{R}$. Then there exists a function $C : \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that if*

$$b \geq C(\alpha, \sigma),$$

then the largest Lyapunov exponent λ_{top} is positive.

The random attractor A is a random strange attractor in this situation, as illustrated in Figure 2 (e)–(h).

2.3 Qualitative changes in the finite-time behaviour indicated by the dichotomy spectrum

The final two main results concern the qualitative changes in the finite-time behaviour. If shear is small, then these changes occur at the deterministic Hopf bifurcation point $\alpha = 0$, since the maximal finite-time Lyapunov exponents are equal to α . If the shear is increased, then there is a transition to unbounded maximal finite-time Lyapunov exponents.

We also link these phenomena to qualitative changes in the dichotomy spectrum [8], which is based on the notion of an exponential dichotomy. We first need the concept of an invariant projector of a linear random dynamical system $(\theta : \mathbb{R} \times \Omega \rightarrow \Omega, \Psi : \mathbb{R} \times \Omega \rightarrow \mathbb{R}^{d \times d})$, which is given by a measurable function $P : \Omega \rightarrow \mathbb{R}^{d \times d}$ with

$$P(\omega) = P(\omega)^2 \quad \text{and} \quad P(\theta_t \omega) \Psi(t, \omega) = \Psi(t, \omega) P(\omega) \quad \text{for all } t \in \mathbb{R} \text{ and } \omega \in \Omega.$$

Definition 2.2 (Exponential dichotomy). Let (θ, Ψ) be a linear random dynamical system and let $\gamma \in \mathbb{R}$ and $P_\gamma : \Omega \rightarrow \mathbb{R}^{d \times d}$ be an invariant projector of (θ, Ψ) . Then (θ, Ψ) is said to admit an *exponential dichotomy* with *growth rate* $\gamma \in \mathbb{R}$, constants $\alpha > 0$, $K \geq 1$ and projector P_γ if for almost all $\omega \in \Omega$, one has

$$\begin{aligned} \|\Psi(t, \omega) P_\gamma(\omega)\| &\leq K e^{(\gamma - \alpha)t} \quad \text{for all } t \geq 0, \\ \|\Psi(t, \omega) (\text{Id} - P_\gamma(\omega))\| &\leq K e^{(\gamma + \alpha)t} \quad \text{for all } t \leq 0. \end{aligned}$$

We additionally define that (θ, Ψ) admits an exponential dichotomy with growth rate ∞ if there exists a $\gamma \in \mathbb{R}$ such that (θ, Ψ) admits an exponential dichotomy with growth rate γ and projector $P_\gamma = \text{Id}$. Analogously, (θ, Ψ) admits an exponential dichotomy with growth rate $-\infty$ if there exists a $\gamma \in \mathbb{R}$ such that (θ, Ψ) admits an exponential dichotomy with growth rate γ and projector $P_\gamma = 0$.

Definition 2.3 (Dichotomy spectrum [8]). Consider the linear random dynamical system (θ, Ψ) . Then the dichotomy spectrum is defined by

$$\Sigma := \{\gamma \in \mathbb{R} \cup \{-\infty, \infty\} : (\theta, \Psi) \text{ does not admit an exponential dichotomy with growth rate } \gamma\}.$$

Under the assumption of small shear, the following result describes a random bifurcation that corresponds to the deterministic Hopf bifurcation. The notions of uniform and finite-time attractivity are given precisely in Section 5.1.

Theorem E (Bifurcation for small shear). *Consider the stochastic differential equation (1.1) with $|b| < a$. Then the random attractor A is given by an attracting random equilibrium for all $\alpha \leq 0$ and all $\alpha > 0$ in a neighbourhood of 0. We observe the following bifurcation at $\alpha = 0$:*

- (i) *For $\alpha < 0$, the random equilibrium is globally uniformly attracting, but for $\alpha > 0$, the random equilibrium is not even locally uniformly attracting.*
- (ii) *Let $\Phi(t, \omega) := D\varphi(t, \omega, A(\omega))$ denote the linearised random dynamical system along the random equilibrium for fixed α . Then the dichotomy spectrum Σ of Φ is given by*

$$\Sigma = [-\infty, \alpha],$$

i.e. hyperbolicity is lost at $\alpha = 0$.

- (iii) *For $\alpha < 0$, the random equilibrium is finite-time attracting, whereas for $\alpha > 0$, it is not finite-time attracting.*

The last result of the paper concerns the impact of shear on finite-time Lyapunov exponents. It implies a bifurcation of the spectrum of finite-time Lyapunov exponents for some critical value of shear $b^* \in [a, 2a]$.

Theorem F (Shear intensity as bifurcation parameter). *Let a, b, σ satisfy $b > 2a > 0$ and $\sigma \neq 0$. Then for any $z \in \mathbb{R}^2$, the finite-time Lyapunov exponents of solutions starting in z can be arbitrarily large and arbitrarily small with positive probability. More precisely, there exists a $T > 0$ such that for all $t \in (0, T]$, we have*

$$\operatorname{ess\,sup}_{\omega \in \Omega} \sup_{\|v\|=1} \frac{1}{t} \ln \|D\varphi(t, \omega, z)v\| = \infty \quad \text{and} \quad \operatorname{ess\,inf}_{\omega \in \Omega} \inf_{\|v\|=1} \frac{1}{t} \ln \|D\varphi(t, \omega, z)v\| = -\infty.$$

3 Generation of the random dynamical system and existence of a random attractor

We prove Theorem A in this section by following methods developed in [17, 18]. We conjugate the SDE (1.1) to a random differential equation via a suitable transformation using an Ornstein–Uhlenbeck process, so that we need to prove the existence of the random dynamical system and its random attractor for the corresponding random differential equation. An advantage in working with random differential equations (in comparison to *stochastic* differential equations) is that we can work with sample path estimates of solutions.

For $c > 0$, consider the stochastic differential equation

$$dZ = -cZdt + dW_t, \tag{3.1}$$

where $Z \in \mathbb{R}^2$. Define the random variable $Z^* := \int_{-\infty}^0 e^{cs} dW_s$. Then $t \mapsto Z^*(\theta_t \omega)$ solves (3.1), i.e.

$$Z^*(\theta_t \omega) = Z^*(\omega) - c \int_0^t Z^*(\theta_s \omega) ds + \omega(t). \quad (3.2)$$

By replacing Ω with a measurable subset $\widehat{\Omega} \subset \Omega$ of full probability that is invariant under θ , there exist two random variables K and L such that

$$|Z^*(\theta_t \omega)|^2 \leq K(\omega) + L(\omega) \ln(1 + |t|) \quad \text{for all } t \in \mathbb{R} \text{ and } \omega \in \widehat{\Omega}, \quad (3.3)$$

see [20]. We define the map $T : \widehat{\Omega} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $T(\omega, Z) := Z + \sigma Z^*(\omega)$. Under the change of variable $Z \mapsto T(\omega, Z)$, the SDE (1.1) is transformed into the random differential equation

$$\dot{Z} = g(\theta_t \omega, Z), \quad (3.4)$$

where $g(\omega, Z) := f(T(\omega, Z)) + c\sigma Z^*(\omega)$. We show later in Lemma 3.2 and the proof of Theorem A that the solution $\Psi(t, \omega, Z)$ of this random differential equation,

$$\Psi(t, \omega, Z) = Z + \int_0^t g(\theta_s \omega, \Psi(s, \omega, Z)) ds,$$

exists for all $t \geq 0$ and forms a random dynamical system. The following lemma holds using this fact.

Lemma 3.1. *The following statements hold.*

(i) *The random dynamical system $\varphi : \mathbb{R}_0^+ \times \widehat{\Omega} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$, defined by*

$$\varphi(t, \omega, Z) := \varphi(t, \omega, Z) := T(\theta_t \omega, \Psi(t, \omega, T(\omega)^{-1} Z)), \quad (3.5)$$

is generated by the stochastic differential equation (1.1).

(ii) *If the random dynamical system Ψ has a random attractor, then also the random dynamical system φ has a random attractor.*

Proof. (i) From (3.5) and the definition of T , we have

$$\varphi(t, \omega, Z) = \Psi(t, \omega, Z - \sigma Z^*(\omega)) + \sigma Z^*(\theta_t \omega),$$

which together with the fact that Ψ is a solution of (3.4) implies that

$$\begin{aligned} \varphi(t, \omega, Z) &= Z - \sigma Z^*(\omega) + \int_0^t g(\theta_s \omega, \Psi(s, \omega, Z - \sigma Z^*(\omega))) ds + \sigma Z^*(\theta_t \omega) \\ &= Z + \int_0^t g(\theta_s \omega, T(\theta_s \omega)^{-1}(\varphi(s, \omega, Z))) ds + \sigma(Z^*(\theta_t \omega) - Z^*(\omega)). \end{aligned}$$

Thus, using (3.2), we obtain that

$$\begin{aligned} \varphi(t, \omega, Z) &= Z + \int_0^t g(\theta_s \omega, T(\theta_s \omega)^{-1}(\varphi(s, \omega, Z))) - \sigma c Z^*(\theta_s \omega) ds + \sigma \omega(t) \\ &= Z + \int_0^t f(\theta_s \omega, \varphi(s, \omega, Z)) ds + \sigma \omega(t), \end{aligned}$$

which completes the proof of this part.

(ii) This follows from the definition of a random attractor and the fact that the shifted term in the transformation $T(\omega, Z)$, namely $Z^*(\theta_t \omega)$, is tempered. \square

We show that the Euclidean norm of the solutions of (3.4) is bounded by the growth of the corresponding solutions of the scalar equation

$$\dot{\zeta} = \gamma_t(\omega) - \sqrt{a}\zeta, \quad (3.6)$$

where the stochastic process $(\gamma_t)_{t \in \mathbb{R}}$ is chosen appropriately. Note that for each initial value $\zeta_0 \in \mathbb{R}$, the explicit solution of (3.6) is given by

$$\zeta(t, \omega, \zeta_0) = e^{-\sqrt{a}t} \zeta_0 + \int_0^t e^{-\sqrt{a}(t-s)} \gamma_s(\omega) ds. \quad (3.7)$$

Lemma 3.2. *There exists a tempered stochastic processes $(\gamma_t)_{t \in \mathbb{R}}$, i.e.*

$$\lim_{t \rightarrow \pm\infty} \frac{|\gamma_t(\omega)|}{e^{\varepsilon|t|}} = 0 \quad \text{for all } \varepsilon > 0 \text{ and } \omega \in \widehat{\Omega}, \quad (3.8)$$

such that for $Z \in \mathbb{R}^2$, we have

$$\|\Psi(t, \omega, Z)\|^2 \leq 2\zeta(t, \omega, \|Z\|^2), \quad (3.9)$$

which implies that the solution $\Psi(t, \omega, Z)$ exists for all $t \geq 0$.

Proof. By replacing Z with $(x, y)^\top$ and Z^* with $(x^*, y^*)^\top$, we rewrite (3.4) as

$$\begin{aligned} \begin{pmatrix} \dot{x}_t \\ \dot{y}_t \end{pmatrix} &= \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} x_t + \sigma x^*(\theta_t \omega) \\ y_t + \sigma y^*(\theta_t \omega) \end{pmatrix} + c\sigma \begin{pmatrix} x^*(\theta_t \omega) \\ y^*(\theta_t \omega) \end{pmatrix} \\ &\quad - \left\| \begin{pmatrix} x_t + \sigma x^*(\theta_t \omega) \\ y_t + \sigma y^*(\theta_t \omega) \end{pmatrix} \right\|^2 \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x_t + \sigma x^*(\theta_t \omega) \\ y_t + \sigma y^*(\theta_t \omega) \end{pmatrix}. \end{aligned}$$

Let $r_t := \frac{1}{2}(x_t^2 + y_t^2)$. Then a direct computation yields that

$$\begin{aligned} \dot{r}_t &= x_t \dot{x}_t + y_t \dot{y}_t \\ &= 2\alpha r_t + \sigma x^*(\theta_t \omega) ((\alpha + c)x_t + \beta y_t) - \sigma y^*(\theta_t \omega) (\beta x_t - (\alpha + c)y_t) \\ &\quad - \left\| \begin{pmatrix} x_t + \sigma x^*(\theta_t \omega) \\ y_t + \sigma y^*(\theta_t \omega) \end{pmatrix} \right\|^2 (2\alpha r_t + \sigma x^*(\theta_t \omega) (a x_t + b y_t) - \sigma y^*(\theta_t \omega) (b x_t - a y_t)). \end{aligned}$$

Note that $\max\{(\alpha + c)x_t + \beta y_t, \beta x_t - (\alpha + c)y_t\} \leq \sqrt{((\alpha + c)^2 + \beta^2)2r_t}$. Thus,

$$\begin{aligned} &|x^*(\theta_t \omega) ((\alpha + c)x_t + \beta y_t) - y^*(\theta_t \omega) (\beta x_t - (\alpha + c)y_t)| \\ &\leq \sqrt{((\alpha + c)^2 + \beta^2)2r_t} (|x^*(\theta_t \omega)| + |y^*(\theta_t \omega)|) \\ &\leq 2\sqrt{((\alpha + c)^2 + \beta^2)r_t} \|Z^*(\theta_t \omega)\|. \end{aligned} \quad (3.10)$$

On the other hand, we have

$$\left\| \begin{pmatrix} x_t + \sigma x^*(\theta_t \omega) \\ y_t + \sigma y^*(\theta_t \omega) \end{pmatrix} \right\|^2 = 2r_t + \sigma^2 \|Z^*(\theta_t \omega)\|^2 + 2\sigma x^*(\theta_t \omega) x_t + 2\sigma y^*(\theta_t \omega) y_t,$$

which together with the fact that $|x^*(\theta_t \omega) x_t + y^*(\theta_t \omega) y_t| \leq \|Z^*(\theta_t \omega)\| \sqrt{2r_t}$ implies that

$$\left| \left\| \begin{pmatrix} x_t + \sigma x^*(\theta_t \omega) \\ y_t + \sigma y^*(\theta_t \omega) \end{pmatrix} \right\|^2 - 2r_t - \sigma^2 \|Z^*(\theta_t \omega)\|^2 \right| \leq 2\sigma \|Z^*(\theta_t \omega)\| \sqrt{2r_t}.$$

Consequently,

$$ar_t \left\| \begin{pmatrix} x_t + \sigma x^*(\theta_t \omega) \\ y_t + \sigma y^*(\theta_t \omega) \end{pmatrix} \right\|^2 \geq 2ar_t^2 - 2^{\frac{3}{2}} a \sigma \|Z^*(\theta_t \omega)\| r_t^{\frac{3}{2}} + a \sigma^2 \|Z^*(\theta_t \omega)\|^2 r_t, \quad (3.11)$$

and from the fact that

$$|x^*(\theta_t \omega)(ax_t + by_t) - y^*(\theta_t \omega)(bx_t - ay_t)| \leq \sqrt{a^2 + b^2} \|Z^*(\theta_t \omega)\| \sqrt{2} r_t,$$

we derive that

$$\begin{aligned} & \left| \sigma x^*(\theta_t \omega)(ax_t + by_t) - \sigma y^*(\theta_t \omega)(bx_t - ay_t) \right| \left\| \begin{pmatrix} x_t + \sigma x^*(\theta_t \omega) \\ y_t + \sigma y^*(\theta_t \omega) \end{pmatrix} \right\|^2 \\ & \leq 2^{\frac{3}{2}} \sigma \sqrt{a^2 + b^2} \|Z^*(\theta_t \omega)\| r_t^{\frac{3}{2}} + \sigma^2 \sqrt{2(a^2 + b^2)} \|Z^*(\theta_t \omega)\|^2 r_t \\ & \quad + \sigma^3 \sqrt{2(a^2 + b^2)} \|Z^*(\theta_t \omega)\|^3 \sqrt{r_t}. \end{aligned} \quad (3.12)$$

Using (3.10), (3.11), (3.12) and a comparison argument, we obtain for all $t \geq 0$ and $Z \in \mathbb{R}^2 \setminus \{0\}$ that $\frac{1}{2} \|\Psi(t, \omega, Z)\|^2 \leq \tilde{\zeta}(t, \omega, \|Z\|^2)$, where $t \mapsto \tilde{\zeta}(t, \omega, \|Z\|^2) = \tilde{\zeta}_t$ is the solution of the following scalar differential equation

$$\dot{\tilde{\zeta}}_t = a_t(\omega) \tilde{\zeta}_t^{\frac{1}{2}} + b_t(\omega) \tilde{\zeta}_t + c_t(\omega) \tilde{\zeta}_t^{\frac{3}{2}} - 4a \tilde{\zeta}_t^2,$$

with initial condition $\tilde{\zeta}_0 = \|Z\|^2$. Here the functions a_t, b_t, c_t are defined by

$$\begin{aligned} a_t(\omega) &:= 2\sigma \sqrt{(a+c)^2 + \beta^2} \|Z^*(\theta_t \omega)\| + \sqrt{2} \sigma^3 \sqrt{a^2 + b^2} \|Z^*(\theta_t \omega)\|^2, \\ b_t(\omega) &:= 2\alpha + 4\sigma^2 \sqrt{a^2 + b^2} \|Z^*(\theta_t \omega)\|^2 - 2a\sigma^2 \|Z^*(\theta_t \omega)\|^2, \\ c_t(\omega) &:= \left(2^{3/2} \sqrt{a^2 + b^2} \sigma + 2^{5/2} a \sigma \right) \|Z^*(\theta_t \omega)\|. \end{aligned}$$

From temperdness of $Z^*(\theta_t \omega)$, all stochastic processes $(a_t)_{t \in \mathbb{R}}, (b_t)_{t \in \mathbb{R}}$ and $(c_t)_{t \in \mathbb{R}}$ are also tempered. Note that

$$\begin{aligned} a \tilde{\zeta}_t^2 + \sqrt[3]{\frac{a_t(\omega)^4}{4^4 a}} &\geq |a_t(\omega)| \tilde{\zeta}_t^{\frac{1}{2}}, \\ a \tilde{\zeta}_t^2 + \frac{b_t(\omega)^2}{2a} &\geq |b_t(\omega)| \tilde{\zeta}_t, \\ a \tilde{\zeta}_t^2 + \frac{3^3 c_t(\omega)^4}{4^4 a^3} &\geq |c_t(\omega)| \tilde{\zeta}_t^{\frac{3}{2}}. \end{aligned}$$

Therefore,

$$a_t(\omega) \tilde{\zeta}_t^{\frac{1}{2}} + b_t(\omega) \tilde{\zeta}_t + c_t(\omega) \tilde{\zeta}_t^{\frac{3}{2}} - 4a \tilde{\zeta}_t^2 \leq \sqrt[3]{\frac{a_t(\omega)^4}{4^4 a}} + \frac{b_t(\omega)^2}{2a} + \frac{3^3 c_t(\omega)^4}{4^4 a^3} - a \tilde{\zeta}_t^2 \leq \gamma_t(\omega) - \sqrt{a} \tilde{\zeta}_t,$$

where

$$\gamma_t(\omega) := \frac{1}{4} + \sqrt[3]{\frac{a_t(\omega)^4}{4^4 a}} + \frac{b_t(\omega)^2}{2a} + \frac{3^3 c_t(\omega)^4}{4^4 a^3}$$

is tempered. Hence, using a comparison argument, the solution ζ of (3.6) satisfies (3.9), which finishes the proof of this lemma. \square

Proof of Theorem A. (i) According to [2], there exists a local random dynamical system generated by solutions of (3.4). Due to Lemma 3.2, the solution $\Psi(t, \omega, Z)$ exists for all $t \geq 0$. Hence, this proves the fact that we assumed to prove Lemma 3.1. Lemma 3.1 (i) completes the proof of (i).

(ii) Let $D \in \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d)$ be tempered. Then there exists a tempered random variable $R : \widehat{\Omega} \rightarrow \mathbb{R}^+$ such that $D(\omega) \subset B_{R(\omega)}(0)$. By Lemma 3.2, for all $Z \in D(\theta_{-t}\omega)$, we have

$$\|\Psi(t, \theta_{-t}\omega, Z)\|^2 \leq 2\zeta(t, \theta_{-t}\omega, R(\theta_{-t}\omega)) \leq 2e^{-\sqrt{at}}R(\theta_{-t}\omega) + 2 \int_{-t}^0 e^{\sqrt{as}}\gamma_s(\omega) ds,$$

where we use (3.7) to obtain the last inequality. Since $(\gamma_t)_{t \in \mathbb{R}}$ is tempered, $\int_{-\infty}^0 e^{\sqrt{as}}\gamma_s(\omega) ds$ exists. On the other hand, since R is tempered, it follows that $\lim_{t \rightarrow \infty} e^{-\sqrt{at}}R(\theta_{-t}\omega) = 0$. Define $r(\omega) := \sqrt{1 + 2 \int_{-\infty}^0 e^{\sqrt{as}}\gamma_s(\omega) ds}$. Thus, for each $\omega \in \widehat{\Omega}$, there exists $T > 0$ such that

$$\Psi(t, \theta_{-t}\omega, D(\theta_{-t}\omega)) \subset B_{r(\omega)}(0) \quad \text{for all } t \geq T.$$

This means that $B_{r(\omega)}(0)$ is an absorbing set. Applying Theorem B.2 completes the proof. \square

4 Synchronisation

We prove in this section that the system (1.1) admits synchronisation if the largest Lyapunov exponent is negative (Theorem B), and we show that small shear implies negativity of the largest Lyapunov exponent and thus synchronisation (Theorem C). In addition, we show that the system satisfies the integrability condition of the Multiplicative Ergodic Theorem [2, Theorem 3.4.1], and we prove that the sum of the two Lyapunov exponents is always negative.

4.1 Negativity of the sum of the Lyapunov exponents

Recall that $\Phi : \mathbb{R}_0^+ \times \Omega \times \mathbb{R}^2 \rightarrow \mathbb{R}^{2 \times 2}$ is the linear random dynamical system satisfying $\Phi(0, \omega, Z) = \text{id}$ and

$$\dot{\Phi}(t, \omega, Z) = Df(\varphi(t, \omega, Z))\Phi(t, \omega, Z).$$

We show that Φ satisfies the integrability condition of the Multiplicative Ergodic Theorem with respect to the measure μ and also show that the sum of the Lyapunov exponents of Φ is always negative. In the proof, we make use of the following facts from Section 2.2: firstly, the Markov process solving (1.1) has a unique stationary measure ρ with density $p(x, y)$ given in (2.3). Furthermore, via the relations (2.4) and (2.5), the measure ρ corresponds with the Markov measure μ which is invariant for the skew-product flow $(\Theta_t)_{t \in \mathbb{R}_0^+}$.

Proposition 4.1. *The following statements hold.*

(i) Let $\lambda^+ : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by

$$\lambda^+(Z) := \max_{\|r\|=1} \langle Df(Z)r, r \rangle. \quad (4.1)$$

Then for almost all $\omega \in \Omega$ and all $Z \in \mathbb{R}^2$, we have

$$\|\Phi(t, \omega, Z)\| \leq \exp\left(\int_0^t \lambda^+(\varphi(s, \omega, Z)) ds\right) \quad \text{for all } t \geq 0, \quad (4.2)$$

and the linear random dynamical system Φ satisfies the integrability condition of the Multiplicative Ergodic Theorem.

(ii) Let λ_Σ be the sum of the two Lyapunov exponents of the linear random dynamical system Φ . Then $\lambda_\Sigma < 0$ and the disintegrations of the Markov measure μ are singular with respect to the Lebesgue measure on \mathbb{R}^2 .

Proof. (i) Let $v \in \mathbb{R}^2 \setminus \{0\}$ be arbitrary. By definition of Φ , we have

$$\begin{aligned} \frac{d}{dt} \|\Phi(t, \omega, Z)v\|^2 &= 2 \langle Df(\varphi(t, \omega, Z))\Phi(t, \omega, Z)v, \Phi(t, \omega, Z)v \rangle \\ &= 2 \left\langle Df(\varphi(t, \omega, Z)) \frac{\Phi(t, \omega, Z)v}{\|\Phi(t, \omega, Z)v\|}, \frac{\Phi(t, \omega, Z)v}{\|\Phi(t, \omega, Z)v\|} \right\rangle \|\Phi(t, \omega, Z)v\|^2 \\ &\leq 2\lambda^+(\varphi(t, \omega, Z)) \|\Phi(t, \omega, Z)v\|^2. \end{aligned}$$

This implies that

$$\|\Phi(t, \omega, Z)v\|^2 \leq \|v\|^2 \exp\left(2 \int_0^t \lambda^+(\varphi(s, \omega, Z)) ds\right). \quad (4.3)$$

Since v is arbitrary, (4.2) is proved. Using (4.2), we obtain that

$$\sup_{0 \leq t \leq 1} \ln^+ \|\Phi(t, \omega, Z)\| \leq \int_0^1 |\lambda^+(\varphi(s, \omega, Z))| ds,$$

which implies that

$$\begin{aligned} \int_{\Omega \times \mathbb{R}^2} \sup_{0 \leq t \leq 1} \ln^+ \|\Phi(t, \omega, Z)\| d\mu(\omega, Z) &\leq \int_{\Omega \times \mathbb{R}^2} \int_0^1 |\lambda^+(\varphi(s, \omega, Z))| ds d\mu(\omega, Z) \\ &= \int_0^1 \int_{\Omega \times \mathbb{R}^2} |\lambda^+(\varphi(s, \omega, Z))| d\mu(\omega, Z) ds \\ &= \int_{\mathbb{R}^2} |\lambda^+(Z)| d\rho(Z), \end{aligned} \quad (4.4)$$

where in the last equality, we use the fact that the skew product $\Theta_s(\omega, Z) = (\theta_s\omega, \varphi(s, \omega, Z))$ preserves the probability measure μ . By definition of λ^+ and the explicit form of Df given by

$$Df(Z) = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} - \begin{pmatrix} 3ax^2 + 2bxy + ay^2 & bx^2 + 2axy + 3by^2 \\ -3bx^2 - 2axy - by^2 & ax^2 - 2bxy + 3ay^2 \end{pmatrix},$$

it follows that

$$|\lambda^+(Z)| \leq |\alpha| + 6(|a| + |b|)(x^2 + y^2) \quad \text{for all } Z = (x, y)^\top \in \mathbb{R}^2.$$

Together with (4.4), this implies that

$$\int_{\Omega \times \mathbb{R}^2} \sup_{0 \leq t \leq 1} \ln^+ \|\Phi(t, \omega, Z)\| d\mu(\omega, Z) \leq |\alpha| + 6(|a| + |b|) \int_{\mathbb{R}^2} (x^2 + y^2)p(x, y) dx dy,$$

where $p(x, y)$ is given as in (2.3). Thus, the linear random dynamical system Φ satisfies the integrability condition of the Multiplicative Ergodic Theorem.

(ii) Due to $\lambda_\Sigma = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \det \Phi(t, \omega, Z)$, the sum of the two Lyapunov exponents of the linear random dynamical system generated by (2.2) reads as

$$\lambda_\Sigma = 2\alpha - 4a \int_{\mathbb{R}^2} (x^2 + y^2)p(x, y) dx dy.$$

Using the explicit formula for $p(x, y)$ from (2.3), we obtain that

$$\lambda_\Sigma = 2\alpha - 4a \frac{\int_{\mathbb{R}^2} (x^2 + y^2) \exp\left(\frac{2\alpha(x^2+y^2) - a(x^2+y^2)^2}{2\sigma^2}\right) dx dy}{\int_{\mathbb{R}^2} \exp\left(\frac{2\alpha(x^2+y^2) - a(x^2+y^2)^2}{2\sigma^2}\right) dx dy}.$$

Applying the change of variables $x = \sigma r \sin \phi$, $y = \sigma r \cos \phi$ the previous integral yields that

$$\lambda_\Sigma = 2\alpha - 4a\sigma^2 \frac{\int_0^\infty r^3 \exp\left(\frac{2\alpha r^2 - a\sigma^2 r^4}{2}\right) dr}{\int_0^\infty r \exp\left(\frac{2\alpha r^2 - a\sigma^2 r^4}{2}\right) dr}.$$

A further change of variable $r^2 \mapsto r$ gives that

$$\lambda_\Sigma = 2\alpha - 4a\sigma^2 \frac{\int_0^\infty r \exp\left(\frac{2\alpha r - a\sigma^2 r^2}{2}\right) dr}{\int_0^\infty \exp\left(\frac{2\alpha r - a\sigma^2 r^2}{2}\right) dr},$$

which proves that $\lambda_\Sigma < 0$ if $\alpha \leq 0$. We also show this for $\alpha > 0$ now. Using the change of variable $\sqrt{a}|\sigma|r - \frac{\alpha}{\sqrt{a}|\sigma|} \mapsto r$, we obtain that

$$\begin{aligned} \lambda_\Sigma &= -2\alpha - 4\sqrt{a}|\sigma| \frac{\int_{-\frac{\alpha}{|\sigma|\sqrt{a}}}^\infty r \exp\left(-\frac{r^2}{2}\right) dr}{\int_{-\frac{\alpha}{|\sigma|\sqrt{a}}}^\infty \exp\left(-\frac{r^2}{2}\right) dr} \\ &= -2\alpha - 4\sqrt{a}|\sigma| \frac{\exp\left(-\frac{\alpha^2}{2a\sigma^2}\right)}{\int_{-\frac{\alpha}{|\sigma|\sqrt{a}}}^\infty \exp\left(-\frac{r^2}{2}\right) dr}, \end{aligned}$$

which shows that $\lambda_\Sigma < 0$ for $\alpha > 0$. As a consequence, using [23, Proposition 1] and [4, Theorem 4.15], the disintegration of the Markov measure μ is singular with respect to ρ if $\lambda_\Sigma < 0$. The fact that ρ is equivalent to the Lebesgue measure finishes the proof of this proposition. \square

4.2 Negative largest Lyapunov exponent implies synchronisation

The aim of this subsection is to prove synchronisation of the random dynamical system generated by (1.1) when its largest Lyapunov exponent λ_{top} (2.6) is negative. Our proof consists of two ingredients. The first ingredient is a result from [15] that implies that the fibers of the random attractor are singletons. The second ingredient is the stable manifolds theorem, which we use to verify that this random attractor is also attracting in forward time.

We make use of the following sufficient conditions for the collapse of a random attractor [15, Theorem 2.14].

Theorem 4.2 (Collapse of the random attractor). *We assume that a random dynamical system (θ, φ) is*

- (i) asymptotically stable on a fixed non-empty open set $U \subset \mathbb{R}^2$, in the sense that there exists a sequence $t_n \rightarrow \infty$ such that

$$\mathbb{P}\left(\omega \in \Omega : \lim_{n \rightarrow \infty} \text{diam}(\varphi(t_n, \omega, U)) = 0\right) > 0.$$

(ii) swift transitive, i.e. for all $x, y \in \mathbb{R}^2$ and $r > 0$, there exists a $t > 0$ such that

$$\mathbb{P}(\omega \in \Omega : \varphi(t, \omega, B_r(x)) \subset B_{2r}(y)) > 0.$$

(iii) contracting on large sets, i.e. for all $R > 0$, there exist $y \in \mathbb{R}^2$ and $t > 0$ such that

$$\mathbb{P}(\omega \in \Omega : \text{diam}(\varphi(t, \omega, B_R(y))) \leq \frac{R}{4}) > 0.$$

Suppose further that (θ, φ) has a random attractor A with $\mathcal{F}_{-\infty}^0$ -measurable fibers. Then $A(\omega)$ is a singleton \mathbb{P} -almost surely.

We use this result for the following proposition.

Proposition 4.3. *Suppose that the largest Lyapunov exponent λ_{top} of the random dynamical system generated by (1.1) is negative. Then the fibers of the random attractor are singletons, given by $\mathcal{F}_{-\infty}^0$ -measurable map $A : \Omega \rightarrow \mathbb{R}^2$. Furthermore, the following statements hold:*

(i) A is a random equilibrium of φ , i.e.

$$\varphi(t, \omega, A(\omega)) = A(\theta_t \omega) \quad \text{for all } t \geq 0 \text{ and almost all } \omega \in \Omega.$$

(ii) The random equilibrium is distributed according to the stationary density $(x, y) \mapsto p(x, y)$, see (2.3). More precisely,

$$\mathbb{P}(\{\omega \in \Omega : A(\omega) \in C\}) = \int_C p(x, y) \, dx \, dy \quad \text{for all } C \in \mathcal{B}(\mathbb{R}^2).$$

(iii) The largest Lyapunov exponent of the linearization along the random equilibrium a ,

$$\dot{\xi} = Df(A(\theta_t \omega))\xi,$$

is equal to λ_{top} as given in (2.6).

Proof. In the first part of the proof, we show that the random dynamical system φ generated by (1.1) fulfils the assumptions (i), (ii), and (iii) of Theorem 4.2. Note that (i) follows from the negativity of the largest Lyapunov exponent (see [15, Lemma 4.1 and Corollary 4.4]), and swift transitivity holds for our system according to [15, Proposition 4.9]. Hence, it remains to show contraction on large sets for φ . By definition of f , we have that

$$\begin{aligned} \langle f(x) - f(y), x - y \rangle &\leq (\alpha - a\frac{1}{2}(\|x\|^2 + \|y\|^2)) \|x - y\|^2 \\ &\quad + b(x_1 y_2 - y_1 x_2) (2\langle x - y, y \rangle + \|x - y\|^2). \end{aligned}$$

Fix $r > 0$, and consider $B_r(z)$, where $z = (R, 0)$ for some $R > 0$ to be chosen large enough. For any $x, y \in B_r(z)$, observe that

$$(x_1 y_2 - y_1 x_2) \langle x - y, y \rangle \leq r \|y\| \|x - y\|^2 + r^2 \|x - y\|^2$$

and

$$(x_1 y_2 - y_1 x_2) \|x - y\|^2 \leq 2 \|x - y\|^2 \|y\| \|x - y\|.$$

This implies that for all $x, y \in B_r(z)$,

$$\begin{aligned} \langle f(x) - f(y), x - y \rangle &\leq \|x - y\|^2 \left(\alpha - a \frac{1}{2} (\|x\|^2 + \|y\|^2) + 2b(r\|y\| + r^2 + 2\|y\|r) \right) \\ &< K \|x - y\|^2 \end{aligned}$$

for some $K < 0$ if R is big enough (due to the quadratic terms, K has negative sign). This property is called monotonicity on large sets, which implies contraction on large sets due to [15, Proposition 3.10].

We now prove the statements (i), (ii) and (iii) of the proposition.

(i) This follows immediately from the definition of a random attractor (see Appendix).

(ii) Note that $\omega \mapsto A(\omega)$ is measurable with respect to $\mathcal{F}_{-\infty}^0$, and thus, $\mu_\omega := \delta_{A(\omega)}$ defines a Markov measure. The invariance of μ_ω follows directly from (i). Hence, $\{\mu_\omega\}_{\omega \in \Omega}$ is the disintegration of the ergodic invariant measure μ associated with the ergodic stationary measure ρ , and we obtain from (2.5) that for all $C \in \mathcal{B}(\mathbb{R}^2)$

$$\mathbb{P}(\{\omega \in \Omega : A(\omega) \in C\}) = \int_{\Omega} \delta_{A(\omega)}(C) \, d\mathbb{P}(\omega) = \int_{\Omega} \mu_\omega(C) \, d\mathbb{P}(\omega) = \rho(U) = \int_C p(x, y) \, dx \, dy.$$

(iii) According to the Multiplicative Ergodic Theorem, the existence of the Lyapunov spectrum holds for a set $M \subset \Omega \times \mathbb{R}^2$ of full μ -measure. We observe that the set

$$D = \bigcup_{\omega \in \Omega} \{(\omega, A(\omega))\}$$

has full μ -measure, since

$$\mu(D) = \int_{\Omega} \mu_\omega(\{A(\omega)\}) \, d\mathbb{P}(\omega) = \int_{\Omega} \delta_{A(\omega)}(\{A(\omega)\}) \, d\mathbb{P}(\omega) = 1.$$

Hence, $\mu(M \cap D) = 1$. Since the Oseledets space associated with the second Lyapunov exponent has zero Lebesgue measure for any $(\omega, x) \in M \cap D$, the claim follows. \square

Finally, we prove Theorem B.

Proof of Theorem B. The existence of the attracting random equilibrium $A : \Omega \rightarrow \mathbb{R}^2$ has been shown in Proposition 4.3. Define $\psi : \mathbb{R}_0^+ \times \Omega \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$\psi(t, \omega, x) = \varphi(t, \omega, A(\omega) + x) - \varphi(t, \omega, A(\omega)).$$

Obviously, $\psi(t, \omega, 0) = 0$ and $\psi(t, \omega, x)$ is the solution of the random differential equation

$$\dot{\xi} = Df(A(\theta_t \omega))\xi + R(t, \omega, \xi), \tag{4.5}$$

where

$$R(t, \omega, \xi) := f(A(\theta_t \omega) + \xi) - f(A(\theta_t \omega)) - Df(A(\theta_t \omega))\xi.$$

Note that for $R \equiv 0$, the largest Lyapunov exponent of the homogeneous equation (4.5) is negative. Using the stable manifold theorem [2, Theorems 7.5.5 and 7.5.16], there exists $r(\omega) > 0$ such that for almost all $\omega \in \Omega$ and $x \in B_{r(\omega)}(0)$, one has

$$\lim_{t \rightarrow \infty} e^{-\frac{\lambda_{\text{top}}}{2}t} \|\psi(t, \omega, x)\| = \lim_{t \rightarrow \infty} e^{-\frac{\lambda_{\text{top}}}{2}t} \|\varphi(t, \omega, x + A(\omega)) - A(\theta_t \omega)\| = 0. \tag{4.6}$$

Choose and fix an arbitrary initial value $x \in \mathbb{R}^2$, and define

$$V := \left\{ \omega \in \Omega : \limsup_{t \rightarrow \infty} e^{-\frac{\lambda_{\text{top}}}{2}t} \|\varphi(t, \omega, x) - A(\theta_t \omega)\| = 0 \right\}.$$

It remains to show that $\mathbb{P}(V) = 1$. For each $n \in \mathbb{N}$, we define

$$\Omega_n := \left\{ \omega \in \Omega : \varphi(t, \theta_{-t} \omega, x) \in B_{r(\omega)}(A(\omega)) \text{ for all } t \geq n \right\}.$$

Note that $(\Omega_n)_{n \in \mathbb{N}}$ is an increasing sequence of measurable sets. By virtue of Proposition 4.3, the random equilibrium a is the random attractor of φ , which implies $\lim_{n \rightarrow \infty} \mathbb{P}(\Omega_n) = 1$. From the definition of Ω_n , we derive that $\varphi(n, \theta_{-n} \omega, x) \in B_{r(\omega)}(A(\omega))$ for all $\omega \in \Omega_n$. Together with (4.6), this implies that for all $\omega \in \Omega_n$, one has

$$\begin{aligned} 0 &= \limsup_{t \rightarrow \infty} e^{-\frac{\lambda_{\text{top}}}{2}t} \|\varphi(t, \omega, \varphi(n, \theta_{-n} \omega, x)) - A(\theta_t \omega)\| \\ &= \limsup_{t \rightarrow \infty} e^{-\frac{\lambda_{\text{top}}}{2}t} \|\varphi(t+n, \theta_{-n} \omega, x) - A(\theta_t \omega)\|. \end{aligned}$$

Consequently, $\theta_{-n} \Omega_n \subset V$, and thus, $\mathbb{P}(V) = 1$, which finishes the proof. \square

4.3 Small shear implies synchronisation

We prove Theorem C in this subsection, which says that small shear implies negativity of the largest Lyapunov exponent. The main ingredient for the proof of Theorem C is the inequality in Proposition 4.1(i).

We first need the following estimate on the function λ^+ defined as in (4.1).

Lemma 4.4. *For any $Z = (x, y)^\top \in \mathbb{R}^2$, we have*

$$\lambda^+(Z) \leq \alpha + (\sqrt{a^2 + b^2} - 2a)(x^2 + y^2),$$

and equality holds if and only if $xy = 0$.

Proof. Using the following explicit form of $Df(Z)$,

$$Df(Z) = \begin{pmatrix} \alpha - ay^2 - 3ax^2 - 2byx & -\beta - 2axy - bx^2 - 3by^2 \\ \beta - 2axy + by^2 + 3bx^2 & \alpha - ax^2 - 3ay^2 + 2byx \end{pmatrix},$$

we obtain for any $r \in \mathbb{R}^2$ with $\|r\| = 1$ that

$$\begin{aligned} \langle Df(x, y)r, r \rangle &= r_1^2(\alpha - ay^2 - 3ax^2) + r_1 r_2(-\beta - 2axy) + r_1 r_2(\beta - 2axy) + r_2^2(\alpha - ax^2 - 3ay^2) \\ &\quad - 2byx r_1^2 + 2byx r_2^2 + r_1 r_2(2bx^2 - 2by^2) \\ &= \alpha - a(x^2 + y^2) + 2b(r_1 r_2 x^2 - r_1 r_2 y^2) + (r_2^2 - r_1^2)xy - 2a(r_1 x + r_2 y)^2. \end{aligned}$$

Since $r_1^2 + r_2^2 = 1$, it is possible to write that $r_1 = \sin \phi$ and $r_2 = \cos \phi$ for some $\phi \in [0, 2\pi)$. Thus, a simple calculation yields that

$$\begin{aligned} \langle Df(x, y)r, r \rangle &= \alpha - 2a(x^2 + y^2) + (ax^2 + bxy - ay^2) \cos 2\phi + (bx^2 - 2axy - by^2) \sin 2\phi \\ &\leq \alpha - 2a(x^2 + y^2) + \sqrt{(ax^2 + 2bxy - ay^2)^2 + (bx^2 - 2axy - by^2)^2} \\ &= \alpha - 2a(x^2 + y^2) + \sqrt{(a^2 + b^2)(x^2 - y^2)^2 + 4b^2 x^2 y^2} \\ &\leq \alpha - 2a(x^2 + y^2) + \sqrt{a^2 + b^2}(x^2 + y^2), \end{aligned}$$

which completes the proof. \square

Proof of Theorem C. From inequality (4.2), we derive that

$$\lambda_{\text{top}} = \limsup_{t \rightarrow \infty} \frac{1}{t} \ln \|\Phi(t, \omega, s)\| \leq \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \lambda^+(\varphi(s, \omega, x)) \, ds.$$

Note that the skew product flow $\Theta_s(\omega, Z) = (\theta_s \omega, \varphi(s, \omega, Z))$ preserves the probability measure μ , and λ^+ is integrable. By using Birkhoff's Ergodic Theorem [31, Theorem 1.14], we obtain that

$$\lambda_{\text{top}} \leq \int_{\mathbb{R}^2} \lambda^+(x, y) p(x, y) \, dx \, dy,$$

where the density function p is as in (2.3). Thus, by virtue of Lemma 4.4, we arrive at

$$\lambda_{\text{top}} < \alpha + (\sqrt{a^2 + b^2} - 2a) \int_{\mathbb{R}^2} (x^2 + y^2) p(x, y) \, dx \, dy.$$

Inserting the explicit form of the density function p in the preceding inequality gives that

$$\lambda_{\text{top}} < \alpha + (\sqrt{a^2 + b^2} - 2a) K \int_{\mathbb{R}^2} (x^2 + y^2) \exp\left(\frac{2\alpha(x^2 + y^2) - a(x^2 + y^2)^2}{2\sigma^2}\right) \, dx \, dy, \quad (4.7)$$

with the normalization constant $K = \frac{2\sqrt{2a}}{\sqrt{\pi}\sigma \operatorname{erfc}(-\alpha/\sqrt{2a}\sigma^2)}$. Using polar coordinates, we obtain that

$$\begin{aligned} & K \int_{\mathbb{R}^2} \left(\alpha - a(x^2 + y^2) \exp\left(\frac{2\alpha(x^2 + y^2) - a(x^2 + y^2)^2}{2\sigma^2}\right) \right) \, dx \, dy \\ &= 2\pi K \int_0^\infty (\alpha - ar^2) r \exp\left(\frac{2\alpha r^2 - ar^4}{2\sigma^2}\right) \, dr \\ &= -\pi\sigma^2 K. \end{aligned}$$

This implies

$$\int_{\mathbb{R}^2} (x^2 + y^2) \exp\left(\frac{2\alpha(x^2 + y^2) - a(x^2 + y^2)^2}{2\sigma^2}\right) \, dx \, dy = \frac{\alpha}{Ka} + \frac{\pi\sigma^2}{a},$$

which together with (4.7) implies that

$$\lambda_{\text{top}} < \alpha + (\sqrt{a^2 + b^2} - 2a) \left(\frac{\alpha}{a} + \frac{\pi K \sigma^2}{a} \right).$$

Consequently,

$$\lambda_{\text{top}} < -\pi K \sigma^2 + \left(\sqrt{1 + \frac{b^2}{a^2}} - 1 \right) (\alpha + \pi K \sigma^2). \quad (4.8)$$

Note that by definition of K it is easy to see that $\alpha + \pi K \sigma^2 > 0$. Therefore, for all $|b| \leq \kappa$, we have

$$\lambda_{\text{top}} < -\pi K \sigma^2 + \left(\sqrt{1 + \frac{\kappa^2}{a^2}} - 1 \right) (\alpha + \pi K \sigma^2) = 0,$$

which completes the proof of this theorem. \square

5 Random Hopf bifurcation

We analyse random bifurcations for the stochastic differential equation (1.1) in this section, which captures qualitative changes in the asymptotic as well as the finite-time behaviour.

We first need the following preparatory proposition.

Proposition 5.1. *Consider (1.1) such that $|b| \leq \kappa$. Then for any $y \in \mathbb{R}^2$, $\varepsilon > 0$ and $T \geq 0$, there exists a set $E \in \mathcal{F}_{-\infty}^T$ with $\mathbb{P}(E) > 0$ such that*

$$A(\theta_s \omega) \in B_\varepsilon(y) \quad \text{for all } s \in [0, T] \text{ and } \omega \in E,$$

where $\{A(\omega)\}$ is the unique random equilibrium for (1.1) from Proposition 4.3.

Proof. Let $\varepsilon > 0$ and $T \geq 0$. Since $\Omega = \bigcup_{x \in \mathbb{Q}^2} \{\omega \in \Omega : A(\omega) \in B_{\varepsilon/4}(x)\}$, there exists an $x \in \mathbb{R}^2$ such that

$$A_0 := \{\omega \in \Omega : A(\omega) \in B_{\varepsilon/4}(x)\}$$

has positive measure. From [15, Proposition 3.10] we know that there exists $t_0 > 0$ such that

$$B_0 := \{\omega \in \Omega : \varphi(t_0, \omega, x') \in B_{\varepsilon/2}(y) \text{ for all } x' \in B_{\varepsilon/4}(x)\}$$

has positive measure. Since θ is measure preserving, the two sets

$$A_1 := \theta_{t_0} A_0 = \{\omega \in \Omega : A(\theta_{-t_0} \omega) \in B_{\varepsilon/4}(x)\},$$

$$B_1 := \theta_{t_0} B_0 = \{\omega \in \Omega : \varphi(t_0, \theta_{-t_0} \omega, x') \in B_{\varepsilon/2}(y) \text{ for all } x' \in B_{\varepsilon/4}(x)\}$$

have positive measure. Due to the Markov property of the random dynamical system, we observe that B_1 and A_1 are independent, and hence, $\mathbb{P}(B_1 \cap A_1) > 0$. Thus, the set

$$E_0 = \{\omega \in \Omega : A(\omega) \in B_{\varepsilon/2}(y)\} \supset A_1 \cap B_1$$

has positive measure and clearly lies in $\mathcal{F}_{-\infty}^0$. Fix $\omega \in E_0$. Similarly to the proof of [15, Proposition 3.10], define

$$h(t) := -\frac{tf(A(\omega))}{\sigma} \quad \text{for all } t \in [0, T],$$

where f denotes the vector field of the drift in (2.1). We write $\varphi(t, g, z)$, $t \in [0, T]$, for the solution of (2.1) with initial condition z and path $g \in C_0^T := \{\bar{g} \in C([0, T], \mathbb{R}^2) : \bar{g}(0) = 0\}$. We can infer that $\varphi(t, h, A(\omega)) = A(\omega)$ for all $t \in [0, T]$. Recall that the map $g \mapsto \varphi(\cdot, g, z)$ is continuous from C_0^T to $C([0, T], \mathbb{R}^2)$ with respect to the supremum norm $\|\cdot\|_\infty$. Hence, there is a $\delta > 0$ such that for all $g \in C_\delta := \{\bar{g} \in C_0^T : \|\bar{g} - h\| \leq \delta\}$, we have

$$\|\varphi(t, g, A(\omega)) - \varphi(t, h, A(\omega))\| < \varepsilon/2 \quad \text{for all } t \in [0, T].$$

Since the set $E_+ := \{\omega : \omega|_{[0, T]} \in C_\delta\}$ has positive measure and is independent of E_0 , the set $E = E_0 \cap E_+ \in \mathcal{F}_{-\infty}^T$ has positive measure and satisfies

$$A(\theta_t \omega) \in B_\varepsilon(y) \quad \text{for all } t \in [0, T] \text{ and } \omega \in E,$$

by the above construction. □

5.1 Bifurcation for small shear

In this subsection, we consider the stochastic differential equation (1.1) with small enough shear such there exists a random equilibrium for α close to zero. We prove in Theorem 5.2 that the random equilibrium $A : \Omega \rightarrow \mathbb{R}^2$ loses uniform attractivity at the deterministic bifurcation point $\alpha = 0$. On the other hand, we will observe a loss of hyperbolicity at the bifurcation point in the dichotomy spectrum associated with the random equilibrium. Moreover, we can show that $A : \Omega \rightarrow \mathbb{R}^2$ is finite-time attracting before, but not after the bifurcation point, indicated by a transition from zero to positive probability of positive finite-time Lyapunov exponents.

We call the random attractor A *locally uniformly attracting* if there exists a $\delta > 0$ such that

$$\lim_{t \rightarrow \infty} \sup_{x \in B_\delta(0)} \text{ess sup}_{\omega \in \Omega} \|\varphi(t, \omega, A(\omega) + x) - A(\theta_t \omega)\| = 0.$$

We call it *globally uniformly attracting* if the above holds for any $\delta > 0$.

Theorem 5.2. *Consider the stochastic differential equation (1.1) such that there is a unique attracting random equilibrium $A : \Omega \rightarrow \mathbb{R}^2$ (see Proposition 4.3 and Theorem C). Then for $\alpha < 0$ and $|b| \leq a$, the random attractor $A : \Omega \rightarrow \mathbb{R}^2$ is globally uniformly attracting. Furthermore, for all pairs of initial conditions $U, V \in \mathbb{R}^2$, we have*

$$\|\phi(t, \omega, U) - \phi(t, \omega, V)\| \leq e^{\alpha t} \|U - V\| \quad \text{for all } t \geq 0.$$

For $\alpha > 0$, the random attractor $A : \Omega \rightarrow \mathbb{R}^2$ is not even locally uniformly attracting.

Proof. Fix $\alpha < 0$, and choose arbitrary $U, V \in \mathbb{R}^2$, $\omega \in \Omega$. Define

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} := \phi(t, \omega, U) \quad \text{and} \quad \begin{pmatrix} \hat{x}_t \\ \hat{y}_t \end{pmatrix} := \phi(t, \omega, V).$$

From (1.1), we derive that

$$\begin{aligned} \frac{d}{dt} \begin{pmatrix} x_t - \hat{x}_t \\ y_t - \hat{y}_t \end{pmatrix} &= \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} x_t - \hat{x}_t \\ y_t - \hat{y}_t \end{pmatrix} - \\ &\quad (x_t^2 + y_t^2) \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix} + (\hat{x}_t^2 + \hat{y}_t^2) \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} \hat{x}_t \\ \hat{y}_t \end{pmatrix}. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left\| \begin{pmatrix} x_t - \hat{x}_t \\ y_t - \hat{y}_t \end{pmatrix} \right\|^2 &= (x_t - \hat{x}_t) \frac{d}{dt} (x_t - \hat{x}_t) + (y_t - \hat{y}_t) \frac{d}{dt} (y_t - \hat{y}_t) \\ &= \alpha \left\| \begin{pmatrix} x_t - \hat{x}_t \\ y_t - \hat{y}_t \end{pmatrix} \right\|^2 - R(x_t, \hat{x}_t, y_t, \hat{y}_t), \end{aligned}$$

where

$$R(x_t, y_t, \hat{x}_t, \hat{y}_t) := a(r_t^2 + \hat{r}_t^2 - (x_t \hat{x}_t + y_t \hat{y}_t)(r_t + \hat{r}_t)) + b(x_t \hat{y}_t - \hat{x}_t y_t)(r_t - \hat{r}_t)$$

with $r_t := x_t^2 + y_t^2$ and $\hat{r}_t := \hat{x}_t^2 + \hat{y}_t^2$. To show global uniform attractivity, it is sufficient to establish that $R(x_t, y_t, \hat{x}_t, \hat{y}_t) \geq 0$. From the inequality $(|xy| + |uv|)^2 \leq (x^2 + u^2)(y^2 + v^2)$, we derive that

$$\begin{aligned} &|(x_t \hat{x}_t + y_t \hat{y}_t)(r_t + \hat{r}_t)| + |(x_t \hat{y}_t - \hat{x}_t y_t)(r_t - \hat{r}_t)| \\ &\leq \sqrt{(x_t \hat{x}_t + y_t \hat{y}_t)^2 + (x_t \hat{y}_t - \hat{x}_t y_t)^2} \sqrt{(r_t + \hat{r}_t)^2 + (r_t - \hat{r}_t)^2} \\ &= \sqrt{2r_t \hat{r}_t (r_t^2 + \hat{r}_t^2)} \\ &\leq r_t^2 + \hat{r}_t^2. \end{aligned}$$

Together with the fact that $|b| \leq a$, this implies that $R(x_t, y_t, \hat{x}_t, \hat{y}_t) \geq 0$, which establishes global uniform attractivity for $\alpha < 0$.

We assume now that $\alpha > 0$. Suppose to the contrary that there exists $\delta > 0$ such that

$$\lim_{t \rightarrow \infty} \sup_{x \in B_\delta(0)} \text{ess sup}_{\omega \in \Omega} \|\varphi(t, \omega, A(\omega) + x) - A(\theta_t \omega)\| = 0.$$

This implies that there exists an $N \in \mathbb{N}$ such that for all $t > N$, we have

$$\sup_{x \in B_\delta(0)} \text{ess sup}_{\omega \in \Omega} \|\varphi(t, \omega, A(\omega) + x) - A(\theta_t \omega)\| < \frac{1}{4} \sqrt{\frac{\alpha}{a}}.$$

Due to Proposition 5.1, there exists a positive measure set $E_0 \in \mathcal{F}_{-\infty}^0$ such that $A(\omega) \in B_{\delta/4}(0)$ for all $\omega \in E_0$. Let $\phi(\cdot, x_0)$ denote the solution of the deterministic equation (1.1) for $\sigma = 0$ with initial condition $x(0) = x_0$. Then there exists a $T > N$ such that

$$\|\phi(T, (\pm \frac{1}{4}\delta, 0))\| > \frac{1}{2} \sqrt{\frac{\alpha}{a}},$$

and at the same time

$$\|\phi(T, (\frac{1}{4}\delta, 0)) - \phi(T, (-\frac{1}{4}\delta, 0))\| > \sqrt{\frac{\alpha}{a}}.$$

Recall from the proof of Proposition 5.1 that $\omega \mapsto \varphi(\cdot, \omega, x)$ is continuous from C_0^T to $C([0, T], \mathbb{R}^2)$ with respect to the supremum norm. This implies that there exists an $\varepsilon > 0$ such that for all $\omega \in E_\varepsilon = \{\omega \in \Omega : \sup_{t \in [0, T]} \|\omega(t)\| < \varepsilon\} \in \mathcal{F}_0^T$, we obtain

$$\|\phi(T, (\frac{1}{4}\delta, 0)) - \varphi(T, \omega, (\frac{1}{4}\delta, 0))\| < \frac{1}{4} \sqrt{\frac{\alpha}{a}} \quad \text{and} \quad \|\phi(T, (-\frac{1}{4}\delta, 0)) - \varphi(T, \omega, (-\frac{1}{4}\delta, 0))\| < \frac{1}{4} \sqrt{\frac{\alpha}{a}}.$$

This implies that

$$\|\varphi(T, \omega, (\frac{1}{4}\delta, 0)) - \varphi(T, \omega, (-\frac{1}{4}\delta, 0))\| > \frac{1}{2} \sqrt{\frac{\alpha}{a}}.$$

Since E_ε and E_0 are independent sets of positive measure, we get that $\mathbb{P}(E) > 0$ where $E = E_\varepsilon \cap E_0$. However, for all $\omega \in E$, we conclude

$$\begin{aligned} & \sup_{x \in B_\delta(0)} \|\varphi(t, \omega, A(\omega) + x) - A(\theta_t \omega)\| \\ & \geq \max \left\{ \left\| \varphi(t, \omega, (\frac{1}{4}\delta, 0)) - A(\theta_t \omega) \right\|, \left\| \varphi(t, \omega, (-\frac{1}{4}\delta, 0)) - A(\theta_t \omega) \right\| \right\} > \frac{1}{4} \sqrt{\frac{\alpha}{a}}, \end{aligned}$$

which contradicts our assumption. \square

We show now that this loss of uniform attractivity at the deterministic bifurcation point is associated with a change of sign in the dichotomy spectrum.

Theorem 5.3. *Consider the stochastic differential equation (1.1) such that there exists a unique attracting random equilibrium $A : \Omega \rightarrow \mathbb{R}^2$ (see Proposition 4.3 and Theorem C). Let $\Phi(t, \omega) := D\varphi(t, \omega, A(\omega))$ denote the linearized random system along the random equilibrium. Then for $|b| < a$ and $\alpha \in \mathbb{R}$ small enough such that the random equilibrium $A : \Omega \rightarrow \mathbb{R}^2$ exists, the dichotomy spectrum Σ of Φ is given by*

$$\Sigma = [-\infty, \alpha].$$

Proof. Recall from Proposition 4.1 that we have

$$\|D\varphi(t, \omega, x)\| \leq \exp \left(\int_0^t \lambda^+(\varphi(s, \omega, x)) ds \right).$$

Since Lemma 4.4 implies that $\lambda^+(x) \leq \alpha - (a - |b|)\|x\|^2$, we have

$$\|\Phi(t, \omega)\| \leq \exp\left(\int_0^t (\alpha - (a - |b|)\|A(\theta_s \omega)\|^2) ds\right). \quad (5.1)$$

Similarly, with $\lambda^-(x) := \min_{\|r\|=1} \langle Df(x)r, r \rangle$, we have

$$\|\mathbb{D}\varphi(t, \omega, x)\| \geq \exp\left(\int_0^t \lambda^-(\varphi(s, \omega, x)) ds\right).$$

It is easy to see that $\lambda^-(x) \geq \alpha - 4a\|x\|^2$, which implies

$$\|\Phi(t, \omega)\| \geq \exp\left(\int_0^t (\alpha - 4a\|A(\theta_s \omega)\|^2) ds\right). \quad (5.2)$$

From (5.1) we can deduce immediately that for almost all $\omega \in \Omega$, we have

$$\|\Phi(t, \omega)\| \leq e^{\alpha|t|} \quad \text{for all } t \in \mathbb{R}.$$

This implies that $\Sigma \subset (-\infty, \alpha]$.

We now show that $(-\infty, \alpha] \subset \Sigma$. Choose $\gamma \in (-\infty, \alpha]$, and suppose to the contrary that Φ admits an exponential dichotomy with growth rate γ with an invariant projector P_γ and constants $K, \varepsilon > 0$. We consider the following three cases (note that the rank of the invariant projector $\text{rk } P_\gamma$ does not depend on ω , see [8]):

(i) $P_\gamma \equiv \text{id}$. This means that for almost all $\omega \in \Omega$,

$$\|\Phi(t, \omega)\| \leq Ke^{(\gamma - \varepsilon)t} \quad \text{for all } t \geq 0.$$

Fix $T > 0$ such that $e^{\frac{1}{5}\varepsilon T} > K$. According to Proposition 5.1, there exists a positive measure set E such that for all $\omega \in E$ and $s \in [0, T]$, we have $A(\theta_s \omega) \in B_{\sqrt{\varepsilon/(5a)}}(0)$. We derive from (5.2) that for such $\omega \in E$, we have

$$\|\Phi(T, \omega)\| \geq e^{T(\alpha - \frac{4}{5}\varepsilon)} \geq Ke^{(\gamma - \varepsilon)T}.$$

This contradicts the assumption.

(ii) $\text{rk } P_\gamma \equiv 1$. The argument is the same as in the previous case, since our estimates do not depend on the tangent vector v , but hold for the norm $\|\Phi(t, \omega)\|$.

(iii) $P_\gamma \equiv 0$. This means that for almost all $\omega \in \Omega$, we have

$$\|\Phi(t, \omega)\| \geq \frac{1}{K}e^{(\gamma + \varepsilon)t} \quad \text{for all } t \geq 0.$$

Together with (5.1), this implies that

$$\frac{\ln K + (\alpha - \varepsilon - \gamma)t}{a - |b|} \geq \int_0^t \|A(\theta_s \omega)\|^2 ds.$$

Choose some $T > 1$ and $y \in \mathbb{R}^2$ such that

$$\|y\|^2 > 4 \max\left\{\frac{\ln K}{a - |b|}, \frac{\alpha - \varepsilon - \gamma}{a - |b|}\right\}.$$

Take $\delta < \frac{\|y\|}{2}$. Then by Proposition 5.1, there exists a set $E \in \mathcal{F}_{-\infty}^T$ such that

$$A(\theta_s \omega) \in B_\delta(y) \quad \text{for all } s \in [0, T] \text{ and } \omega \in E.$$

This implies

$$\int_0^t \|A(\theta_s \omega)\|^2 ds > T \frac{\|y\|^2}{4} > \frac{\ln K + (\alpha - \varepsilon - \gamma)T}{a - |b|},$$

which is a contradiction.

This finishes the proof of this theorem. \square

We demonstrate now that the change of sign in the dichotomy spectrum is mirrored by finite-time properties of the system. To see this, consider a compact time interval $[0, T]$ and the corresponding *finite-time largest Lyapunov exponents* associated with the attracting random equilibrium $A : \Omega \rightarrow \mathbb{R}^2$, given by

$$\lambda^{T, \omega} := \sup_{\|v\|=1} \frac{1}{T} \ln \|\Phi(T, \omega)v\| \quad \text{for all } \omega \in \Omega.$$

From Proposition 4.3 (iii), we obviously have $\lambda_{\text{top}} = \lim_{T \rightarrow \infty} \lambda^{T, \omega}$ almost surely, where λ_{top} is the largest Lyapunov exponent of (1.1) as given in (2.6).

Proposition 5.4. *Consider the stochastic differential equation (1.1) with $|b| < a$ and $\alpha \in \mathbb{R}$ such that there exists a unique attracting random equilibrium $A : \Omega \rightarrow \mathbb{R}^2$. The following statements hold.*

- (i) *For $\alpha < 0$, we have $\lambda^{T, \omega} \leq \alpha < 0$ for all $\omega \in \Omega$, which means that the random attractor $A : \Omega \rightarrow \mathbb{R}^2$ is finite-time attracting.*
- (ii) *For $\alpha > 0$, we have $\mathbb{P}(\omega \in \Omega : \lambda^{T, \omega} > 0) > 0$, which means that the random attractor $A : \Omega \rightarrow \mathbb{R}^2$ is not finite-time attracting.*

Proof. (i) Recall from (5.1) that

$$\|\Phi(t, \omega)\| \leq \exp \left(\int_0^t (\alpha - (a - |b|)\|A(\theta_s \omega)\|^2) ds \right),$$

which implies that

$$\lambda^{T, \omega} \leq \frac{1}{T} \int_0^T (\alpha - (a - |b|)\|A(\theta_s \omega)\|^2) ds \leq \alpha < 0.$$

(ii) Recall from (5.2) that

$$\|\Phi(t, \omega)\| \geq \exp \left(\int_0^t (\alpha - 4a\|A(\theta_s \omega)\|^2) ds \right).$$

Choose $\varepsilon := \sqrt{\frac{\alpha}{5a}} > 0$. According to Proposition 5.1, there exists a set $E \in \mathcal{F}_{-\infty}^T$ of positive measure such that $A(\theta_s \omega) \in B_\varepsilon(0)$ for all $s \in [0, T]$ and $\omega \in E$. Then

$$\lambda^{T, \omega} \geq \frac{1}{T} \int_0^T (\alpha - 4a\|A(\theta_s \omega)\|^2) ds \geq \alpha - \frac{4\alpha}{5} = \frac{\alpha}{5} > 0 \quad \text{for all } \omega \in E.$$

This shows the claim. \square

Proof of Theorem E. The claims follow from Theorem 5.2, Theorem 5.3 and Proposition 5.4. \square

The proofs of Theorem 5.3 and Proposition 5.4 explain in detail how the change of finite-time attractivity is connected to the loss of hyperbolicity in the dichotomy spectrum. Due to [8, Theorem 4.5], we obtain

$$\lim_{T \rightarrow \infty} \operatorname{ess\,sup}_{\omega \in \Omega} \lambda^{T, \omega} = \lim_{T \rightarrow \infty} \operatorname{ess\,sup}_{\omega \in \Omega} \sup_{\|v\|=1} \frac{1}{T} \ln \|\Phi(T, \omega)v\| = \sup \Sigma. \quad (5.3)$$

A similar statement holds for the infimum of the dichotomy spectrum. This means that the finite-time Lyapunov exponents are, at least asymptotically, supported on the dichotomy spectrum, and having positive values in the spectrum implies that, at least asymptotically, we can observe positive finite-time Lyapunov exponents.

5.2 Shear intensity as bifurcation parameter

We now do not assume the existence of an attracting random equilibrium, and we aim at proving Theorem F in this subsection. We first show a statement that corresponds to Proposition 5.1 in this more general context.

Proposition 5.5. *Let (θ, φ) be the random dynamical system generated by (1.1), and let $x, y \in \mathbb{R}^2$, $\varepsilon > 0$ and $T > 0$. Then for any $t_0 \in (0, T]$, there exists a set $E \in \mathcal{F}$ with $\mathbb{P}(E) > 0$ such that*

$$\varphi(s, \omega, x) \in B_\varepsilon(y) \quad \text{for all } s \in [t_0, T] \text{ and } \omega \in E.$$

Proof. Similarly as in the proof of [15, Proposition 3.10], fix $t_0 \in (0, T]$ and define

$$\psi(t) := x + \frac{t}{t_0}(y - x) \quad \text{for all } t \in [0, t_0],$$

and

$$h(t) := \frac{1}{\sigma} \left(\psi(t) - x - \int_0^t f(\psi(s)) \, ds \right) \quad \text{for all } t \in [0, T],$$

where f denotes the vector field of the drift in (1.1). As in the proof of Proposition 5.1, we write $\varphi(t, g, z)$ for the solution of (1.1) with initial condition z and path $g \in C_0^T$. We can infer that $\varphi(t, h, x) = \psi(t)$ for all $t \in [0, t_0]$, and in particular $\varphi(t_0, h, x) = y$. Recall that the map $g \mapsto \varphi(\cdot, g, z)$ is continuous from C_0^T to $C([0, t_0], \mathbb{R}^2)$ with respect to the supremum norm $\|\cdot\|_\infty$. This implies that there exists a $\delta > 0$ such that for all $g \in C_\delta := \{b \in C_0^T : \|b - h\| \leq \delta\}$, we have

$$\|\varphi(t, g, x) - \varphi(t, h, x)\| < \frac{1}{2}\varepsilon \quad \text{for all } t \in [0, t_0].$$

Hence, we have established that there is a positive measure set $E_1 := \{\omega \in \Omega : \omega|_{[0, t_0]} \in C_\delta\}$ such that for all $\omega \in E_1$, we have $\varphi(t_0, \omega, x) \in B_{\varepsilon/2}(y)$.

Similar to this argument, one can construct a set E_2 of positive measure that is independent from E_1 (by the Markov property) such that for all $\omega \in E := E_1 \cap E_2$, we have

$$\varphi(t, \omega, x) \in B_\varepsilon(y) \quad \text{for all } t \in [t_0, T].$$

This finishes the proof of this proposition. \square

Proof of Theorem F. For $(\omega, z) \in \Omega \times \mathbb{R}^2$ and $\alpha \in \mathbb{R}$, the linear random dynamical system $t \mapsto \Phi(t, \omega, z)$ is solution of the variational equation

$$\frac{d}{dt} \Phi(t, \omega, z) = Df(\varphi(t, \omega, z)) \Phi(t, \omega, z), \quad \text{where } \Phi(0, \omega, z) = \text{Id}.$$

Define $s_t(\omega, z, v) := \frac{\Phi(t, \omega, z)v}{\|\Phi(t, \omega, z)v\|}$ and observe that for $v \in \mathbb{R}^2 \setminus \{0\}$,

$$\begin{aligned} \frac{d}{dt} \|\Phi(t, \omega, z)v\|^2 &= 2 \langle Df(\varphi(t, \omega, z)) \Phi(t, \omega, z)v, \Phi(t, \omega, z)v \rangle \\ &= 2 \langle Df(\varphi_t(\omega, z)) s_t(\omega, z, v), s_t(\omega, z, v) \rangle \|\Phi(t, \omega, z)v\|^2. \end{aligned}$$

Let $\mu > 0$, and let $z' = (w, w) \in \mathbb{R}^2$ be such that $\frac{b-2a}{2} \|z'\|^2 = (b-2a)w^2 \geq \mu$ and $w > 1$. Note that

$$Df(x, y) = \begin{pmatrix} \alpha - ay^2 - 3ax^2 - 2byx & -\beta - 2axy - bx^2 - 3by^2 \\ \beta - 2axy + by^2 + 3bx^2 & \alpha - ax^2 - 3ay^2 + 2byx \end{pmatrix}.$$

With $\tilde{r} = (0, 1)$, we get

$$\langle Df(z') \tilde{r}, \tilde{r} \rangle = \alpha + 2(b-2a)w^2 \geq \alpha + 2\mu.$$

Let

$$\varepsilon = \min \left\{ 1, \frac{1}{16} \frac{b-2a}{bw}, \frac{\sqrt{b-2a}}{4a} \right\} \quad \text{and} \quad \delta = \frac{1}{8} \frac{b-2a}{4b}.$$

Then by Proposition 5.5, there is a positive measure set $E_1 \subset \Omega$ such that for all $\omega \in E_1$

$$\varphi(t, \omega, z') \in B_\varepsilon(z') \quad \text{for all } t \in [0, 1].$$

This implies that the coefficients of $Df(\varphi(t, \omega, z'))$ are bounded uniformly in $\omega \in E_1$ for $t \in [0, 1]$. Because Φ is continuous, there is a $T \in (0, 1]$ such that

$$\|s_t(\omega, z', \tilde{r}) - \tilde{r}\| = \left\| \frac{\Phi(t, \omega, z') \tilde{r}}{\|\Phi(t, \omega, z') \tilde{r}\|} - \tilde{r} \right\| < \delta \quad \text{for all } t \in [0, T] \text{ and } \omega \in E_1$$

Note that we obtain for any $r \in \mathbb{R}^2$ with $\|r\| = 1$

$$\begin{aligned} \langle Df(x, y)r, r \rangle &= r_1^2(\alpha - ay^2 - 3ax^2) + r_1r_2(-\beta - 2axy) + r_1r_2(\beta - 2axy) + r_2^2(\alpha - ax^2 - 3ay^2) \\ &\quad - 2byxr_1^2 + 2byxr_2^2 + r_1r_2(2bx^2 - 2by^2) \\ &= \alpha - a(x^2 + y^2) + 2b(r_1r_2x^2 - r_1r_2y^2 + yx(r_2^2 - r_1^2)) - 2a(r_1x + r_2y)^2. \end{aligned}$$

This means that for all $t \in (0, T]$ and $\omega \in E_1$, we have by the choice of ε and δ above that

$$\begin{aligned} &\langle Df(\varphi_t(\omega, z)) s_t(\omega, z, \tilde{r}), s_t(\omega, z, \tilde{r}) \rangle \\ &\geq \alpha - 2a(w + \varepsilon)^2 + 2b(w - \varepsilon)^2(1 - 2\delta) - 2b\delta[(w + \varepsilon)^2 - (w - \varepsilon)^2] - 2a(w + \varepsilon)^2 \\ &= \alpha + (b-2a)w^2 + ((b-2a)w^2 - 4a(2w\varepsilon + \varepsilon^2) - 4bw\varepsilon - \delta 4b(w - \varepsilon)^2 - 2\delta 4bw\varepsilon) \\ &\geq \alpha + \mu. \end{aligned}$$

Hence, we get that for all $\omega \in E_1$ and $t \in (0, T]$, the finite-time largest Lyapunov exponent of trajectories starting in z' satisfies

$$\lambda^{t, \omega, z'} := \sup_{\|v\|=1} \frac{1}{t} \ln \|\Phi(t, \omega, z')v\| \geq \alpha + \mu.$$

Since $\mu > 0$ was arbitrary, we obtain with positive probability arbitrarily large finite-time Lyapunov exponents when starting in z' .

We now show that for any $z \in \mathbb{R}^2$ and $t_0 \in (0, T]$, the finite-time largest Lyapunov exponent $\lambda^{t, \omega, z}$, $t \in [t_0, T]$, can be arbitrarily large for ω from a set of positive measure. By Proposition 5.5, there exists a set $E_2 \in \mathcal{F}$ with $\mathbb{P}(E_2) > 0$ such that

$$\varphi(s, \omega, z) \in B_\varepsilon(z') \quad \text{for all } s \in [t_0, T] \text{ and } \omega \in E_2,$$

where the values of $\varphi(t, \omega, z)$, $t \in [0, t_0]$, stay close to the line between z and z' (see proof of Proposition 5.5). Since t_0 can be chosen arbitrarily small and the solutions stay in a compact set for $t \in [0, t_0]$, we obtain with similar arguments as before that with positive probability there are arbitrarily large finite-time Lyapunov exponents.

Let $\mu^- < 0$. Then by choosing $z'' = (w, -w)$, we obtain with similar arguments as above that for some $T \in (0, 1]$

$$\inf_{\|v\|=1} \frac{1}{t} \ln \|\Phi(t, \omega, z'')v\| \leq \alpha + \mu^- \quad \text{for all } t \in [0, T] \text{ and } \omega \text{ from a set of positive probability.}$$

By using Proposition 5.5 again, we can then deduce that with positive probability, there are arbitrarily small finite-time Lyapunov exponents for any initial conditions. \square

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Appendix

A Lyapunov spectrum

A random dynamical system (θ, φ) is called *linear* if the map $\varphi(t, \omega) : \mathbb{R}^d \rightarrow \mathbb{R}^d$, $x \mapsto \varphi(t, \omega, x)$, is linear for any $(t, \omega) \in \mathbb{R} \times \Omega$. Define $\Phi : \mathbb{R} \times \Omega \rightarrow \mathbb{R}^{d \times d}$ by $\Phi(t, \omega)x := \varphi(t, \omega, x)$. Suppose that Φ

satisfies the integrability condition

$$\sup_{0 \leq t \leq 1} \ln^+ \|\Phi(t, \omega)\| \in L^1(\mathbb{P}),$$

where $\ln^+(x) := \max\{\ln(x), 0\}$. Then the Multiplicative Ergodic Theorem [2, Theorem 3.4.1] guarantees the existence of a θ -forward invariant set $\widehat{\Omega} \subset \Omega$ with $\mathbb{P}(\widehat{\Omega}) = 1$, the Lyapunov exponents $\lambda_1 > \dots > \lambda_p$, and an invariant measurable filtration

$$\mathbb{R}^d = V_1(\omega) \supsetneq V_2(\omega) \supsetneq \dots \supsetneq V_p(\omega) \supsetneq V_{p+1}(\omega) = \{0\},$$

such that for all $0 \neq x \in \mathbb{R}^d$, the *Lyapunov exponent* $\lambda(\omega, x)$, defined by

$$\lambda(\omega, x) = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \|\Phi(t, \omega)x\|$$

exists, and we have

$$\lambda(\omega, x) = \lambda_i \iff x \in V_i(\omega) \setminus V_{i+1}(\omega) \quad \text{for all } i \in \{1, \dots, p\}.$$

B Random attractors

A random variable $R : \Omega \rightarrow \mathbb{R}$ is called *tempered* if

$$\lim_{t \rightarrow \pm\infty} \frac{1}{|t|} \ln^+ R(\theta_t \omega) = 0 \quad \text{for almost all } \omega \in \Omega,$$

see also [2, p. 164]. A set $D \in \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d)$ is called *tempered* if there exists a tempered random variable R such that

$$D(\omega) \subset B_{R(\omega)}(0) \quad \text{for almost all } \omega \in \Omega,$$

where $D(\omega) := \{x \in \mathbb{R}^d : (\omega, x) \in D\}$. D is called *compact* if $D(\omega) \subset \mathbb{R}^d$ is compact for almost all $\omega \in \Omega$. Denote by \mathcal{D} the set of all compact tempered sets $D \in \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d)$. We now define the notion of a random attractor with respect to \mathcal{D} , see also [21, Definition 14.3].

Definition B.1 (Random attractor). A set $A \in \mathcal{D}$ is called a *random attractor* (with respect to \mathcal{D}) if the following two properties are satisfied.

- (i) A is φ -invariant, i.e.

$$\varphi(t, \omega)A(\omega) = A(\theta_t \omega) \quad \text{for all } t \geq 0 \text{ and almost all } \omega \in \Omega.$$

- (ii) For all $D \in \mathcal{D}$, we have

$$\lim_{t \rightarrow \infty} \text{dist}(\varphi(t, \theta_{-t} \omega)D(\theta_{-t} \omega), A(\omega)) = 0 \quad \text{for almost all } \omega \in \Omega,$$

$$\text{where } \text{dist}(E, F) := \sup_{x \in E} \inf_{y \in F} \|x - y\|.$$

Note that we require that the random attractor is measurable with respect to $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d)$, in contrast to a weaker statement normally used in the literature (see also [11, Remark 4]).

The existence of random attractors is proved via so-called absorbing sets. A set $B \in \mathcal{D}$ is called an *absorbing set* if for almost all $\omega \in \Omega$ and any $D \in \mathcal{D}$, there exists a $T > 0$ such that

$$\varphi(t, \theta_{-t} \omega)D(\theta_{-t} \omega) \subset B(\omega) \quad \text{for all } t \geq T.$$

A proof of the following theorem can be found in [16, Theorem 3.5].

Theorem B.2 (Existence of random attractors). *Suppose that (θ, φ) is a continuous random dynamical system with an absorbing set B . Then there exists a unique random attractor A , given by*

$$A(\omega) := \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} \varphi(t, \theta_{-t}\omega) B(\theta_{-t}\omega)} \quad \text{for almost all } \omega \in \Omega.$$

Furthermore, $\omega \mapsto A(\omega)$ is measurable with respect to $\mathcal{F}_{-\infty}^0$, i.e. the past of the system.